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Quantum Speed Limit and Divisibility of the Dynamical Map

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Abstract: The quantum speed limit (QSL) is the theoretical lower limit of the time for a quantum system to evolve from a given state to another one. Interestingly, it has been shown that non-Markovianity can be used to speed-up the dynamics and to lower the QSL time, although this behaviour is not universal. In this paper, we further carry on the investigation on the connection between QSL and non-Markovianity by looking at the effects of P- and CP-divisibility of the dynamical map to the quantum speed limit. We show that the speed-up can also be observed under P- and CP-divisible dynamics, and that the speed-up is not necessarily tied to the transition from P-divisible to non-P-divisible dynamics.

Keywords: quantum speed limit; open quantum system; dynamical map

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1. Introduction

The quantum speed limit (QSL) is the theoretical lower bound to the time that is needed for a state to be transformed into another. The concept of QSL was first introduced in [1] as a lower time limit of the evolution between two orthogonal pure states for the harmonic oscillator and it ss shown to be bounded by the variance of energy $\tau_{MT} \geq h/4\Delta E$. This initial perspective was then further developed and connected to the maximal rates of computations for a quantum computer in [2]. In that paper, it was concluded that the minimum interaction time is bounded by the average energy as $\tau_{ML} \geq h/4E$. It can be shown that the two bounds are not ordered and the actual QSL should be the maximum of the two bounds. Since then, the study of QSL has been extended to include mixed states [3] and more general dynamics [4–8].

More recently, the study of the quantum speed limit has gained renewed interest after discovering that it can be lowered by means of memory effects, thus theoretically speeding up the process. Specifically, in [4], it was shown that the quantum speed limit is lowered under certain non-Markovian dynamics in an open qubit system. This result was then experimentally confirmed in [9]. A more thorough analysis on the role of non-Markovianity was performed in [10], where it was shown that its connection with QSL is not as straightforward and the speed-up can be present, even when the dynamics is Markovian.

In this paper, we deepen our investigation by considering other aspects of non-Markovianity, specifically the lack of P-divisibility and CP-divisibility of dynamics. We show that the speed-up, which was previously widely credited to information backflow, as defined in [11], can also be observed with P-divisible and even with CP-divisible dynamics. As a paradigmatic example of dynamics, we consider the phase-covariant master equation, since it includes well-known maps, such as amplitude damping and pure dephasing. The conditions for P-divisibility of the phase-covariant master equation were recently studied in [12]. We consider a specific phase-covariant model that can describe the crossover between P-divisible and non-P-divisible dynamics by tuning a certain parameter.

The paper is structured, as follows. In Section 2, we recall the basic definitions and concepts that were used in this paper, and present the dynamics of the example systems that we used. In Sections 3 and 4, we present the results for the QSL of CP- and P-divisible dynamics. Finally, Section 5 summarises the results and presents conclusions.

2. Open Quantum Systems, Dynamical Maps, Divisibility, and QSL

In textbooks, many elementary examples of a quantum system are of idealised closed system. However, in reality, every quantum system is interacting with its environment, whick makes it an open quantum system. When we study an open quantum system, we are usually interested in the reduced dynamics of the smaller system, for example, a qubit, rather than the environment.

A quantum dynamical map Φ_t is a map describing the time evolution of a quantum system, which is $\rho(t) = \Phi_t(\rho(0))$, where $\rho(t)$ is a time dependent density matrix. In an open quantum system with the system of interest (S) and the environment (E), the reduced dynamics of the system is given by $\rho_S(t) = \Phi_t(\rho_s(0)) = \mathrm{tr}_E[U^\dagger_{SE}\rho_S(t) \otimes \rho_E(0)U_{SE}]$, where U_{SE} is a unitary operator describing the time evolution of the total system, with $\rho_S(0)$ and $\rho_E(0)$ being the system and environment states at $t = 0$, respectively.

A dynamical map Φ_t is said to be k-positive if the the map $\Phi_t \otimes \mathbb{I}_k$, where $\mathbb{I}_k$ is the identity operator for a k-dimensional ancillary Hilbert space, is positive. If a map is positive for all k, it is called completely positive (CP) and, if a map is 1-positive, it is called positive (P). A dynamical map is called P- or CP- divisible, if the map can be written using a positive or completely positive intermediate map $V_{s,t}$, s.t. $\Phi_t = V_{s,t}\Phi_s$, for $0 \geq s \geq t$.

The explicit dynamics that are considered in this paper arise from a class of master equations in the time-local GKSL form:

$$\frac{d\rho_S(t)}{dt} = L_t(\rho_S(t)) = \frac{i}{\hbar}[\rho_S(t), H(t)] + \sum_i \gamma_i(t)\left(A_i\rho_s(t)A_i^\dagger - \frac{1}{2}\left\{A_i^\dagger A_i, \rho_S(t)\right\}\right), \tag{1}$$

where H is the system Hamiltonian, $\gamma_i(t)$ the time-dependent decay rates, and A_i the Lindblad operators. The GKSL-theorem implies that, for master equations in the form of Equation (1), with $\gamma_i(t) \geq 0$, the resulting dynamics is always completely positive and trace preserving (CPTP) and, thus, always physical [13–15]. One should keep in mind that, in the framework of a microscopic description of system plus environment, the GKSL master equation is the result of a number of approximations. When these approximations do not hold, this master equation fails to grasp some—possibly relevant—features of the studied dynamics. Our examples come from the family of so-called phase-covariant master equations [16–19]:

$$\begin{aligned} L_t(\rho(t)) = {} & i\omega(t)[\rho(t), \sigma_3] + \frac{\gamma_1(t)}{2}\left(\sigma_+\rho(t)\sigma_- - \frac{1}{2}\{\sigma_-\sigma_+, \rho(t)\}\right) \\ & + \frac{\gamma_2(t)}{2}\left(\sigma_-\rho(t)\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho(t)\}\right) + \frac{\gamma_3(t)}{2}(\sigma_3\rho(t)\sigma_3 - \rho_t), \end{aligned} \tag{2}$$

where σ_1, σ_2 and σ_3 are the Pauli x, y, and z matrices, respectively, with $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$, and $\gamma_1(t)$, $\gamma_2(t)$, and $\gamma_3(t)$ being the heating, dissipation, and dephasing rates, respectively. This class of master equations contains some widely used models, such as amplitude damping and pure dephasing [16,18,19].

In this paper, we use the definitions of the QSL for open quantum systems, as defined in [4]:

$$\tau_{QSL} = \frac{1}{\Lambda_\tau^{\mathrm{op}}}\sin^2(\mathcal{L}(\rho(0), \rho(\tau))), \tag{3}$$

where $\mathcal{L}(\rho(0),\rho(\tau))$ is the Bures angle between the initial pure state $\rho(0) = |\Phi_0\rangle\langle\Phi_0|$ and the time evolved state $\rho(t)$, defined as

$$\mathcal{L}(\rho(0),\rho(\tau)) = \arccos(\sqrt{\langle\Phi_0|\rho(t)|\Phi_0\rangle})\,, \tag{4}$$

and

$$\Lambda_\tau^{\mathrm{op}} = \frac{1}{\tau}\int_0^\tau ||L_t(\rho(t))||_{op}\; dt\,, \tag{5}$$

where

$$||L_t(\rho(t))||_{\mathrm{op}} = \max_i\{s_i\}\,, \tag{6}$$

is the operator norm, with s_i being the singular values of $L_t(\rho(t))$.

In [4], it was shown that, for an amplitude damping system, as given by master Equation (2) with $\gamma_1(t) = \gamma_3(t) = 0$ and $\gamma_2(t) = \gamma(t)$, the QSL is directly dependent on the information backflow as

$$\tau_{QSL}/\tau = \frac{1-|b(\tau)|^2}{1-|b(\tau)|+\mathcal{N}}\,, \tag{7}$$

where $\Phi_t(|1\rangle\langle 1|) = |b(t)|^2|1\rangle\langle 1|$ and $\mathcal{N}$ is the Breuer–Laine–Piilo (BLP) non-Markovianity measure, as given by

$$\mathcal{N}(\Phi) = \int_{\partial_t|b(t)|^2>0} \partial_t|b(t)|^2 dt\,. \tag{8}$$

This connection was later studied in more detail, and it was found that the speed-up is not always dependent on the information backflow and can sometimes be present without any non-Markovian effects [10]. In this case, the presence of information backflow coincides with the loss of P-divisibility.

3. QSL for the Non-Monotonic Populations

In [12], the authors introduce an always-CP-divisible model with oscillations in the populations. This model can be written in the form of a master Equation (2), with

$$\gamma_1(t) = \nu + \frac{\nu}{\sqrt{4\nu^2+\omega^2}}\left(2\nu\sin(\omega t) + \omega\cos(\omega t)\right), \tag{9}$$

$$\gamma_2(t) = \nu - \frac{\nu}{\sqrt{4\nu^2+\omega^2}}\left(2\nu\sin(\omega t) + \omega\cos(\omega t)\right), \tag{10}$$

$$\gamma_3(t) = 0\,, \tag{11}$$

where $\nu, \omega \geq 0$. For simplicity, we use a general pure qubit state and parametrize our initial state as

$$\rho(0) = \begin{pmatrix} a & \sqrt{a}\sqrt{1-a} \\ \sqrt{a}\sqrt{1-a} & 1-a \end{pmatrix}, \tag{12}$$

where $a \in [0,1]$. We omit the phase parameter, since it does not affect the results in the phase-covariant case. The time-evolved density matrix is

$$\rho(t) = \begin{pmatrix} 1 - e^{\nu t}\left(1 - a + \frac{\nu}{16}f(\nu,\omega,t)\right) & \sqrt{a(a-1)}e^{-\nu t/2} \\ \sqrt{a(a-1)}e^{-\nu t/2} & e^{\nu t}\left(1 - a + \frac{\nu}{16}f(\nu,\omega,t)\right) \end{pmatrix}, \tag{13}$$

where

$$f(\nu,\omega,t) = -1 + e^{8t} + \frac{-16(\nu-4)\omega + 8e^{8t}\left(2(\nu-4)\omega\cos(\omega t) - (16\nu+\omega^2)\sin(\omega t)\right)}{(64+\omega^2)\sqrt{4\nu^2+\omega^2}}\,. \tag{14}$$

As an example, in Figure 1 we show the QSL as a function of the interaction time τ and of a, for some exemplary values of the parameters ν and ω. We see that the QSL oscillates wildly and it is almost always below $\tau_{QSL}/\tau = 1$. Figure 2 shows the state dynamics of this model, as well as the fidelity between $\rho(0)$ and $\rho(t)$ and the QSL for $a = 1$.

Note that the oscillations and the speed-up in QSL are connected to the oscillations of the fidelity (defined as $F(\rho(0),\rho(t)) = \mathrm{Tr}\left[\sqrt{\sqrt{\rho(t)}\rho(0)\sqrt{\rho(t)}}\right]^2$), even in the absence of non-Markovian effects. Indeed, this example shows that, when fidelity increases, the QSL also decreases.

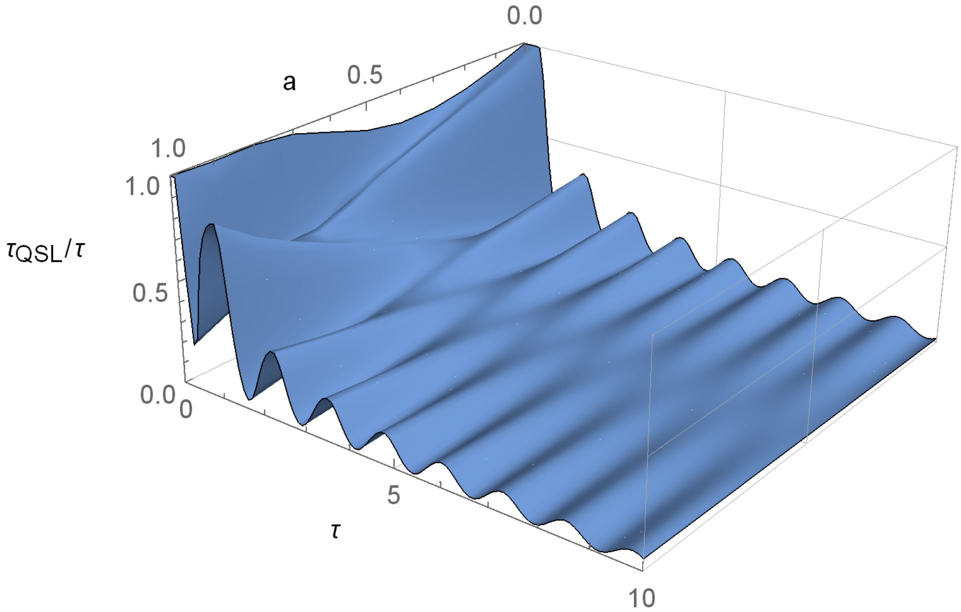

Figure 1. The quantum speed limit (QSL) for the phase-covariant system defined in Equations (9)–(11) for $\nu = 8$ and $\omega = 5$. This system is completely positive (CP)-divisible at all times, but clearly there is significant change in τ_{QSL}/τ for all pure initial states of the form of Equation (12).

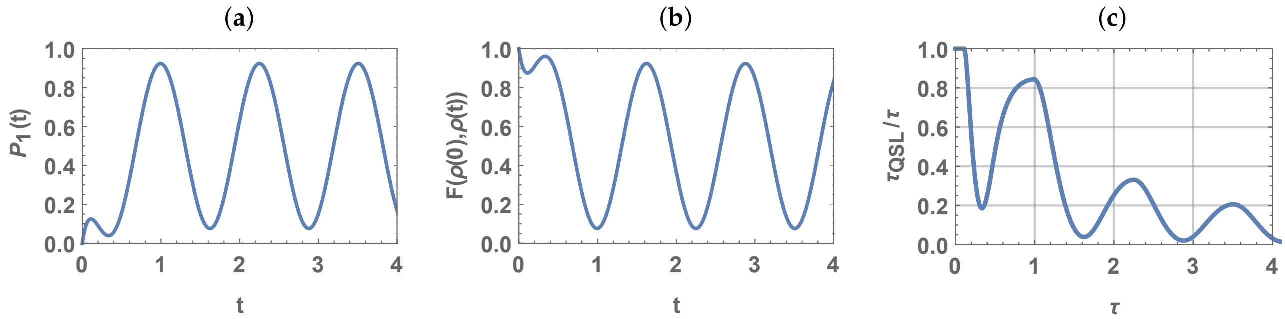

Figure 2. (**a**) The probability for the qubit to be in the excited state of the model used in Figure 1 for initial ground state ($a = 1$), (**b**) the fidelity between $\rho(0)$ and $\rho(t)$, and (**c**) the QSL. The populations undergo oscillations, which results in oscillations in fidelity as well as in QSL. The coherences always remain equal to their initial zero value.

4. P-Divisibility of the Phase-Covariant System

The P-divisibility of this system was studied in [12]. The requirement for P-divisibility is

$$\gamma_{1,2}(t) \geq 0\,, \tag{15}$$

$$\sqrt{\gamma_1(t)\gamma_2(t)} + 2\gamma_3(t) > 0\,, \tag{16}$$

where $\gamma_{1,2,3}(t)$ are the decay rates from the master Equation (2). For unital phase-covariant dynamics, which is when $\gamma_1(t) = \gamma_2(t)$, these are equivalent to the BLP non-Markovianity [16]. In the borderline case $\sqrt{\gamma_1(t)\gamma_2(t)} + 2\gamma_3(t) = 0$, a stricter rule

$$\frac{d\gamma_3(t)}{dt} > \gamma_3(t)\big(\gamma_1(t) + \gamma_2(t)\big)\,, \tag{17}$$

can be used to determine P-divisibility [12].

As an example, we can use the master Equation (2), with:

$$\gamma_1(t) = e^{-t/2}, \tag{18}$$
$$\gamma_2(t) = e^{-t/4}, \tag{19}$$
$$\gamma_3(t) = \frac{\kappa}{2} e^{-3t/8} \cos(2t) \qquad (k \geq 0), \tag{20}$$

which is P-divisible according to Equations (15) and (16) when $\kappa < 1$ and non-P-divisible when $\kappa \geq 1$, whcih is $\exists t \geq 0$ such that $\sqrt{\gamma_1(t)\gamma_2(t)} + 2\gamma_3(t) > 0$. Figure 3 shows the ratio τ_{QSL}/τ as a function of the initial state parameter a and the total interaction time τ for the P-divisible model of Equations (18)–(20) for $\kappa = 0.5$. When the ratio drops below $\tau_{QSL}/\tau = 1$, we know that the theoretical lower limit is lower than the chosen τ and it is possible to speed-up the evolution.

$\kappa = 0.5$, P-divisible

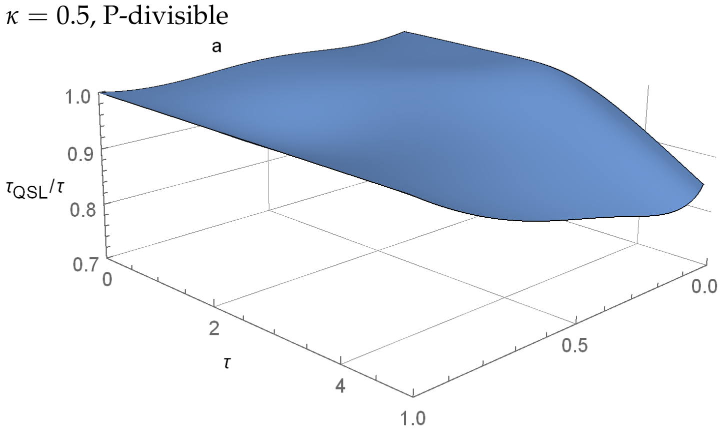

Figure 3. The QSL values for the initial states of (12) with $a \in [0, 1]$ and dynamics described by Equations (18)–(20), with $\kappa = 0.5$. Despite being P-divisible according to Equations (15) and (16), we see that the evolution is sped up from the so-called optimal $\tau_{QSL}/\tau = 1$ case for most values of a, similar to the results presented in [4] for non-Markovian dynamics. For $a = 1$, we have $\tau_{QSL}/\tau = 1$ for all values of τ.

Figure 4 shows the same plot with $\kappa = 1$, i.e., when the map is not P-divisible. We see a similar speedup as in Figure 3, with some amplified oscillations. However, the regions where $\tau_{QSL}/\tau = 1$ remains the same in both cases.

We can also break the P-divisibility by choosing $\gamma_1(t)$ and $\gamma_2(t)$, such that Equation (15) is violated, for example:

$$\gamma_1(t) = \gamma_2(t) = e^{-t/2}\big(\kappa + \cos(2t)\big) \tag{21}$$
$$\gamma_3(t) = e^{-3/8t}. \tag{22}$$

In this case, when $\kappa < 1$, $\exists t > 0$, such that $\gamma_{1,2}(t) < 0$, which implies non-P-divisible dynamics because of the violation of (15). However, in this case, the dynamics is non-Markovian and the previous results regarding non-Markovianity and quantum speed-up hold [4,16].

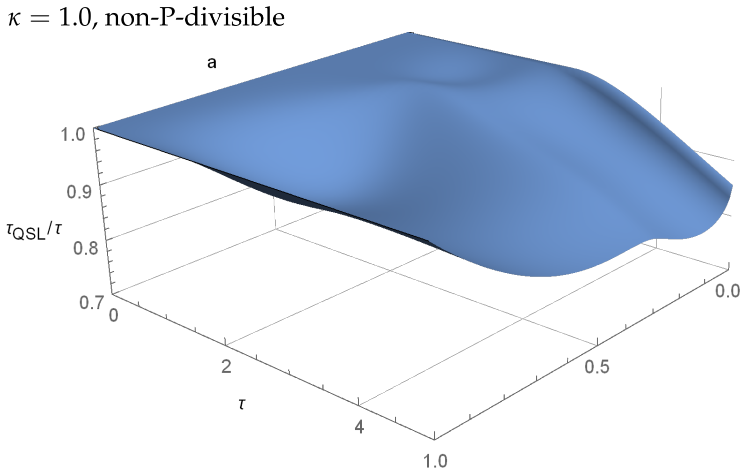

Figure 4. A similar plot as in Figure 3, but with $\kappa = 1$, making the model non-P-divisible. For $a = 1$ the ratio $\tau_{QSL}/\tau = 1$ for all τ, but for other values we can see similar speed-up effects as in Figure 3. All of the areas where $\tau_{QSL}/\tau = 1$ coincide with Figure 3, and changes can only be found when $\tau_{QSL}/\tau < 1$.

In general, for the model that is described by Equations (18)–(20), there is no significant connection between the P-divisbility or non-P-divisible dynamics and the optimality, or non-optimality of the evolution (see Figures 3 and 4 for reference). In both cases, there exists regions where $\tau_{QSL}/\tau = 1$ coincide, as well as the regions where $\tau_{QSL}/\tau < 1$. However, we can numerically find a slight difference between $\kappa = 1/2$ and $\kappa = 1$ for $a = 0.3$, where, for the P-divisible case $\tau_{QSL}/\tau = 1$, and for the non-P-divisible $\tau_{QSL}/\tau < 1$.

In the case of Equations (21) and (22), we see the speedup when κ is greater than the critical value. In Figure 5, we see the QSL as a function of a and τ for $\kappa = 0.5$ and $\kappa = 1.0$. For $a = 1$, we can clearly see that $\tau_{QSL}/\tau = 1$ in the $\kappa = 1$ case, while, for $\kappa < 1$, we have $\tau_{QSL}/\tau = 1$. In this case, the results are consistent with the previous result in [16], since, in this case, $\gamma(t) < 0$ implies BLP non-Markovian dynamics that has been studied and proved to speed up the evolution.

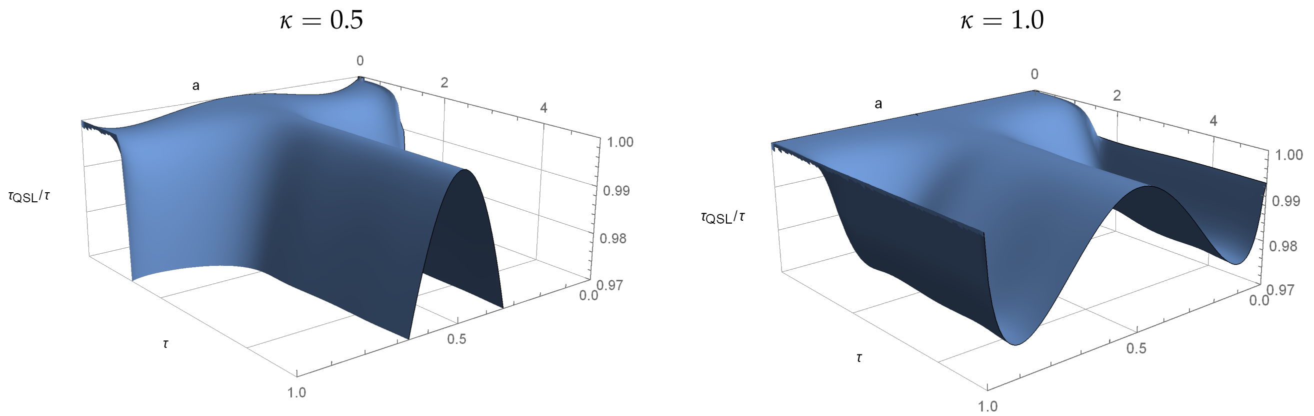

Figure 5. QSL for the dynamics given by Equations (21)–(22). We can see a clear difference for both $a = 0$ and $a = 1$. However, in this case we can explain this using the previous results, since the dynamics is clearly BLP non-Markovian in the left plot, which is when $\kappa = 0.5$, according to [16].

5. Discussion

In this paper, we have studied the quantum speed limit under different phase-covariant dynamics, with both P-divisible and non-P-divisible examples. We have observed that the speed-up effect, which is indicated by $\tau_{QSL}/\tau < 1$, can be seen with non-P-divisible, P-divisible, and even CP-divisible dynamics, further concluding that the speed-up is not simply linked to non-Markovian dynamics. Based on our results, the speed-up is not necessarily connected to non-P- or non-CP-divisible dynamics, and it is possibly linked to oscillations in the populations of a two-level system, which are often present in non-Markovian dynamics.

For the examples that are considered here, there seems to be no difference between P-divisible or non-P-divisible dynamics when considering optimal evolution, which is when $\tau_{QSL}/\tau = 1$. The value of the ratio τ_{QSL}/τ for the regions where $\tau_{QSL}/\tau < 1$ varies, depending on the choice of κ in our examples, but the regions with $\tau_{QSL}/\tau = 1$ are the same. Concluding, we have presented evidence that the speed-up is not generally the result of non-P-divisible dynamics. Moreover, for the model studied, the transition from P-divisible to non-P-divisible dynamics causes speed-up when the transition coincides with the transition between BLP Markovian and non-Markovian.

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