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A Novel Interpretation of the Electromagnetic Fields of Lightning Return Strokes

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Abstract: Electric and/or magnetic fields are generated by stationary charges, uniformly moving charges and accelerating charges. These field components are described in the literature as static fields, velocity fields (or generalized Coulomb field) and radiation fields (or acceleration fields), respectively. In the literature, the electromagnetic fields generated by lightning return strokes are presented using the field components associated with short dipoles, and in this description the one–to-one association of the electromagnetic field terms with the physical process that gives rise to them is lost. In this paper, we have derived expressions for the electromagnetic fields using field equations associated with accelerating (and moving) charges and separated the resulting fields into static, velocity and radiation fields. The results illustrate how the radiation fields emanating from the lightning channel give rise to field terms varying as $1/r$ and $1/r^2$, the velocity fields generating field terms varying as $1/r^2$, and the static fields generating field components varying as $1/r^2$ and $1/r^3$. These field components depend explicitly on the speed of propagation of the current pulse. However, the total field does not depend explicitly on the speed of propagation of the current pulse. It is shown that these field components can be combined to generate the field components pertinent to the dipole technique. However, in this conversion process the connection of the field components to the physical processes taking place at the source that generate these fields (i.e., static charges, uniformly moving charges and accelerating charges) is lost.

Keywords: Electromagnetic fields; return strokes; dipole fields; accelerating charges; radiation fields; static fields; velocity fields

1. Introduction

Estimating the field strength and temporal features of electromagnetic fields from lightning return strokes at a given distance is of interest both in engineering studies, where the protection of electrical installations from induced voltages from lightning is concerned [1–3], and in physics studies, where the properties of lightning return strokes are extracted from the features of electromagnetic fields measured under conditions where the propagation effects are minimal [4,5]. The procedure used in these studies is to specify the spatial and temporal variation of the return stroke current $I(z,t)$ by appealing to a return stroke model and from that calculate the electromagnetic fields [6,7]. Once $I(z,t)$ is specified, there are four methods to estimate the electromagnetic fields. Three of these methods, namely, the dipole technique, monopole technique and the apparent charge density technique, are described in [6,7]. The fourth method, based on the field equations pertinent to the moving and accelerating charges is described in [8]. For a given $I(z,t)$, all these techniques generate the same total electromagnetic fields but the various components that constitute the total fields differ in different techniques.

The goals of the present paper are the following. First, it is standard practice today to describe the electromagnetic fields of lightning return strokes in terms of static (field terms decreasing with distance as $1/r^3$, where r is the distance from the source to the point of observation), induction (field terms decreasing with distance as $1/r^2$) and radiation (field terms decreasing as $1/r$) [9]. In the sections to follow we will call these field components dipole-static, dipole-induction and dipole-radiation. Except in the case of distant radiation fields, this division of the field components cannot be directly attached to the physical processes that generate the electromagnetic fields. In reality, there are only two types of electromagnetic fields. These are the Coulomb and radiation fields. Coulomb fields are produced by stationary and uniformly moving charges. The Coulomb field produced by stationary or static charges is called the static field. When the charges are moving, the Coulomb field has to be modified to take into account their motion and this modified field, to separate it from the static field, is called the velocity field (or generalized Coulomb field). The radiation fields (or acceleration fields) are produced by accelerating charges. The magnetic field is generated either by moving charges or by accelerating charges. Thus, the magnetic field consists of either the velocity fields or the radiation fields, or both. Unfortunately, when one divides the total field into dipole-static (i.e., the field components changing as $1/r^3$), dipole-induction (i.e., the field components changing as $1/r^2$) and dipole-radiation (i.e., the field components changing as $1/r$), as is the standard practice, except in the case of distant radiation field, the direct association of the field components with the physical process that generate electromagnetic fields is lost. In this paper, we derive expressions for the electromagnetic fields of a return stroke where each field component is directly associated with the physical process that gives rise to it, i.e., stationary charges, moving charges and accelerating charges. Second, we will show analytically and illustrate by example, that the resulting field components can be combined together to produce the field components that are identified in the literature as dipole-static, dipole-induction and dipole-radiation.

2. Mathematical Analysis

Some of the mathematical formulations used in the present paper are identical to those already presented previously by Cooray and Cooray [10,11]. For example, the approximations listed in Equations (1)–(11) and the field Equations (12)–(23) can be extracted directly from references [10] and [11]. However, since the other equations given in this paper are constructed using these equations, for the sake of completeness and for easy reference, they are also given here.

The problem under consideration is the calculation of electromagnetic fields from a return stroke channel when $I(z,t)$ is specified. The normal procedure for such a calculation is to divide the channel into infinitesimal channel sections of length dz and first estimate the electromagnetic fields from such a channel element located at height z, where z is the height of the channel element. Once the electric and magnetic fields produced by this channel element are known, the total field can be obtained by summing the contributions from all these elements. Let us consider the element dz located at height z. The first step is to estimate the electromagnetic fields generated by the channel element. For convenience we will treat the problem in frequency domain first and later convert it into time domain. Assume that the current flowing along the channel element is given by $i(z)e^{j\omega t}$. We consider the case where this current travels along the channel element with speed u and is absorbed at the other end of the channel element. In other words, the current that appears at the bottom of the channel element at any time t will appear at the top of the channel element after a time delay of dz/u. The geometry necessary for the calculation of the electromagnetic fields from this channel element using the technique pertinent to the moving and accelerating charges is shown in Figure 1.

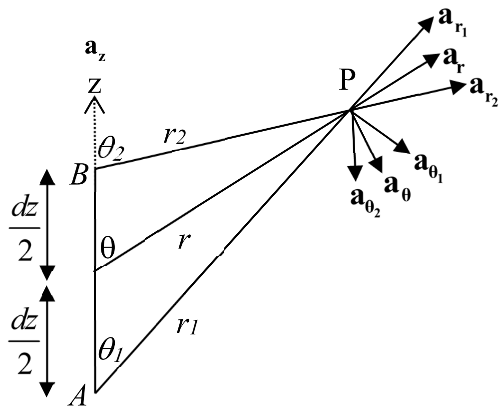

Figure 1. Geometry, angles and unit vectors pertinent to the evaluation of electromagnetic fields generated by a channel element. The unit vector in the direction of positive z-axis is denoted by $\mathbf{a_z}$. The unit vectors in the radial directions r, r_1 and r_2 are denoted by $\mathbf{a}_r$, $\mathbf{a}_{r_1}$ and $\mathbf{a}_{r_2}$, respectively. The unit vectors $\mathbf{a}_\theta$, $\mathbf{a}_{\theta_1}$ and $\mathbf{a}_{\theta_2}$ are defined as $\mathbf{a}_r \times (\mathbf{a}_r \times \mathbf{a}_z)$, $\mathbf{a}_{r_1} \times (\mathbf{a}_{r_1} \times \mathbf{a}_z)$ and $\mathbf{a}_{r_2} \times (\mathbf{a}_{r_2} \times \mathbf{a}_z)$, respectively. The unit vector $\mathbf{a}_\phi$ is in the direction of the vector $\mathbf{a}_r \times \mathbf{a}_\theta$ (i.e., into the page). Note that point P can be located anywhere in space.

Before we proceed with the analysis, let us consider some of the geometrical simplifications that can be used in the analysis. First of all, we assume that the distance to the point of observation, r, is such that $r >> dz$. When this condition is satisfied one can also make the following simplifications:

$$\delta\theta_2 = (\theta_2 - \theta) = \frac{dz \sin\theta}{2r} \; ; \; \delta\theta_1 = (\theta - \theta_1) = \frac{dz \sin\theta}{2r} \tag{1}$$

$$\cos\delta\theta_1 \approx 1 \; ; \; \cos\delta\theta_2 \approx 1 \tag{2}$$

$$r_1 = r + \frac{dz\cos\theta}{2}; \; r_2 = r - \frac{dz\cos\theta}{2} \tag{3}$$

$$\frac{1}{r_1} = \frac{1}{r}\left\{1 - \frac{dz\cos\theta}{2r}\right\}; \; \frac{1}{r_2} = \frac{1}{r}\left\{1 + \frac{dz\cos\theta}{2r}\right\} \tag{4}$$

$$\frac{1}{r_1^2} = \frac{1}{r^2}\left\{1 - \frac{dz\cos\theta}{r}\right\}; \; \frac{1}{r_2^2} = \frac{1}{r^2}\left\{1 + \frac{dz\cos\theta}{r}\right\} \tag{5}$$

$$\sin\theta_1 = \sin\theta\left\{1 - \frac{dz\cos\theta}{2r}\right\}; \; \sin\theta_2 = \sin\theta\left\{1 + \frac{dz\cos\theta}{2r}\right\} \tag{6}$$

$$\cos\theta_1 = \cos\theta + \frac{dz\sin^2\theta}{2r}; \; \cos\theta_2 = \cos\theta - \frac{dz\sin^2\theta}{2r} \tag{7}$$

$$1 - \frac{u\cos\theta_1}{c} = \left\{1 - \frac{u}{c}\cos\theta\right\}\left\{1 - \frac{udz\sin^2\theta}{2rc\left(1 - \frac{u}{c}\cos\theta\right)}\right\} \tag{8}$$

$$1 - \frac{u\cos\theta_2}{c} = \left\{1 - \frac{u}{c}\cos\theta\right\}\left\{1 + \frac{udz\sin^2\theta}{2rc\left(1 - \frac{u}{c}\cos\theta\right)}\right\} \tag{9}$$

$$\frac{1}{1 - \frac{u\cos\theta_1}{c}} = \frac{1}{1 - \frac{u\cos\theta}{c}}\left\{1 + \frac{udz\sin^2\theta}{2rc\left(1 - \frac{u}{c}\cos\theta\right)}\right\} \tag{10}$$

$$\frac{1}{1 - \frac{u\cos\theta_2}{c}} = \frac{1}{1 - \frac{u\cos\theta}{c}}\left\{1 - \frac{udz\sin^2\theta}{2rc\left(1 - \frac{u}{c}\cos\theta\right)}\right\} \tag{11}$$

Now we are in a position to write down the expressions for the electromagnetic fields. The electromagnetic fields generated by the channel element can be divided into different components as follows: (a) The electric and magnetic radiation fields generated at the initiation and termination of the current at the end points of the channel element due to charge acceleration and deceleration, respectively; (b) the electric and magnetic velocity fields generated by the movement of charges along the channel element; (c) the static fields generated by the accumulation of charges at the two ends of the channel element. Let us consider these different field components separately. In writing down these field components, we will depend heavily on the results published previously by Cooray and Cooray [10,11]. The field expressions identified by Equations (12)–(23) can be constructed easily from the results presented in [10].

2.1. Radiation Field Generated by the Charge Acceleration and Deceleration at the Ends of the Channel Element

The electric radiation field generated by the initiation of current at the bottom of the channel element and by the termination of that current at the top of the channel element is given by

$$d\mathbf{e}_{rad} = \frac{i(z)u}{4\pi\varepsilon_o c^2}\left\{ \frac{e^{j\omega(t-r_1/c)}\sin\theta_1}{r_1\left[1-\frac{u\cos\theta_1}{c}\right]}\mathbf{a}_{\theta_1} - \frac{e^{j\omega(t-dz/u-r_2/c)}\sin\theta_2}{r_2\left[1-\frac{u\cos\theta_2}{c}\right]}\mathbf{a}_{\theta_2}\right\} \tag{12}$$

The above expression is exact and does not contain any approximations. In order to extract the electric fields of an infinitesimal current element, we will write down the components of this electric field in the direction of $\mathbf{a_r}$ and $\mathbf{a}_\theta$ using the geometrical approximations listed in Equations (1)–(11), which are valid when $r >> dz$. Moreover, we also assume that $\omega r/c << 1$ and $\omega dz/u << 1$. Using these approximations and keeping only the first order terms with respect to dz, the components of this field in the directions of $\mathbf{a_r}$ and $\mathbf{a}_\theta$ become (with $t' = t - r/c$; see Appendix A for the derivation):

$$d\mathbf{e}_{rad,\theta} = \frac{i(z)e^{j\omega t'}u\sin\theta}{4\pi\varepsilon_o c^2 r(1-\frac{u}{c}\cos\theta)}\left\{\frac{j\omega dz}{u} - \frac{j\omega dz\cos\theta}{c} - \frac{2dz\cos\theta}{r} + \frac{udz\sin^2\theta}{rc(1-\frac{u}{c}\cos\theta)}\right\}\mathbf{a}_\theta \tag{13}$$

$$d\mathbf{e}_{rad,r} = \frac{i(z)e^{j\omega t'}}{4\pi\varepsilon_o c^2 r^2}\left\{\frac{udz\sin^2\theta}{(1-\frac{u}{c}\cos\theta)}\right\}\mathbf{a}_r \tag{14}$$

These two equations define the total radiation field produced by the channel element.

2.2. Velocity Field Generated by the Charges Moving from A to B

The velocity field generated as the current pulse propagates along the channel element can be written as

$$d\mathbf{e}_{vel} = \frac{i(z)e^{j\omega t'}dz}{4\pi\varepsilon_o r^2 u\left[1-\frac{u}{c}\cos\theta\right]^2}\left[1-\frac{u^2}{c^2}\right]\mathbf{a_r} - \frac{i(z)e^{j\omega t'}dz}{4\pi\varepsilon_o r^2 c\left[1-\frac{u}{c}\cos\theta\right]^2}\left[1-\frac{u^2}{c^2}\right]\mathbf{a_z} \tag{15}$$

The components of this field in the direction of $\mathbf{a_r}$ and $\mathbf{a}_\theta$ after some mathematical manipulations are given by

$$d\mathbf{e}_{vel,r} = \frac{i(z)e^{j\omega t'}dz}{4\pi\varepsilon_o r^2 u\left[1-\frac{u}{c}\cos\theta\right]}\left[1-\frac{u^2}{c^2}\right]\mathbf{a_r} \tag{16}$$

$$d\mathbf{e}_{vel,\theta} = -\frac{i(z)e^{j\omega t'}\sin\theta dz}{4\pi\varepsilon_o r^2 c\left[1-\frac{u}{c}\cos\theta\right]^2}\left[1-\frac{u^2}{c^2}\right]\mathbf{a}_\theta \tag{17}$$

2.3. Electrostatic Field Generated by the Accumulation of Charge at A and B

As the positive current leaves point A, negative charge accumulates at A and when the current is terminated at B, positive charge is accumulated there. The static Coulomb field produced by these stationary charges is given by

$$d\mathbf{e}_{stat}(t) = -\frac{i(z)}{4\pi\varepsilon_o j\omega}\left[\frac{e^{j\omega(t-r_1/c)}}{r_1^2}\mathbf{a}_{\mathbf{r}_1} - \frac{e^{j\omega(t-dz/u-r_2/c)}}{r_2^2}\mathbf{a}_{\mathbf{r}_2}\right] \tag{18}$$

After using the approximations given earlier and following a procedure similar to that given in Appendix A, the components of this field in the direction of $\mathbf{a_r}$ and $\mathbf{a_\theta}$ can be written as

$$d\mathbf{e}_{stat,r} = \frac{i(z)e^{j\omega t'}dz}{4\pi\varepsilon_o r^2 j\omega}\left\{\frac{j\omega\cos\theta}{c} + \frac{\cos\theta}{r} - \frac{j\omega}{u}\right\}\mathbf{a_r} \tag{19}$$

$$d\mathbf{e}_{stat,\theta}(t) = \frac{i(z)e^{j\omega t'}dz}{4\pi\varepsilon_o r^2 j\omega}\left\{\frac{\sin\theta}{r}\right\}\mathbf{a_\theta} \tag{20}$$

2.4. Magnetic Radiation Field Generated during the Initiation and Termination of the Current

The magnetic radiation field generated during the initiation and termination of the current at the ends of the channel element is given by

$$d\mathbf{b}_{rad,\phi} = \frac{i(z)u}{4\pi\varepsilon_o c^3}\left\{\frac{\sin\theta_1 e^{j\omega(t-r_1/c)}}{r_1\left[1-\frac{u\cos\theta_1}{c}\right]} - \frac{\sin\theta_2 e^{j\omega(t-dz/u-r_2/c)}}{r_2\left[1-\frac{u\cos\theta_2}{c}\right]}\right\}\mathbf{a}_\phi \tag{21}$$

Utilizing the geometrical approximations mentioned earlier and following procedure almost identical to that presented in Appendix A (and keeping only the second order terms in dz), one obtains

$$d\mathbf{b}_{rad,\phi} = \frac{i(z)e^{j\omega t'}u\sin\theta dz}{4\pi\varepsilon_o c^3 r}\frac{1}{\left[1-\frac{u\cos\theta}{c}\right]}\left\{\frac{j\omega}{u} - \frac{j\omega\cos\theta}{c} - \frac{2\cos\theta}{r} + \frac{u\sin^2\theta}{rc(1-\frac{u}{c}\cos\theta)}\right\}\mathbf{a_\varphi} \tag{22}$$

2.5. Magnetic Velocity Field Generated as the Current Pulse Propagates Along the Channel Element

The magnetic velocity field generated during the passage of the current along the channel element is given by

$$d\mathbf{b}_{vel,\phi} = \frac{i(z)e^{j\omega t'}dz\sin\theta}{4\pi\varepsilon_o r^2 c^2\left[1-\frac{u}{c}\cos\theta\right]^2}\left[1-\frac{u^2}{c^2}\right]\mathbf{a_\varphi} \tag{23}$$

3. Electromagnetic Fields of a Channel Element in the Time Domain

From the frequency domain equations, the time domain equations for the electric and magnetic fields generated by the channel element can be written directly. The results are the following.

3.1. Radiation Fields

$$d\mathbf{E}_{rad,\theta}(t) = \frac{dz\sin\theta}{4\pi\varepsilon_o c^2 r}\left\{\frac{\partial I(z,t')}{\partial t} - \frac{2u\cos\theta}{r(1-\frac{u}{c}\cos\theta)}I(z,t') + \frac{u^2\sin^2\theta}{rc(1-\frac{u}{c}\cos\theta)^2}I(z,t')\right\}\mathbf{a_\theta} \tag{24}$$

$$d\mathbf{E}_{rad,r}(t) = \frac{I(z,t')dz}{4\pi\varepsilon_o c^2 r^2}\left\{\frac{u\sin^2\theta}{(1-\frac{u}{c}\cos\theta)}\right\}\mathbf{a}_r \tag{25}$$

$$d\mathbf{B}_{rad,\phi}(t) = \frac{dz\sin\theta}{4\pi\varepsilon_o c^3 r}\left\{\frac{\partial I(z,t')}{\partial t} - \frac{2u\cos\theta}{r(1-\frac{u}{c}\cos\theta)}I(z,t') + \frac{u^2\sin^2\theta}{rc(1-\frac{u}{c}\cos\theta)^2}I(z,t')\right\}\mathbf{a}_{\boldsymbol{\varphi}} \tag{26}$$

3.2. Velocity Fields

$$d\mathbf{E}_{vel,\theta}(t) = \frac{I(z,t')dz\sin\theta}{4\pi\varepsilon_o r^2 c\left[1-\frac{u}{c}\cos\theta\right]^2}\left[1-\frac{u^2}{c^2}\right]\mathbf{a}_\theta \tag{27}$$

$$d\mathbf{E}_{vel,r}(t) = \frac{I(z,t')dz}{4\pi\varepsilon_o r^2 u\left[1-\frac{u}{c}\cos\theta\right]}\left[1-\frac{u^2}{c^2}\right]\mathbf{a}_r \tag{28}$$

$$d\mathbf{B}_{vel,\phi}(t) = \frac{I(z,t')dz\sin\theta}{4\pi\varepsilon_o r^2 c^2\left[1-\frac{u}{c}\cos\theta\right]^2}\left[1-\frac{u^2}{c^2}\right]\mathbf{a}_{\boldsymbol{\varphi}} \tag{29}$$

3.3. Static Fields

$$d\mathbf{E}_{stat,r}(t) = \frac{dz}{4\pi\varepsilon_o r^2}\left\{\frac{\cos\theta}{c}I(z,t') - \frac{1}{u}I(z,t') + \frac{2\cos\theta}{r}\int_0^t I(z,\tau')d\tau\right\}\mathbf{a}_\mathbf{r} \tag{30}$$

$$d\mathbf{E}_{stat,\theta}(t) = \frac{dz}{4\pi\varepsilon_o r^2}\left\{\frac{\sin\theta}{r}\int_0^t I(z,\tau')d\tau\right\}\mathbf{a}_{\boldsymbol{\theta}} \tag{31}$$

In the above equations $\tau' = \tau - r/c$.

4. Comparison of the Fields of the Channel Element with Fields of a Short Dipole

The set of equations given in the previous section describes the radiation, velocity and static fields generated by the channel element. Each of these field terms are associated with the particular process that generates these fields. Note also that each term is associated with the speed of propagation of the current pulse. Now, let us sum up the fields in the directions of $\mathbf{a_r}$ and $\mathbf{a}_\theta$ without any regard to the physical mechanism of the field generation. This generates the following field components

$$d\mathbf{E}_\theta(t) = \frac{dz\sin\theta}{4\pi\varepsilon_o}\left\{\frac{1}{c^2 r}\frac{\partial I(z,t')}{\partial t} + \frac{1}{r^3}\int_0^t I(z,\tau')d\tau + \frac{I(z,t')\ u^2\sin^2\theta}{r^2c^3(1-\frac{u}{c}\cos\theta)^2} - \frac{I(z,t')\,2u\cos\theta}{c^2r^2(1-\frac{u}{c}\cos\theta)} + \frac{I(z,t')\,(1-u^2/c^2)}{cr^2(1-\frac{u}{c}\cos\theta)^2}\right\}\mathbf{a}_{\boldsymbol{\theta}} \tag{32}$$

$$d\mathbf{E}_r(t) = \frac{dz}{4\pi\varepsilon_o}\left\{\frac{2\cos\theta}{r^3}\int_0^t I(z,\tau')d\tau + \frac{I(z,t')\cos\theta}{r^2c} - \frac{I(z,t')}{ur^2} + \frac{I(z,t')\,u\sin^2\theta}{c^2r^2(1-\frac{u}{c}\cos\theta)} + \frac{I(z,t')\,(1-u^2/c^2)}{ur^2(1-\frac{u}{c}\cos\theta)}\right\}\mathbf{a}_\mathbf{r} \tag{33}$$

$$d\mathbf{B}_\phi(t) = \frac{dz\sin\theta}{4\pi\varepsilon_o}\left\{\frac{1}{c^3 r}\frac{\partial I(z,t')}{\partial t} - \frac{I(z,t')\,2u\cos\theta}{c^3r^2(1-\frac{u}{c}\cos\theta)} + \frac{I(z,t')\,u^2\sin^2\theta}{r^2c^4(1-\frac{u}{c}\cos\theta)^2} + \frac{I(z,t')\,(1-u^2/c^2)}{c^2r^2(1-\frac{u}{c}\cos\theta)^2}\right\}\mathbf{a}_\phi \tag{34}$$

With some mathematical manipulations, one can show that these equations will reduce to (the mathematical steps necessary for this reduction are given in Appendix B)

$$d\mathbf{E}_\theta(t) = \frac{dz\sin\theta}{4\pi\varepsilon_o}\left\{\frac{1}{r^3}\int_0^t I(z,\tau')d\tau + \frac{I(z,t')}{cr^2} + \frac{1}{c^2 r}\frac{\partial I(z,t')}{\partial t}\right\}\mathbf{a}_{\boldsymbol{\theta}} \tag{35}$$

$$d\mathbf{E}_r(t) = \frac{dz}{4\pi\varepsilon_o}\left\{\frac{2\cos\theta}{r^3}\int_0^t I(z,\tau')d\tau + \frac{2\cos\theta}{r^2c}\frac{\partial I(z,t')}{\partial t}\right\}\mathbf{a}_\mathbf{r} \tag{36}$$

$$d\mathbf{B}_{\phi}(t) = \frac{dz\sin\theta}{4\pi\varepsilon_o}\left\{\frac{1}{c^3 r}\frac{\partial I(z,t')}{\partial t} + \frac{I(z,t')}{r^2c^2}\right\}\mathbf{a}_{\phi} \tag{37}$$

The Equations (35)–(37) are identical to the fields of a short dipole [7,12]. This proves the equivalence of the results pertinent to dipole technique and the results obtained using equations of accelerating and moving charges. This equivalence of the two techniques was previously shown for frequency domain fields in reference [10].

Observe that the field components given by Equations (24)–(31) depend on the speed of propagation of the pulse along the channel element, i.e., u. These field components will change their amplitudes if the speed of propagation is changed. For example, note that the velocity fields go to zero when the speed of propagation becomes equal to the speed of light in free space. However, the field components given by Equations (35)–(37), which were obtained by summing all the field terms given by Equations (24)–(31), are independent of the speed of propagation of the pulse. They depend only on the current waveform exciting the current element. This is the reason why it is not necessary to specify the speed of propagation of the current when writing down the electromagnetic fields of a short dipole. This also means that the field terms belonging to the radiation, velocity and electrostatics, which depend on the speed of propagation of the pulse, cancel out during the summation, leaving behind a total field which is independent of the speed of propagation. Thus, the speed of propagation enters into the electromagnetic field expressions of the channel element only if we wish to separate the total field into its physical constituents.

5. The Time Domain Fields of the Lightning Channel

The electromagnetic fields generated by the lightning channel (without the contribution of the ground) can be obtained by summing the contributions of all the channel elements located between $z = 0$ and $z = H$. The results are the following.

5.1. Radiation Fields

$$\mathbf{E}_{rad,\theta}(t) = \int_0^H \frac{\sin\theta\, dz}{4\pi\varepsilon_o c^2 r}\left\{\frac{\partial I(z,t')}{\partial t} - \frac{2u\cos\theta}{r(1-\frac{u}{c}\cos\theta)}I(z,t') + \frac{u^2\sin^2\theta}{rc(1-\frac{u}{c}\cos\theta)^2}I(z,t')\right\}\mathbf{a}_{\theta} \tag{38}$$

$$\mathbf{E}_{rad,r}(t) = \int_0^H \frac{I(z,t')dz}{4\pi\varepsilon_o c^2 r^2}\left\{\frac{u\sin^2\theta}{(1-\frac{u}{c}\cos\theta)}\right\}\mathbf{a}_r \tag{39}$$

$$\mathbf{B}_{rad,\phi}(t) = \int_0^H \frac{dz\sin\theta}{4\pi\varepsilon_o c^3 r}\left\{\frac{\partial I(z,t')}{\partial t} - \frac{2u\cos\theta}{r(1-\frac{u}{c}\cos\theta)}I(z,t') + \frac{u^2\sin^2\theta}{rc(1-\frac{u}{c}\cos\theta)^2}I(z,t')\right\}\mathbf{a}_{\varphi} \tag{40}$$

5.2. Velocity Fields

$$\mathbf{E}_{vel,\theta}(t) = \int_0^H \frac{I(z,t')dz\sin\theta}{4\pi\varepsilon_o r^2 c\left[1-\frac{u}{c}\cos\theta\right]^2}\left[1-\frac{u^2}{c^2}\right]\mathbf{a}_{\theta} \tag{41}$$

$$\mathbf{E}_{vel,r}(t) = \int_0^H \frac{I(z,t')dz}{4\pi\varepsilon_o r^2 u\left[1-\frac{u}{c}\cos\theta\right]}\left[1-\frac{u^2}{c^2}\right]\mathbf{a}_r \tag{42}$$

$$\mathbf{B}_{vel,\phi}(t) = \int_0^H \frac{I(z,t')dz\sin\theta}{4\pi\varepsilon_o r^2 c^2\left[1-\frac{u}{c}\cos\theta\right]^2}\left[1-\frac{u^2}{c^2}\right]\mathbf{a}_{\varphi} \tag{43}$$

5.3. Static Fields

$$\mathbf{E}_{stat,r}(t) = \int_0^H \frac{dz}{4\pi\varepsilon_o r^2}\left\{\frac{\cos\theta}{c}I(z,t') - \frac{1}{u}I(z,t') + \frac{2\cos\theta}{r}\int_0^t I(z,\tau')d\tau\right\}\mathbf{a_r} \tag{44}$$

$$\mathbf{E}_{stat,\theta}(t) = \int_0^H \frac{dz}{4\pi\varepsilon_o r^2}\left\{\frac{\sin\theta}{r}\int_0^t I(z,\tau')d\tau\right\}\mathbf{a_\theta} \tag{45}$$

6. Electromagnetic Fields of the Lightning Channel over Perfectly Conducting Ground

The electric field at any point over perfectly conducting ground can be obtained by replacing the ground plane by an image channel. The relevant geometry is shown in Figure 2. The electric field at point P located at the surface of the perfectly conducting ground has only a z component and the corresponding radiation, velocity and static field terms are given by

$$\mathbf{E}_{rad,z}(t) = -\int_0^H \frac{dz}{2\pi\varepsilon_o c^2 r} \left\{\frac{\partial I(z,t')\sin^2\theta}{\partial t} - \frac{2u\sin^2\theta\cos\theta}{r(1-\frac{u}{c}\cos\theta)}I(z,t') + \frac{u^2\sin^4\theta}{rc(1-\frac{u}{c}\cos\theta)^2}I(z,t') - \frac{u\cos\theta\sin^2\theta}{(1-\frac{u}{c}\cos\theta)}I(z,t')\right\}\mathbf{a_z} \tag{46}$$

$$\mathbf{E}_{vel}(t) = -\int_0^H \frac{I(z,t')dz}{2\pi\varepsilon_o r^2 c\left[1-\frac{u}{c}\cos\theta\right]^2}\left\{\frac{\sin^2\theta}{c} - \frac{\cos\theta}{u}\right\}\left[1-\frac{u^2}{c^2}\right]a_z \tag{47}$$

$$\mathbf{E}_{stat}(t) = -\int_0^H \frac{dz}{2\pi\varepsilon_o r^2}\left\{-\frac{\cos^2\theta}{c}I(z,t') + \frac{\cos\theta}{u}I(z,t') - \frac{\{3\sin^2\theta - 2\}}{r}\int_0^t I(z,\tau')d\tau\right\}\mathbf{a_z} \tag{48}$$

$$\mathbf{B}_{rad}(t) = \int_0^H \frac{dz\sin\theta}{2\pi\varepsilon_o c^3 r}\left\{\frac{\partial I(z,t')}{\partial t} - \frac{2u\cos\theta}{r(1-\frac{u}{c}\cos\theta)}I(z,t') + \frac{u^2\sin^2\theta}{rc(1-\frac{u}{c}\cos\theta)^2}I(z,t')\right\}\mathbf{a_\varphi} \tag{49}$$

$$\mathbf{B}_{vel}(t) = \int_0^H \frac{I(z,t')dz\sin\theta}{4\pi\varepsilon_o r^2 c^2\left[1-\frac{u}{c}\cos\theta\right]^2}\left[1-\frac{u^2}{c^2}\right]\mathbf{a_\varphi} \tag{50}$$

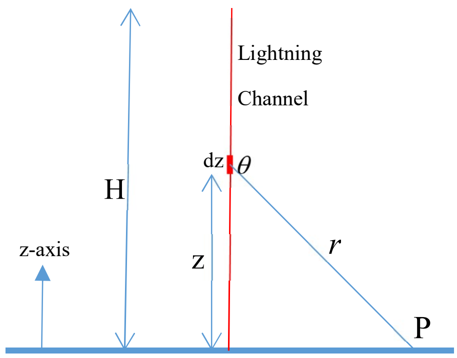

Figure 2. Geometry relevant to the calculation of electromagnetic fields from a return stroke.

These field expressions can be further simplified as

$$\mathbf{E}_{rad}(t) = -\int_0^H \frac{dz}{2\pi\varepsilon_o c^2}\left\{\frac{\partial I(z,t')\sin^2\theta}{r\,\partial t} + \frac{I(z,t')}{r^2}\left[\frac{u^2\sin^4\theta}{c(1-\frac{u}{c}\cos\theta)^2} + \frac{u\cos\theta\sin^2\theta\{\cos\theta - 2\}}{(1-\frac{u}{c}\cos\theta)}\right]\right\}\mathbf{a_z} \tag{51}$$

$$\mathbf{E}_{vel}(t) = -\int_0^H \frac{I(z,t')dz}{2\pi\varepsilon_o r^2 c\left[1-\frac{u}{c}\cos\theta\right]^2}\left[1-\frac{u^2}{c^2}\right]\left\{\cos\theta - \sin^2\theta\right\}\mathbf{a_z} \tag{52}$$

$$\mathbf{E}_{stat}(t) = -\int_0^H \frac{dz}{2\pi\varepsilon_o}\left\{\frac{\cos\theta}{ur^2}\left\{1-\frac{u}{c}\cos\theta\right\}I(z,t') - \frac{\left\{3\sin^2\theta - 2\right\}}{r^3}\int_0^t I(z,\tau')d\tau\right\}\mathbf{a_z} \tag{53}$$

$$\mathbf{B}_{rad}(t) = \int_0^H \frac{dz\sin\theta}{2\pi\varepsilon_o c^3}\left\{\frac{\partial I(z,t')}{r\,\partial t} - \frac{uI(z,t')}{r^2(1-\frac{u}{c}\cos\theta)}\left\{\frac{u\sin\theta}{c(1-\frac{u}{c}\cos\theta)} - 2\cos\theta\right\}\right\}\mathbf{a_\varphi} \tag{54}$$

$$\mathbf{B}_{vel}(t) = \int_0^H \frac{I(z,t')dz\sin\theta}{4\pi\varepsilon_o r^2 c^2\left[1-\frac{u}{c}\cos\theta\right]^2}\left[1-\frac{u^2}{c^2}\right]\mathbf{a_\varphi} \tag{55}$$

Based on the analysis presented in Section 4, these field terms can be reduced to dipole fields, resulting in

$$\mathbf{E}_{rad}(t) = -\int_0^H \frac{dz}{2\pi\varepsilon_o c^2}\left\{\frac{\partial I(z,t')\sin^2\theta}{r\,\partial t} + \frac{c\left\{3\sin^2\theta - 2\right\}I(z,t')}{r^2} + \frac{c^2\left\{3\sin^2\theta - 2\right\}}{r^3}\int_0^t I(z,\tau')d\tau\right\}\mathbf{a_z} \tag{56}$$

$$\mathbf{B}_{rad}(t) = \int_0^H \frac{dz\sin\theta}{2\pi\varepsilon_o c^3}\left\{\frac{\partial I(z,t')}{r\,\partial t} + \frac{cI(z,t')}{r^2}\right\}\mathbf{a_\varphi} \tag{57}$$

In order to illustrate this point further, let us consider the electric fields generated by a lightning return stroke as simulated using the modified transmission line model with exponential current attenuation (MTLE) [13,14]. The spatial and temporal distribution of the current, $I(z,t)$, associated with this model are given by

$$I(z,t) = 0, \text{ if } t < z/u \tag{58}$$

$$I(z,t) = e^{-z/\lambda}I(0,t), \text{ if } t > z/u \tag{59}$$

In the above equations, z is the height of the point of observation along the vertical channel, λ is the current decay height constant, u is the speed of propagation of the return stroke front and $I(0,t)$ is the current at the channel base. The three field terms and the total field associated with the accelerating charge technique (Equations (51)–(53)) and the dipole technique (Equation (56)) calculated using the above current distribution are shown in Figure 3. In the calculations, the distance to the point of observation located at ground from the strike point is taken to be 5 km, λ =2 km and $u = 1.5 \times 10^8$ m/s. The channel base current with a peak 12 kA is identical to the one suggested by Nucci et al. [15] for a subsequent return stroke.

Note that the total field generated by both techniques is identical, whereas the three terms that build up this total field is different in the two techniques. For example, observe that the dipole-radiation term at this distance is bipolar, whereas the radiation field generated by accelerating charges is unipolar. As one can observe the rest of the terms are also different in the two techniques.

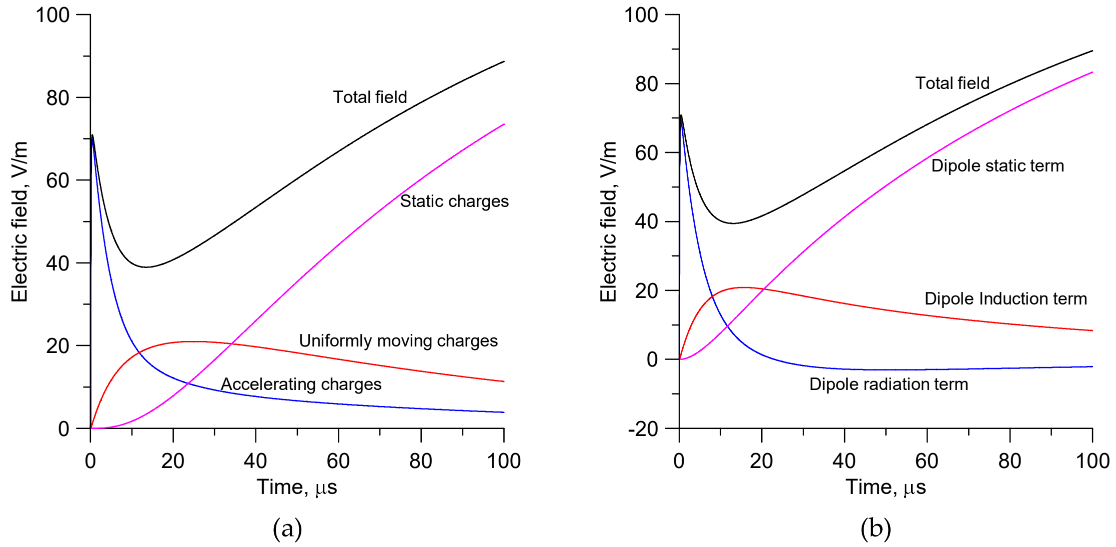

Figure 3. (**a**) The three field components and the total field associated with accelerating and moving charges, and (**b**) the three field components and the total field associated with dipole fields. The electric fields at 5 km distance from a lightning channel are obtained using the MTLE model with a 12 kA current, 1.5×10^8 m/s return stroke speed and 2 km current decay height constant. Note that in depicting the electric fields, the field components directed into the ground (i.e., directed along the negative z-axis) are considered positive.

7. Discussion

In converting the field equations pertinent to pure radiation (fields produced by accelerating charges) and pure Coulomb (electric fields produced by stationary and uniformly moving charges) to those pertinent to dipoles, the connection of the field components to the processes which generate them is lost. Dipole fields have three field components, namely, dipole-static, dipole-induction and dipole-radiation. None of these separate field components completely describe either the radiation (fields generated by accelerating charges), velocity or static field components (Coulomb fields). Of course, the distant dipole-radiation field describes the energy radiated to infinity by the accelerating charges. However, this is not the only field component that is being produced by accelerating charges. The dipole-static term describes only part of the static field, whereas the dipole-induction field is a combination of static, velocity and radiation terms, and the dipole-radiation field describes only part of the radiation field. Note that the pure radiation field has terms both varying as $1/r$ and $1/r^2$.

It is important to point out that not only the $1/r$ field terms carry energy from one place to another, but the field terms varying as $1/r^2$ terms do as well. The difference is that while the $1/r$ terms carry energy to infinity, the $1/r^2$ terms carry energy into space in the vicinity of the dipole. As the current flows along the channel, charges are transported along it and these displaced charges generate electric fields in space. These electric fields (and also magnetic fields) carry energy and it is the $1/r^2$ terms that supply this energy. However, if the displaced charges reunite by coming back to the original location, these field components will transport this energy—located in space—back to the source. That means the flow of energy caused by the $1/r^2$ terms can change direction when the current flowing along the channel reverses its direction of flow or polarity. Indeed, in the case of oscillating dipoles, these terms carry energy back and forth from the source to space and vice-versa.

Finally, note that the total field calculated for a channel element or a lightning return stroke is the same irrespective of whether the equations pertinent to accelerating charges or the dipole equations are used. Even though this point has been demonstrated numerically in previous publications [3], it is demonstrated analytically in the present paper.

8. Conclusions

In this paper we have presented analytical expressions for the electromagnetic fields generated by return strokes in terms of the field components generated by accelerating charges, moving charges and stationary charges. It is shown analytically and demonstrated using return stroke fields that these three field components can be reduced to those of the field expressions pertinent to the dipole technique, but in the reduction process the association of the individual field components to the physical processes that generate the electromagnetic fields is lost.

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Appendix A. Mathematical Steps Necessary to Convert (Equation (12) to Equations (13) and (14))

Let us start with Equation (12).

$$d\mathbf{e}_{rad} = \frac{i(z)u}{4\pi\varepsilon_o c^2}\left\{\frac{e^{j\omega(t-r_1/c)}\sin\theta_1}{r_1\left[1-\frac{u\cos\theta_1}{c}\right]}\mathbf{a}_{\theta_1} - \frac{e^{j\omega(t-dz/u-r_2/c)}\sin\theta_2}{r_2\left[1-\frac{u\cos\theta_2}{c}\right]}\mathbf{a}_{\theta_2}\right\} \tag{A1}$$

This can be written as

$$d\mathbf{e}_{rad} = \frac{i(z)u}{4\pi\varepsilon_o c^2}\{\mathbf{F}_1 - \mathbf{F}_2\} \tag{A2}$$

Let us start with the expression for $\mathbf{F}_1$.

$$\mathbf{F}_1 = \frac{e^{j\omega(t-r_1/c)}\sin\theta_1}{r_1\left[1-\frac{u\cos\theta_1}{c}\right]}\mathbf{a}_{\theta_1} \tag{A3}$$

Since dz is an elementary length, the exponential term after substituting for r_1 from Equation (3) can be expanded as follows:

$$e^{j\omega(t-r_1/c)} = e^{j\omega(t-r/c)}\left\{1-\frac{j\omega dz\cos\theta}{2c}\right\} \tag{A4}$$

Using this expression and substituting into $1/r_1$, $\sin\theta_1$ and $1/[1-u\cos\theta_1/c]$ from Equations (4), (6) and (10) respectively, we obtain

$$\mathbf{F}_1 = \frac{e^{j\omega(t-r/c)}\left\{1-\frac{j\omega dz\cos\theta}{2c}\right\}\sin\theta\left\{1-\frac{dz\cos\theta}{2r}\right\}\frac{1}{r}\left\{1-\frac{dz\cos\theta}{2r}\right\}\left\{1+\frac{udz\sin^2\theta}{2rc\left(1-\frac{u}{c}\cos\theta\right)}\right\}}{\left[1-\frac{u\cos\theta}{c}\right]}\mathbf{a}_{\theta_1} \tag{A5}$$

Reorganizing the terms, we can write

$$\mathbf{F}_1 = \frac{e^{j\omega(t-r/c)}\sin\theta\left\{1-\frac{j\omega dz\cos\theta}{2c}\right\}\left\{1-\frac{dz\cos\theta}{2r}\right\}\left\{1-\frac{dz\cos\theta}{2r}\right\}\left\{1+\frac{udz\sin^2\theta}{2rc\left(1-\frac{u}{c}\cos\theta\right)}\right\}}{r\left[1-\frac{u\cos\theta}{c}\right]}\mathbf{a}_{\theta_1} \tag{A6}$$

Keeping only the first order terms with respect to dz, we obtain

$$\mathbf{F}_1 = \frac{e^{j\omega(t-r/c)}\sin\theta\left\{1-\frac{j\omega dz\cos\theta}{2c}-\frac{dz\cos\theta}{r}+\frac{udz\sin^2\theta}{2rc\left(1-\frac{u}{c}\cos\theta\right)}\right\}}{r\left[1-\frac{u\cos\theta}{c}\right]}\mathbf{a}_{\theta_1} \tag{A7}$$

This vector can be resolved into the directions of $\mathbf{a}_\theta$ and $\mathbf{a}_r$ using Equations (1) and (2) (note that $\sin\delta\theta_1 \approx \delta\theta_1$) as follows:

$$\mathbf{F}_{11} = \frac{e^{j\omega(t-r/c)}\sin\theta\left\{1 - \frac{j\omega dz\cos\theta}{2c} - \frac{dz\cos\theta}{r} + \frac{udz\sin^2\theta}{2rc\left(1-\frac{u}{c}\cos\theta\right)}\right\}}{r\left[1 - \frac{u\cos\theta}{c}\right]}\mathbf{a}_\theta \tag{A8}$$

$$\mathbf{F}_{12} = \frac{e^{j\omega(t-r/c)}\sin\theta\left\{1 - \frac{j\omega dz\cos\theta}{2c} - \frac{dz\cos\theta}{r} + \frac{udz\sin^2\theta}{2rc\left(1-\frac{u}{c}\cos\theta\right)}\right\}\left(\frac{dz\sin\theta}{2r}\right)}{r\left[1 - \frac{u\cos\theta}{c}\right]}\mathbf{a}_r \tag{A9}$$

Again, keeping only the first order terms with respect to dz we obtain

$$\mathbf{F}_{12} = \frac{e^{j\omega(t-r/c)}\sin^2\theta}{2r\left[1 - \frac{u\cos\theta}{c}\right]}\mathbf{a}_r \tag{A10}$$

Let us now consider the term $\mathbf{F}_2$.

$$\mathbf{F}_2 = \frac{e^{j\omega(t-dz/u-r_2/c)}\sin\theta_2}{r_2\left[1 - \frac{u\cos\theta_2}{c}\right]}\mathbf{a}_{\theta_2} \tag{A11}$$

Let us consider the exponential term. Substituting for r_2 from Equation (3) and expanding the exponential term (since dz is an elementary length) we obtain

$$e^{j\omega(t-r_2/c-dz/u)} = e^{j\omega(t-r/c+\frac{dz\cos\theta}{c}-\frac{dz}{u})} = e^{j\omega(t-r/c)}\left\{1 - \frac{j\omega dz}{u} + \frac{j\omega dz\cos\theta}{2c}\right\} \tag{A12}$$

This can be written as

$$e^{j\omega(t-r_2/c-dz/u)} = e^{j\omega(t-r/c)}\left\{1 - \frac{j\omega dz}{u}(1 - \frac{u}{2c}\cos\theta)\right\} \tag{A13}$$

Using this expression and substituting to $1/r_2$, $\sin\theta_2$ and $1/[1 - u\cos\theta_2/c]$ from Equations (4), (6) and (11) respectively, we obtain

$$\mathbf{F}_2 = \frac{e^{j\omega(t-r/c)}\left\{1 - \frac{j\omega dz}{u}(1 - \frac{u}{2c}\cos\theta)\right\}\sin\theta\left\{1 + \frac{dz\cos\theta}{2r}\right\}\left\{1 - \frac{udz\sin^2\theta}{2rc\left(1-\frac{u}{c}\cos\theta\right)}\right\}\left\{1 + \frac{dz\cos\theta}{2r}\right\}}{r\left\{1 - \frac{u}{c}\cos\theta\right\}}\mathbf{a}_{\theta_2} \tag{A14}$$

Keeping only the first order terms in dz we obtain

$$\mathbf{F}_2 = \frac{e^{j\omega(t-r/c)}\sin\theta\left\{1 - \frac{j\omega dz}{u}(1 - \frac{u}{2c}\cos\theta) + \frac{dz\cos\theta}{r} - \frac{udz\sin^2\theta}{2rc\left(1-\frac{u}{c}\cos\theta\right)}\right\}}{r\left[1 - \frac{u\cos\theta}{c}\right]}\mathbf{a}_{\theta_2} \tag{A15}$$

This vector can be resolved into the directions of $\mathbf{a}_\theta$ and $\mathbf{a}_r$ using Equations (1) and (2) (note that $\sin\delta\theta_2 \approx \delta\theta_2$) as follows:

$$\mathbf{F}_{21} = \frac{e^{j\omega(t-r/c)}\sin\theta\left\{1 - \frac{j\omega dz}{u}(1 - \frac{u}{2c}\cos\theta) + \frac{dz\cos\theta}{r} - \frac{udz\sin^2\theta}{2rc\left(1-\frac{u}{c}\cos\theta\right)}\right\}}{r\left[1 - \frac{u\cos\theta}{c}\right]}\mathbf{a}_\theta \tag{A16}$$

$$\mathbf{F}_{22} = \frac{e^{j\omega(t-r/c)}\sin\theta\left\{1 - \frac{j\omega dz}{u}\left(1 - \frac{u}{2c}\cos\theta\right) + \frac{dz\cos\theta}{r} - \frac{udz\sin^2\theta}{2rc\left(1-\frac{u}{c}\cos\theta\right)}\right\}\left(-\frac{dz\sin\theta}{2r}\right)}{r\left[1 - \frac{u\cos\theta}{c}\right]}\mathbf{a}_\theta \tag{A17}$$

Keeping only the first order terms in dz we obtain

$$\mathbf{F}_{22} = -\frac{e^{j\omega(t-r/c)}\sin^2\theta dz}{2r\left[1 - \frac{u\cos\theta}{c}\right]}\mathbf{a}_r \tag{A18}$$

Now, the net field component in the $\mathbf{a}_\theta$ direction is given by

$$d\mathbf{e}_{rad,\theta} = \frac{i(z)u}{4\pi\varepsilon_o c^2}\{\mathbf{F}_{11} - \mathbf{F}_{21}\} \tag{A19}$$

After substituting for $\mathbf{F}_{11}$ and $\mathbf{F}_{21}$ and replacing $t - r/c$ by t' the expression reduces to Equation (13), i.e.,

$$d\mathbf{e}_{rad,\theta} = \frac{i(z)e^{j\omega t'}u\sin\theta}{4\pi\varepsilon_o c^2 r(1 - \frac{u}{c}\cos\theta)}\left\{\frac{j\omega dz}{u} - \frac{j\omega dz\cos\theta}{c} - \frac{2dz\cos\theta}{r} + \frac{udz\sin^2\theta}{rc(1 - \frac{u}{c}\cos\theta)}\right\}\mathbf{a}_\theta \tag{A20}$$

The net component in the $\mathbf{a}_r$ direction is given by

$$d\mathbf{e}_{rad,r} = \frac{i(z)u}{4\pi\varepsilon_o c^2}\{\mathbf{F}_{12} - \mathbf{F}_{22}\} \tag{A21}$$

After substituting for $\mathbf{F}_{12}$ and $\mathbf{F}_{22}$ the expression reduces to Equation (14), i.e.,

$$d\mathbf{e}_{rad,r} = \frac{i(z)e^{j\omega t'}}{4\pi\varepsilon_o c^2 r^2}\left\{\frac{udz\sin^2\theta}{(1 - \frac{u}{c}\cos\theta)}\right\}\mathbf{a}_r \tag{A22}$$

Appendix B. Mathematical Steps Necessary for the Reduction of Electromagnetic Fields Obtained From Accelerating and Moving Charges to the Dipole Fields

(a) Reduction of $d\mathrm{E}_\theta$ to the dipole fields:

The mathematical steps necessary for the reduction starting from Equation (32) are shown below.

$$d\mathbf{E}_\theta(t) = \frac{dz\sin\theta}{4\pi\varepsilon_o}\left\{\frac{1}{c^2 r}\frac{\partial i(z,t')}{\partial t} + \frac{1}{r^3}\int_0^t I(z,\tau')d\tau + \frac{i(z,t')\,u^2\sin^2\theta}{r^2c^3(1-\frac{u}{c}\cos\theta)^2} - \frac{i(z,t')\,2u\cos\theta}{c^2r^2(1-\frac{u}{c}\cos\theta)} + \frac{i(z,t')\,(1-u^2/c^2)}{cr^2(1-\frac{u}{c}\cos\theta)^2}\right\}\mathbf{a}_\theta \tag{A23}$$

Let us combine the third and fifth terms inside the bracket. The result is given by

$$d\mathbf{E}_\theta(t) = \frac{dz\sin\theta}{4\pi\varepsilon_o}\left\{\frac{1}{c^2 r}\frac{\partial i(z,t')}{\partial t} + \frac{1}{r^3}\int_0^t I(z,\tau')d\tau + \frac{i(z,t')}{r^2c(1-\frac{u}{c}\cos\theta)^2}\left[\frac{u^2\sin^2\theta}{c^2} + 1 - \frac{u^2}{c^2}\right] - \frac{i(z,t')\,2u\cos\theta}{c^2r^2(1-\frac{u}{c}\cos\theta)}\right\}\mathbf{a}_\theta \tag{A24}$$

$$d\mathbf{E}_\theta(t) = \frac{dz\sin\theta}{4\pi\varepsilon_o}\left\{\frac{1}{c^2 r}\frac{\partial i(z,t')}{\partial t} + \frac{1}{r^3}\int_0^t I(z,\tau')d\tau + \frac{i(z,t')}{r^2c(1-\frac{u}{c}\cos\theta)^2}\left[\frac{u^2}{c^2} - \frac{u^2\cos^2\theta}{c^2} + 1 - \frac{u^2}{c^2}\right] - \frac{i(z,t')\,2u\cos\theta}{c^2r^2(1-\frac{u}{c}\cos\theta)}\right\}\mathbf{a}_\theta \tag{A25}$$

$$d\mathbf{E}_\theta(t) = \frac{dz\sin\theta}{4\pi\varepsilon_o}\left\{\frac{1}{c^2 r}\frac{\partial i(z,t')}{\partial t} + \frac{1}{r^3}\int_0^t I(z,\tau')d\tau + \frac{i(z,t')}{r^2c(1-\frac{u}{c}\cos\theta)^2}\left[1 - \frac{u^2\cos^2\theta}{c^2}\right] - \frac{i(z,t')\,2u\cos\theta}{c^2r^2(1-\frac{u}{c}\cos\theta)}\right\}\mathbf{a}_\theta \tag{A26}$$

$$d\mathbf{E}_\theta(t) = \frac{dz\sin\theta}{4\pi\varepsilon_o} \left\{ \frac{1}{c^2 r}\frac{\partial i(z,t')}{\partial t} + \frac{1}{r^3}\int_0^t I(z,\tau')d\tau + \frac{i(z,t')}{r^2 c(1-\frac{u}{c}\cos\theta)^2}\left[(1-\frac{u}{c}\cos\theta)(1+\frac{u}{c}\cos\theta)\right] - \frac{i(z,t')\,2u\cos\theta}{c^2 r^2(1-\frac{u}{c}\cos\theta)} \right\}\mathbf{a_\theta} \quad \text{(A27)}$$

$$d\mathbf{E}_\theta(t) = \frac{dz\sin\theta}{4\pi\varepsilon_o} \left\{ \frac{1}{c^2 r}\frac{\partial i(z,t')}{\partial t} + \frac{1}{r^3}\int_0^t I(z,\tau')d\tau + \frac{i(z,t')(1+\frac{u}{c}\cos\theta)}{r^2 c(1-\frac{u}{c}\cos\theta)} - \frac{i(z,t')\,2u\cos\theta}{c^2 r^2(1-\frac{u}{c}\cos\theta)} \right\}\mathbf{a_\theta} \quad \text{(A28)}$$

$$d\mathbf{E}_\theta(t) = \frac{dz\sin\theta}{4\pi\varepsilon_o} \left\{ \frac{1}{c^2 r}\frac{\partial i(z,t')}{\partial t} + \frac{1}{r^3}\int_0^t I(z,\tau')d\tau + \frac{i(z,t')}{r^2 c(1-\frac{u}{c}\cos\theta)}\left[\left(1+\frac{u}{c}\cos\theta\right) - \frac{2u\cos\theta}{c}\right] \right\}\mathbf{a_\theta} \quad \text{(A29)}$$

$$d\mathbf{E}_\theta(t) = \frac{dz\sin\theta}{4\pi\varepsilon_o} \left\{ \frac{1}{c^2 r}\frac{\partial i(z,t')}{\partial t} + \frac{1}{r^3}\int_0^t I(z,\tau')d\tau + \frac{i(z,t')}{r^2 c(1-\frac{u}{c}\cos\theta)}\left[\left(1-\frac{u}{c}\cos\theta\right)\right] \right\}\mathbf{a_\theta} \quad \text{(A30)}$$

$$d\mathbf{E}_\theta(t) = \frac{dz\sin\theta}{4\pi\varepsilon_o} \left\{ \frac{1}{c^2 r}\frac{\partial i(z,t')}{\partial t} + \frac{1}{r^3}\int_0^t I(z,\tau')d\tau + \frac{i(z,t')}{r^2 c} \right\}\mathbf{a_\theta} \quad \text{(A31)}$$

(b) Reduction of $d\mathrm{E}_r$ to the dipole fields:

The mathematical steps necessary for the reduction starting from Equation (33) are shown below.

$$d\mathbf{E}_r(t) = \frac{dz}{4\pi\varepsilon_o} \left\{ \frac{2\cos\theta}{r^3}\int_0^t i(z,\tau')d\tau + \frac{i(z,t')\cos\theta}{r^2 c} - \frac{i(z,t')}{ur^2} + \frac{i(z,t')\,u\sin^2\theta}{c^2 r^2(1-\frac{u}{c}\cos\theta)} + \frac{i(z,t')\,(1-u^2/c^2)}{ur^2(1-\frac{u}{c}\cos\theta)} \right\}\mathbf{a_r} \quad \text{(A32)}$$

$$d\mathbf{E}_r(t) = \frac{dz}{4\pi\varepsilon_o} \left\{ \frac{2\cos\theta}{r^3}\int_0^t i(z,\tau')d\tau + \frac{i(z,t')}{ur^2}\left[\frac{u\cos\theta}{c} - 1 + \frac{u^2\sin^2\theta}{c^2(1-\frac{u}{c}\cos\theta)} + \frac{(1-u^2/c^2)}{(1-\frac{u}{c}\cos\theta)}\right] \right\}\mathbf{a_r} \quad \text{(A33)}$$

$$d\mathbf{E}_r(t) = \frac{dz}{4\pi\varepsilon_o} \left\{ \frac{2\cos\theta}{r^3}\int_0^t i(z,\tau')d\tau + \frac{i(z,t')}{ur^2}\left[\frac{u\cos\theta}{c} - 1 + \frac{1}{c^2(1-\frac{u}{c}\cos\theta)}\left(u^2\sin^2\theta + c^2 - u^2\right)\right] \right\}\mathbf{a_r} \quad \text{(A34)}$$

$$d\mathbf{E}_r(t) = \frac{dz}{4\pi\varepsilon_o} \left\{ \frac{2\cos\theta}{r^3}\int_0^t i(z,\tau')d\tau + \frac{i(z,t')}{ur^2}\left[\frac{u\cos\theta}{c} - 1 + \frac{1}{c^2(1-\frac{u}{c}\cos\theta)}\left(c^2 - u^2\cos^2\theta\right)\right] \right\}\mathbf{a_r} \quad \text{(A35)}$$

$$d\mathbf{E}_r(t) = \frac{dz}{4\pi\varepsilon_o} \left\{ \frac{2\cos\theta}{r^3}\int_0^t i(z,\tau')d\tau + \frac{i(z,t')}{ur^2}\left[\frac{u\cos\theta}{c} - 1 + \frac{1}{(1-\frac{u}{c}\cos\theta)}\left(1 - \frac{u^2}{c^2}\cos^2\theta\right)\right] \right\}\mathbf{a_r} \quad \text{(A36)}$$

$$d\mathbf{E}_r(t) = \frac{dz}{4\pi\varepsilon_o} \left\{ \frac{2\cos\theta}{r^3}\int_0^t i(z,\tau')d\tau + \frac{i(z,t')}{ur^2}\left[\frac{u\cos\theta}{c} - 1 + \left(1 + \frac{u\cos\theta}{c}\right)\right] \right\}\mathbf{a_r} \quad \text{(A37)}$$

$$d\mathbf{E}_r(t) = \frac{dz}{4\pi\varepsilon_o} \left\{ \frac{2\cos\theta}{r^3}\int_0^t i(z,\tau')d\tau + \frac{i(z,t')}{r^2}\left[\frac{2\cos\theta}{c}\right] \right\}\mathbf{a_r} \quad \text{(A38)}$$

(c) Reduction of $d\mathbf{B}_\phi$ to the dipole fields:

The mathematical steps necessary for the reduction starting from Equation (34) are shown below.

$$d\mathbf{B}_\phi(t) = \frac{dz\sin\theta}{4\pi\varepsilon_o} \left\{ \frac{1}{c^3 r}\frac{\partial i(z,t')}{\partial t} - \frac{i(z,t')\,2u\cos\theta}{c^3 r^2(1-\frac{u}{c}\cos\theta)} + \frac{i(z,t')\,u^2\sin^2\theta}{r^2 c^4(1-\frac{u}{c}\cos\theta)^2} + \frac{i(z,t')\,(1-u^2/c^2)}{c^2 r^2(1-\frac{u}{c}\cos\theta)^2} \right\}\mathbf{a}_\phi \quad \text{(A39)}$$

$$d\mathbf{B}_\phi(t) = \frac{dz\sin\theta}{4\pi\varepsilon_o} \left\{ \frac{1}{c^3 r}\frac{\partial i(z,t')}{\partial t} - \frac{i(z,t')\,2u\cos\theta}{c^3 r^2(1-\frac{u}{c}\cos\theta)} + \frac{i(z,t')}{r^2 c^2(1-\frac{u}{c}\cos\theta)^2}\left(\frac{u^2}{c^2}\sin^2\theta + (1-\frac{u^2}{c^2})\right) \right\}\mathbf{a}_\phi \quad \text{(A40)}$$

$$d\mathbf{B}_{\phi}(t) = \frac{dz\sin\theta}{4\pi\varepsilon_o}\left\{\frac{1}{c^3 r}\frac{\partial i(z,t')}{\partial t} - \frac{i(z,t')\,2u\cos\theta}{c^3 r^2(1-\frac{u}{c}\cos\theta)} + \frac{i(z,t')}{r^2c^2(1-\frac{u}{c}\cos\theta)^2}\left(1-\frac{u^2}{c^2}\cos^2\theta\right)\right\}\mathbf{a}_{\phi} \quad (A41)$$

$$d\mathbf{B}_{\phi}(t) = \frac{dz\sin\theta}{4\pi\varepsilon_o}\left\{\frac{1}{c^3 r}\frac{\partial i(z,t')}{\partial t} - \frac{i(z,t')\,2u\cos\theta}{c^3 r^2(1-\frac{u}{c}\cos\theta)} + \frac{i(z,t')}{r^2c^2(1-\frac{u}{c}\cos\theta)}\left(1+\frac{u}{c}\cos\theta\right)\right\}\mathbf{a}_{\phi} \quad (A42)$$

$$d\mathbf{B}_{\phi}(t) = \frac{dz\sin\theta}{4\pi\varepsilon_o}\left\{\frac{1}{c^3 r}\frac{\partial i(z,t')}{\partial t} + \frac{i(z,t')}{c^2 r^2(1-\frac{u}{c}\cos\theta)}\left(1+\frac{u}{c}\cos\theta-\frac{2u}{c}\cos\theta\right)\right\}\mathbf{a}_{\phi} \quad (A43)$$

$$d\mathbf{B}_{\phi}(t) = \frac{dz\sin\theta}{4\pi\varepsilon_o}\left\{\frac{1}{c^3 r}\frac{\partial i(z,t')}{\partial t} + \frac{i(z,t')}{c^2 r^2}\right\}\mathbf{a}_{\phi} \quad (A44)$$

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