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Single-Manufacturer Multi-Retailer Supply Chain Models with Discrete Stochastic Demand

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Abstract: Although there are highly discrete stochastic demands in practical supply chain problems, they are seldom considered in the research on supply chain systems, especially the single-manufacturer multi-retailer supply chain systems. There are no significant differences between continuous and discrete demand supply chain models, but the solutions for discrete random demand models are more challenging and difficult. This paper studies a supply chain system of a single manufacturer and multiple retailers with discrete stochastic demands. Each retailer faces a random discrete demand, and the manufacturer utilizes different wholesale prices to influence each retailer's ordering decision. Both Make-To-Order and Make-To-Stock scenarios are considered. For each scenario, the corresponding Stackelberg game model is constructed respectively. By proving a series of theorems, we transfer the solution of the game model into non-linear integer programming model, which can be easily solved by a dynamic programming method. However, with the increase in the number of retailers and the production capacity of manufacturers, the computational complexity of dynamic programming drastically increases due to the Dimension Barrier. Therefore, the Fast Fourier Transform (FFT) approach is introduced, which significantly reduces the computational complexity of solving the supply chain model.

Keywords: decentralized supply chain; multiple retailers; discrete demand; Fast Fourier Transform

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1. Introduction

Many studies in supply chain management focus on one manufacturer/supplier and one retailer setting; those studies have produced abundant results in the supply chain integration, supply chain coordination, and supply chain inventory management, etc. The studies in one manufacturer/supplier and multiple retailers in the decentralized setting are more complicated and challenging. Though there are attempts at supply chain integration, many decentralized supply chain decisions remain vitally important in the practices.

Most supply chain models consider the uncertain demand as a continuous random variable, but demands for certain products such as luxury automobiles, large appliances, expensive jewelry, watches, etc., are discrete random variables. Though continuous and discrete demand–supply chain models have no significant differences, the discrete random demand models are considered as stochastic, dynamic games between supplier and retailers. The solutions for such models are more challenging and difficult.

This paper studies a single-manufacturer multiple-retailers supply chain system. Each retailer faces a different random discrete demand, and the manufacturer offers differential wholesale prices to each retailer to influence each retailer's ordering quantity, thereby maximizing their expected profit. Both Make-To-Order and Make-To-Stock scenarios are considered. For each scenario, the original stochastic game problems are transferred into discrete optimization problems. Discrete optimization problems can be generally solved by dynamic programming methods. However, as the problem dimension increases, the amount of computation drastically increases and makes them hard to solve. This paper uses the Fast Fourier Transform (FFT) approach (cf., Golub and Van Loan [1]) to solve discrete optimization problems. Birbil et al. [2] use this approach to study the airline seat allocation problem. For the supply chain model in this paper, when there are one manufacturer with a production capacity, C, and m retailers; the computation complexity for dynamic programming is $O(mC^2)$, whereas the computation complexity for FFT is only $O(mC)$.

The rest of the paper is as follows: Section 2 provides a brief review on the related literature. Section 3 presents the model notations and assumptions. The Make-To-Order model is discussed in Section 4, and the Make-To-Stock model is discussed in Section 5, respectively. Numerical examples are presented in Section 6. Concluding remarks and future work suggestions are presented in Section 7.

2. Literature Review

Most current research in single-manufacturer/supplier and multiple-retailers supply chains focuses on supply chain integration and VMI (Vendor Managed Inventory). Darwish and Odah [3] developed a model for a supply chain with single vendor and multiple retailers under VMI mode of operation. This model explicitly includes the VMI contractual agreement between the vendor and retailers. The developed model can easily describe supply chains with capacity constraints by selecting a high penalty cost. Theorems were established to alleviate the complexity of the model and render the mathematics tractable. Moreover, an efficient algorithm was introduced to find the global optimal solution. This algorithm reduces the computational efforts significantly. Mateen and Chatterjee [4] developed analytical models for a single-vendor multiple retailers system, showing various approaches in which the system may be coordinated through VMI. They also discussed the conditions under which each of these approaches may be preferred. Mateen et al. [5] presented an approximate expression for minimizing the expected total cost for a VMI system with one vendor and multiple retailers and tested the expression via simulations. Chen and Chang [6] dealt with the problem of jointly determining the optimal retail price, the replenishment cycle, and the number of shipments for exponentially deteriorating items under conditions of channel coordination, joint replenishment program, and pricing policy. Two profit-maximization models including the non-integrated policy and the integrated policy were formulated with the objective of maximizing the channel-wide profit. The study demonstrated the optimal properties of the models and developed a search algorithm to obtain the optimal solutions.

Glock and Kim [7] studied a single-vendor multi-retailer supply chain and considered the case where the vendor merges with one of its retailers. After the merger, the vendor supplies products to the market both through a direct (integrated) sales channel and the remaining retailers. They compared the pre-merger situation to the post-merger situation and showed under which conditions the merger is beneficial to the vendor, the retailers, the supply chain, and the consumers. The results indicated that the type of competition is of major importance for the structure of the supply chain after the merger, and that under certain conditions, the merger could benefit all parties involved. Yang et al. [8] established the coordination mechanism of dual-channel supply chain. This paper analyzes the complex mechanism of retailer's innovation input level affecting supply chain operation and designs a dual-coordination mechanism. The results show that the optimal combination of wholesale price, retail price, and innovation input level can optimize the operation efficiency of the supply chain; the non-cooperation among channel members affects the retailer's product pricing, reduces the market share of the physical channel, and increases the manufacturer's market demand; this dual coordination mechanism can alleviate the channel conflict and improve the operation efficiency of the supply chain. Monthatipkul and Yenradee [9,10] studied appropriate levels and locations of safety stock of a single product in a supply chain consisting of one centralized warehouse and multiple retailers. It is controlled by the optimal inventory/distribution plan which is obtained by solving a proposed linear programming model. The experimental results showed that the suitable location to carry safety stock depend on the operating environment. The safety stock should be carried at the retailers for all cases. Carrying safety stock at the warehouse is beneficial when the warehouse has no cycle stock due to lot-for-lot ordering. The suitable level of safety stock is dependent on the supply chain cost structure and can be determined by an experiment. Islam et al. [11] considered a three-tier supply chain with a single supplier, a single manufacturer, and multiple retailers. The manufacturer collects raw materials from the supplier to produce finished products and then delivers the finished products to the retailer according to the demand of the retailer. They proposed a manufacturer-managed consignment policy and compared it with the traditional policy. It was found that, compared with the traditional policy, the profits of all supply chain members under the consignment policy are increased, and the three-tier consignment mode is better than the two-tier consignment mode.

Li [12] and Zhang [13] studied information sharing in single-manufacturer multiple-retailer supply chain. Li [12] examined the incentives for firms to share information vertically in a two-level supply chain in which there are an upstream firm (a manufacturer) and many downstream firms (retailers). Vertical information sharing has two effects: "direct effect" due to the changes in strategy by the parties involved in sharing the information and "indirect effect" (or "leakage effect") due to the changes in strategy by other competing firms (who may infer the information from the actions of the informed parties). Both changes would affect the profitability of the firms. The authors showed that the leakage effect discourages the retailers from sharing their demand information with the manufacturer while encouraging them to share their cost information. On the other hand, the direct effect always discourages the retailers from sharing their information. When voluntary information sharing is not possible, they identified conditions under which information can be traded and showed how price should be determined to facilitate such information exchange. They also examined the impact of vertical information sharing on the total supply chain profits and social benefits. Zhang [13] considered a supply chain with one manufacturer in the upstream and two competing retailers in the downstream. Retailers compete with Cournot (quantity) or Bertrand (price), and their products can be substitutes or supplements. This model is more restrictive than Li [12] because it only deals with two retailers, while Li [12] considers any number of retailers. On the other hand, it allows differentiated goods and/or Bertrand model, which Li [12] assumes homogeneous goods and Cournot model. Zhang [13] studied the incentive mechanism of vertical information sharing with a strict non-cooperative game. The results show what the consequences could be if coordination and cooperation efforts fail. Darwish et al. [14] present a newsvendor supply chain model with single supplier and multiple retailers, in which the newsvendor has two ordering opportunities. At the beginning of the sales season, the retailer orders a certain number of products from the supplier in order to achieve the predetermined service level. At the second ordering moment, retailers learn more about demand patterns and use new available demand data to update future demand using the Bayesian method. Based on the updated demand, the retailer evaluates the new service level for the rest of the sales season. If this service level is below a specific value, order the second batch. They establish a general demand distribution model and determine the optimal quantity at the beginning of the sales season and the second order opportunity. Su and Geunes [15] considered a two-stage supply chain in which a supplier serves a set of stores in a retail chain. They considered a two-stage Stackelberg game in which the supplier must set price discounts for each period of a finite planning horizon under uncertainty in retail-store demand. As a mechanism to stimulate sales, the supplier offers periodic off-invoice price discounts to the retail chain. Based on the price discounts offered by the supplier and after store demand uncertainty is resolved, the retail chain determines individual store order quantities in each period. Sarkar et al. [16] and Malik and Kim [17] considered the supply chain with variable productivity. Sarkar et al. [16] established a single-supplier multi-buyer supply chain model with variable productivity and incomplete product quality. Unit production cost is a function of productivity. By establishing three different production functions, process quality and productivity are linked. The purpose of this study is to explore the impact of productivity elasticity on product quality and supply chain cost under a single set of multiple delivery strategies. The results show that variable productivity has a great influence on the total cost of the supply chain model. Aiming at the single-supplier single-buyer supply chain model of complex products, Malik and Kim [17] studied a flexible production system with variable productivity as an alternative method to overcome the shortage risk caused by fuzzy random demand uncertainty in the supply chain model. The relationship between productivity and quality of products under random demand and uncertain demand is established. The mathematical function of unit production cost is established, which depends on productivity and changes with the change of optimal productivity. The results show that considering the fuzziness of demand and budget, service level and spatial constraints can help management to reduce supply chain cost and improve customer service level by shortening lead time.

Malik and Sarkar [18] proposed a supply-chain coordination method based on the lead-time crashing, that adopts shortening transportation lead time as the coordination scheme among supply chain members. They considered setting cost as a variable and using discrete investment function to reduce setting cost. In order to make the model more practical, they consider that the lead time demand is stochastic, with unknown distribution function and limited known information. Therefore, the author adopts a distribution-free approach to solve this problem. Yang et al. [19] studied the optimal spare parts inventory management problem considering discrete Weibull distribution. Wei and Wei [20] studied a centralized supply chain model with one supplier and multiple retailers, in which demand is discrete and stochastic.

As shown in Table 1, we summarize the related literature regarding multi-retailer supply chain.

Table 1. Summary of the related literature regarding multi-retailer supply chain.

Literature	Decision-Making	Multiple Retailers	Discrete Stochastic
[3–6]	Centralized	Yes	No
[7–17]	Decentralized	Yes	No
[18,19]	Centralized	No	Yes
[20]	Centralized	Yes	Yes
Our study	Decentralized	Yes	Yes

Most of the above studies are aimed at deterministic demand or stochastic continuous demand, but there are few studies on single-manufacturer multi-retailer supply chain with stochastic discrete demand. Though continuous and discrete demand supply chain models have no significant differences, the discrete random demand decentralized models are considered as stochastic, dynamic games between supplier and retailers. The solutions for such models are more challenging and difficult.

3. Model Notation and Assumption

C: Manufacturer's production capacity;

Q: Manufacturer's production quantity, $Q \le C$;

m: Number of retailers;

q_j: Retailer j's order quantity, retailer j's decision variable, $q = (q_1, q_2, \cdots, q_m)$;

$Q_j(q, Q)$: Product quantity allocated to retailer j, $Q_j(q, Q) \le \min\{q_j, Q\}$, $q = (q_1, q_2, \cdots, q_m)$. When the retailers' total ordering quantity, $\sum_{j=1}^{m} q_j$, is no more than manufacture's production quantity Q, all orders are satisfied, and $Q_j(q, Q) = q_j$; when the retailers' total ordering quantity, $\sum_{j=1}^{m} q_j$ is more than manufacture's production quantity, Q, the manufacture allocates the Q unit products to m retailers according to a predetermined allocation rule, $Y_j(q, Q)$, where $Y_j(q, Q) \leq q_j$ is strictly decreasing in q_j and $\sum_{j=1}^{m} Y_j(q, Q) = Q$. For example, a common allocation rule is proportional allocation, $Y_j(q, Q) = \frac{Q}{\sum_{j=1}^{m} q_j} \cdot q_j$

c_p: Manufacturer's unit production cost;

c_j: Manufacturer's allocation cost to retailer j, $j = 1, 2 \ldots, m$;

w_j: Manufacturer's wholesale price to retailer j, manufacturer's decision variables, $w = (w_1, w_2, \cdots, w_m)$;

Pr_j: Retailer j's retail price;

v: Salvage value after the selling season, where $Pr_j - c_j > w_j > c_p > v$ to avoid any triviality.

D_j: Retailer j's random demand with a discrete probability distribution,$p_{jk} = P\{D_j = k\}$, $k = 1, 2 \ldots$.

For a random demand with a continuous probability distribution function $F(x)$, Lariviere and Porteus [21] defines a generalized failure rate function $e(x) = \frac{xdF(x)/dx}{1-F(x)}$ and assumes it is strictly increasing to ensure profit function's unimodality. This assumption has since been widely used in the supply chain management, revenue management, and queuing theory (see a summary about this assumption in Ziya et al. [22] and further discussions in Wei et al. [23]).

Similarly, define the following general failure rate function for a discrete demand, D_j,

$$e_j(n) = \frac{n \cdot \left(\mathrm{P}\{D_j \leq n\} - \mathrm{P}\{D_j \leq n-1\}\right)}{\mathrm{P}\{D_j > n\}} \tag{1}$$

Without loss of generality, assume that

Assumption 1. *There is an increasing generalized failure rate, that is,* $e_j(n)$ *is strictly increasing in* n.

We summarize the notations used throughout the paper in Table 2 for ease of reference.

Table 2. List of the notation.

Notation	Explanations	Remark
C	Manufacturer's production capacity	-
Q	Manufacturer's production quantity	$Q \leq C$ Manufacturer's decision in case of MTO scenario
m	Number of retailers	-
q_j	Retailer j's order quantity	Retailer j's decision variable, $q = (q_1, q_2, \cdots, q_m)$
$Q_j(q, Q)$	Product quantity allocated to retailer j	If $\sum_{j=1}^{m} q_j \leq Q$, then $Q_j(q, Q) = q$, else $Q_j(q, Q) = Y_j(q, Q)$, a predetermined allocation rule.
c_p	Manufacturer's unit production cost	-
c_j	Manufacturer's allocation cost to retailer j	-
w_j	Manufacturer's wholesale price to retailer j	Manufacturer's decision variables $w = (w_1, w_2, \cdots, w_m)$
Pr_j	Retailer j's retail price	-
v	Salvage value after the selling season	$Pr_j - c_j > w_j > c_p > v$
D_j	Retailer j's random demand	$p_{jk} = P\{D_j = k\}$
$e_j(n)$	General failure rate function	$e_j(n)$ is strictly increasing in n

4. Make-To-Order

The manufacturer first announces wholesale price $(w_1, w_2, \ldots, w_m)$ to each retailer; then, each retailer decides his/her order quantity accordingly. After receiving each retailer's order, the manufacturer starts production and then delivers the products to each retailer based on a pre-determined policy.

4.1. Model

When the retailers' total ordering quantity, $\sum_{j=1}^{m} q_j$, is less than the manufacturer's production capacity, C, then the manufacturer production quantity $Q = \sum_{j=1}^{m} q_j$ and would then allocate $Q_j(q, Q) = q_j$ to retailer j. When the retailers' total ordering quantity, $\sum_{j=1}^{m} q_j$, is more than the manufacturer's production capacity, C, then the manufacturer production quantity equals its production capacity, C, and would then allocate $Q_j(q, Q) = Y_j(q, C)$ to retailer j.

With random demand D_j and allocation quantity Q_j, retailer j's expected profit is

$$\pi_r^j(Q_j : w_j) = (Pr_j - v) \cdot \mathrm{E}\min\{Q_j, D_j\} - (w_j - v)Q_j \tag{2}$$

Correspondingly, the manufacturer's expected profit is

$$\pi_s(w, Q_1, \ldots, Q_m) = \sum_{j=1}^{m} (w_j - c_j) \cdot Q_j + v \cdot \left(Q - \sum_{j=1}^{m} Q_j\right) - c_p Q = \sum_{j=1}^{m} (w_j - c_j - v)Q_j + (v - c_p)Q \tag{3}$$

Let $Q_j(w_j)$ denote the allocation quantity that maximizes retailer's expected profit at wholesale price w_j.

Theorem 1.

(1) *$\pi_r^j(Q_j : w_j)$ is concave in Q_j;*

(2) *$Q_j(w_j) = \max_{l \in Z^+}\{\pi_r^j(l : w_j) - \pi_r^j(l - 1 : w_j) > 0\}$;*

(3) *$Q_j(w_j)$ is strictly decreasing in w_j.*

Proof of Theorem 1. (1) The first order and second order differences for retailer j's expected profit function $\pi_r^j(Q_j)$ are

$$\Delta\pi_r^j(Q_j : w_j) = \pi_r^j(Q_j : w_j) - \pi_r^j(Q_j - 1 : w_j) = (\mathrm{Pr}_j - v) \cdot P\{D_j \geq Q_j\} - (w_j - v)$$

$$\Delta^2\pi_r^j(Q_j : w_j) = \Delta\pi_r^j(Q_j : w_j) - \Delta\pi_r^j(Q_j - 1 : w_j) = -(\mathrm{Pr}_j - v) \cdot p_{j,Q_j-1} \leq 0$$

Thus, $\pi_r^j(Q_j : w_j)$ is concave in Q_j;

(2) It is easy to see because of concavity of $\pi_r^j(Q_j : w_j)$ in Q_j;

(3) Proof by contradiction, assume there are two points, $w_j^1 < w_j^2$, such that $Q_j(w_j^1) < Q_j(w_j^2)$, then

$$\begin{aligned}
\pi_r^j\left(Q_j(w_j^2) : w_j^1\right) &= (Pr_j - v) \cdot \mathrm{E}\min\left\{Q_j(w_j^2), D_j\right\} - (w_j^1 - v)Q_j(w_j^2) \\
&= \pi_r^j\left(Q_j(w_j^2) : w_j^2\right) - (w_j^1 - v)Q_j(w_j^2) + (w_j^2 - v)Q_j(w_j^2) \\
&\geq \pi_r^j\left(Q_j(w_j^1) : w_j^2\right) - (w_j^1 - v)Q_j(w_j^2) + (w_j^2 - v)Q_j(w_j^2) \\
&= \pi_r^j\left(Q_j(w_j^1) : w_j^1\right) + (w_j^2 - w_j^1) \cdot \left(Q_j(w_j^2) - Q_j(w_j^1)\right) \\
&\geq \pi_r^j\left(Q_j(w_j^1) : w_j^1\right)
\end{aligned}$$

The first inequality derives from the definition of $Q_j(w_j^2)$; the second inequality derives from $w_j^1 < w_j^2$ and $Q_j(w_j^1) < Q_j(w_j^2)$.

This result contradicts the definition of $Q_j(w_j^1)$; therefore, $Q_j(w_j)$ is strictly decreasing in w_j. □

Although $Q_j(w_j)$ is the quantity that maximizes retailer *j*'s expected profit, the retailer may not be able to obtain $Q_j(w_j)$ unit in full.

(1) If $\sum_{j=1}^{m} Q_j(w_j) \leq C$, then retailer *j*'s orders $Q_j(w_j)$, and is allocated $Q_j(w_j)$, unit of product, to achieve his maximum profit $\pi_r^j(Q_j(w_j) : w_j)$;

(2) If $\sum_{j=1}^{m} Q_j(w_j) > C$, then retailers' order cannot be fully satisfied, and some retailers are not able to obtain $Q_j(w_j)$ unit of product. However, the following theorem shows that this situation is uneconomical for the manufacturer, and the manufacturer adjusts the wholesale price to avoid this situation.

Theorem 2. *If there is a wholesale strategy, $w^1 = (w_1^1, w_2^1, \ldots, w_m^1)$, such that $\sum_{j=1}^{m} Q_j(w_j^1) > C$, then w^1 is manufacturer's strictly dominated strategy.*

Proof of Theorem 2. When $\sum_{j=1}^{m} Q_j(w_j^1) > C$, let retailer *j*'s equilibrium order quantity be q_j' and its corresponding allocation quantity be Q_j'. There must exists a wholesale strategy $w^2 = (w_1^2, w_2^2, \ldots, w_m^2)$ with $Q_j(w_j^2) = Q_j'$ and $w_j^2 \geq w_j^1$, so that $\pi_s(w^1, Q_1', \ldots, Q_m') < \pi_s(w^2, Q_1(w_1^2), \ldots, Q_m(w_m^2))$. Therefore, w^1 is a strictly dominated strategy for the manufacturer. □

Since $\sum_{j=1}^{m} Q_j(w_j) > C$ is the manufacturer's strictly dominated strategy, we need only consider the condition where $\sum_{j=1}^{m} Q_j(w_j) \leq C$. When the optimal order quantity for retailer *j*, $q_j = Q_j(w_j)$, the manufacturer's production quantity equals the sum of retailers' order quantities, $\sum_{j=1}^{m} Q_j(w_j)$. The manufacturer's decision model becomes

$$\begin{aligned} \max_{w} \pi_s(w) = & \sum_{j=1}^{m} (w_j - c_j - c_p) Q_j(w_j) \\ s.t. \quad & \sum_{j=1}^{m} Q_j(w_j) \leq C \end{aligned} \tag{4}$$

$$\text{Let } w_j(Q_j) = \left(Pr_j - v\right) \cdot \sum_{k=Q_j}^{\infty} p_{jk} + v, Q_j = 0, 1, 2 \cdots \tag{5}$$

Theorem 3. *The manufacturer's equilibrium wholesale price w_j^* must be attainable in the set of $\left\{ w_j(Q_j),\ Q_j = 0, 1, 2 \cdots \right\}$.*

Proof of Theorem 3. By contradiction, assume there is a non-negative integer, g_j, such that

$$\left(Pr_j - v\right) \cdot \sum_{k=g_j+1}^{\infty} p_{jk} + v < w_j^* < \left(Pr_j - v\right) \cdot \sum_{k=g_j}^{\infty} p_{jk} + v = w_j(g_j)$$

From Theorem 1, $Q_j(w_j^*) = Q_j(w_j(g_j)) = g_j$. Then,

$$\begin{aligned} \pi^S\left(w_j^*; j = 1, \ldots, m\right) &= \sum_{j=1}^{m} (w_j^* - v) Q_j(w_j^*) - (c - v) Q \\ &= \sum_{j=1}^{m} (w_j^* - v) Q_j(w_j(g_j)) - (c - v) Q \\ &< \sum_{j=1}^{m} (w_j(g_j) - v) Q_j(w_j(g_j)) - (c - v) Q \\ &= \pi^S\left(w_j(g_j); j = 1, \ldots, m\right) \end{aligned}$$

This contradicts the definition of w_j^*. □

Theorem 3 makes the manufacturer's decision set from continuous set to a discrete set of $\left\{w_j(Q_j),\ Q_j = 0, 1, 2 \cdots\right\}$, and there is a one-to-one correspondence between wholesale price w_j and allocation quantity Q_j. The manufacturer's decision changes to Q_j instead of w_j. The manufacturer's decision model becomes

$$
\begin{aligned}
&\max_{Q_j} \pi_s(Q_1, Q_2, \ldots, Q_m) = \sum_{j=1}^{m} \left[(Pr_j - v) \cdot \sum_{k=Q_j}^{\infty} p_{jk} + v - c_j - c_p \right] Q_j \\
&s.t. \qquad \sum_{j=1}^{m} Q_j \le C \\
&\qquad\qquad Q_j \in Z^+, j = 1, 2, \ldots, m
\end{aligned}
\tag{6}
$$

Denote the optimal solution as $Q^* = (Q_1^*, Q_2^*, \ldots, Q_m^*)$.

Let $a_{jQ_j} = \pi_s\left(Q_1, \ldots, Q_j, \ldots, Q_m\right) - \pi_s\left(Q_1, \ldots, Q_j - 1, \ldots, Q_m\right)$ be the marginal profit for the manufacturer when the allocation quantity to retailer j from $Q_j - 1$ to Q_j, then,

$$
a_{jQ_j} = (Pr_j - v) \cdot \left(\sum_{k=Q_j-1}^{\infty} p_{jk} - Q_j p_{j,(Q_j-1)} \right) - (c_j + c_p - v) \tag{7}
$$

Theorem 4. *Under Assumption 1, a_{jQ_j} decreases in Q_j, i.e., $\pi_s(Q_1, Q_2, \ldots, Q_m)$ is concave in Q_j.*

Proof of Theorem 4. Since,

$$
\begin{aligned}
a_{jQ_j} &= (Pr_j - v) \cdot \left(\sum_{k=Q_j}^{\infty} p_{jk} - (Q_j - 1) p_{j,(Q_j-1)} \right) - (c_j + c_p - v) \\
&= (Pr_j - v) \cdot \textstyle\sum_{Q_j+1}^{\infty} p_{jk} \cdot (1 - e_j(Q_j - 1)) - (c_j + c_p - v)
\end{aligned}
$$

$\sum_{Q_j+1}^{\infty} p_{jk}$ decreases in Q_j and $e_j(Q_j)$ increases in Q_j, it is easy to show that a_{jQ_j} decreases in Q_j. □

4.2. Simplified Solution

The non-linear integer programming problem in (6) is difficult to solve using integer programming method. For this type of the model, it is common to convert it to a dynamic programming model and make it easy to solve.

Let $V_l(h)$ be the manufacturer's maximum expected profit when allocating h units' products to retailers, $l, l+1, \ldots, m$, then, $V_1(C) = \max \pi_s(Q_1, Q_2, \ldots, Q_m)$. Using dynamic programming method, $V_l(h)$ (l = 1, 2, … , m) satisfies the following Bellman equation:

$$
\begin{aligned}
&V_{m+1}(h) = 0; \\
&V_l(h) = \max_{Q_l \in Z^+} \left\{ \left[(Pr_j - v) \cdot \sum_{k=Q_l}^{\infty} p_{lk} + v - c_l - c_p \right] Q_l + V_{l+1}(h - Q_l) \right\}, l = 1, 2, \ldots, m
\end{aligned}
\tag{8}
$$

Using recursion to solve (8) is much simpler than directly solving (6). However, the amount of computation for this dynamic program is $O(mC^2)$, and it becomes harder when there are more states and decision variables. We propose to use the FFT approach and construct the following matrix for the marginal profit:

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1C} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2C} \\ \cdots & \vdots & \cdots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mC} \end{bmatrix} \tag{9}$$

Using marginal matrix (9), it is easy to obtain the optimal solution for Model (5). If the number of positive values, denoted Q, is less than C, then $Q_j^* = \max\limits_{l} \{l \in Z^+, a_{jl} > 0\}$ and the manufacturer's total production, $\sum_j^m Q_j^* = Q$. If $Q > C$, then the optimal solution is determined by the C largest values, and the sum of those values is the maximum expected value for the manufacturer. In addition, in these numbers of C values, the occurrences of subscript j is Q_j^*. Since α_{jQ_j} strictly deceases in Q_j, in selecting these C largest values, we only need to compare with the non-selected value in the far left in each row. There are m rows and C comparisons; therefore, the amount of computation becomes $\mathrm{O}(mC)$.

5. Make-To-Stock

In this situation, the manufacturer produces a fixed quantity of goods in advance (the production is no longer a decision variable) and determines the wholesale price $(w_1, w_2, \ldots, w_m)$ for each retailer; then, each of m retailers decides his/her own order quantity q_j according to the wholesale price w_j. The actual quantity of goods $Q_j(q, Q)$ are assigned to the retailer j according to the pre-set rules $Y_j(q, Q)$. Accordingly, the retailer's expected profit is

$$\pi_r^j(Q_j : w_j) = (Pr_j - v) \cdot \mathrm{E}\min\{Q_j, D_j\} - (w_j - v)Q_j \tag{10}$$

Correspondingly, the manufacture's expected profit is

$$\pi^S(w, Q_1, \ldots, Q_m) = \sum_{j=1}^{m} (w_j - c_j - v)Q_j + (v - c_p)Q \tag{11}$$

It can be seen from Theorem 1 that the retailer j's expected profit function $\pi_r^j(Q_j : w_j)$ is concave in Q_j. Let $Q_j(w_j) = \max\limits_{l \in Z^+}\{\pi_r^j(l : w_j) - \pi_r^j(l-1 : w_j) > 0\}$ and denote the optimal assigned quantity that maximizes the retailer's profit; then, $Q_j(w_j)$ is strictly decreasing in w_j. Similar to the discussion in the previous section, the following properties of manufacturer's wholesale price can be obtained:

Theorem 5. *If there is a wholesale price strategy, $w^1 = (w_1^1, w_2^1, \ldots, w_m^1)$, such that $\sum_{j=1}^m Q_j(w_j^1) > Q$ and then w^1 is manufacturer's strictly dominated strategy.*

Proof of Theorem 5. It is similar to that of Theorem 2. □

According to Theorem 5, the model of the manufacture can be simplified as

$$\begin{aligned} &\max \pi^S(w_1, w_2, \ldots, w_m) = \sum_{j=1}^{m} (w_j - c_j - v)q_j(w_j) + (v - c_p)Q \\ &s.t. \quad \sum_{j=1}^{m} q_j(w_j) \le Q \end{aligned} \tag{12}$$

Using the function of $w_j(Q_j) = (Pr_j - v) \cdot \sum\limits_{k=Q_j}^{\infty} p_{jk} + v$, we obtain the following theorem.

Theorem 6. *The manufacturer's equilibrium wholesale price w_j^* must be attainable in the set of* $\left\{w_j(Q_j),\ Q_j = 0, 1, 2\cdots\right\}$.

Proof of Theorem 6. It is similar to that of Theorem 3. □

From Theorem 6, it can be seen that the manufacturer's decision-making set changes from a continuous set to a discrete set of $\left\{w_j(Q_j),\ Q_j = 0, 1, 2\cdots\right\}$, and there is a one-to-one correspondence between the wholesale price w_j and assigned quantity of goods Q_j. Therefore, we can change the manufacturer's decision-making from w_j to Q_j, and then the manufacturer's model becomes,

$$\begin{aligned} \max_{Q_j} \pi^S(Q_j; j = 1, \ldots, m) &= \sum_{j=1}^{m}\left[(Pr_j - v)\cdot\sum_{k=Q_j}^{\infty} p_{jk} - c_j\right]\cdot Q_j - (c_p - v)Q \\ s.t. \quad & \sum_{j=1}^{m} Q_j \le Q \\ & Q_j \in Z^+, j = 1, 2, \ldots, m \end{aligned} \tag{13}$$

Denote the optimal solution as $(Q_1^{**}, Q_2^{**}, \ldots, Q_m^{**})$.

When the allocated quantity of goods to retailer j increases from $Q_j - 1$ to Q_j, the marginal profit of manufacturer, denoted as α'_{jQ_j}, increases.

$$\alpha'_{jQ_j} = \pi^S\left(Q_1, \ldots, Q_j, \ldots, Q_m\right) - \pi^S\left(Q_1, \ldots, Q_j - 1, \ldots, Q_m\right) = (Pr_j - v)\cdot\left(\sum_{k=Q_j-1}^{\infty} p_{jk} - Q_j p_{j,(Q_j-1)}\right) - c_j \tag{14}$$

By comparing Equations (14) and (7), we can find that there is only one constant difference $(c_p - v)$ between α'_{jQ_j} and a_{jQ_j}; therefore, we can easily obtain the following property.

Corollary 1. *Under assumption 1, α'_{jQ_j} is decreasing in Q_j.*

Proof. It can be easily obtained according to Theorem 4. □

According to Corollary 1, we can construct a $m \times Q$ marginal profit matrix

$$\begin{bmatrix} \alpha'_{11} & \alpha'_{12} & \cdots & \alpha'_{1Q} \\ \alpha'_{21} & \alpha'_{22} & \cdots & \alpha'_{2Q} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha'_{m1} & \alpha'_{m2} & \cdots & \alpha'_{mQ} \end{bmatrix} \tag{15}$$

The equilibrium profit of supply chain can be easily obtained from this matrix. If the number of positive values in Matrix (14) exceeds Q, the largest Q values are selected, and the sum of these Q values is the manufacturer's equilibrium profit. In these Q values, the number of times that subscript j appears is the retailer j's equilibrium purchase quantity Q_j^{**}, and the corresponding equilibrium wholesale price is $w_j(Q_j^{**})$. If the number of positive values Q in Matrix (14) is less than Q, then the sum of these Q values is the manufacturer's equilibrium profit, and in these Q values, the number of times that subscript j appears is the retailer j's equilibrium purchase quantity Q_j^{**}.

6. Numerical Example

Considering a small furniture supply chain system, there is one furniture manufacturer and three retailers in a supply chain system. The production cost for the manufacturer is \$1000 per set, and the salvage value for the unsold furniture is \$600 per set. The distribution costs from the manufacturer to the retailers 1, 2, and 3 are \$300, \$200, and \$400 per set, respectively. The retail prices for retailers 1, 2, and 3 are \$2000, \$2200, and \$2300, respectively. The random discrete demands follow Poisson distributions, and the mean demands for retailers 1, 2, and 3 are 6, 7, and 8 sets of furniture per week, respectively. Table 3 has the summary of the parameter values.

Table 3. Value of parameters.

Parameter	c_p	v	c_1	c_2	c_3	Pr_1	Pr_2	Pr_3	λ_1	λ_2	λ_3
Value	1000	600	300	200	400	2000	2200	2300	6	7	8

In the MTO scenario, bringing the above parameter values into Formula (7), one can obtain the marginal revenue under different order quantities. All that is needed is to calculate the part with the value greater than 0 and bring it into the marginal revenue Matrix (9).

$$\begin{pmatrix} 696.53 & 654.89 & 488.31 & 113.53 & -448.65 & \\ 998.54 & 978.11 & 881.09 & 618.95 & 139.37 & -502.87 \\ 899.43 & 890.31 & 840.12 & 681.96 & 341.31 & -203.73 \end{pmatrix} \tag{16}$$

At this time, only 17 values need to be calculated, of which 14 are positive values, which means that the manufacturer's production quantity does not exceed 14 sets. The value of each row of matrix (16) decreases from left to right. In this case, the optimal order quantity can be easily found from the matrix for any manufacturer's production capacity by using FFT method. For example, when the production capacity is 10, select the largest 10 positive values in the matrix in turn: if two values are selected in the first row, then the optimal order quantity of retailer 1 is two sets; if four values are selected in the second line, the optimal order quantity of retailer 2 is four sets; if four values are selected in the third line, the optimal order quantity of retailer 3 is four sets. By introducing the retailer's optimal order quantity into Formula (5), the manufacturer's equilibrium wholesale price strategy can be easily obtained. Table 4 lists equilibrium strategy for the supply chain under different production capabilities by the manufacturer.

Table 4. Equilibrium strategy under different manufacturers' production capacity in MTO scenario.

C	w_1	w_2	w_3	q_1	q_2	q_3	Q
5	2000	2152.58	2294.87	0	3	2	5
6	2000	2152.58	2276.62	0	3	3	6
7	2000	2152.58	2227.95	0	3	4	7
8	1996.53	2152.58	2227.95	1	3	4	8
9	1975.71	2152.58	2227.95	2	3	4	9
10	1975.71	2069.18	2227.95	2	4	4	10
11	1913.24	2069.18	2227.95	3	4	4	11
12	1913.24	2069.18	2130.62	3	4	5	12
13	1913.24	1923.21	2130.62	3	5	5	13
≥14	1788.31	1923.21	2130.62	4	5	5	14

In the MTS scenario, substituting all the parameter values into the MTS model, recall that $\alpha'_{jQ_j} = a_{jQ_j} + (c_p - v) = a_{jQ_j} + 400$, and the manufacturer's marginal profit matrix (15) is expressed as α'_{jQ_j}

$$\begin{pmatrix} 1096.53 & 1054.89 & 888.31 & 513.53 & -48.65 & & \\ 1398.54 & 1378.11 & 1281.09 & 1018.95 & 539.37 & -102.87 & \\ 1299.43 & 1290.31 & 1240.12 & 1081.96 & 741.31 & 196.27 & -478.55 \end{pmatrix} \tag{17}$$

The above matrix has 15 positive values, which means the maximum production quantity for the manufacturer is 15 sets. Similar to the MTO scenario, the supply chain equilibrium strategy can be easily obtained by using FFT method. Table 5 summarizes the retailers' equilibrium orders and the manufacturer's marginal profit under different production quantities.

Table 5. Equilibrium strategy under different manufacturers' production quantity in MTS Scenario.

Q	(w_1,w_2,w_3)	(Q_1,Q_2,Q_3)
$\leq$14	Same as in MTO Scenario	Same as in MTO Scenario
$\geq$15	(1788.31, 1923.21, 1974.90)	(4,5,6)

From the above analysis, one can see that for the above numerical example, whether it is MTO scenario or MTS scenario, the equilibrium strategy of the supply chain can be easily calculated in 20 steps by using the method in this paper. Moreover, when the production capacity in MTO scenario or the output of MTS scenario changes, the marginal profit matrix is not affected, only a few steps are needed to calculate the new equilibrium strategy. For the same example, when the manufacturer's production capacity is 15, the computational complexity of the dynamic programming method is $O(675)$. Although the actual calculation may not take so many steps, it is far beyond the method in this paper. Moreover, the more the number of retailers and the greater the manufacturer's production capacity, the more advantages of this method.

7. Conclusions

This paper studies a supply chain system of single manufacturer and multiple retailers with discrete stochastic demands. The scenario of Make-To-Order model is discussed firstly, and the corresponding Stackelberg game model is established. Through a series of mathematical deduction and proof, the game model of wholesale price and purchase quantity is transferred into an integer programming model of quantity allocation, which is usually solved by dynamic programming. However, with the increase in the number of retailers and the production capacity of manufacturers, the computational complexity of dynamic programming increases sharply due to the Dimension Barrier. Therefore, based on the quantity allocation model, the Fast Fourier Transform (FFT) method is introduced to solve the integer programming, which can greatly reduce the complexity of the model. When manufacturer capacity is C and retailer number is m, the computational complexity for the dynamic programming approach is $O(mC^2)$ and for the FFT approach is $O(mC)$. Furthermore, the other scenario of Make-To-Order is discussed, and a similar conclusion is obtained.

This study provides a reference for the multi-retailer supply chain with discrete demands, but it also has the following limitations: Firstly, the generalized failure rate function $e_j(n)$ is assumed strictly increasing in n, and future research can consider more general demand function. Secondly, retail price is considered as an exogenous variable, which can be considered controllable retail price in future work.

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