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Considering In-App Advertising Mode, Platform-App Channel Coordination by a Sustainable Cooperative Advertising Mechanism

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Abstract: Regardless of the platform or Apps the number of users is a prerequisite for monetization. Considering in-App advertising, through the optimal control theory, this paper establishes a dynamic advertising strategy model of one platform and n Apps under the decentralized and integrated systems. For each system, the model is constrained by the states dynamics of the number of the platform's users and App's users. Our research has obtained some management insights and findings as follows. Firstly, Apps don't have to worry about the negative effects of in-App advertising caused by other Apps when making advertising decisions. Interestingly, the platform will use the advertising subsidy policy to limit the delivery density of in-App advertising. Specifically, the higher the negative impact of in-App advertising caused by App i, the lower the advertising subsidies provided to App i. Additionally, when determining the advertising subsidy rate, the platform will also comprehensively consider the profitability of Apps and the costs and benefits of in-App advertising. Secondly, our proposed sustainable cooperative advertising mechanism can simultaneously coordinate the Platform-App channel and implement the optimal integrated objective. Finally, we theoretically prove that under some mild conditions, the mechanism can both improve the profits of the platform and n Apps.

Keywords: platform; App; advertising; mechanism; system coordination; sustainability; optimal control theory

1. Introduction

As a two-side market, the mobile platform not only provides users with a variety of Apps, but also places App developers into a broad user base, effectively realizing the interconnection between the users and Apps. Statistics report shows that in 2018, the global mobile App download volume has reached 194 billion times, and the global App store user expenditure has reached 101 billion dollars. (Source: The State of Mobile in 2019—The Most Important Trends to Know. Retrieved 16 January 2019. https://www.appannie.com/en/insights/market-data/the-state-of-mobile-2019/.) In a sense, the emergence of mobile platform really enables App developers to realize data monetization. What's important is that, both for the platform and Apps, the number of users is a key factor in gaining benefits. Therefore, the platform and Apps will use various advertising channels to hype the features of their products in an attempt to expand their respective user base.

In addition to the sales revenue generated by the functional services provided by the platform owner, most of the platform's revenue comes from the transfer revenue of Apps, namely Apps' sales revenue sharing. For example, Apple and Android will force the third-party App developers to share 30% of their sales revenue with the platform owner in return. In contrast, the revenue source of an App is more from in-App advertising than from sales. In-App advertising is a mode in which an App developer embeds advertisers' ads into App, and when the users click or watch the ads, the advertiser will pay App developer by a certain billing way [1]. However, even if in-App advertising brings huge advertising revenue to the developers, it also carries an immeasurable risk of damaging the user experience and causing annoyance to users, which is the great dilemma brought by this mode to the App developers [2].

In this paper, we study the dynamic advertising strategies of one platform and n Apps in the context of in-App advertising mode where App can serve as an advertiser to display ads in the other Apps for promotion. This problem is complicated and challenging because of the interplay between the platform and n Apps. Specifically, in a system consisting of one platform and n App, the increase of the number of platform users means that the possibility of Apps being downloaded is increased. And the increase in Apps' advertising investment will prompt more new users to register as the platform users before downloading the desired Apps. Therefore, the platform owner is very willing to provide some advertising subsidies for App developers, which is in line with the actual business situation. In addition, due to the existence of in-App advertising, an App can serve as an advertiser to deliver ads into other Apps and can also serve as an advertisement display platform to accept ads from other Apps. In order to maintain a good mutually beneficial relationship among Apps, while earning advertising revenue, Apps will also provide a certain percentage of advertising subsidies to each other. Note that, it's important for App developers to balance the relationship between revenue from in-App advertising and user experience.

In such a complicated economic environment, we try to explore the following problems:

1. Considering the impact of in-App advertising, how should App developers effectively adjust their advertising strategies? In addition, what impacts will the in-App advertising mode has on the user number and total profits of Apps?
2. Whether the platform owner will intervene against the negative impact of in-App advertising, and if so, what measures.
3. Is there a sustainable advertising mechanism to eliminate possible system inefficiency?

To solve the above problems rigorously, we develop a dynamic advertising model between the platform and n Apps using the optimal control theory. Consider two systems: an integrated system where the platform owner and App developers are integrated as one company, and a decentralized system where the platform owner is the leader and App developers are the followers of Stackelberg Game. Importantly, our model takes into account the in-App advertising mode, which makes Apps act as either advertisers by delivering ads into other Apps, or as advertising display platforms to earn advertising revenue. In addition, we also consider the negative impact of this mode on the number of App users. Under the above model setting, we discuss the optimal advertising decisions between one platform and n Apps.

Some of the important results of this paper include the following.

1. When making advertising decisions, App i does not need to consider the negative impact of in-App advertising caused by the other Apps. Interestingly, the platform owner will adjust the proportion of advertising subsidy provided to App i, in order to balance the negative impact of in-App advertising caused by App i, himself. Specifically, the greater App i's negative impact of in-App advertising, the less advertising subsidy the platform will provide to App i. Additionally, when determining the advertising subsidy rate, the platform owner will take into account the profitability of Apps, as well as the costs and benefits of in-App advertising mode, which means that there exists a competitive relationship among Apps.
2. The sustainable cooperative advertising mechanism we proposed can effectively coordinate the decentralized system with the integrated system, and as expected, successfully implement the optimal objective of the integrated system.

3. As for the system members, we theoretically find that under the effect of a sustainable cooperative advertising mechanism, the profits of the platform and n Apps can be improved together under some mild conditions.

The rest of this paper is organized as follows. Section 2 presents a related literature review. Section 3 develops the model. Section 4 gives the equilibrium analysis. In Section 5, a sustainable cooperative advertising mechanism is proposed to coordinate the decentralized system with the integrated system. Section 6 discusses the research results and puts forward the future research direction. All proofs in this paper are provided in Appendix A.

2. Literature Review

The relevant literature of this paper is mainly derived from the following two research streamlines: cooperative advertising in supply chain and online advertising.

2.1. Cooperative Advertising in Supply Chain

The model of this paper is to explore the dynamic optimal advertising strategy of one platform and n Apps, and finally achieve system coordination, which is very relevant to the research on the cooperative advertising in the supply chain. The cooperative advertising, we discuss here is vertical cooperative advertising, which is the most common comprehension. The cooperative advertising describes a financial arrangement in which a manufacturer bears a certain percentage of a retailer's advertising costs [3]. The existing articles on cooperative advertising are mainly divided into two research streams, static and dynamic model setting. The dynamic model is to reflect the time dependence of advertising decision, including the discrete-time model and continuous-time model [4]. The continuous-time model is solved by the optimal control theory or differential game. The traditional dynamic model settings are built on the framework of Stackelberg game, with the manufacturer as the leader and the retailer as the follower. Berger et al. examine integration decisions from a cooperative advertising perspective to compare the profitability in online channel [5]. Karray and Zaccour implement cooperative advertising programs to offset the harmful effects of retailers' private branding on manufacturers [6]. Yue et al. consider what happens when the manufacturer offers a price discount to the customers [7]. Yang et al. investigate the effect of the retailer's fairness concerns [8]. Interestingly, a few articles consider a retailer Stackelberg game, Xie and Neyret (for the first static model) [9] and Buratto et al. (for the first dynamic model) [10] analyze the decision framework of retailers dominating manufacturers. There are also studies that introduce price decisions into models [11–16]. In addition, Wang et al. extend the model to one manufacturer and two retailers [17]. Karray and Amin consider the competitive relationship between the retailers [18].

Demand function is one of the important features of the dynamic model, which reflects the influence of members' advertising efforts on consumer demand. Nerlove and Arrow develop the classical Nerlove-Arrow model, they introduce a so-called goodwill stock, which depends on members' advertising efforts [19]. Furthermore, other variables like pricing and quality can also be included into this Nerlove-Arrow model [20,21]. Another demand model, proposed by Vidale and Wolfe [22] and expanded by Sethi [23], reflects the impact of advertising on consumer awareness. Additionally, based on a duopoly setting, Kimball introduce the Lanchester model to analyze competitive advertising problems, which captures the dynamics of the market share [24]. As for some durable goods, the demand is more dependent on the social influence (diffusion process). Bass propose a diffusion model in which the possibility of individual purchase mainly depends on two factors, innovation and imitation [25]. Nikolopoulos and Yannacopoulos extend the Bass model to explore the optimal advertising strategy on the new product diffusion in a stochastic setting [26].

The background of cooperative advertising in this paper is different from that in traditional supply chain. We develop a dynamic advertising model where the platform as a two-side market makes Apps and users interconnected. In terms of the profit distribution rules, App developers need to share a part of the sales revenue to the platform in return. Based on the above background, we re-explore the possibility of implementing cooperative advertising in the Platform-App channel.

2.2. Online Advertising

Our research has also contributed to the study of online advertising model. The literature on online advertising covers a wide range of topics. Previous studies focus on the optimal advertising strategies [27,28], auctions of ads plot [29–31], targeted behavioral advertising [32–34], and so on. Due to the rapid development of mobile Internet, a large number of studies focus on in-App advertising. The research on in-App advertising can be divided into empirical analysis and theoretical analysis.

For the empirical studies, Cheung and to extend the theory of planned behavior to take trust propensity and trust as antecedents of mobile users' attitudes toward in-App advertising. A structural equation model is used to test 480 young Chinese mobile phone users. The results show that users' tendency to trust influences their attitude towards in-App ads and their willingness to view in-App ads [35]. Tongaonkar et al. propose a new method to identify Android Apps in network traces by in-App advertising and analyze the official Android market from an advertising perspective [36]. Logan examines social media fatigue through the lens of rational choice theory to better understand users' attitudes toward in-App advertising and how it affects brands [37]. Cicek et al. test the effect of banner ads location, App type, and App direction on recall of in-App advertising through an experimental design. They find that users are better able to recall the content of advertising when the banner ads are at the top [38]. Lee and Shin investigate how to select potential active users for in-App advertising based on previous App usage behavior by large-scale field data on game Apps. Their study shows that usage behavior in game Apps, the level of user participation, and especially daily purchasing activities are important factors [39]. In addition, there are also studies on young children and adolescents' attitudes and behavioral responses to in-App advertising when they use mobile phones or play App games [40–42].

For the theoretical model, the existing research on in-App advertising is relatively few. Guo et al. explore the mechanism of reward advertising in game Apps, and find that only when the marginal revenue of advertising is rapidly decreasing should the number of ads per user be limited [43]. Oh et al. propose a new bargaining model (the improved apex game) which analyzes the revenue sharing mechanism of platform and App, and explore the appropriate revenue distribution between platform and App [44]. Based on the 'sojourn' and 'exposure' effects, Sun et al. study the ad-sequencing problem of fading ads shown to App users, without considering the role of the platform [45]. Hao et al. discuss paid Apps and the revenue-sharing policy of the platform under in-App advertising mode [1]. Chen et al. extend the model to discuss the strategic choice between paid Apps and free Apps with in-advertising mode when App developers face multiple platforms [46]. However, the above literatures are all based on the static model. Under the setting of the dynamic model, Kumar and Sethi use dynamic pricing to weigh the relationship between the subscription revenue of web content and the advertising revenue [47]. Ji et al. are focusing on the research of exactly what mechanism will enable Platform-App channel to achieve coordination, but do not consider the situation that Apps, as advertisers or advertising display platforms, deliver ads among Apps (i.e., in-App advertising mode) [48]. Even though Wang et al. propose an advertising contract that could coordinate the system, they only showed it through numerical examples without giving any theoretical proofs [49].

Considering in-App advertising mode, our research attempts to expand the dynamic advertising model into one platform and n Apps. Importantly, we theoretically give the conditions under which the platform and Apps need to meet in different situations under the influence of sustainable cooperative advertising mechanism.

3. Model Development

3.1. Platform and App User Growth

We consider a system consisting of one platform and n Apps (for convenience, through this paper, we refer to the platform owner as 'she' and App developer as 'he'). The number of users is the basis for the platform owner and App developers to realize data monetization. Therefore, the amount of revenue mainly depends on the size of user base.

It is worth noting that the most effective way to expand the market and gain potential users is advertising. Platform owner and App developers use various advertising channels to promote the characteristics of their own products in order to get more new users. Specifically, App developers can choose to advertise within or outside the platform, as well as deliver ads into other well-known Apps, namely, in-App advertising mode. Denote the platform owner's advertising effort by $u_P(t)$, and App i developer's advertising effort by $u_{Ai}(t)$, I € {1, ... , n}. Consistent with common sense, the advertising can increase or maintain the number of platform users and App users. Let $x(t)$ and $y_i(t)$ denote the number of platform users and App i's users, respectively. Then the states dynamics of platform's user growth and App i's user growth can be respectively written as:

$$\dot{x}(t) = \alpha u_P(t) - \delta x(t),\ x(0) = x_0 > 0 \tag{1}$$

$$\dot{y}_i(t) = \gamma_i u_{Ai}(t) + \beta_i x(t) - \delta y_i(t) - \sum_{j=1, j\neq i}^{n} \eta_j u_{Aj}(t),\ y_i(0) = y_{i0} > 0,\ i,\ j \in \{1, \ldots, n\} \text{ and } i \neq j \tag{2}$$

where α, γ_i, β_i, η_j, and δ are all positive parameters.

The meanings of all parameters are explained as follows:

1. By Equations (1) and (2), the constants α and γ_i measure the effectiveness of the platform owner's advertising effort (u_P) and App i developer's advertising effort (u_{Ai}) on their own user growth respectively. Note that only u_P has a positive impact on the platform's user growth $\dot{x}(t)$ by Equation (1).
2. Different from Equation (1), the term $\beta_i x(t)$ in Equation (2) characterizes the positive impact of the number of platform users on App i user growth $\dot{y}_i(t)$, which reflects that when App i advertises within the platform, the larger the user base of the platform, the higher the likelihood that App i will be downloaded, which naturally increases the number of App i's users.
3. In addition, the term $\eta_j u_{Aj}(t)$ in Equation (2) captures the in-App advertising mode where App j delivers ads into App i for promotion as an advertiser. On the one hand, in-App advertising can create additional advertising revenue for App i, but on the other hand, in-App advertising may also cause users to feel annoyed and reduce the user experience, resulting in the loss of the number of App i's users [2]. In such a situation, η_j implies that the greater the advertising effort paid by App j, the higher the negative impact on the user growth of App i will be [47]. Of course, no App developers will endlessly add a lot of in-App advertising to destroy their brand reputation, so we assume that the negative impact of in-App advertising will certainly not exceed the positive impact of App i's own advertising effort, that is,

$$\gamma_i u_{Ai}^* - \sum_{j=1, j\neq i}^{n} \eta_j u_{Aj}^* > 0.$$

4. Due to the lack of technological innovation or the gradual decline of user experience, both platform and Apps will face the loss of existing users. Let δ represent the decay coefficient in Equations (1) and (2), indicating the negative effects on the platform and Apps user growth. Note that we use the same decay coefficient δ for both platform and Apps in order to reduce the mathematic difficulty. Our main results will not be affected by this choice.

3.2. Profits of the Platform and App

The platform and Apps will inevitably incur advertising costs if they exert advertising efforts to increase the number of users. Following previous studies [14,48], we assume that the advertising cost takes a quadratic form, that is

$$C_P(t) = \frac{1}{2}u_P^2(t) \text{ and } C_{Ai}(t) = \frac{1}{2}u_{Ai}^2(t) \tag{3}$$

which implies an increasing marginal advertising cost.

In such a system, the revenue of App developers mainly comes from two sources, one is sales revenue, the other is in-App advertising revenue. Of the two, the sales revenue comes mainly from some purchase transactions that the users make within Apps, i.e., in-App purchases. Let p_{Ai} denote the marginal sales revenue per App i's user, then App i developer can get a revenue of $p_{Ai}y_i(t)$ from this source. However, when purchasing within App occurs, App developers have to share a certain proportion of the sales revenue with the platform owner, among which the sharing revenue rate is set by the platform owner. Thus, we denote λ_i as the sharing revenue rate of App i, then the platform owner will get a revenue of $\lambda_i p_{Ai}y_i(t)$ from in-App purchases and the rest, $(1 - \lambda_i)p_{Ai}y_i(t)$ is for App i developer himself.

In-App advertising is another important revenue source for App developers. When App j chooses to embed the ads into App i for display, App i need to share a part of App j's sales revenue as the in-App advertising revenue. We define θ_{ji} as the proportion of App j's payment to App i, and then App i's in-App advertising revenue paid by App j is $\theta_{ji}p_{Aj}y_j$. Similarly, $\theta_{ij}p_{Ai}y_i$ means the cost that App i need to pay to App j when App i chooses to deliver the ads into App j for display. Interestingly, in order to get more advertising volume from App j, App i is very willing to help App j to undertake a part of the advertising cost as an advertising subsidy. We specify the proportion of advertising costs that App i bears for App j as ξ_{ij} (namely, App j's advertising subsidy rate set by App i), and then App j's advertising subsidy provided by App i is $\xi_{ij}C_{Aj}(t)$, which is an expenditure for App i. Similarly, $\xi_{ji}C_{Ai}(t)$ serves as App i's advertising subsidy provided by App j, which is an expenditure for App j.

As for the platform owner, the revenue mainly derived from the users in-platform purchases, including some paid services, ancillary functions, music, and so on. Let p_P denote the marginal sales revenue per platform user, then the platform owner can get the sales revenue $p_P x(t)$. Further, due to the revenue sharing policy, the platform will receive an additional revenue from App i, i.e., $\lambda_i p_{Ai}y_i(t)$. Thus, the platform owner is very willing to bear a part of the advertising cost of App i as the advertising subsidy on the purpose of encouraging App i to increase the advertising investment. We define ϕ_i as the proportion of advertising costs that the platform owner bears for App i (namely, the advertising subsidy rate set by the platform owner). Therefore, $\phi_i C_{Ai}(t)$ can be regarded as the advertising subsidy provided by the platform owner for App i, which is an expenditure for the platform owner.

By combining the revenues and costs discussed above, we can obtain the instantaneous profits of platform owner and App i developer respectively as follows

$$\pi_P(t) = p_P x(t) - \frac{1}{2}u_P^2(t) + \sum_{i=1}^{n}\left[\lambda_i p_{Ai}y_i(t) - \frac{1}{2}\phi_i(t)u_{Ai}^2(t)\right] \tag{4}$$

$$\begin{aligned} \pi_{Ai}(t) &= (1-\lambda_i)p_{Ai}y_i(t) + \tfrac{1}{2}\phi_i(t)u_{Ai}^2(t) - \tfrac{1}{2}u_{Ai}^2(t) \\ &+ \sum_{j=1, i\neq j}^{n}\left[\theta_{ji}p_{Aj}y_j(t) + \tfrac{1}{2}\xi_{ji}u_{Ai}^2(t) - \theta_{ij}p_{Ai}y_i(t) - \tfrac{1}{2}\xi_{ij}u_{Aj}^2(t)\right]. \end{aligned} \tag{5}$$

Thus, the total profit of whole system, namely, the integrated system is

$$\pi_I(t) = p_P x(t) - \frac{1}{2}u_P^2(t) + \sum_{i=1}^{n}\left[p_{Ai}y_i(t) - \frac{1}{2}u_{Ai}^2(t)\right] \tag{6}$$

It's worth noting that λ_i, ϕ_i, θ_{ij} (θ_{ji}), and ξ_{ji} (ξ_{ij}) are all exogenous, and we assume that

$$0 \le \lambda_i + \sum_{j=1, j\neq i}^{n}\theta_{ij} \le 1 \text{ and } 0 \le \phi_i + \sum_{j=1, j\neq i}^{n}\xi_{ji} \le 1$$

which means that the sum of the proportion of App i's sales revenue shared by the platform and other Apps is between 0 and 1. Similarly, the sum of the advertising subsidy rate provided by the platform and other Apps is also between 0 and 1.

In summary, in this system, the revenue and expenditure flows of the platform and Apps, and the revenue and expenditure flows of in-App advertising between Apps are respectively shown in Figures 1 and 2.

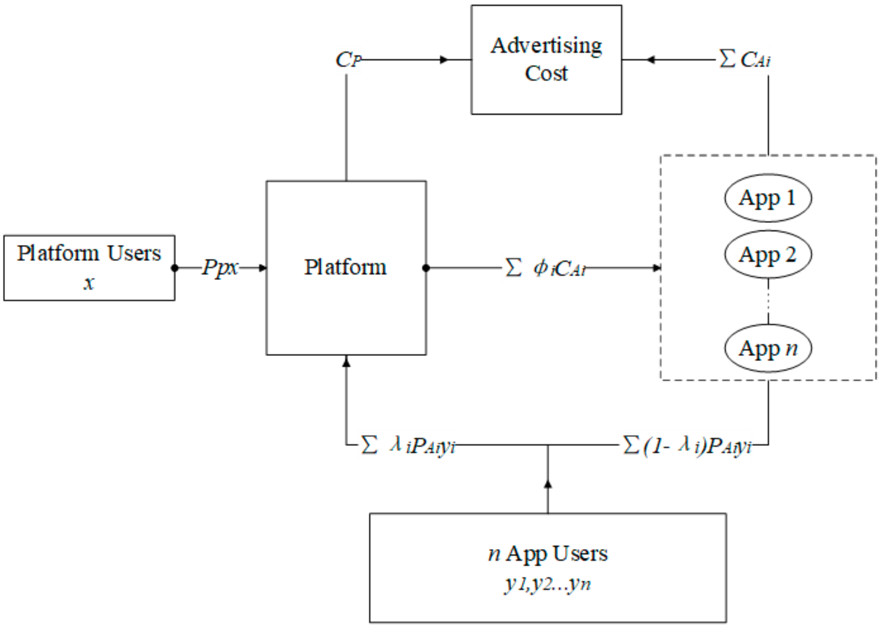

Figure 1. Revenue and expenditure flows of platform and Apps.

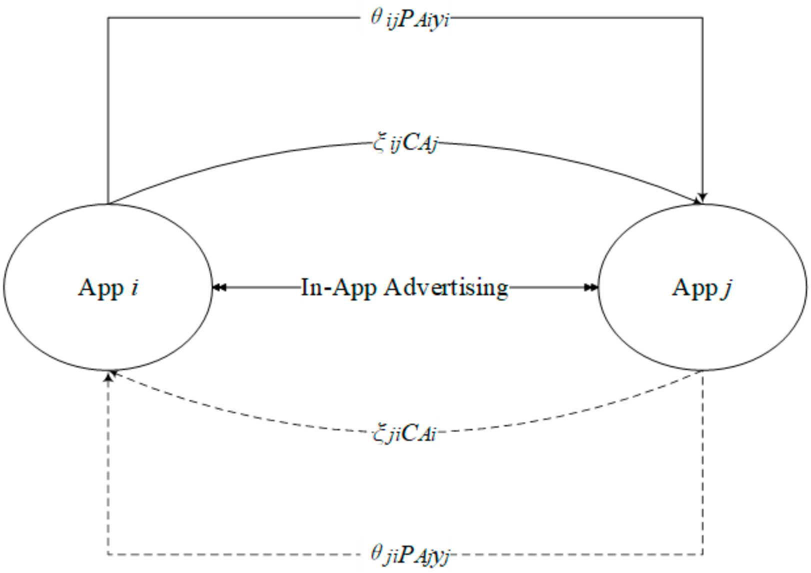

Figure 2. Revenue and expenditure flows of in-App Advertising between Apps.

3.3. Optimal Problems Faced by Platform and App

We consider an infinite time horizon problem with a positive discount rate of ρ. When the platform owner and App i developer make advertising decisions separately, namely, in a decentralized system, the objective of the platform owner is to determine the optimal advertising effort and advertising subsidy rate to maximize the present value of profit. Thus, the problem faced by the platform owner is

$$V_P(x,y) = \max_{u_P(t)\geq 0,\phi_i(t)\geq 0} \int_0^\infty e^{-\rho t}\left\{p_P x(t) - \frac{1}{2}u_P^2(t) + \sum_{i=1}^{n}\left[\lambda_i p_{Ai} y_i(t) - \frac{1}{2}\phi_i(t) u_{Ai}^2(t)\right]\right\}dt \quad \text{s.t. (1) and (2).} \tag{7}$$

While in the decentralized system, the objective of App i developer is also to maximize the present value of profit, but he only needs to determine his optimal advertising effort. Meanwhile, App i developer allows the other App (App j) as an advertiser to deliver ads into himself for promotion, that is, in-App advertising mode. Thus, the problem faced by App i developer is

$$\begin{aligned} V_{Ai}(x,y) = \max_{u_{Ai}(t)\geq 0} \int_0^\infty e^{-\rho t}\Big\{&(1-\lambda_i)p_{Ai}y_i(t) + \frac{1}{2}\phi_i(t)u_{Ai}^2(t) - \frac{1}{2}u_{Ai}^2(t) \\ &+ \sum_{j=1,j\neq i}^{n}\left[\theta_{ji}p_{Aj}y_j(t) + \frac{1}{2}\xi_{ji}u_{Ai}^2(t) - \theta_{ij}p_{Ai}y_i(t) - \frac{1}{2}\xi_{ij}u_{Aj}^2(t)\right]\Big\}dt \\ &\text{s.t. (1) and (2).} \end{aligned} \tag{8}$$

Further, when the platform owner and App developers coordinate as a vertically integrated system, their objective is

$$V_I(x,y) = \max_{u_P(t)\geq 0,u_{Ai}(t)\geq 0} \int_0^\infty e^{-\rho t}\left\{p_P x(t) - \frac{1}{2}u_P^2(t) + \sum_{i=1}^{n}\left[p_{Ai}y_i(t) - \frac{1}{2}u_{Ai}^2(t)\right]\right\}dt \quad \text{s.t. (1) and (2).} \tag{9}$$

In the following sections, we will analyze the equilibrium solutions of the decentralized system and integrated system respectively, and then discuss the sustainable cooperative advertising mechanism that can effectively coordinate the decentralized system with the integrated system.

For simplicity, in the rest of the paper, we suppress t in time-dependent variables. The notations used in this paper are summarized in Table 1.

Table 1. Summary of notations.

t	**Time t, $t \geq 0$**
Decision variables	
$u_P(t)$	Advertising effort by the platform owner at time t
$u_{Ai}(t)$	Advertising effort by App i developer at time t
$\phi_i(t)$	App i's advertising subsidy rate set by the platform owner at time t
State variables	
$x(t)$	The number of platform users at time t
$y_i(t)$	The number of App i's users at time t
Parameters	
α	The platform's advertising effectiveness on her user growth
γ_i	The App i's advertising effectiveness on his user growth
β_i	The platform's user base effectiveness on App i's user growth
η_j	The negative effect on App i' user growth caused by App j delivering ads to App i
δ	User growth decay parameter of the platform and App i
ρ	Decay coefficient
λ_i	App i' sales revenue sharing rate set by the platform owner
θ_{ji} (θ_{ij})	When App j (App i) adopts in-App advertising mode, the proportion of App j's (App i's) sales revenue shared by App i (App j)
ξ_{ji} (ξ_{ij})	When App i (App j) adopts in-App advertising mode, the proportion of App i's (App j's) advertising costs borne by App j (App i)
p_P, p_{Ai}	Marginal sales revenue of the platform and App i, respectively
$\pi_P(t)$, $\pi_{Ai}(t)$, $\pi_i(t)$	Instantaneous profit of the platform, App i, and the integrated system, respectively
V_I	Value function of the integrated system
V_P	Value function of the platform in the decentralized system
V_{Ai}	Value function of App i in the decentralized system

4. Equilibrium Analysis of Two Systems

4.1. Decentralized System (Stackelberg Game)

In the decentralized system, when the platform owner and App i developer make advertising decisions separately to maximize the present values of their respective profits, we consider the decision structure as a Stackelberg game. The platform owner as a leader first determines her advertising effort $u_P(x, y)$ and announces the advertising subsidy rate $\phi_i(x, y)$. After the actions of the platform owner, App i developer acts as a follower in determining his own advertising effort $u_{Ai}(x, y)$, independently.

The optimal problems faced by the platform owner and App i developer are given by (7) and (8), respectively. Thus, the Hamilton-Jacobi-Bellman (HJB) equations [50] for Equations (7) and (8) are

$$\begin{aligned} \rho V_P(x,y) = \max_{u_P\geq 0, \phi_i\geq 0} \Big\{ & p_P x - \tfrac{1}{2}u_P^2 + +V_{Px}(\alpha u_P - \delta x) \\ & + \sum_{i=1}^{n} [(\lambda_i p_{Ai} y_i - \tfrac{1}{2}\phi_i u_{Ai}^2) + V_{Py_i}(\gamma_i u_{Ai} + \beta_i x - \delta y_i - \sum_{j=1, j\neq i}^{n} \eta_j u_{Aj})] \Big\}, \end{aligned} \tag{10}$$

$$\begin{aligned} \rho V_{Ai}(x,y) = \max_{u_{Ai}\geq 0} \Big[& (1-\lambda_i) p_{Ai} y_i + \sum_{j=1, j\neq i}^{n} (\theta_{ji} p_{Aj} y_j + \tfrac{1}{2}\xi_{ji} u_{Ai}^2 - \theta_{ij} p_{Ai} y_i - \tfrac{1}{2}\xi_{ij} u_{Aj}^2) \\ & - \tfrac{1}{2}(1-\phi_i) u_{Ai}^2 + V_{Aix}(\alpha u_P - \delta x) + V_{Aiy_i}(\gamma_i u_{Ai} + \beta_i x - \delta y_i - \sum_{j=1, j\neq i}^{n} \eta_j u_{Aj}) \Big] \\ & i,\ j \in \{1, \ldots, n\} \text{ and } i \neq j. \end{aligned} \tag{11}$$

In (10) and (11), $V_{Px} = \partial V_P/\partial x$, $V_{Pyi} = \partial V_P/\partial y_i$, $V_{Aix} = \partial V_{Ai}/\partial x$ and $V_{Aiyi} = \partial V_{Ai}/\partial y_i$, which can be interpreted as the change in the present value of profit due to the increase in the number of the platform users and App i's users. Solving (10) and (11), the optimal solutions of the platform owner and App i developer are proposed in Proposition 1.

Proposition 1. *In the decentralized system, the equilibrium results of the platform owner and App i developer are as follows:*

Case 1: When no App developer adopts in-App advertising mode, that is, $\eta_i = \xi_{ji} = \theta_{ij} = 0$, *then the optimal advertising effort of the platform owner is*

$$u_P^* = \alpha \frac{(\rho+\delta)p_P + \sum_{i=1}^{n} \lambda_i \beta_i p_{Ai}}{(\rho+\delta)^2} \tag{12}$$

And the optimal advertising subsidy rate set by the platform owner is

$$\phi_i{}^* = \frac{3\lambda_i - 1}{1+\lambda_i} \tag{13}$$

The optimal advertising effort of App i developer is

$$\begin{gathered} u_{Ai}^* = \tfrac{(1+\lambda_i)\gamma_i p_{Ai}}{2(\rho+\delta)} \\ i,\ j \in \{1, \ldots, n\} \text{ and } i \neq j. \end{gathered} \tag{14}$$

Case 2: When every App developer adopts in-App advertising mode, meanwhile given the condition $\eta_i \leq \frac{[(3\lambda_i - 1 + \sum_{j=1, j\neq i}^{n} \theta_{ij})]\gamma_i p_{Ai}}{2\sum_{j=1, j\neq i}^{n} \lambda_j p_{Aj}}$ *holds, then the optimal advertising effort of the platform owner is still Equation (12). But the optimal advertising subsidy rate set by the platform owner is*

$$\phi_i{}^* = (1 - \sum_{j=1, j\neq i}^{n} \xi_{ji}) \frac{(3\lambda_i - 1 + \sum_{j=1, j\neq i}^{n} \theta_{ij})\gamma_i p_{Ai} - 2\eta_i \sum_{j=1, j\neq i}^{n} \lambda_j p_{Aj}}{(1 + \lambda_i - \sum_{j=1, j\neq i}^{n} \theta_{ij})\gamma_i p_{Ai} - 2\eta_i \sum_{j=1, j\neq i}^{n} \lambda_j p_{Aj}} \quad (15)$$

$$i,\ j \in \{1, \ldots, n\} \text{ and } i \neq j.$$

Furthermore, the optimal advertising effort of App i developer is

$$u_{Ai}^{*} = \frac{(1 + \lambda_i - \sum_{j=1, j\neq i}^{n} \theta_{ij})\gamma_i p_{Ai} - 2\eta_i \sum_{j=1, j\neq i}^{n} \lambda_j p_{Aj}}{2(\rho + \delta)(1 - \sum_{j=1, j\neq i}^{n} \xi_{ji})} \quad (16)$$

$$i,\ j \in \{1, \ldots, n\} \text{ and } i \neq j.$$

As for Case 1:

By Equation (12), the optimal platform owner's advertising effort $u_P{}^*$ increases with her own advertising effectiveness α and marginal sales revenue p_P. Meanwhile, the effectiveness of the platform's user base on App i's user growth, β_i, has a positive effect on $u_P{}^*$. Moreover, App's marginal sales revenue p_{Ai} and App i's revenue sharing rate λ_i both have positive effects on $u_P{}^*$. In addition, the more the number of Apps entering the platform, the more the platform owner wants to increase her optimal advertising effort $u_P{}^*$.

In Equation (13), there is a positive correlation between the App i's advertising subsidy rate ϕ_i and App i's revenue sharing rate λ_i (Differentiating Equation (13) with respect to λ_i, we have $\frac{\partial \phi_i{}^*}{\partial \lambda_i} = \frac{4}{(1+\lambda_i)^2} > 0$). Further, the equivalent condition of $0 \leq \phi_i{}^* \leq 1$ is $1/3 \leq \lambda_i \leq 1$, implying that when App i doesn't adopt in-App advertising mode, even if no advertising subsidies are provided ($\phi_i{}^* = 0$), the platform owner will still share at least 1/3 of App i's sales revenue. Relatively, as for the platform owner, bearing all the advertising costs of App i ($\phi_i{}^* = 1$) means sharing all the sales revenue of App i ($\lambda_i = 1$).

By Equation (14), App i's optimal advertising effort $u_{Ai}{}^*$ increases with γ_i and p_{Ai}. Interestingly, the revenue sharing proportion λ_i has a positive effect on $u_{Ai}{}^*$. The reason behind this is that $\partial \phi_i{}^*/\partial \lambda_i > 0$, which indicates that a higher revenue sharing rate λ_i corresponds to a higher optimal advertising subsidy rate $\phi_i{}^*$, which will eventually encourage App i developer to increase his optimal advertising effort $u_{Ai}{}^*$.

As for Case 2:

The above condition is to guarantee that the optimal solutions are all positive. In particular, when $\eta_i = 0$, that means in-App advertising caused by App i will not have a negative impact on App j's user growth, then Equation (15) becomes

$$\phi_i{}^* = (1 - \sum_{j=1, j\neq i}^{n} \xi_{ji}) \frac{3\lambda_i - 1 + \sum_{j=1, j\neq i}^{n} \theta_{ij}}{1 + \lambda_i - \sum_{j=1, j\neq i}^{n} \theta_{ij}}$$

Thus, the equivalence of $0 \leq \phi_i{}^* \leq 1$ is

$$\frac{1}{3}(1 - \sum_{j=1, j\neq i}^{n} \theta_{ij}) \leq \lambda_i \leq \frac{2 - \sum_{j=1, j\neq i}^{n} \xi_{ji}}{2 - 3\sum_{j=1, j\neq i}^{n} \xi_{ji}}(1 - \sum_{j=1, j\neq i}^{n} \theta_{ij})$$

Since $0 \leq \lambda_i \leq 1$, at this time the condition that ξ_{ji} has to satisfy is

$$\sum_{j=1, j\neq i}^{n} \xi_{ji} \leq \frac{2\sum_{j=1, j\neq i}^{n} \theta_{ij}}{2 + \sum_{j=1, j\neq i}^{n} \theta_{ij}}$$

which means that $\sum\limits_{j=1,j\neq i}^{n} \xi_{ji}$ will not exceed 2/3.

Next, we make the comparative static analysis of App i's advertising subsidy rate ϕ_i* and optimal advertising effort u_{Ai}* as follows.

By Equation (15):

1. The parameter η_i characterizes the negative effect when App i delivers ads into the other Apps. Differentiating Equation (15) with respect to η_i, we have $\partial\phi_i$*/$\partial\eta_i$ < 0, implying that the lower the negative effectiveness of in-App advertising caused by App i, the higher the advertising subsidy rate set by the platform owner to App i (Since $0 \leq \lambda_i + \sum\limits_{j=1,j\neq i}^{n} \theta_{ij} \leq 1$, we obtain $\frac{\partial\phi_i^*}{\partial\eta_i} = (1 - \sum\limits_{j=1,j\neq i}^{n} \xi_{ji}) \frac{-4(1-\lambda_i - \sum\limits_{j=1,j\neq i}^{n} \theta_{ij})\gamma_i p_{Ai} \sum\limits_{j=1,j\neq i}^{n} \lambda_j p_{Aj}}{[(1+\lambda_i - \sum\limits_{j=1,j\neq i}^{n} \theta_{ij})\gamma_i p_{Ai} - 2\eta_i \sum\limits_{j=1,j\neq i}^{n} \lambda_j p_{Aj}]^2} < 0$). Therefore, it can be explained that in determining the advertising subsidy rate, the platform owner definitely takes into account the negative impact that App i will have on the other Apps, that is, the loss of other Apps users. Interestingly, ϕ_i* is independent of η_j, namely, the negative effectiveness of in-App advertising caused by App j when App j delivers ads into App i. This happens because the platform owner sets App i developer's advertising subsidy rate only according to the negative effectiveness of in-App advertising from App i himself, regardless of the other Apps.
2. Just like the proof of $\partial\phi_i$*/$\partial\eta_i$ < 0, we can prove that $\partial\phi_i$*/∂p_{Aj} < 0, implying that a higher App j's marginal sales revenue means a lower App i's advertising subsidy rate. Similarly, we have $\partial\phi_i$*/$\partial\lambda_j$ < 0, implying that the more the platform owner shares the sales revenue of App j, the less the advertising subsidies will be provided to App i. The above two scenarios are consistent with intuition, indicating that there is a competitive relationship between Apps, and the platform owner will adjust her advertising subsidy policies appropriately according to the profitability performance of Apps.
3. Differentiating Equation (15) with respect to θ_{ij}, we have $\partial\phi_i$*/$\partial\theta_{ij}$ > 0, implying that the higher the cost of in-App advertising paid by App i to App j (θ_{ij}), the higher the advertising subsidy provided by the platform to App i (ϕ_i). In addition, it's clear that $\partial\phi_i$*/$\partial\xi_{ji}$ < 0, implying that the more advertising subsidies App j provides to App i, the less advertising subsidies the platform provides to App i. The above two situations indicate that, when determining the advertising subsidy rate, the platform owner will comprehensively consider the advertising cost and advertising subsidies from other Apps under the in-App advertising mode.

By (16), we find that when faced with a higher η_i, App i developer prefers to decrease his optimal advertising effort u_{Ai}*. The reason behind this is that $\partial\phi_i$*/$\partial\eta_i$ < 0, which means when the negative effectiveness of in-App advertising caused by App i is relatively high (η_i), the platform owner will appropriately reduce App i's advertising subsidy rate (ϕ_i*), leading to a lower u_{Ai}*. Besides, u_{Ai}* is independent of η_j, which indicates that when making the advertising decision, App i does not need to consider the negative impact of in-App advertising caused by other Apps. Interestingly, u_{Ai}* decreases in p_{Aj} and λ_j. The reason for this is that when more Apps join the platform, App j's higher marginal sales revenue p_{Aj} or higher revenue sharing rate λ_j will cause the platform owner to reduce the advertising subsidy rate ϕ_i ($\partial\phi_i$*/∂p_{Aj} < 0 and $\partial\phi_i$*/$\partial\lambda_j$ < 0 by Equation (15)), leading to the corresponding reduction of u_{Ai}*. In addition, it's clear that ∂u_{Ai}*/$\partial\theta_{ij}$ < 0, implying that the more in-App advertising costs that App i pays to App j, the less optimal advertising effort App i exerts. Meanwhile, we have ∂u_{Ai}*/$\partial\xi_{ji}$ > 0, implying that u_{Ai}* increases with the advertising subsidies App j provides to App i.

Proposition 2. *In the decentralized system, when every App displays the other Apps' ads:*

1. *Suppose that* $\gamma_i u^*_{Ai} - \sum\limits_{j=1, j\neq i}^{n} \eta_j u^*_{Aj} > 0$. *Then, the unique steady-state of the number of the platform users and App i's users, represented by* x_{SS} *and* y_{SSi}, *are given by*

$$x_{SS} = \frac{\alpha u^*_P}{\delta} = \alpha^2 \frac{(\rho+\delta)p_P + \sum\limits_{i=1}^{n} \lambda_i \beta_i p_{Ai}}{\delta(\rho+\delta)^2} \tag{17}$$

$$\begin{aligned} y_{SSi} &= \tfrac{1}{\delta}\left(\tfrac{\alpha\beta_i}{\delta} u^*_P + \gamma_i u^*_{Ai} - \sum\limits_{j=1,j\neq i}^{n} \eta_j u^*_{Aj}\right) \\ &= \frac{1}{\delta(\rho+\delta)} \left\{ \alpha^2 \beta_i \left(\frac{p_P}{\delta} + \frac{\sum\limits_{i=1}^{n} \lambda_i \beta_i p_{Ai}}{\delta(\rho+\delta)} \right) + \frac{\gamma_i [(1+\lambda_i - \sum\limits_{j=1,j\neq i}^{n} \theta_{ij})\gamma_i p_{Ai} - 2\eta_i \sum\limits_{j=1,j\neq i}^{n} \lambda_j p_{Aj}]}{2(1 - \sum\limits_{j=1,j\neq i}^{n} \xi_{ji})} \right. \\ &\quad \left. - \sum\limits_{j=1,j\neq i}^{n} \frac{\eta_j}{2(1-\sum\limits_{i=1,i\neq j}^{n} \xi_{ij})} [(1+\lambda_j - \sum\limits_{i=1,i\neq j}^{n} \theta_{ji})\gamma_j p_{Aj} - 2\eta_j \sum\limits_{i=1,i\neq j}^{n} \lambda_i p_{Ai}] \right\} \\ i,\ j &\in \{1,\ldots,n\} \text{ and } i \neq j. \end{aligned} \tag{18}$$

2. *The optimal trajectories of the number of platform users and App i's users are*

$$x(t) = (x_0 - x_{ss})e^{-\delta t} + x_{ss} \tag{19}$$

$$y_i(t) = [y_0 - y_{SSi} + \beta_i(x_0 - x_{SS})t]e^{-\delta t} + y_{SSi} \tag{20}$$

The condition in Part 1 is to guarantee the steady-state (x_{SS}, y_{SSi}) is positive. Next, we make the comparative static analysis of x_{SS} and y_{SSi}, respectively.

By Equation (17), x_{SS} increases with α and p_P, implying that the advertising effectiveness α and the marginal sales revenue p_P both have a positive effect on the number of platform users x_{SS}. More importantly, x_{SS} also increases with the number of Apps joining the platform.

By Equation (18), it's not difficult to find that the platform's marginal sales revenue p_P, the platform's user base on App i's user growth β_i, as well as the platform's and App i's advertising effectiveness, α and γ_i, all have positive impacts on the number of App i's users y_{SSi}. Further, since $\partial\phi_i^*/\partial\eta_i < 0$ and $\partial u_{Ai}^*/\partial\eta_i < 0$, we have $\partial y_{SSi}/\partial\eta_i < 0$. That means the greater the negative effectiveness of in-App advertising caused by App i (η_i), the lower the App i's advertising subsidy rate set by the platform owner (ϕ_i^*), which leads to the decrease of App i's optimal advertising effort (u_{Ai}^*) and the number of App i's users (y_{SSi}).

Note that the number of the platform users and App i's users in Equations (19) and (20) will ultimately achieve their own steady-state x_{SS} and y_{SSi} when $t \to +\infty$.

4.2. Integrated System

In this section, we assume that the platform owner and App developers are vertically integrated, and the objective of integrated system is given by Equation (9). Then the HJB equation for Equation (9) is

$$\rho V_I = p_P x - \frac{u_P^2}{2} + V_{Ix}(\alpha u_P - \delta x) + \sum_{i=1}^{n} [p_{Ai} y_i - \frac{u_{Ai}^2}{2} + V_{Iyi}(\gamma_i u_{Ai} + \beta_i x - \delta y_i - \sum_{j=1,j\neq i}^{n} \eta_j u_{Aj})] \tag{21}$$

where $V_{Ix} = \partial V_I/\partial x$ and $V_{Iyi} = \partial V_I/\partial y_i$. Aiming at the above optimization problems, the optimal advertising efforts of the platform and App i are proposed in Proposition 3.

Proposition 3. *Suppose that* $\eta_i < \frac{\gamma_i p_{Ai}}{\sum\limits_{j=1,j\neq i}^{n} p_{Aj}}$. *Then, in the integrated system, the respective optimal advertising efforts of the platform owner and App i developer are given by*

$$\overline{u}_P^* = \alpha \frac{(\rho+\delta)p_P + \sum\limits_{i=1}^{n} \beta_i p_{Ai}}{(\rho+\delta)^2} \tag{22}$$

$$\overline{u}_{Ai}^* = \frac{\frac{\gamma_i p_{Ai} - \eta_i \sum\limits_{j=1,j\neq i}^{n} p_{Aj}}{\rho+\delta}}{} \quad i,\ j \in \{1,\ldots,n\} \text{ and } i \neq j. \tag{23}$$

The above condition is to guarantee App i's optimal advertising effort is positive. Different from Equations (12) and (14), here Equations (22) and (23) don't have the parameter λ, which means in the integrated system the optimal advertising efforts have nothing to do with the sales revenue sharing rate.

Proposition 4. *In the integrated system, when every App displays the other Apps' ads:*

1. *Suppose that* $\gamma_i \overline{u}_{Ai}^* - \sum\limits_{j=1,j\neq i}^{n} \eta_j \overline{u}_{Aj}^* > 0$. *Then, the unique steady-state of the number of platform users and App i's users, represented by* $\overline{x}_{SS}$ *and* $\overline{y}_{SSi}$, *are given by*

$$\overline{x}_{SS} = \frac{\alpha \overline{u}_P^*}{\delta} = \alpha^2 \frac{(\rho+\delta)p_P + \sum\limits_{i=1}^{n} \beta_i p_{Ai}}{\delta(\rho+\delta)^2} \tag{24}$$

$$\begin{aligned} \overline{y}_{SSi} &= \tfrac{1}{\delta}\left(\tfrac{\alpha\beta_i}{\delta}\overline{u}_P^* + \gamma_i \overline{u}_{Ai}^* - \sum\limits_{j=1,j\neq i}^{n} \eta_j \overline{u}_{Aj}^*\right) = \tfrac{1}{\delta(\rho+\delta)}\Bigg[\tfrac{\alpha^2\beta_i p_P}{\delta} + \tfrac{\alpha^2\beta_i \sum\limits_{i=1}^{n} \beta_i p_{Ai}}{\delta(\rho+\delta)} \\ &+ \gamma_i\Big(p_{Ai} - \eta_i \sum\limits_{j=1,j\neq i}^{n} p_{Aj}\Big) - \sum\limits_{j=1,j\neq i}^{n} \eta_j\Big(\gamma_j p_{Aj} - \eta_j \sum\limits_{i=1,i\neq j}^{n} p_{Ai}\Big)\Bigg] \\ & i,\ j \in \{1,\ldots,n\} \text{ and } i \neq j. \end{aligned} \tag{25}$$

2. *The optimal trajectories of the number of platform users and App i's users are*

$$x(t) = (x_0 - \overline{x}_{SS})e^{-\delta t} + \overline{x}_{SS} \tag{26}$$

$$y_i(t) = [y_0 - \overline{y}_{SSi} + \beta_i(x_0 - \overline{x}_{SS})t]e^{-\delta t} + \overline{y}_{SSi} \tag{27}$$

The condition in Part 1 is to guarantee that the steady-state of the integrated system is positive. Next, we will calculate the present values of profits of the decentralized system and the integrated system to compare the efficiency of the two systems.

4.3. Optimal Present Values of Profits in Two Systems

Proposition 5. *When every App displays the other Apps' ads:*

1. *In the decentralized system, the optimal present values of profits of the platform owner and n App developers, are given by (Here, n Apps are regarded as a whole mathematically, which can obtain some regular results and facilitate the comparison with subsequent results):*

$$V_P^* = \frac{u_P^{*2} + \sum_{i=1}^n u_{Ai}^{*2}}{2\rho} + \frac{(\rho+\delta)p_P + \sum_{i=1}^n \lambda_i\beta_i p_{Ai}}{(\rho+\delta)^2}x_0 + \sum_{i=1}^n \frac{\lambda_i p_{Ai}}{\rho+\delta}y_{i0} \tag{28}$$

$$\sum_{i=1}^n V_{Ai}^* = \frac{(\overline{u}_P^* - u_P^*)u_P^* + \sum_{i=1}^n u_{Ai}^*(\overline{u}_{Ai}^* - u_{Ai}^*)}{\rho} + \frac{\sum_{i=1}^n (1-\lambda_i)p_{Ai}}{(\rho+\delta)^2}[\beta_i x_0 + (\rho+\delta)y_{i0}] \tag{29}$$

And the sum of V_P and $\sum V_{Ai}$* represented by V_D*, is given by*

$$V_D^* = V_P^* + \sum_{i=1}^n V_{Ai}^* = \frac{u_P^*(2\overline{u}_P^* - u_P^*) + \sum_{i=1}^n u_{Ai}^*(2\overline{u}_{Ai}^* - u_{Ai}^*)}{2\rho} + \frac{\overline{u}_P^*}{\alpha}x_0 + \sum_{i=1}^n \frac{p_{Ai}}{\rho+\delta}y_{i0} \tag{30}$$

2. *In the integrated system, the optimal present value of total profit, is given by*

$$V_I{}^* = \frac{\overline{u}_P^{*2} + \sum_{i=1}^n \overline{u}_{Ai}^{*2}}{2\rho} + \frac{\overline{u}_P^*}{\alpha}x_0 + \sum_{i=1}^n \frac{p_{Ai}}{\rho+\delta}y_{i0} \tag{31}$$

Using Equations (30) and (31), we compare the optimal present value of profit between the decentralized system and integrated system, that is, V_D* and V_I*. The results are presented in the following corollary.

Corollary 1. *The optimal present value of total profit in the integrated system is always greater than or equal to the sum of the optimal present values of profits of the platform owner and n App developers in the decentralized system, that is V_I* ≥ V_D*.*

Corollary 1 indicates that the decentralized system is less efficient than the integrated system. The next research question to address is how to improve the efficiency of the decentralized system.

5. Sustainable Cooperative Advertising Mechanism

5.1. Platform-App Channel Coordination

The focus of this section is on how to improve the efficiency of the decentralized system with the expectation that the objective of integrated system can be implemented (V_I*).

To achieve the above purposes, we propose a sustainable cooperative advertising mechanism, in which the platform owner and App developers need to share each other's advertising costs at the participation rate of ψ_i and φ_i respectively. Note that, under this mechanism, the platform owner's participation rate ψ_i, is exactly App i's advertising subsidy rate set by the platform owner, ϕ_i mentioned in Section 4. The difference is that the platform owner's participation rate ψ_i is a parameter to be determined, while App i's advertising subsidy rate ϕ_i appears as a control variable. For any given sustainable cooperative advertising mechanism (ψ_i, φ_i), I € {1, . . . , n}, the optimization problems of the platform owner and App i developer are

$$\begin{gathered}\hat{V}_P(x,y) = \max_{\hat{u}_P \geq 0}\int_0^\infty e^{-\rho t}\left[p_P x - \tfrac{1}{2}\hat{u}_P^2 + \sum_{i=1}^n(\lambda_i p_{Ai}y_i + \tfrac{1}{2}\varphi_i\hat{u}_P^2 - \tfrac{1}{2}\psi_i\hat{u}_{Ai}^2)\right]dt \\ \text{s.t. (1) and (2),}\end{gathered} \tag{32}$$

$$\begin{aligned}\hat{V}_{Ai}(x,y) = \max_{\hat{u}_{Ai}\geq 0}\int_0^\infty e^{-\rho t}\Big[&(1-\lambda_i)p_{Ai}y_i - \tfrac{1}{2}\hat{u}_{Ai}^2 + \tfrac{1}{2}\psi_i\hat{u}_{Ai}^2 - \tfrac{1}{2}\varphi_i\hat{u}_P^2 \\ &+ \sum_{j=1,j\neq i}^n(\theta_{ji}p_{Aj}y_j + \tfrac{1}{2}\xi_{ji}\hat{u}_{Ai}^2 - \theta_{ij}p_{Ai}y_i - \tfrac{1}{2}\xi_{ij}\hat{u}_{Aj}^2)\Big]dt \\ \text{s.t. (1) and (2).}\end{aligned} \tag{33}$$

If the decentralized system wants to be coordinated, then under the effect of the sustainable cooperative advertising mechanism, the optimal advertising efforts of the platform owner and App i developer in the decentralized system must be equal to the corresponding ones in the integrated system, i.e., $\hat{u}_P^* = \overline{u}_P^*$ and $\hat{u}_{Ai}^* = \overline{u}_{Ai}^*$.

When the participation rates ψ_i and φ_i are fixed, we obtain the optimal advertising efforts of the platform owner and App i developer as shown in the following proposition.

Proposition 6. *Under the sustainable cooperative advertising mechanism, when the participation rates ψ_i and φ_i are fixed, the optimal advertising efforts of the platform owner and App i developer are*

$$\hat{u}_P^* = \alpha \frac{(\rho+\delta)p_P + \sum\limits_{i=1}^{n} \lambda_i \beta_i p_{Ai}}{(\rho+\delta)^2 (1 - \sum\limits_{i=1}^{n} \varphi_i)} \tag{34}$$

$$\hat{u}_{Ai}^* = \frac{(1 - \lambda_i - \sum\limits_{j=1, j\neq i}^{n} \theta_{ij})\gamma_i p_{Ai}}{(\rho+\delta)(1 - \psi_i - \sum\limits_{j=1, j\neq i}^{n} \xi_{ji})} \tag{35}$$

Once there exists a specific pair of (ψ_i, φ_i), I € {1, … ,n}, which makes Equation (22) equal to Equation (34), and Equation (23) equal to Equation (35), then the decentralized system can be effectively coordinated. Solving $\hat{u}_P^* = \overline{u}_P^*$ and $\hat{u}_{Ai}^* = \overline{u}_{Ai}^*$, we get the following proposition.

Proposition 7. *Given the condition $\eta_i < \frac{(\lambda_i + \sum\limits_{j=1, j\neq i}^{n} \theta_{ij})\gamma_i p_{Ai}}{\sum\limits_{j=1, j\neq i}^{n} p_{Aj}}$ holds, then there exists a unique set of participation rates $(\hat{\psi}_i, \hat{\varphi}_i)$ to coordinate the decentralized system, when*

$$\hat{\psi}_i = \frac{(\lambda_i + \sum\limits_{j=1, j\neq i}^{n} \theta_{ij})\gamma_i p_{Ai} - \eta_i \sum\limits_{j=1, j\neq i}^{n} p_{Aj}}{\gamma_i p_{Ai} - \eta_i \sum\limits_{j=1, j\neq i}^{n} p_{Aj}} - \sum_{j=1, j\neq i}^{n} \xi_{ji} \tag{36}$$

$$\sum_{i=1}^{n} \hat{\varphi}_i = \frac{\sum\limits_{i=1}^{n} (1-\lambda_i)\beta_i p_{Ai}}{(\rho+\delta)p_P + \sum\limits_{i=1}^{n} \beta_i p_{Ai}} \tag{37}$$

The above condition is to ensure that the participation rates satisfy between 0 and 1.

By Equation (36), when determining the participation rate of App i, the platform owner will consider the negative impact of in-App advertising caused by App i on the other Apps (η_i). Especially, when no App developer adopts in-App advertising mode, that is, $\eta_i = \xi_{ji} = \theta_{ij} = 0$, the platform owner's participation rate is just equal to the sales revenue sharing rate of App i, that is

$$\hat{\psi}_i = \lambda_i$$

By Equation (37), since $0 < \lambda_i < 1$, I € {1, … ,n}, we have $0 \leq \sum\limits_{i=1}^{n} \hat{\varphi}_i \leq 1$, which implies the achievement of coordination if and only if the sum of the participation rates of n Apps is shown as

Equation (37). This may cause free-riding behavior among Apps, that is, one App may have a high participation rate, while the other App may have a low participation rate.

Once the decentralized system has been coordinated with the integrated system, the number of the platform users and App i users in the decentralized system will respectively equal to the corresponding number of users in the integrated system. Substituting Equations (26) and (27) into Equations (32) and (33) respectively. We get the present values of profits of the platform owner and n App developers under the sustainable cooperative advertising mechanism, as shown in the following proposition.

Proposition 8. *Under the sustainable cooperative advertising mechanism, the optimal present values of profits of the platform owner and n App developers, are given by:*

$$\hat{V}_P^* = \frac{u_P^* \hat{u}_P^* + \sum_{i=1}^n (2u_{Ai}^* - \hat{u}_{Ai}^*)\hat{u}_{Ai}^*}{2\rho} + \frac{(\rho+\delta)p_P + \sum_{i=1}^n \lambda_i \beta_i p_{Ai}}{(\rho+\delta)^2} x_0 + \sum_{i=1}^n \frac{\lambda_i p_{Ai}}{\rho+\delta} y_{i0} \tag{38}$$

$$\sum_{i=1}^n \hat{V}_{Ai}^* = \frac{(\hat{u}_P^* - u_P^*)\hat{u}_P^* + 2\sum_{i=1}^n (\hat{u}_{Ai}^* - u_{Ai}^*)\hat{u}_{Ai}^*}{2\rho} + \frac{\sum_{i=1}^n (1-\lambda_i) p_{Ai}}{(\rho+\delta)^2} [\beta_i x_0 + (\rho+\delta) y_{i0}] \tag{39}$$

and the sum of $\hat{V}_P^*$ and $\sum_{i=1}^n \hat{V}_{Ai}^*$, represented by $V_M{}^*$, is given by

$$V_M^* = \hat{V}_P^* + \sum_{i=1}^n \hat{V}_{Ai}^* = \frac{\hat{u}_P^{*\,2} + \sum_{i=1}^n \hat{u}_{Ai}^{*\,2}}{2\rho} + \frac{\hat{u}_P^*}{\alpha} x_0 + \sum_{i=1}^n \frac{p_{Ai}}{\rho+\delta} y_{i0}. \tag{40}$$

Using Equations (31) and (40), we compare the optimal present value of profit between the integrated system and the new decentralized system (under the sustainable cooperative advertising mechanism), that is, $V_I{}^*$ and $V_M{}^*$. The results are presented in the following corollary.

Corollary 2. *Under the sustainable cooperative advertising mechanism, the sum of the optimal present value of profits of the platform owner and n App developers is definitely equal to the optimal present value of total profit in the integrated system that is $V_M{}^* = V_I{}^*$.*

Corollary 2 shows that the sustainable cooperative advertising mechanism not only improves the efficiency of the original decentralized system, but also successfully implements the objective of the integrated system while ensuring that the new decentralized system has been coordinated with the integrated system.

Under the action of the sustainable cooperative advertising mechanism, the efficiency of the decentralized system has been improved and implements the objective of the integrated system ($V_I{}^*$). However, as for the system members, whether the present values of profits of the platform owner and n App developers have been increased simultaneously or only one party increases while the other party decreases. In response to the above questions, we present the relevant results in the following proposition.

Proposition 9. *By comparing the present values of profits of the platform owner and n App developers in the decentralized system with and without the sustainable cooperative advertising mechanism, the following results are obtained:*

1. *As for the platform owner, the necessary and sufficient condition for $\hat{V}_P^* \geq V_P^*$ is*

$$\sum_{i=1}^n (\hat{u}_{Ai}^* - u_{Ai}^*)^2 \leq u_P^* (\hat{u}_P^* - u_P^*)$$

2. *As for n App developers, the necessary and sufficient condition for* $\sum_{i=1}^{n} \hat{V}_{Ai}^* \geq \sum_{i=1}^{n} V_{Ai}^*$ *is*

$$\sum_{i=1}^{n} (\hat{u}_{Ai}^* - u_{Ai}^*)^2 \geq (u_P^* - \frac{\hat{u}_P^*}{2})(\hat{u}_P^* - u_P^*)$$

Proposition 9 proposes what conditions need to be satisfied so that the present values of the profits of the platform owner and n App developers can be respectively improved under the sustainable cooperative advertising mechanism. Further, in combination with the above conditions, we propose what conditions need to be satisfied when the platform owner and App developers face different situations together.

Corollary 3. *Under the effect of the sustainable cooperative advertising mechanism:*

1. *The necessary and sufficient condition for* $\hat{V}_P^* \geq V_P^*$ *and* $\sum_{i=1}^{n} \hat{V}_{Ai}^* \leq \sum_{i=1}^{n} V_{Ai}^*$ *is*

$$\sum_{i=1}^{n} (\hat{u}_{Ai}^* - u_{Ai}^*)^2 \leq (u_P^* - \frac{\hat{u}_P^*}{2})(\hat{u}_P^* - u_P^*)$$

2. *The necessary and sufficient condition for*$\hat{V}_P^* \leq V_P^*$ *and* $\sum_{i=1}^{n} \hat{V}_{Ai}^* \geq \sum_{i=1}^{n} V_{Ai}^*$ *is*

$$\sum_{i=1}^{n} (\hat{u}_{Ai}^* - u_{Ai}^*)^2 \geq u_P^*(\hat{u}_P^* - u_P^*)$$

3. *The necessary and sufficient condition for*$\hat{V}_P^* \geq V_P^*$ *and* $\sum_{i=1}^{n} \hat{V}_{Ai}^* \geq \sum_{i=1}^{n} V_{Ai}^*$ *is*

$$(u_P^* - \frac{\hat{u}_P^*}{2})(\hat{u}_P^* - u_P^*) \leq \sum_{i=1}^{n} (\hat{u}_{Ai}^* - u_{Ai}^*)^2 \leq u_P^*(\hat{u}_P^* - u_P^*)$$

4. *The scenario where* $\hat{V}_P^* \leq V_P^*$ *and* $\sum_{i=1}^{n} \hat{V}_{Ai}^* \leq \sum_{i=1}^{n} V_{Ai}^*$ *cannot occur simultaneously.*

Parts 1, 2, and 4 show that one of the platform owner and n App developers will suffer the loss of benefits, but it is impossible for both parties to sacrifice the benefits. The platform and Apps need to satisfy the above-mentioned condition in Part 3, if they want to improve the profits together under the sustainable cooperative advertising mechanism. It is worth noting that n Apps are regarded as a whole, $\sum V_{Ai}{}^*$ mentioned here is the sum of the present values of profits of n Apps. The purpose of such mathematical treatment is to obtain some regular results and to facilitate the comparison of $\sum V_{Ai}{}^*$ in the two systems to form important conclusions. As for whether the present value of profit of each App has been improved or not, it is uncertain.

In order to more intuitively highlight the role of the sustainable cooperative advertising mechanism, next, we will conduct a parameters sensitivity analysis to characterize the changes in the optimal present values of profits in terms of the platform's advertising effectiveness.

5.2. Parameters Sensitivity Analysis

Taking one platform and two Apps as examples, we analyze the relationship between the advertising effectiveness of the platform and the present values of profits. That means the platform's advertising effectiveness α varies and all the other parameters are fixed as follows: $p_P = 25$, $p_{A1} = 18$,

$p_{A2} = 20$, $\gamma_1 = 0.8$, $\gamma_2 = 1$, $\beta_1 = 0.1$, $\beta_2 = 0.15$, $\eta_1 = 0.1$, $\eta_2 = 0.15$, $\rho = 0.1$, $\delta = 0.1$, $\lambda_1 = 0.35$, $\lambda_2 = 0.4$, $\theta_{12} = 0.1$, $\theta_{21} = 0.15$, $\xi_{12} = 0.02$, $\xi_{21} = 0.05$, $x_0 = 200$, $y_{10} = 100$ and $y_{20} = 150$.

Firstly, we draw the tendency of V_D* and V_I* in Figure 3, where V_D* means the sum of the optimal present values of profits of the platform owner and two Apps developers in the decentralized system, while V_I* means the optimal present value of total profit in the integrated system. As shown in Figure 3, both V_D* and V_I* increase with α, but V_I* is always above V_D*, which verifies the conclusion of Corollary 1 (V_I* ≥ V_D*).

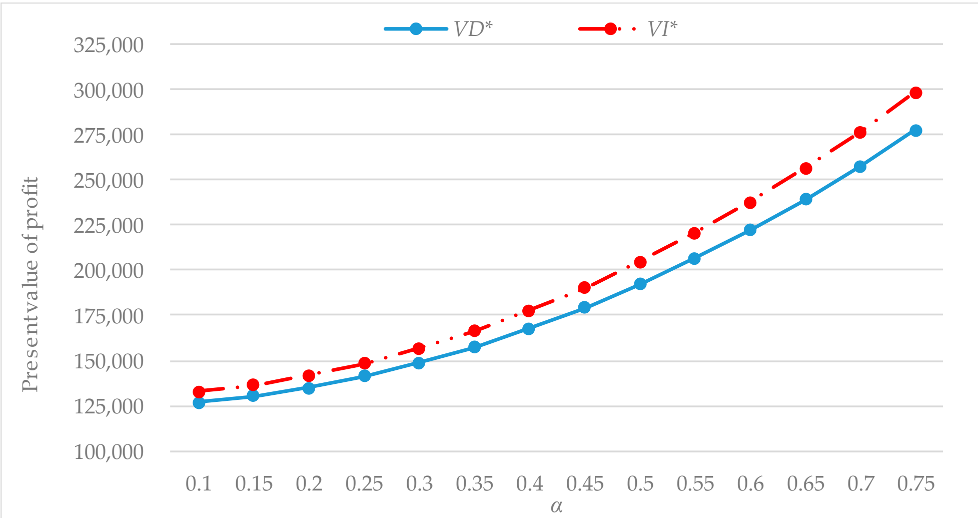

Figure 3. Present value of profits comparison between the decentralized and integrated system.

Secondly, we use the numerical analysis to evaluate the performance of the platform and two Apps with and without the sustainable cooperative advertising mechanism. In Figure 4, $\Delta V_P^* = \hat{V}_P^* - V_P^*$ and $\Delta V_A^* = \hat{V}_{A1}^* + \hat{V}_{A2}^* - V_{A1}^* - V_{A2}^*$. It is not difficult to find that with the increase of the advertising effectiveness of the platform, α, the additional benefits obtained by the platform owner continues to increase, and the difference also changes from negative to positive. On the contrary, the additional benefits obtained by two Apps show a decreasing trend, and may even suffer a risk of loss (when $\Delta V_A^* < 0$).

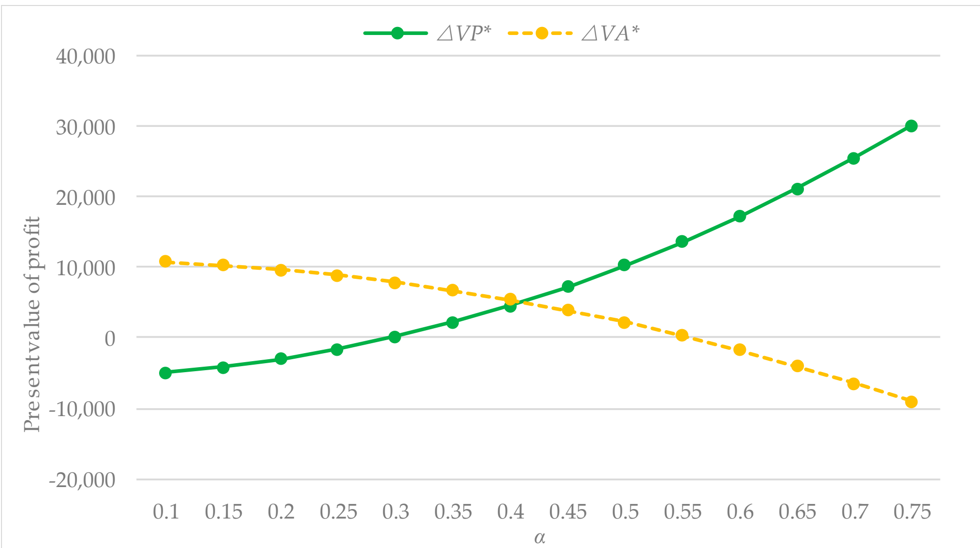

Figure 4. Present value of profits comparison with and without the sustainable cooperative advertising mechanism.

Finally, it is further clarified that the conditions of Part 3 in Corollary 3 can be satisfied, so that the benefits of platform and two Apps can be both improved under the sustainable cooperative advertising mechanism. As shown in Figure 5, with the increase of the advertising effectiveness of the platform, α, both $\Delta V_P^* \geq 0$ and $\Delta V_A^* \geq 0$, implying that under the mechanism, the optimal present values of profits of the platform and two Apps have been increased together.

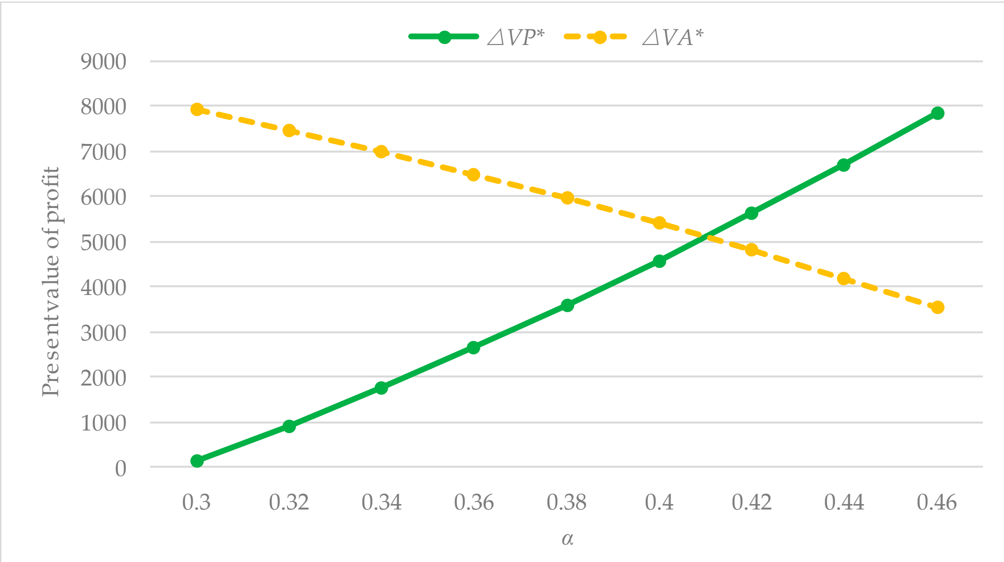

Figure 5. Both the platform and n Apps benefit from the sustainable cooperative advertising mechanism.

6. Discussion

User base is the cornerstone of the platform and Apps to achieve data monetization, and the most efficient way to acquire users is through advertising. As for the platform owner and App developers, how to make the optimal advertising decision is what we pay attention to. Considering one platform and n Apps, the optimal advertising efforts, the steady-state of the user number and the optimal trajectories of the user number, are respectively calculated in the decentralized system and integrated system by utilizing the optimal control theory. Meanwhile, we propose a sustainable cooperative advertising mechanism, requiring the platform and Apps to share each other's advertising costs, so as to effectively make the decentralized system coordinate with the integrated system and implement the optimal objective value of the integrated system.

The main research results of this paper include the following.

1. App i developer just need to take into account the negative impact of in-App advertising caused by himself, regardless of the other Apps when determining his optimal advertising effort. Because the platform owner can use the advertising subsidy policy to adjust. More specifically, the higher the negative effectiveness of in-App advertising caused by App i, the lower the advertising subsidy rate set by the platform owner to App i. Thus, App i developer may reduce her optimal advertising effort accordingly, and the number of App users may also decline. In addition, the advertising subsidy rate set by the platform owner is also closely related to the profitability performance of Apps, as well as the costs and benefits of in-App advertising mode, which means that there exists a competitive relationship among Apps.
2. The optimal present value of total profit in the integrated system is always greater than or equal to the sum of the optimal present values of profits of the platform owner and n App developers in the decentralized system, which means that the decentralized system is inefficient.
3. Our proposed sustainable cooperative advertising mechanism not only can coordinate the decentralized system with the integrated system, but also can implement the optimal objective

value of the integrated system. However, in contrast to the previous research results [49], we theoretically conclude that one of the platform and n Apps may suffer profits loss under the sustainable cooperative advertising mechanism compared with the corresponding ones in the original decentralized system. However, as long as some mild conditions are satisfied, the profits of the platform owner and n App developers can both be improved under the operation of the mechanism.

Our research mainly focuses on the complicated benefit relationship between the platform and App. Due to the complexity and cumbersomeness of mathematical solution caused by involving n Apps, on the issue of the improvement of the present value of profits, n Apps are treated as a whole mathematically which can obtain regular conclusions and facilitate the comparison of the results between the two systems. In subsequent research, we will focus on the change in the present value of profit of each App, trying to confirm whether each App developer can benefit from the sustainable cooperative advertising mechanism and what conditions are required. Additionally, some valuable extensions of this paper can be further explored. Firstly, the payment model of in-App advertising can be expanded to include click rate, display rate and other indicators. Secondly, it would be more interesting to introduce an auction mechanism to filter in-App advertising so as to maximize profits. Finally, this paper only considers the case of one platform and n Apps. Therefore, our model can be extended to a more complex situation of n platforms and n Apps with competing relationships.

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Appendix A

Proof of Proposition 1. Backward induction is used to solve the equilibrium of Stackelberg game. Given the platform owner's decisions $u_P(x, y)$ and $\phi_i(x, y)$, we assume that App i developer' value functions $V_{Ai}(x, y)$: $\Re_+ \to \Re$, $I \in \{1, \ldots, n\}$ are bounded and continuously differentiable and there is a unique and absolutely continuous solution pair $(x(t), y(t))$ to the initial value problem ((1) and (2)), which is the sufficient condition for a stationary Nash equilibrium between n Apps.

Through the first-order condition of (11) with respect to u_{Ai} yields

$$u_{Ai}{}^*(x, y|u_P, \phi_i) = \frac{\gamma_i V_{Aiyi}}{1 - \phi_i - \sum\limits_{j=1, j\neq i}^{n} \xi_{ji}} \tag{A1}$$

Substituting (A1) into (10), the platform owner's HJB equation can be rewritten as

$$\begin{aligned}\rho V_P = &\max_{u_P \geq 0, \phi_i \geq 0} \{p_P x - \tfrac{1}{2}u_P^2 + V_{Px}(\alpha u_P - \delta x) \\ &+ \sum_{i=1}^{n} [\lambda_i p_{Ai} y_i - \frac{\phi_i \gamma_i{}^2 V^2{}_{Aiyi}}{2(1-\phi_i - \sum\limits_{j=1, j\neq i}^{n} \xi_{ji})^2} + V_{Py_i}(\frac{\gamma_i{}^2 V_{Aiyi}}{1-\phi_i - \sum\limits_{j=1, j\neq i}^{n} \xi_{ji}} + \beta_i x - \delta y_i - \sum_{j=1, j\neq i}^{n} \frac{\eta_j \gamma_j V_{Ajyj}}{1-\phi_j - \sum\limits_{i=1, i\neq j}^{n} \xi_{ij}})\}.\end{aligned} \tag{A2}$$

Maximizing the right-hand side of (A2) over u_P and ϕ_i, we obtain the platform owner's equilibrium decisions:

$$u_P^* = \alpha V_{Px} \tag{A3}$$

$$\phi_i{}^* = (1 - \sum_{j=1, j\neq i}^{n} \xi_{ji}) \frac{2\gamma_i V_{Py_i} - \gamma_i V_{Aiyi} - 2\eta_i \sum_{j=1, j\neq i}^{n} V_{Pyj}}{2\gamma_i V_{Py_i} + \gamma_i V_{Aiyi} - 2\eta_i \sum_{j=1, j\neq i}^{n} V_{Pyj}} \quad i,\ j \in \{1, \ldots, n\} \text{ and } i \neq j. \tag{A4}$$

Taking (A1), (A3), and (A4) into account, Equations (10) and (11) can be rewritten as

$$\rho V_P = (p_P - \delta V_{Px} + \sum_{i=1}^{n} \beta_i V_{Py_i})x + \sum_{i=1}^{n} (\lambda_i p_{Ai} - \delta V_{Py_i}) y_i + C_1 \tag{A5}$$

$$\rho V_{Ai} = (\beta_i V_{Aiy_i} - \delta V_{Aix})x + [(1 - \lambda_i) p_{Ai} - \delta V_{Aiyi} - \sum_{j=1, j\neq i}^{n} \theta_{ij} p_{Ai}] y_i + C_2 \tag{A6}$$

where C_1 and C_2 are the remaining constant terms. We conjecture linear functions

$$V_P(x, y) = gx + \sum_{i=1}^{n} g_i y_i + C_1 \tag{A7}$$

$$V_{Ai}(x, y) = h_{Ai1} x + h_{Ai2} y_i + C_2 \tag{A8}$$

Thus, $V_{Px} = g$, $V_{Pyi} = g_i$, $V_{Aix} = h_{Ai1}$ and $V_{Aiyi} = h_{Ai2}$. Substituting the above partial derivatives into (A5) and (A6) and equating the coefficient of x and y_i with zero, we obtain

$$g = \frac{p_P(\rho + \delta) + \sum_{i=1}^{n} \lambda_i \beta_i p_{Ai}}{(\rho + \delta)^2} \tag{A9}$$

$$g_i = \frac{\lambda_i p_{Ai}}{\rho + \delta} \tag{A10}$$

$$h_{Ai1} = \frac{(1 - \lambda_i - \sum_{j=1, j\neq i}^{n} \theta_{ij}) \beta_i}{(\rho + \delta)^2} p_{Ai} \tag{A11}$$

$$h_{Ai2} = \frac{1 - \lambda_i - \sum_{j=1, j\neq i}^{n} \theta_{ij}}{(\rho + \delta)} p_{Ai} \tag{A12}$$

Inserting (A9)–(A12) into (A3), (A4), and (A1) respectively, we obtain the equilibrium results:

$$u_P^* = \alpha \frac{(\rho + \delta) p_P + \sum_{i=1}^{n} \lambda_i \beta_i p_{Ai}}{(\rho + \delta)^2} \tag{A13}$$

$$\phi_i{}^* = (1 - \sum_{j=1, j\neq i}^{n} \xi_{ji}) \frac{(3\lambda_i - 1 + \sum_{j=1, j\neq i}^{n} \theta_{ij}) \gamma_i p_{Ai} - 2\eta_i \sum_{j=1, j\neq i}^{n} \lambda_j p_{Aj}}{(1 + \lambda_i - \sum_{j=1, j\neq i}^{n} \theta_{ij}) \gamma_i p_{Ai} - 2\eta_i \sum_{j=1, j\neq i}^{n} \lambda_j p_{Aj}} \tag{A14}$$

$$u_{Ai}^* = \frac{(1 + \lambda_i - \sum_{j=1, j\neq i}^{n} \theta_{ij}) \gamma_i p_{Ai} - 2\eta_i \sum_{j=1, j\neq i}^{n} \lambda_j p_{Aj}}{2(\rho + \delta)(1 - \sum_{j=1, j\neq i}^{n} \xi_{ji})} \quad i,\ j \in \{1, \ldots, n\} \text{ and } i \neq j, \tag{A15}$$

which are the same results as Case 2 of Proposition 1. It can be verified that when the condition of $(3\lambda_i - 1 + \sum_{j=1, j\neq i}^{n} \theta_{ij})\gamma_i p_{Ai} - 2\eta_i \sum_{j=1, j\neq i}^{n} \lambda_j p_{Aj} > 0$ holds, the above optimal solutions are positive.

Specially, when $\eta_i = \xi_{ji} = \theta_{ij} = 0$, we can obtain the optimal solutions as shown in Case 1 of Proposition 1. □

Proof of Proposition 2.

1. Substituting Equations (12) and (16) into Equations (1) and (2), we get

$$\begin{cases} \dot{x}(t) = \alpha u_P{}^* + \delta x(t),\ x(0) = x_0 > 0. \\ \dot{y}_i(t) = \gamma_i u_{Ai}{}^* + \beta_i x(t) - \delta y_i(t) - \sum_{j=1, j\neq i}^{n} \eta_j u_{Aj}{}^*,\ y_i(0) = y_{i0} > 0,\ i,\ j \in \{1, \ldots, n\} \text{ and } i \neq j. \end{cases} \tag{A16}$$

   Solving $\dot{x}(t) = 0$ and $\dot{y}_i(t) = 0$ yields the unique intertemporal equilibrium (x_{SS}, y_{SSi}), as shown in Equations (17) and (18) in Part 1 of Proposition 2.
2. Equation (A16) is a system of differential equations. By solving the system (A16), we obtain the optimal trajectories of the decentralized system state, as shown in (19) and (20). □

Proof of Proposition 4 is similar to Proof of Proposition 2, so we omit it in the following appendix.

Proof of Proposition 3. By the first order condition with respect to u_P and u_{Ai} in (21), we have

$$\overline{u}_P^* = \alpha V_{Ix} \tag{A17}$$

and

$$\overline{u}_{Ai}^* = \gamma_i V_{Iyi} - \eta_i \sum_{j=1, j\neq i}^{n} V_{Iyj},\ i,\ j \in \{1, \ldots, n\} \text{ and } i \neq j \tag{A18}$$

Substituting (A17) and (A18) into (21), we can rewrite it as

$$\rho V_I = (p_P - \delta V_{Ix} + \sum_{i=1}^{n} \beta_i V_{Iy_i})x + \sum_{i=1}^{n} (p_{Ai} - \delta V_{Iy_i})y_i + C_3 \tag{A19}$$

where C_3 is the remaining constant term, conjecture a linear function

$$V_I(x, y) = fx + \sum_{i=1}^{n} f_i y_i + C_3 \tag{A20}$$

where $V_{Ix} = f$, $V_{Iyi} = f_i$. Inserting these partial derivatives into (A20) and equating the coefficient of x and y_i with zero, we can obtain

$$f = \frac{p_P(\rho + \delta) + \sum_{i=1}^{n} \beta_i p_{Ai}}{(\rho + \delta)^2} \tag{A21}$$

$$f_i = \frac{p_{Ai}}{\rho + \delta} \tag{A22}$$

Substituting (A21) and (A22) into (A17) and (A18) yields the platform owner's and App i developer's optimal advertising efforts as shown in Proposition 3. □

Proof of Proposition 5. Substituting Equations (19) and (20) into Equations (7) and (8) respectively, we get

$$\begin{aligned} V_P(x,y) = \int_0^\infty e^{-\rho t}\{p_P[(x_0-x_{SS})e^{-\delta t}+x_{SS}] - \tfrac{1}{2}{u_P^*}^2 \\ + \sum_{i=1}^{n}\Big[\lambda_i p_{Ai}\Big\{[y_{i0}-y_{SSi}+\beta_i(x_0-x_{SS})t]e^{-\delta t}+y_{SSi}\Big\} - \tfrac{1}{2}\phi_i^* {u_{Ai}^*}^2\Big]\}dt, \end{aligned} \tag{A23}$$

$$\sum_{i=1}^{n} V_{Ai}(x,y) = \sum_{i=1}^{n}\int_0^\infty e^{-\rho t}\left\{(1-\lambda_i)p_{Ai}\Big\{[y_{i0}-y_{SSi}+\beta_i(x_0-x_{SS})t]e^{-\delta t}+y_{SSi}\Big\} - \frac{(1-\phi_i^*){u_{Ai}^*}^2}{2}\right\} \tag{A24}$$

Solving Equations (A23) and (A24), we obtain

$$\begin{aligned} V_P &= p_P\left(\tfrac{x_0-x_{SS}}{\rho+\delta}+\tfrac{x_{SS}}{\rho}\right) - \tfrac{{u_P^*}^2}{2\rho} + \sum_{i=1}^{n}\Big[\lambda_i p_{Ai}\big[\tfrac{y_{i0}-y_{SSi}}{\rho+\delta}+\tfrac{\beta_i(x_0-x_{SS})}{(\rho+\delta)^2}+\tfrac{y_{SSi}}{\rho}\big] - \tfrac{\phi_i^*{u_{Ai}^*}^2}{2\rho}\Big] \\ &= p_P\tfrac{\delta x_{SS}}{\rho(\rho+\delta)} - \tfrac{{u_P^*}^2}{2\rho} + \sum_{i=1}^{n}\Big[\lambda_i p_{Ai}\big[\tfrac{\delta y_{SSi}}{\rho(\rho+\delta)} - \tfrac{\beta_i x_{SS}}{(\rho+\delta)^2}\big] - \tfrac{\phi_i^*{u_{Ai}^*}^2}{2\rho}\Big] \\ &+ \tfrac{x_0}{(\rho+\delta)}\big[p_P + \sum_{i=1}^{n}\tfrac{\lambda_i\beta_i p_{Ai}}{(\rho+\delta)}\big] + \sum_{i=1}^{n}\lambda_i p_{Ai}\tfrac{y_{i0}}{(\rho+\delta)} \end{aligned} \tag{A25}$$

and

$$\begin{aligned} \sum_{i=1}^{n} V_{Ai} &= \sum_{i=1}^{n}\Big[(1-\lambda_i)p_{Ai}\big[\tfrac{y_{i0}-y_{SSi}}{\rho+\delta}+\tfrac{\beta_i(x_0-x_{SS})}{(\rho+\delta)^2}+\tfrac{y_{SSi}}{\rho}\big] - \tfrac{(1-\phi_i^*){u_{Ai}^*}^2}{2\rho}\Big] \\ &= \sum_{i=1}^{n}\Big[(1-\lambda_i)p_{Ai}\big[\tfrac{\delta y_{SSi}}{\rho(\rho+\delta)} - \tfrac{\beta_i x_{SS}}{(\rho+\delta)^2}\big] - \tfrac{(1-\phi_i^*){u_{Ai}^*}^2}{2\rho}\Big] + \sum_{i=1}^{n}\tfrac{(1-\lambda_i)p_{Ai}}{(\rho+\delta)^2}[\beta_i x_0 + (\rho+\delta)y_{i0}]. \end{aligned} \tag{A26}$$

Plugging Equations (17) and (18) into Equations (A25) and (A26) respectively, and considering Equations (12) and (16), after simplification, we obtain

$$V_P^* = \frac{{u_P^*}^2 + \sum_{i=1}^{n}{u_{Ai}^*}^2}{2\rho} + \frac{(\rho+\delta)p_P + \sum_{i=1}^{n}\lambda_i\beta_i p_{Ai}}{(\rho+\delta)^2}x_0 + \sum_{i=1}^{n}\frac{\lambda_i p_{Ai}}{\rho+\delta}y_{i0}$$

and

$$\sum_{i=1}^{n} V_{Ai}^* = \frac{(\overline{u}_P^* - u_P^*)u_P^* + \sum_{i=1}^{n}u_{Ai}^*(\overline{u}_{Ai}^* - u_{Ai}^*)}{\rho} + \frac{\sum_{i=1}^{n}(1-\lambda_i)p_{Ai}}{(\rho+\delta)^2}[\beta_i x_0 + (\rho+\delta)y_{i0}]$$

which are consistent with Part (1) in Proposition 5.

Proof of Part 2 in Proposition 5 and Proposition 8 are similar to Part 1 in Proposition 5, so we omit them in the following appendix. □

Proof of Corollary 1. Using Equations (30) and (31), we have

$$\begin{aligned} {V_I}^* - V_D^* &= \tfrac{{\overline{u}_P^*}^2 + \sum_{i=1}^{n}{\overline{u}_{Ai}^*}^2 - u_P^*(2\overline{u}_P^* - u_P^*) - \sum_{i=1}^{n}u_{Ai}^*(2\overline{u}_{Ai}^* - u_{Ai}^*)}{2\rho} \\ &= {\overline{u}_P^*}^2 - 2u_P^*\overline{u}_P^* + {u_P^*}^2 + \sum_{i=1}^{n}{\overline{u}_{Ai}^*}^2 - \sum_{i=1}^{n}2u_{Ai}^*\overline{u}_{Ai}^* + \sum_{i=1}^{n}{u_{Ai}^*}^2 \\ &= (\overline{u}_P^* - u_P^*)^2 + \sum_{i=1}^{n}(\overline{u}_{Ai}^* - u_{Ai}^*)^2 \geq 0, \end{aligned}$$

which means ${V_I}^* \geq {V_D}^*$. □

Proof of Proposition 6. Similar to Proposition 1, when the platform owner and App i developer make decisions separately, under the Stackelberg game. we first analyze the object of App i by using backward recursion. As a follower, the problem faced by App i developer is Equation (33), so his HJB equation is

$$\rho\hat{V}_{Ai}(x,y)=\max_{u_{Ai}\geq 0}\Bigg[(1-\lambda_i)p_{Ai}y_i-\frac{(1-\psi_i)u_{Ai}^2}{2}-\frac{\varphi_i u_P^2}{2}+\sum_{j=1,j\neq i}^{n}(\theta_{ji}p_{Aj}y_j+\frac{\xi_{ji}u_{Ai}^2}{2}-\theta_{ij}p_{Ai}y_i-\frac{\xi_{ij}u_{Aj}^2}{2}) \\ +\hat{V}_{Aix}(\alpha u_P-\delta x)+\hat{V}_{Aiyi}(\gamma_i u_{Ai}+\beta_i x-\delta y_i-\sum_{j=1,j\neq i}^{n}\eta_j u_{Aj})\Bigg], \tag{A27}$$

where $\hat{V}_{Aix}=\partial\hat{V}_{Ai}/\partial x$ and $\hat{V}_{Aiyi}=\partial\hat{V}_{Ai}/\partial yi$. To solve the above problem, maximizing the right-hand side of (A27) yields

$$\hat{u}_{Ai}{}^{*}(x,y\big|u_P)=\frac{\gamma_i\hat{V}_{Aiyi}}{1-\psi_i-\sum\limits_{j=1,j\neq i}^{n}\xi_{ji}} \tag{A28}$$

Substituting (A28) into (32), the platform owner's HJB equation can be written as

$$\rho\hat{V}_P(x,y)=\max_{u_P\geq 0}\left\{p_Px-\frac{1}{2}u_P^2+\sum_{i=1}^{n}[\lambda_i p_{Ai}y_i+\frac{\varphi_i u_P^2}{2}-\frac{\psi_i\gamma_i{}^2\hat{V}^2{}_{Aiyi}}{2(1-\psi_i-\sum\limits_{j=1,j\neq i}^{n}\xi_{ji})^2}] \\ +\hat{V}_{Px}(\alpha u_P-\delta x)+\sum_{i=1}^{n}[\hat{V}_{Py_i}(\frac{\gamma_i{}^2\hat{V}_{Aiyi}}{1-\psi_i-\sum\limits_{j=1,j\neq i}^{n}\xi_{ji}}+\beta_i x-\delta y_i-\sum_{j=1,j\neq i}^{n}\eta_j\frac{\gamma_j\hat{V}_{Ajyj}}{1-\psi_j-\sum\limits_{j=1,j\neq i}^{n}\xi_{ji}})]\right\}, \tag{A29}$$

where $\hat{V}_{Px}=\partial\hat{V}_P/\partial x$ and $\hat{V}_{Pyi}=\partial\hat{V}_P/\partial yi$. Through the first order condition of (A29) with respect to u_P, the platform owner's optimal advertising effort is

$$\hat{u}_P^*=\frac{\alpha\hat{V}_{Px}}{1-\sum\limits_{i=1}^{n}\varphi_i} \tag{A30}$$

With (A28) and (A30), the platform owner's and App i developer's HJB equation can be rewritten as

$$\rho\hat{V}_{Ai}=(\beta_i\hat{V}_{Aiy_i}-\delta\hat{V}_{Aix})x+[(1-\lambda_i)p_{Ai}-\delta\hat{V}_{Aiyi}-\sum_{j=1,j\neq i}^{n}\theta_{ij}p_{Ai}]y_i+C_4 \tag{A31}$$

$$\rho\hat{V}_P=(p_P-\delta\hat{V}_{Px}+\sum_{i=1}^{n}\beta_i\hat{V}_{Py_i})x+\sum_{i=1}^{n}(\lambda_i p_{Ai}-\delta\hat{V}_{Py_i})y_i+C_5 \tag{A32}$$

where C_4 and C_5 are remaining constant terms, according their functional form, we conjecture linear functions

$$\hat{V}_P(x,y)=zx+\sum_{i=1}^{n}z_i y_i+C_4 \tag{A33}$$

$$\hat{V}_{Ai}(x,y)=q_{Ai1}x+q_{Ai2}y_i+C_5 \tag{A34}$$

Thus, $\hat{V}_{Px}=z$, $\hat{V}_{Pyi}=zi$, $\hat{V}_{Ax}=q_{Ai1}$, $\hat{V}_{Ayi}=q_{Ai2}$. Substituting these partial derivatives into (A31) and (A32), we have

$$z=g,z_i=g_i,q_{Ai1}=h_{Ai1},q_{Ai2}=h_{Ai2} \tag{A35}$$

Plugging (A35) into (A28) and (A30) respectively, we will obtain the optimal advertising efforts as shown in Proposition 6. □

Proof of Proposition 7. If the decentralized system wants to be coordinated, we must make the optimal advertising efforts in the decentralized system under the sustainable cooperative advertising mechanism equal to the corresponding ones in the integrated system, that is $\hat{u}_P^*=\overline{u}_P^*$ and $\hat{u}_{Ai}^*=\overline{u}_{Ai}^*$.

By equating Equation (22) with Equation (34), Equation (23) with Equation (35), we get the unique set of participation rates $(\hat{\psi}_i, \hat{\varphi}_i)$ as shown in Proposition 8. □

Proof of Corollary 2. Since $\hat{u}_P^* = \overline{u}_P^*$ and $\hat{u}_{Ai}^* = \overline{u}_{Ai}^*$ after the coordination, it's easy to prove that Equation (40) is equal to Equation (31), that is $V_M{}^* = V_I{}^*$. □

Proof of Proposition 9.

1. By subtracting Equation (28) from Equation (38), we have

$$\begin{aligned}\hat{V}_P^* - V_P^* &= \frac{u_P^* \hat{u}_P^* + \sum_{i=1}^n (2u_{Ai}^* - \hat{u}_{Ai}^*)\hat{u}_{Ai}^* - u_P^{*2} - \sum_{i=1}^n u_{Ai}^{*2}}{2\rho} \\ &= \frac{u_P^*(\hat{u}_P^* - u_P^*) + \sum_{i=1}^n \hat{u}_{Ai}^*(u_{Ai}^* - \hat{u}_{Ai}^*)}{2\rho}.\end{aligned}$$

Thus, the necessary and sufficient condition for $\hat{V}_P^* \geq V_P^*$ is

$$u_P^*(\hat{u}_P^* - u_P^*) + \sum_{i=1}^n \hat{u}_{Ai}^*(u_{Ai}^* - \hat{u}_{Ai}^*) \geq 0$$

2. By subtracting Equation (29) from Equation (39), we get

$$\sum_{i=1}^n \hat{V}_{Ai}^* - \sum_{i=1}^n V_{Ai}^* = \frac{(\hat{u}_P^* - u_P^*)\hat{u}_P^* + 2\sum_{i=1}^n (\hat{u}_{Ai}^* - u_{Ai}^*)\hat{u}_{Ai}^* - 2(\overline{u}_P^* - u_P^*)u_P^* - 2\sum_{i=1}^n u_{Ai}^*(\overline{u}_{Ai}^* - u_{Ai}^*)}{2\rho}$$

Since $\overline{u}_P^* = \hat{u}_P^*$ and $\overline{u}_{Ai}^* = \hat{u}_{Ai}^*$, simplifying we have

$$\sum_{i=1}^n \hat{V}_{Ai}^* - \sum_{i=1}^n V_{Ai}^* = \frac{(\hat{u}_P^* - u_P^*)(\frac{\hat{u}_P^*}{2} - u_P^*) + \sum_{i=1}^n (\hat{u}_{Ai}^* - u_{Ai}^*)(\hat{u}_{Ai}^* - u_{Ai}^*)}{\rho}$$

Thus, the necessary and sufficient condition for $\sum\limits_{i=1}^n \hat{V}_{Ai}^* \geq \sum\limits_{i=1}^n V_{Ai}^*$ is

$$(\hat{u}_P^* - u_P^*)(\frac{\hat{u}_P^*}{2} - u_P^*) + \sum_{i=1}^n (\hat{u}_{Ai}^* - u_{Ai}^*)^2 \geq 0$$

□

Proof of Corollary 3.

1. $\hat{V}_P^* \geq V_P^*$ and $\sum\limits_{i=1}^n \hat{V}_{Ai}^* \leq \sum\limits_{i=1}^n V_{Ai}^*$ means

$$\begin{cases} \sum_{i=1}^n (\hat{u}_{Ai}^* - u_{Ai}^*)^2 \leq u_P^*(\hat{u}_P^* - u_P^*) \\ \sum_{i=1}^n (\hat{u}_{Ai}^* - u_{Ai}^*)^2 \leq (u_P^* - \frac{\hat{u}_P^*}{2})(\hat{u}_P^* - u_P^*) \end{cases} \Rightarrow \sum_{i=1}^n (\hat{u}_{Ai}^* - u_{Ai}^*)^2 \leq (u_P^* - \frac{\hat{u}_P^*}{2})(\hat{u}_P^* - u_P^*)$$

2. $\hat{V}_P^* \leq V_P^*$ and $\sum\limits_{i=1}^n \hat{V}_{Ai}^* \geq \sum\limits_{i=1}^n V_{Ai}^*$ means

$$\begin{cases} \sum_{i=1}^n (\hat{u}_{Ai}^* - u_{Ai}^*)^2 \geq u_P^*(\hat{u}_P^* - u_P^*) \\ \sum_{i=1}^n (\hat{u}_{Ai}^* - u_{Ai}^*)^2 \geq (u_P^* - \frac{\hat{u}_P^*}{2})(\hat{u}_P^* - u_P^*) \end{cases} \Rightarrow \sum_{i=1}^n (\hat{u}_{Ai}^* - u_{Ai}^*)^2 \geq u_P^*(\hat{u}_P^* - u_P^*)$$

3. $\hat{V}_P^* \geq V_P^*$ and $\sum\limits_{i=1}^{n} \hat{V}_{Ai}^* \geq \sum\limits_{i=1}^{n} V_{Ai}^*$ means

$$\begin{cases} \sum_{i=1}^{n} (\hat{u}_{Ai}^* - u_{Ai}^*)^2 \leq u_P^*(\hat{u}_P^* - u_P^*) \\ \sum_{i=1}^{n} (\hat{u}_{Ai}^* - u_{Ai}^*)^2 \geq (u_P^* - \frac{\hat{u}_P^*}{2})(\hat{u}_P^* - u_P^*) \end{cases} \Rightarrow (u_P^* - \frac{\hat{u}_P^*}{2})(\hat{u}_P^* - u_P^*) \leq \sum_{i=1}^{n} (\hat{u}_{Ai}^* - u_{Ai}^*)^2 \leq u_P^*(\hat{u}_P^* - u_P^*)$$

4. $\hat{V}_P^* \geq V_P^*$ and $\sum\limits_{i=1}^{n} \hat{V}_{Ai}^* \geq \sum\limits_{i=1}^{n} V_{Ai}^*$ means

$$\begin{cases} \sum_{i=1}^{n} (\hat{u}_{Ai}^* - u_{Ai}^*)^2 \geq u_P^*(\hat{u}_P^* - u_P^*) \\ \sum_{i=1}^{n} (\hat{u}_{Ai}^* - u_{Ai}^*)^2 \leq (u_P^* - \frac{\hat{u}_P^*}{2})(\hat{u}_P^* - u_P^*) \end{cases} \Rightarrow \varnothing$$

This implies that Case 4 won't happen. □

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