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Minimal Linear Codes Constructed from Sunflowers

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Abstract: Sunflower in coding theory is a class of important subspace codes and can be used to construct linear codes. In this paper, we study the minimality of linear codes over $\mathbb{F}_q$ constructed from sunflowers of size s in all cases. For any sunflower, the corresponding linear code is minimal if $s \geq q+1$, and not minimal if $2 \leq s \leq 3 \leq q$. In the case where $3 < s \leq q$, for some sunflowers, the corresponding linear codes are minimal, whereas for some other sunflowers, the corresponding linear codes are not minimal.

Keywords: linear code; minimal code; sunflower

MSC: 94B05; 94A62

1. Introduction

Let $\mathbb{F}_q$ be the finite field with q elements and $\mathbb{F}_q^n$ the vector space with dimension n over $\mathbb{F}_q$. For a vector $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{F}_q^n$, let $\mathrm{Suppt}(\mathbf{v}) := \{1 \leq i \leq n : v_i \neq 0\}$ be the support of $\mathbf{v}$. The *Hamming weight* of $\mathbf{v}$ is $\mathrm{wt}(\mathbf{v})$:=#$\mathrm{Suppt}(\mathbf{v})$. For any two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{F}_q^n$, if $\mathrm{Suppt}(\mathbf{u}) \subseteq \mathrm{Suppt}(\mathbf{v})$, we say that $\mathbf{v}$ covers $\mathbf{u}$ (or $\mathbf{u}$ is covered by $\mathbf{v}$) and write $\mathbf{u} \preceq \mathbf{v}$. Clearly, $a\mathbf{v} \preceq \mathbf{v}$ for all $a \in \mathbb{F}_q$.

An $[n, m]_q$ linear code $\mathcal{C}$ over $\mathbb{F}_q$ is an m-dimensional subspace of $\mathbb{F}_q^n$. A codeword $\mathbf{c}$ in a linear code $\mathcal{C}$ is called *minimal* if $\mathbf{c}$ covers only the codewords $a\mathbf{c}$ for all $a \in \mathbb{F}_q$, but no other codewords in $\mathcal{C}$. If every codeword in $\mathcal{C}$ is minimal, then $\mathcal{C}$ is said to be a *minimal linear code*. Minimal linear codes have interesting applications in secret sharing [1–5] and secure two-party computation [6,7], and could be decoded with a minimum distance decoding method [8].

Up to now, there are two approaches to studying minimal linear codes. One is the algebraic method and the other is the geometric method. The algebraic method is based on the Hamming weights of the codewords. In [8], Ashikhmin and Barg gave a sufficient condition for a linear code to be minimal. Many minimal linear codes satisfying the condition $\frac{w_{\min}}{w_{\max}} > \frac{q-1}{q}$ are obtained from linear codes with few weights; for example [9,10]. Cohen et al. [7] provided an example to show that the condition $\frac{w_{\min}}{w_{\max}} > \frac{q-1}{q}$ is not necessary for a linear code to be minimal. Ding, Heng, and Zhou [11,12] derived a sufficient and necessary condition on all Hamming weights for a given linear code to be minimal.

When using the algebraic method to prove the minimality of a given linear code, one needs to know all the Hamming weights in the code, which is very difficult in general. Even if all the Hamming weights are known, it is hard to use the algebraic method to prove the minimality. In this paper, we will use the geometric approaches to study the minimality of some linear codes. Based on the geometric approaches (see [13–15]) it is easier to construct minimal linear codes or to prove the minimality of some linear codes (see [16–21]).

Sunflower in coding theory is a class of important subspace codes and can be used to construct linear codes, see [22]. Let s be the number of the elements in a sunflower. In [23], (Theorem 10), the authors proved that if $s \geq p+1$, then the corresponding linear code over $\mathbb{F}_p$ is minimal, where p is a prime number.

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In this paper, we will use the approach used in [14] to consider the minimality of linear codes over $\mathbb{F}_q$ constructed from sunflowers for all s. We obtain the following three results: (1) when $s \geq q+1$, for any sunflower, the corresponding linear code is minimal; (2) when $2 \leq s \leq 3 \leq q$, for any sunflower, the corresponding linear code is not minimal; (3) when $3 < s \leq q$, for some sunflowers, the corresponding linear codes are minimal, wherea for some other sunflowers, the corresponding linear codes are not minimal.

This paper is organized as follows. In Section 2, we introduce some basic knowledge about sunflowers, Euclidean inner product, and minimal linear codes. In Section 3, we consider the linear codes constructed from sunflowers and discuss the minimality of these linear codes in three cases. In Section 4, we conclude this paper.

2. Preliminaries

2.1. Sunflower

Throughout this paper, let k and t_0 be two positive integers, $m = 2k + t_0$ and $l = k + t_0$. Let $2 \leq s \leq q^k + 1$ be a positive integer, $T_0 \leq \mathbb{F}_q^m$ be a subspace of $\mathbb{F}_q^m$, and $\dim T_0 = t_0$. We denote $\mathcal{G}_q(l, m)$ the set of l-dimensional vector subspaces of $\mathbb{F}_q^m$. We define

$$\Phi = \{E_i \leq \mathbb{F}_q^m : \dim E_i = l,\ E_i \cap E_j = T_0,\ 1 \leq i \neq j \leq s\}.$$

Then, $\Phi \subseteq \mathcal{G}_q(l, m)$ is a *sunflower* of $\mathbb{F}_q^m$ and the space T_0 is called the center of the sunflower Φ.

Lemma 1. *Let $\Phi \subseteq \mathcal{G}_q(l, m)$ be a sunflower and T_0 the center of Φ. For any E_i, $E_j \in \Phi$ with $1 \leq i \neq j \leq s$, we have $\mathbb{F}_q^m = E_i + E_j$.*

Proof. Since

$$\begin{aligned} \dim(E_i + E_j) &= \dim(E_i) + \dim(E_j) - \dim(E_i \cap E_j) \\ &= \dim(E_i) + \dim(E_j) - \dim(T_0) \\ &= l + l - t_0 = m \end{aligned}$$

and $E_i + E_j \leq \mathbb{F}_q^m$, we have $\mathbb{F}_q^m = E_i + E_j$. $\square$

Lemma 2. *Let $\Phi \subseteq \mathcal{G}_q(l, m)$ be a sunflower and T_0 the center of Φ. For any E_i, $E_j \in \Phi$ with $1 \leq i \neq j \leq s$, we have $E_i^\perp \cap E_j^\perp = \{\mathbf{0}\}$.*

Proof. Assume that $\mathbf{z} \in E_i^\perp \cap E_j^\perp$. It follows from Lemma 1 that $\mathbf{z} \in \left(\mathbb{F}_q^m\right)^\perp$, which implies $\mathbf{z} = \mathbf{0}$. $\square$

2.2. Euclidean Inner Product

Let m be a positive integer. For $\mathbf{x} = (x_1, x_2, \ldots, x_m)$, $\mathbf{y} = (y_1, y_2, \ldots, y_m) \in \mathbb{F}_q^m$, the *Euclidean inner product* of $\mathbf{x}$ and $\mathbf{y}$ is given by

$$< \mathbf{x}, \mathbf{y} > := \mathbf{x}\mathbf{y}^T = \sum_{i=1}^{m} x_i y_i.$$

For any $S \subseteq \mathbb{F}_q^m$, we define

$$\mathrm{Span}(S) := \left\{ \sum_{i=1}^{r} \lambda_i \mathbf{s}_i \mid r \in \mathbb{N}, \mathbf{s}_i \in S, \lambda_i \in \mathbb{F}_q \right\},$$

$$S^\perp := \{\mathbf{v} \in \mathbb{F}_q^m \mid \mathbf{v}\mathbf{s}^T = 0,\ \text{for any } \mathbf{s} \in S\}.$$

Then, $\text{Span}(S)$ and $S^{\perp}$ are vector spaces over $\mathbb{F}_q$ and

$$\dim(\text{Span}(S)) + \dim(S^{\perp}) = m. \tag{1}$$

2.3. Minimal Linear Codes

All linear codes can be constructed by the following way. Let $m \leq n$ be two positive integers. Let $G := [\mathbf{d}_1, \ldots, \mathbf{d}_n]$ be an $m \times n$ matrix over $\mathbb{F}_q$ and $D := \{\mathbf{d}_1, \ldots, \mathbf{d}_n\}$ be a multiset. Let $r(D) = r(G)$ denote the rank of G, which is equal to the dimension of the vector space $\text{Span}(D)$ over $\mathbb{F}_q$. Let

$$\mathcal{C}(D) := \left\{\mathbf{c}(\mathbf{x}) = \mathbf{x}G = (\mathbf{x}\mathbf{d}_1^T, \ldots, \mathbf{x}\mathbf{d}_n^T), \mathbf{x} \in \mathbb{F}_q^m\right\}.$$

Then, $\mathcal{C}(D)$ is an $[n, r(D)]_q$ linear code with generator matrix G. We always study the minimality of $\mathcal{C}(D)$ by considering some appropriate multisets D.

To present the sufficient and necessary condition for minimal linear codes in [14], some concepts are needed. For any $\mathbf{y} \in \mathbb{F}_q^m$, we define

$$H(\mathbf{y}) := \mathbf{y}^{\perp} = \{\mathbf{x} \in \mathbb{F}_q^m \mid \mathbf{x}\mathbf{y}^T = 0\},$$
$$H(\mathbf{y}, D) := D \cap H(\mathbf{y}) = \{\mathbf{x} \in D \mid \mathbf{x}\mathbf{y}^T = 0\},$$
$$V(\mathbf{y}, D) := \text{Span}(H(\mathbf{y}, D)).$$

It is obvious that $H(\mathbf{y}, D) \subseteq V(\mathbf{y}, D) \subseteq H(\mathbf{y})$.

Proposition 1 ([14]). *For any $\mathbf{x}, \mathbf{y} \in \mathbb{F}_q^m$, $\mathbf{c}(\mathbf{x}) \preceq \mathbf{c}(\mathbf{y})$ if and only if $H(\mathbf{y}, D) \subseteq H(\mathbf{x}, D)$.*

Let $\mathbf{y} \in \mathbb{F}_q^m \backslash \{\mathbf{0}\}$. The following lemma gives a sufficient and necessary condition for the codeword $\mathbf{c}(\mathbf{y}) \in \mathcal{C}(D)$ to be minimal.

Lemma 3 ([14] (Theorem 3.1)). *Let $\mathbf{y} \in \mathbb{F}_q^m \backslash \{\mathbf{0}\}$. Then, the following three conditions are equivalent:*

(1) *$\mathbf{c}(\mathbf{y})$ is minimal in $\mathcal{C}(D)$;*
(2) *$\dim V(\mathbf{y}, D) = m - 1$;*
(3) *$V(\mathbf{y}, D) = H(\mathbf{y})$.*

The following lemma gives a sufficient and necessary condition for linear codes over $\mathbb{F}_q$ to be minimal.

Lemma 4 ([14] (Theorem 3.2)). *The following three conditions are equivalent:*

(1) *$\mathcal{C}(D)$ is minimal;*
(2) *for any $\mathbf{y} \in \mathbb{F}_q^m \backslash \{\mathbf{0}\}$, $\dim V(\mathbf{y}, D) = m - 1$;*
(3) *for any $\mathbf{y} \in \mathbb{F}_q^m \backslash \{\mathbf{0}\}$, $V(\mathbf{y}, D) = H(\mathbf{y})$.*

By the following lemma, we can obtain infinity of many minimal linear codes from any known minimal linear codes.

Lemma 5 ([14] (Proposition 4.1)). *Let $D_1 \subseteq D_2$ be two multisets with elements in $\mathbb{F}_q^m$ and $r(D_1) = r(D_2) = m$. If $\mathcal{C}(D_1)$ is minimal, then $\mathcal{C}(D_2)$ is minimal.*

The following corollary is trivial.

Corollary 1. *Let $D_1 \subseteq D_2$ be two multisets with elements in $\mathbb{F}_q^m$ and $r(D_1) = r(D_2) = m$. If $\mathcal{C}(D_2)$ is not minimal, then $\mathcal{C}(D_1)$ is not minimal.*

In the following section, we will use the above lemmas to consider the minimality of linear codes constructed from sunflowers.

3. The Minimality of Linear Codes Constructed from Sunflowers

In this section, we consider the linear codes constructed from sunflowers and discuss the minimality of these linear codes.

Let

$$\Phi = \{E_i \leq \mathbb{F}_q^m : \dim E_i = l, E_i \cap E_j = T_0, 1 \leq i \neq j \leq s\}.$$

be a sunflower of $\mathbb{F}_q^m$ and T_0 the center of Φ.

Let

$$D := \left(\bigcup_{i=1}^{s} E_i\right) \backslash T_0 = \bigcup_{i=1}^{s} (E_i \backslash T_0). \tag{2}$$

It is easy to see that $\mathcal{C}(D)$ is a $[s(q^l - q^{t_0}), m]_q$ linear code.

The following lemmas are important in the proofs of this section.

Lemma 6 ([24] (Lemma 3.1))**.** *For all* $\mathbf{y} \in \mathbb{F}_q^m \backslash \{\mathbf{0}\}$, $E \leq \mathbb{F}_q^m$ *and* $dim(E) = r$, *we have* $H(\mathbf{y}, E) = V(\mathbf{y}, E)$ *and*

$$\dim V(\mathbf{y}, E) = \begin{cases} r, & if \quad \mathbf{y} \in E^{\perp}; \\ r-1, & if \quad \mathbf{y} \notin E^{\perp}. \end{cases}$$

By linear algebra, we can obtain the following lemma.

Lemma 7. *Let* $\mathbf{y} \in \mathbb{F}_q^m \backslash \{\mathbf{0}\}$. *If for any* $E_i \in \Phi$, $\mathbf{y} \notin E_i{}^{\perp}$, $1 \leq i \leq s$. *For any* E_{i_0}, $E_{j_0} \in \Phi$, $E_{i_0} \neq E_{j_0}$, *let* $D_1 = (E_{i_0} \cup E_{j_0}) \backslash T_0$. *We have*

$$\mathrm{rank} H(\mathbf{y}, D_1) = \begin{cases} m-2, & if \quad \mathbf{y} \in T_0{}^{\perp}; \\ m-1, & if \quad \mathbf{y} \notin T_0{}^{\perp}. \end{cases}$$

Proof. Since $\mathbf{y} \notin E_i^{\perp}$, it follows from Lemma 6 that $\dim H(\mathbf{y}, E_{i_0}) = \dim H(\mathbf{y}, E_{j_0}) = l - 1$. Note that $H(\mathbf{y}, T_0) \leq H(\mathbf{y}, E_{i_0})$ and $H(\mathbf{y}, T_0) \leq H(\mathbf{y}, E_{j_0})$.

If $\mathbf{y} \in T_0^{\perp}$, then $H(\mathbf{y}, T_0) = T_0$. Suppose that

$$\begin{aligned} H(\mathbf{y}, T_0) &= T_0 = \mathrm{Span}\{\gamma_1, \gamma_2, \ldots, \gamma_{t_0}\}, \\ H(\mathbf{y}, E_{i_0}) &= \mathrm{Span}\{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}, \gamma_1, \gamma_2, \ldots, \gamma_{t_0}\}, \\ H(\mathbf{y}, E_{j_0}) &= \mathrm{Span}\{\beta_1, \beta_2, \ldots, \beta_{k-1}, \gamma_1, \gamma_2, \ldots, \gamma_{t_0}\}. \end{aligned}$$

Then, we have

$$\begin{aligned} H(\mathbf{y}, E_{i_0}) \setminus T_0 &\supseteq \{\alpha_1, \alpha_2 \cdots, \alpha_{k-1}, \alpha_1 + \gamma_1, \alpha_1 + \gamma_2, \ldots, \alpha_1 + \gamma_{t_0}\}, \\ H(\mathbf{y}, E_{j_0}) \setminus T_0 &\supseteq \{\beta_1, \beta_2, \ldots, \beta_{k-1}, \beta_1 + \gamma_1, \beta_1 + \gamma_2, \ldots, \beta_1 + \gamma_{t_0}\}. \end{aligned}$$

Since $H(\mathbf{y}, D_1) = (H(\mathbf{y}, E_{i_0}) \cup H(\mathbf{y}, E_{j_0})) \setminus T_0$, the above equations lead to

$$H(\mathbf{y}, D_1) \supseteq \{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}, \beta_1, \beta_2, \ldots, \beta_{k-1}, \alpha_1 + \gamma_1, \alpha_1 + \gamma_2, \ldots, \alpha_1 + \gamma_{t_0}\},$$

i.e., $\mathrm{rank} H(\mathbf{y}, D_1) = m - 2$.

If $\mathbf{y} \notin T_0^{\perp}$, then $\dim H(\mathbf{y}, T_0) = t_0 - 1$ by Lemma 6. Suppose that

$$\begin{aligned} H(\mathbf{y}, T_0) &= \operatorname{Span}\{\gamma_1, \gamma_2, \ldots, \gamma_{t_0-1}\}, \\ T_0 &= \operatorname{Span}\{\gamma_1, \gamma_2, \ldots, \gamma_{t_0-1}, \gamma_{t_0}\}, \\ H(\mathbf{y}, E_{i_0}) &= \operatorname{Span}\{\alpha_1, \alpha_2, \ldots, \alpha_k, \gamma_1, \gamma_2, \ldots, \gamma_{t_0-1}\}, \\ H(\mathbf{y}, E_{j_0}) &= \operatorname{Span}\{\beta_1, \beta_2, \ldots, \beta_k, \gamma_1, \gamma_2, \ldots, \gamma_{t_0-1}\}. \end{aligned}$$

Then, we have

$$\begin{aligned} H(\mathbf{y}, E_{i_0}) \setminus T_0 &\supseteq \{\alpha_1, \alpha_2 \cdots, \alpha_k, \alpha_1 + \gamma_1, \alpha_1 + \gamma_2, \ldots, \alpha_1 + \gamma_{t_0-1}\}, \\ H(\mathbf{y}, E_{j_0}) \setminus T_0 &\supseteq \{\beta_1, \beta_2, \ldots, \beta_k, \beta_1 + \gamma_1, \beta_1 + \gamma_2, \ldots, \beta_1 + \gamma_{t_0-1}\}. \end{aligned}$$

Since $H(\mathbf{y}, D_1) = (H(\mathbf{y}, E_{i_0}) \cup H(\mathbf{y}, E_{j_0})) \setminus T_0$, the above equations yield

$$H(\mathbf{y}, D_1) \supseteq \{\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \beta_2, \ldots, \beta_k, \alpha_1 + \gamma_1, \alpha_1 + \gamma_2, \ldots, \alpha_1 + \gamma_{t_0-1}\},$$

i.e., $\operatorname{rank} H(\mathbf{y}, D_1) = m - 1$. The proof is completed. □

Now, we consider the minimality of $\mathcal{C}(D)$ in three cases. First, when $s \geq q + 1$, we have

Theorem 1. *Let $\Phi = \{E_1, \ldots, E_s\}$ be a sunflower of $\mathbb{F}_q^m$ with center T_0 of dimension t_0. If $s \geq q + 1$, then $\mathcal{C}(D)$ is an $[s(q^l - q^{t_0}), m]_q$ minimal linear code.*

Proof. According to Lemma 4, we only need to prove that for any $\mathbf{y} \in \mathbb{F}_q^m \setminus \{\mathbf{0}\}$, $\dim V(\mathbf{y}, D) = m - 1$. By (2), we obtain

$$H(\mathbf{y}, D) = D \cap H(\mathbf{y}) = \bigcup_{i=1}^{s} (H(\mathbf{y}, E_i) \setminus T_0). \tag{3}$$

There are three cases:

(1) If there exists $E_{i_0} \in \Phi$ such that $\mathbf{y} \in E_{i_0}^{\perp}$, then we have $\dim H(\mathbf{y}, E_{i_0}) = l$ from Lemma 6. According to Lemma 2, for any $E_{j_0} \in \Phi$ with $E_{j_0} \neq E_{i_0}$, we have $\mathbf{y} \notin E_{j_0}^{\perp}$. Then, it follows from Lemma 6 that $\dim H(\mathbf{y}, E_{j_0}) = l - 1$. Since $\mathbf{y} \in E_{i_0}^{\perp} \subseteq T_0^{\perp}$, we have $H(\mathbf{y}, T_0) = T_0$. We set

$$H(\mathbf{y}, T_0) = T_0 = \operatorname{Span}\{\gamma_1, \gamma_2, \ldots, \gamma_{t_0}\}.$$

When $k = 1$, we set

$$H(\mathbf{y}, E_{i_0}) = \operatorname{Span}\{\alpha_1, \gamma_1, \gamma_2, \ldots, \gamma_{t_0}\}.$$

By (3), we have $H(\mathbf{y}, D) \supseteq \{\alpha_1, \alpha_1 + \gamma_1, \alpha_1 + \gamma_2, \ldots, \alpha_1 + \gamma_{t_0}\}$, and so $\dim V(\mathbf{y}, D) = m - 1$.

When $k > 1$, we set

$$H(\mathbf{y}, E_{i_0}) = \operatorname{Span}\{\alpha_1, \alpha_2, \ldots, \alpha_k, \gamma_1, \gamma_2, \ldots, \gamma_{t_0}\},$$

and

$$H(\mathbf{y}, E_{j_0}) = \operatorname{Span}\{\beta_1, \beta_2, \ldots, \beta_{k-1}, \gamma_1, \gamma_2, \ldots, \gamma_{t_0}\}.$$

By (3), we have

$$H(\mathbf{y}, D) \supseteq \{\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \beta_2, \ldots, \beta_{k-1}, \alpha_1 + \gamma_1, \alpha_1 + \gamma_2, \ldots, \alpha_1 + \gamma_{t_0}\}.$$

Since

$$\operatorname{rank}\{\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \beta_2, \ldots, \beta_{k-1}, \alpha_1 + \gamma_1, \alpha_1 + \gamma_2, \ldots, \alpha_1 + \gamma_{t_0}\} = m - 1,$$

it is easy to obtain $\dim V(\mathbf{y}, D) = m - 1$.

(2) If for any $E_i \in \Phi$, $1 \leq i \leq s$, we have $\mathbf{y} \notin E_i^\perp$ and $\mathbf{y} \notin T_0^\perp$, then $\dim H(\mathbf{y}, E_{i_0}) = \dim H(\mathbf{y}, E_{j_0}) = l - 1$ for any $E_{i_0}, E_{j_0} \in \Phi$ with $E_{i_0} \neq E_{j_0}$. Since $\mathbf{y} \notin T_0^\perp$, $\dim H(\mathbf{y}, T_0) = t_0 - 1$. We set

$$H(\mathbf{y}, T_0) = \mathrm{Span}\{\gamma_1, \gamma_2, \ldots, \gamma_{t_0-1}\},\ T_0 = \mathrm{Span}\{\gamma_1, \gamma_2, \ldots, \gamma_{t_0}\}.$$

When $k = 1$, we set

$$H(\mathbf{y}, E_{i_0}) = \mathrm{Span}\{\alpha_1, \gamma_1, \gamma_2, \ldots, \gamma_{t_0-1}\},\ H(\mathbf{y}, E_{j_0}) = \mathrm{Span}\{\beta_1, \gamma_1, \gamma_2, \ldots, \gamma_{t_0-1}\}.$$

Then,

$$H(\mathbf{y}, D) \supseteq \{\alpha_1, \beta_1, \alpha_1 + \gamma_1, \alpha_1 + \gamma_2, \ldots, \alpha_1 + \gamma_{t_0-1}\}.$$

Since

$$\mathrm{rank}\{\alpha_1, \beta_1, \alpha_1 + \gamma_1, \alpha_1 + \gamma_2, \ldots, \alpha_1 + \gamma_{t_0-1}\} = m - 1,$$

it is easy to obtain $\dim V(\mathbf{y}, D) = m - 1$.

When $k > 1$, let $D_1 = (E_{i_0} \cup E_{j_0}) \backslash T_0$. By Lemma 7, we have $\mathrm{rank}(H(\mathbf{y}, D_1)) = m - 1$, thus $\dim V(\mathbf{y}, D) = m - 1$.

(3) If for any $E_i \in \Phi$, $1 \leq i \leq s$, we have $\mathbf{y} \notin E_i^\perp$ and $\mathbf{y} \in T_0^\perp$; then, it follows from Lemma 6 that $\dim H(\mathbf{y}, E_i) = l - 1$ and $\dim H(\mathbf{y}, T_0) = t_0$.

When $k = 1$, we obtain $\dim T_0^\perp$=2 and $\dim E_i^\perp$=1, $1 \leq i \leq s$, then $E_i^\perp$ is the one-dimensional subspace of $T_0^\perp$. There are $q + 1$ one dimensional subspace of $T_0^\perp$, since $s \geq q + 1$, we obtain $s = q + 1$. By Lemma 2, for any E_i, $E_j \in \Phi$, $E_i \neq E_j$, we have $E_i^\perp \cap E_j^\perp = \{0\}$. Thus,

$$T_0^\perp = \bigcup_{i=1}^{s} E_i^\perp. \tag{4}$$

Since $\mathbf{y} \in T_0^\perp$, by (4), there exists $E_j \in \Phi$, such that $\mathbf{y} \in E_j^\perp$, a contradiction. So $k \neq 1$.

When $k > 1$, we have $\dim H(\mathbf{y}, E_1)$=$\dim H(\mathbf{y}, E_2) = l - 1$. Let $D_1 = (E_1 \cup E_2) \backslash T_0$. By Lemma 7, we have $\mathrm{rank}(H(\mathbf{y}, D_1)) = m - 2$. We set

$$T_0 = H(\mathbf{y}, T_0) = \mathrm{Span}\{\gamma_1, \gamma_2, \ldots, \gamma_{t_0}\}.$$

$$E_1 = \mathrm{Span}\{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}, \alpha_k, \gamma_1, \gamma_2, \ldots, \gamma_{t_0}\},\ H(\mathbf{y}, E_1) = \mathrm{Span}\{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}, \gamma_1, \gamma_2, \ldots, \gamma_{t_0}\}.$$

$$E_2 = \mathrm{Span}\{\beta_1, \beta_2, \ldots, \beta_{k-1}, \beta_k, \gamma_1, \gamma_2, \ldots, \gamma_{t_0}\},\ H(\mathbf{y}, E_2) = \mathrm{Span}\{\beta_1, \beta_2, \ldots, \beta_{k-1}, \gamma_1, \gamma_2, \ldots, \gamma_{t_0}\}.$$

Let

$$B = \{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}, \beta_1, \beta_2, \ldots, \beta_{k-1}, \alpha_1 + \gamma_1, \alpha_1 + \gamma_2, \ldots, \alpha_1 + \gamma_{t_0}\}.$$

Then, $\mathrm{rank} B = m - 2$ and $B \subseteq H(\mathbf{y}, D)$. Let $V = \mathbb{F}_q^m$, $W = \mathrm{Span(B)}$ and $\overline{V} = V/W$ the quotient space of V over W. We have $\dim \overline{V} = 2$ and V =Span$\{\overline{\alpha_k}, \overline{\beta_k}\}$. Let π be the standard map from V to $\overline{V}$. For any $E_i \in \Phi, 1 \leq i \leq s$, $\pi(E_i)$ is a subspace of $\overline{V}$. It is easily seen that $\dim \pi(E_i) = 1$ or 2. There are the following two cases.

(i) If there exists $E_{i_0} \in \Phi$ such that $\dim \pi(E_{i_0}) = 2$, then $\pi(E_{i_0}) = \bar{V}$. There must exist $\alpha \in E_{i_0}$ such that

$$\pi(\alpha) = \overline{\alpha_k - b\beta_k},\ \text{where } b = (\alpha_k \mathbf{y}^T)/(\beta_k \mathbf{y}^T).$$

So, $\alpha = \alpha_k - b\beta_k + \mathbf{w}$, where $\mathbf{w} \in W$. It is simply checked that $\alpha \notin T_0$, $\alpha \in H(\mathbf{y})$ and $\alpha \notin W$. We obtain

$$(B \cup \{\alpha\}) \subseteq H(\mathbf{y}, D),\ \mathrm{rank}(B \cup \{\alpha\}) = m - 1.$$

Thus, $\dim V(\mathbf{y}, D) = m - 1$.

(ii) If for any $E_i \in \Phi$ we have $\dim\pi(E_i) = 1$, combining that $V = E_i + E_j$ for any E_i, $E_j \in \Phi$ with $E_i \neq E_j$ in accordance with Lemma 1, we have

$$\overline{V} = \pi(V) = \pi(E_i) + \pi(E_j) \text{ and } \pi(E_i) \neq \pi(E_j).$$

Since $\overline{V}$ has only $q+1$ one-dimensional subspace and $s \geq q+1$, we have $s = q+1$ and $\overline{V} = \bigcup_{i=1}^{s} \pi(E_i)$. There must exist $E_{j_0} \in \Phi$ such that

$$\pi(E_{j_0}) = \text{Span}\{\overline{\alpha_k - b\beta_k}\}, \text{ where } b = (\alpha_k \mathbf{y}^T)/(\beta_k \mathbf{y}^T).$$

Hence, there exists $\alpha = \alpha_k - b\beta_k + \mathbf{w} \in E_{j_0}$, where $\mathbf{w} \in W$, such that $\pi(\alpha) = \overline{\alpha_k - b\beta_k}$. One can easily deduce that $\alpha \notin T_0$, $\alpha \in H(\mathbf{y})$ and $\alpha \notin W$. We obtain

$$(B \cup \{\alpha\}) \subseteq H(\mathbf{y}, D), \text{ rank}(B \cup \{\alpha\}) = m-1.$$

Thus, $\dim V((y), D) = m-1$.

In conclusion, for any $\mathbf{y} \in \mathbb{F}_q^m \backslash \{\mathbf{0}\}$, we have $\dim V(\mathbf{y}, D) = m-1$, so $\mathcal{C}(D)$ is a minimal linear code. □

Remark 1. *In Theorem 1, if $q = p$ is a prime number, then it becomes [23] (Theorem 10). So Theorem 1 is a generalization of [23] (Theorem 10). Our method is different from theirs. When $s \leq q$, our method also can be used to study the minimality of the linear codes, whereas theirs can not.*

Example 1. *Let $\mathbf{e}_1, \ldots, \mathbf{e}_m$ be the standard basis of $\mathbb{F}_q^m$. Let*

$$T_0' = \text{Span}(\{\mathbf{e}_{2k+1}, \mathbf{e}_{2k+2}, \ldots, \mathbf{e}_m\}) = \{(\mathbf{0}, \mathbf{0}, \mathbf{t}) | \mathbf{t} \in \mathbb{F}_q^{t_0}\}. \tag{5}$$

For any $b \in \mathbb{F}_q$, we define

$$E_b = \text{Span}\{\mathbf{e}_1 + b\mathbf{e}_{k+1}, \mathbf{e}_2 + b\mathbf{e}_{k+2}, \ldots, \mathbf{e}_k + b\mathbf{e}_{2k}, \mathbf{e}_{2k+1}, \ldots, \mathbf{e}_m\}. \tag{6}$$

Suppose that

$$\Phi = \{E_b | b \in \mathbb{F}_q\} \cup \text{Span}\{\mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \ldots, \mathbf{e}_{2k}, \mathbf{e}_{2k+1}, \ldots, \mathbf{e}_m\}$$

and

$$D' = \bigcup_{E_i \in \Phi} (E_i \backslash T_0').$$

It is easy to see that Φ is a sunflower of $\mathbb{F}_q^m$ with center T_0' and $s = q+1$. Here, we take $q = 4, k = 3$, and $t_0 = 1$. With the help of Magma, we verify that the code $\mathcal{C}(D')$ is a minimal $[1260, 7]_4$ linear code with minimum distance 768, and

$$\frac{w_{\min}}{w_{\max}} = \frac{4}{5} > \frac{3}{4}.$$

Now, we consider the minimality of $\mathcal{C}(D)$ when $2 \leq s \leq 3 \leq q$. If $s = 3$, we have

Theorem 2. *Let $\Phi = \{E_1, \ldots, E_s\}$ be a sunflower of $\mathbb{F}_q^m$ with center T_0 of dimension t_0. If $s = 3 \leq q$, then $\mathcal{C}(D)$ is not minimal.*

Proof. To prove $\mathcal{C}(D)$ is not minimal, by Lemma 4, we only need to prove there exists $\mathbf{y}_0 \in \mathbb{F}_q^m \backslash \{\mathbf{0}\}$ such that $\dim V(\mathbf{y}_0, D) \leq m-2$.

When $s = 3$, $\Phi = \{E_1, E_2, E_3\}$. By Lemma 2 we know $E_1^{\perp} \cap E_2^{\perp} = \{\mathbf{0}\}$. Then, for any $\mathbf{y}_1 \in E_2^{\perp} \setminus \{\mathbf{0}\}$, we have $\mathbf{y}_1 \notin E_1^{\perp}$ and $\mathbf{y}_1 \in T_0^{\perp}$. Thus, $\dim H(\mathbf{y}_1, E_1) = l - 1$, $\dim H(\mathbf{y}_1, E_2) = l$, and $\dim H(\mathbf{y}_1, T_0) = t_0$. We set

$$T_0 = H(\mathbf{y}_1, T_0) = \mathrm{Span}\{\gamma_1, \gamma_2, \ldots, \gamma_{t_0}\},$$

$$E_1 = \mathrm{Span}\{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}, \alpha_k, \gamma_1, \gamma_2, \ldots, \gamma_{t_0}\},\ H(\mathbf{y}_1, E_1) = \mathrm{Span}\{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}, \gamma_1, \gamma_2, \ldots, \gamma_{t_0}\},$$

$$E_2 = H(\mathbf{y}_1, E_2) = \mathrm{Span}\{\beta_1, \beta_2, \ldots, \beta_k, \gamma_1, \gamma_2, \ldots, \gamma_{t_0}\},$$

where $\alpha_k \mathbf{y}_1^T = 1$. Let

$$E_1' = \mathrm{Span}\{\alpha_1, \alpha_2, \ldots, \alpha_k\},\ E_2' = \mathrm{Span}\{\beta_1, \beta_2, \ldots, \beta_k\},$$

we have $\mathbb{F}_q^m = E_1' \oplus E_2' \oplus T_0$. For any $\eta \in E_3$, there exist unique $\alpha \in E_1', \beta \in E_2', \gamma \in T_0$, such that $\eta = \alpha + \beta + \gamma$. Since $\alpha + \beta = \eta - \gamma \in E_3$, for any $\alpha \in E_1'$, there exists unique $\beta \in E_2'$ such that $\alpha + \beta \in E_3$. Let φ be a map from E_1' to E_2' satisfying $\varphi(\alpha) = \beta$. We can see φ is an isomorphism from E_1' to E_2' and

$$E_3 = \{\mathbf{x} + \varphi(\mathbf{x}) | \mathbf{x} \in E_1'\} \oplus T_0 = E_3' \oplus T_0.$$

Since $\mathbf{y}_1 \notin E_1^{\perp}$, $\mathbf{y}_1 \in T_0^{\perp}$, and $E_1 = E_1' \oplus T_0$, we have $\mathbf{y}_1 \notin (E_1')^{\perp}$, $\dim H(\mathbf{y}_1, E_1') = k - 1$, $\dim \varphi(H(\mathbf{y}_1, E_1')) = k - 1$, and $\dim \varphi(H(\mathbf{y}_1, E_1'))^{\perp} = m - (k-1) = k + t_0 + 1$. Thus,

$$\begin{aligned}
&\dim(\varphi(H(\mathbf{y}_1, E_1'))^{\perp} \cap E_1^{\perp}) \\
=&\dim \varphi(H(\mathbf{y}_1, E_1'))^{\perp} + \dim(E_1^{\perp}) - \dim(\varphi(H(\mathbf{y}_1, E_1'))^{\perp} + E_1^{\perp}) \\
\geq& k + t_0 + 1 + k - m = 1.
\end{aligned}$$

Since $q \geq 3$, there exists $\mathbf{y}_2 \in (\varphi(H(\mathbf{y}_1, E_1'))^{\perp} \cap E_1^{\perp}) \setminus \{\mathbf{0}\}$ such that $\varphi(\alpha_k)\mathbf{y}_2^T \neq -1$. It is easy to see $\mathbf{y}_2 \notin E_2^{\perp}$ and $\mathbf{y}_2 \in T_0^{\perp}$. Let $\mathbf{y}_0 = \mathbf{y}_1 + \mathbf{y}_2$, we obtain $\mathbf{y}_0 \notin E_1^{\perp}$, $\mathbf{y}_0 \notin E_2^{\perp}$, and $\mathbf{y}_0 \in T_0^{\perp}$. Since $\alpha_k + \varphi(\alpha_k) \in E_3$ and

$$\begin{aligned}
(\alpha_k + \varphi(\alpha_k))\mathbf{y}_0^T &= (\alpha_k + \varphi(\alpha_k))(\mathbf{y}_1 + \mathbf{y}_2)^T \\
&= \alpha_k \mathbf{y}_1^T + \alpha_k \mathbf{y}_2^T + \varphi(\alpha_k)\mathbf{y}_1^T + \varphi(\alpha_k)\mathbf{y}_2^T \\
&= 1 + 0 + 0 + \varphi(\alpha_k)\mathbf{y}_2^T \neq 0,
\end{aligned}$$

we obtain $\mathbf{y}_0 \notin E_3^{\perp}$. Thus, $\mathbf{y}_0 \notin E_i^{\perp}$, $1 \leq i \leq 3$ and $\dim H(\mathbf{y}_0, E_i) = l - 1$.

(1) When $k = 1$, $\dim H(\mathbf{y}_0, E_i) = t_0$, since $T_0 \leq H(\mathbf{y}_0, E_i)$, we have $T_0 = H(\mathbf{y}_0, E_i)$. Thus,

$$H(\mathbf{y}_0, D) = \bigcup_{i=1}^{3} (H(\mathbf{y}, E_i) \setminus T_0) = \emptyset.$$

Thus, $\mathcal{C}(D)$ is not minimal.

(2) When $k > 1$, since $E_i = E_i' \oplus T_0$, we have $\mathbf{y}_0 \notin (E_i')^{\perp}$ and $\dim H(\mathbf{y}_0, E_i') = k - 1$. Thus,

$$H(\mathbf{y}_0, E_i) = H(\mathbf{y}_0, E_i') \oplus T_0,\ 1 \leq i \leq 3.$$

By Lemma 6, it is easily verified that

$$H(\mathbf{y}_0, E_1') = H(\mathbf{y}_1, E_1'),$$

$$H(\mathbf{y}_0, E_2') = H(\mathbf{y}_2, E_2') = \varphi(H(\mathbf{y}_1, E_1')) = \varphi(H(\mathbf{y}_0, E_1')),$$

$$H(\mathbf{y}_0, E_3') = \{\mathbf{x} + \varphi(\mathbf{x}) | \mathbf{x} \in H(\mathbf{y}_0, E_1')\} \subseteq \mathrm{Span}(H(\mathbf{y}_0, E_1') \cup H(\mathbf{y}_0, E_2')).$$

Then, $\dim V(\mathbf{y}_0, D) = m - 2$. By Lemma 4, we have that $c(\mathbf{y}_0)$ is not minimal. □

Combining Theorem 2 and Corollary 1, we have

Corollary 2. *Let $\Phi = \{E_1, \ldots, E_s\}$ be a partial spread of $\mathbb{F}_q^m$. If $2 \le s \le 3 \le q$, then $\mathcal{C}(D)$ is not minimal.*

Now, we consider the minimality of $\mathcal{C}(D)$ when $4 \le s \le q$. We recall from (5) that

$$T_0' = \mathrm{Span}(\{\mathbf{e}_{2k+1}, \mathbf{e}_{2k+2}, \ldots, \mathbf{e}_m\}) = \{(\mathbf{0}, \mathbf{0}, \mathbf{t}) | \mathbf{t} \in \mathbb{F}_q^{t_0}\}.$$

We will show that some sunflowers Φ with center T_0', $\mathcal{C}(D)$ are minimal, whereas some other sunflowers Φ with center T_0', $\mathcal{C}(D)$ are not minimal.

First, we construct some sunflowers Φ such that $\mathcal{C}(D)$ are minimal. Let $k \ge 2$, $f(x)$ be an irreducible polynomial in $\mathbb{F}_q[x]$ of degree k and $M \in \mathbb{F}_q^{k \times k}$ be a matrix with characteristic polynomial $f(x)$. We define

$$\begin{aligned} E_1 &= \{(\mathbf{x}, \mathbf{0}, \mathbf{t}) | \mathbf{x} \in \mathbb{F}_q^k, \mathbf{t} \in \mathbb{F}_q^{t_0}\}, E_2 = \{(\mathbf{0}, \mathbf{x}, \mathbf{t}) | \mathbf{x} \in \mathbb{F}_q^k, \mathbf{t} \in \mathbb{F}_q^{t_0}\}, \\ E_3 &= \{(\mathbf{x}, \mathbf{x}, \mathbf{t}) | \mathbf{x} \in \mathbb{F}_q^k, \mathbf{t} \in \mathbb{F}_q^{t_0}\}, E_4 = \{(\mathbf{x}, \mathbf{x}M, \mathbf{t}) | \mathbf{x} \in \mathbb{F}_q^k, \mathbf{t} \in \mathbb{F}_q^{t_0}\}, \end{aligned} \tag{7}$$

and

$$\Phi = \{E_1, E_2, E_3, E_4\}. \tag{8}$$

We can see Φ is a sunflower with center T_0'.

Theorem 3. *For the sunflower Φ defined in (8), the linear code $\mathcal{C}(D)$ is minimal.*

Proof. According to Lemma 4, we only need to prove that for any $\mathbf{y} \in \mathbb{F}_q^m \backslash \{\mathbf{0}\}$, $\dim V(\mathbf{y}, D) = m - 1$. There are three cases:

(1) If there exists $E_{i_0} \in \Phi$ such that $\mathbf{y} \in E_{i_0}^{\perp}$, the proof is similar as that in Theorem 1 (1).

(2) If for any $E_i \in \Phi$, $1 \le i \le s$, we have $\mathbf{y} \notin E_i^{\perp}$ and $\mathbf{y} \notin T_0'^{\perp}$, then the proof is similar to that in Theorem 1 (2).

(3) If for any $E_i \in \Phi$, $1 \le i \le s$, we have $\mathbf{y} \notin E_i^{\perp}$ and $\mathbf{y} \in T_0'^{\perp}$, the proof is as follows. Let $\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3)$ where $\mathbf{y}_1, \mathbf{y}_2 \in \mathbb{F}_q^k, \mathbf{y}_3 \in \mathbb{F}_q^{t_0}$. Next, we define two linear transformations φ, ψ from $\mathbb{F}_q^k$ to $\mathbb{F}_q^k$:

$$\varphi(\mathbf{x}) = \mathbf{x}, \psi(\mathbf{x}) = \mathbf{x}M, \mathbf{x} \in \mathbb{F}_q^k. \tag{9}$$

Then,

$$E_3 = \{(\mathbf{x}, \varphi(\mathbf{x}), \mathbf{t}) | \mathbf{x} \in \mathbb{F}_q^k, \mathbf{t} \in \mathbb{F}_q^{t_0}\}, E_4 = \{(\mathbf{x}, \psi(\mathbf{x}), \mathbf{t}) | \mathbf{x} \in \mathbb{F}_q^k, \mathbf{t} \in \mathbb{F}_q^{t_0}\}. \tag{10}$$

Let

$$\begin{aligned} E_1' &= \{(\mathbf{x}, \mathbf{0}, \mathbf{0}) | \mathbf{x} \in \mathbb{F}_q^k\}, E_2' = \{(\mathbf{0}, \mathbf{x}, \mathbf{0}) | \mathbf{x} \in \mathbb{F}_q^k\}, \\ E_3' &= \{(\mathbf{x}, \varphi(\mathbf{x}), \mathbf{0}) | \mathbf{x} \in \mathbb{F}_q^k\}, E_4' = \{(\mathbf{x}, \psi(\mathbf{x}), \mathbf{0}) | \mathbf{x} \in \mathbb{F}_q^k\}. \end{aligned} \tag{11}$$

It is easy to verify that

$$E_i = E_i' \oplus T_0',\ 1 \le i \le 4.$$

Let

$$\begin{aligned} S :&= \mathrm{Span}\{H(\mathbf{y}, E_1) \cup H(\mathbf{y}, E_2)\} \\ &= \mathrm{Span}\{\{H(\mathbf{y}, E_1) \cup H(\mathbf{y}, E_2)\} \backslash T_0'\} \\ &= \{(\alpha, \beta, \mathbf{0}) | \alpha \in H(\mathbf{y}, E_1), \beta \in H(\mathbf{y}, E_2)\} \oplus \{(\mathbf{0}, \mathbf{0}, \mathbf{t}) | \mathbf{t} \in \mathbb{F}_q^{t_0}\} \\ &= S' \oplus T_0'. \end{aligned} \tag{12}$$

By Lemma 7, we have $\dim S = m - 2$.

Now, we prove $H(\mathbf{y}, E_3) \nsubseteq S$ or $H(\mathbf{y}, E_4) \nsubseteq S$. If not, assume that $H(\mathbf{y}, E_3) \subseteq S$ and $H(\mathbf{y}, E_4) \subseteq S$. By $H(\mathbf{y}, E_3) \subseteq S$, it is obvious that $H(\mathbf{y}, E_3') \subseteq S'$. Since $\mathbf{y} \notin E_3^{\perp}$ and $\mathbf{y} \in T_0'^{\perp}$, we have $\mathbf{y} \notin E_3'^{\perp}$, and then $\dim H(\mathbf{y}, E_3') = k - 1$. There exists $\alpha_1, \ldots, \alpha_{k-1} \in H(\mathbf{y}_1)$,

$\beta_1, \ldots, \beta_{k-1} \in H(\mathbf{y}_2)$ such that $(\alpha_1, \beta_1, \mathbf{0}), \ldots, (\alpha_{k-1}, \beta_{k-1}, \mathbf{0})$ is a basis of $H(\mathbf{y}, E_3')$. Then, (10) yields $\beta_i = \varphi(\alpha_i)$. It is effortlessly demonstrated that $\alpha_1, \ldots, \alpha_{k-1}$ is a basis of $H(\mathbf{y_1})$, and $\beta_1, \ldots, \beta_{k-1}$ is a basis of $H(\mathbf{y_2})$. Thus,

$$\varphi(H(\mathbf{y}_1)) = H(\mathbf{y}_2).$$

Similarly, by $H(\mathbf{y}, E_4) \subseteq S$, we obtain

$$\psi(H(\mathbf{y}_1)) = H(\mathbf{y}_2).$$

Then, we have

$$\psi(H(\mathbf{y}_1)) = H(\mathbf{y}_2) = \varphi(H(\mathbf{y}_1)) = H(\mathbf{y}_1).$$

That is to say, $H(\mathbf{y}_1)$ is the ψ-invariant subspace of $\mathbb{F}_q^k$.

Let $\alpha_1, \ldots, \alpha_{k-1}, \alpha_k$ be a basis of $\mathbb{F}_q^k$, where $\alpha_1, \ldots, \alpha_{k-1}$ is a basis of $H(\mathbf{y}_1)$. Then, the matrix of ψ with respect to this basis is

$$B = \begin{pmatrix} B_1 & B_2 \\ \mathbf{0} & b \end{pmatrix},$$

where B_1 is the matrix of $\psi|H(\mathbf{y}_1)$ with respect to $\alpha_1, \ldots, \alpha_{k-1}$. Note that M is the matrix of ψ with respect to the standard basis, and thus M and B are similar and have the same characteristic polynomial. So

$$f(x) = |xI - B_1|(x - b),$$

a contradiction with the irreducibility of $f(x)$. Hence, $H(\mathbf{y}, E_3) \nsubseteq S$ or $H(\mathbf{y}, E_4) \nsubseteq S$. It is easy to see that $r(\{H(\mathbf{y}, E_1) \cup H(\mathbf{y}, E_2) \cup H(\mathbf{y}, E_3)\} \backslash T_0') = m - 1$ or $r(\{H(\mathbf{y}, E_1) \cup H(\mathbf{y}, E_2) \cup H(\mathbf{y}, E_4)\} \backslash T_0') = m - 1$. So, $\dim V(\mathbf{y}, D) = m - 1$.

In conclusion, for any $\mathbf{y} \in \mathbb{F}_q^m \backslash \{\mathbf{0}\}$, $\dim V(\mathbf{y}, D) = m - 1$. By Lemma 4, $\mathcal{C}(D)$ is minimal. $\square$

Combining Theorem 3 and Lemma 5, we have

Corollary 3. *Let $s \geq 4$ and $\Phi = \{E_1, \ldots, E_s\}$ be a sunflower of $\mathbb{F}_q^m$ with center T_0'. If $\{E_1, E_2, E_3, E_4\}$ are defined as (7), then $\mathcal{C}(D)$ is minimal.*

Example 2. *Take $q = 5, k = 2$, and $t_0 = 1$. Let $f(x) = x^2 + x + 1$ and*

$$M = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

It is easily checked that $f(x) \in \mathbb{F}_q[x]$ is an irreducible polynomial of degree 2 and the characteristic polynomial of M. Then, the code $\mathcal{C}(D)$ constructed based on Theorem 3 is a minimal $[480, 5]_5$ linear code with minimum distance 300, and

$$\frac{w_{\min}}{w_{\max}} = \frac{3}{4} < \frac{4}{5}.$$

Now, we construct some sunflowers Φ with center T_0' such that $\mathcal{C}(D)$ are not minimal. Let us recall from (6) that

$$E_b = \mathrm{Span}\{\mathbf{e}_1 + b\mathbf{e}_{k+1}, \mathbf{e}_2 + b\mathbf{e}_{k+2}, \ldots, \mathbf{e}_k + b\mathbf{e}_{2k}, \mathbf{e}_{2k+1}, \ldots, \mathbf{e}_m\}.$$

Let

$$\Phi = \{E_b | b \in \mathbb{F}_q\}. \tag{13}$$

It is easy to see that Φ is a sunflower of $\mathbb{F}_q^m$ with center T_0'.

Theorem 4. *For the sunflower Φ defined in (13), the linear code $\mathcal{C}(D)$ is not minimal.*

Proof. Let $\mathbf{y}_0 = \mathbf{e}_1$. Then, for any $b \in \mathbb{F}_q$, we obtain

$$\begin{aligned} H(\mathbf{y}_0, E_b) &= \mathrm{Span}\{\mathbf{e}_2 + b\mathbf{e}_{k+2}, \cdots, \mathbf{e}_k + b\mathbf{e}_{2k}, \mathbf{e}_{2k+1}, \ldots, \mathbf{e}_m\} \\ &\subseteq \mathrm{Span}\{\mathbf{e}_2, \ldots, \mathbf{e}_k, \mathbf{e}_{k+2}, \ldots, \mathbf{e}_{2k}, \mathbf{e}_{2k+1}, \ldots, \mathbf{e}_m\}. \end{aligned}$$

By (3), we have

$$H(\mathbf{y}_0, D) \subseteq \mathrm{Span}(\{\mathbf{e}_2, \ldots, \mathbf{e}_k, \mathbf{e}_{k+2}, \ldots, \mathbf{e}_{2k}, \mathbf{e}_{2k+1}, \ldots, \mathbf{e}_m\}).$$

Then, $\dim V(\mathbf{y}_0, D) \leq m - 2$. By Lemma 4, we have that $\mathbf{c}(\mathbf{y}_0)$ is not minimal and $\mathcal{C}(D)$ is not minimal. □

Combining Theorem 4 and Corollary 1, we have

Corollary 4. *Let $3 < s \leq q$ and $S \subseteq \mathbb{F}_q$ where $\#S = s$. Let $\Phi = \{E_b | \, b \in S\}$. Then, $\mathcal{C}(D)$ is not minimal.*

Remark 2. *In Theorem 3, Corollary 3, Theorem 4, and Corollary 4, the center of the sunflower Φ is the special subspace T_0'. When the center is a general subspace, we have not yet proved the minimality of $\mathcal{C}(D)$.*

Example 3. *Take $q = 3, k = 2$, and $t_0 = 2$. Then, the code $\mathcal{C}(D)$ constructed based on Theorem 4 is $[216, 6]_3$ linear code with minimum distance 108, and*

$$\frac{w_{\min}}{w_{\max}} = \frac{2}{3}.$$

According to Magma experiments, there exists $y_1 = [1, 0, 0, 0, 0, 0] \in \mathbb{F}_3^6$ such that $\dim V(y_1, D) = 4$. Then, it follows from Lemma 4 that $\mathcal{C}(D)$ is not minimal.

4. Concluding Remarks

In this paper, we use the approach used in [14] to study the minimality of linear codes constructed from sunflowers in all cases. In [23], the authors proved that if the number s of the elements in a sunflower satisfying $s \geq p + 1$, then the corresponding linear code over $\mathbb{F}_p$ is minimal, where p is a prime number. Our results in this paper generalize [23] (Theorem 10). We discuss the minimality of linear codes constructed from sunflowers for all s. We obtain the following three results: (1) when $s \geq q + 1$, for any sunflower, the corresponding linear code is minimal; (2) when $2 \leq s \leq 3 \leq q$, for any sunflower, the corresponding linear code is not minimal; (3) when $3 < s \leq q$, for some sunflowers, the corresponding linear codes are minimal, whereas for some other sunflowers, the corresponding linear codes are not minimal.

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