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Global Mittag—Leffler Synchronization for Neural Networks Modeled by Impulsive Caputo Fractional Differential Equations with Distributed Delays

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Received: 17 September 2018; Accepted: 2 October 2018; Published: 10 October 2018

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Abstract: The synchronization problem for impulsive fractional-order neural networks with both time-varying bounded and distributed delays is studied. We study the case when the neural networks and the fractional derivatives of all neurons depend significantly on the moments of impulses and we consider both the cases of state coupling controllers and output coupling controllers. The fractional generalization of the Razumikhin method and Lyapunov functions is applied. Initially, a brief overview of the basic fractional derivatives of Lyapunov functions used in the literature is given. Some sufficient conditions are derived to realize the global Mittag–Leffler synchronization of impulsive fractional-order neural networks. Our results are illustrated with examples.

Keywords: fractional-order neural networks; delays; distributed delays; impulses; Mittag–Leffler synchronization; Lyapunov functions; Razumikhin method

1. Introduction

Over the last few decades, fractional differential equations have gained considerable importance and attention due to their applications in science and engineering, i.e., in control, in stellar interiors, star clusters [1], in electrochemistry, in viscoelasticity [2] and in optics [3]. For example, the control of mechanical systems is currently one of the most active fields of research and the use of fractional order calculus increases the flexibility of controlling any system from a point to a space. Applications of fractional quantum mechanics cover dynamics of a free particle and a new representation for a free particle quantum mechanical kernel (see, for example, [4]).

The stability of fractional order systems is quite a recent topic (see, for example, Ref. [5] for the Ulam–Hyers–Mittag–Leffler stability of fractional-order delay differential equations, Ref. [6] for the Mittag–Leffler stability of impulsive fractional neural network, Ref. [7] for the Mittag–Leffler stability of fractional systems, Ref. [8] for the Mittag–Leffler stability for fractional nonlinear systems with delay, and Ref. [9] for the Mittag–Leffler stability of nonlinear fractional systems with impulses). One of the most useful approaches in studying stability for nonlinear fractional differential equations is the Lyapunov approach. Its application to fractional differential equations is connected with several difficulties. One of the main difficulties is connected with the appropriate definition of derivative of Lyapunov functions among thedifferential equations of fractional order. Impulsive differential equations arise in real world problems to describe the dynamics of processes in which sudden, discontinuous jumps occur.

Most research on the synchronization of delayed neural networks has been restricted to the case of discrete delays (see, for example, [10]) Since a neural network usually has a spatial nature due to the presence of an amount of parallel pathways of a variety of axis sizes and lengths, it is desirable to model them by introducing distributed delays. Note in [11] that both time-varying delays and distributed time delays are taken into account in studying fractional neural networks with impulses and constant strengths between two units. In all models of neural networks, one considers the case of constant rate with which the i-th neuron resets its potential to the resting state in isolation, and the constant synaptic connection strength of the i-th neuron to the j-th neuron (see, for example, [10]). In our paper, we consider the general case of time varying coefficients in the model that allows more appropriate modeling of the connections between the neurons. These more complicated mathematical equations lead to an application of new types of fractional derivatives of Lyapunov functions and new stability results.

In this paper, Caputo fractional delay differential equations with impulses and two types of delays-variable in time and distributed ones are studied. Some results for piecewise continuous Lyapunov functions based on the Razumikhin method are obtained. Appropriate derivatives of Lyapunov functions among the studied fractional equations are used. Our results are applied to study the synchronization of neural networks with Caputo fractional derivatives, variable delays, distributed delays, and impulses. We study the case when the lower limit of the fractional derivative is changing after each impulsive time. To the best of our knowledge, this is the first model of neural networks of this type studied in the literature. Additionally, we study the general case of variables in time strengths of the j-th unit on the i-th unit and nonlinear impulsive functions. Both the cases of state coupling controllers and output coupling controllers are considered. Our sufficient conditions naturally depend significantly on the fractional order of the model (compare with sufficient conditions in [11,12]).

2. Impulses in Fractional Delay Differential Equations

Let a sequence $\{t_k\}_{k=1}^{\infty}\ :\ 0 \le t_{k-1} < t_k \le t_{k+1},\ \lim_{k\to\infty} t_k = \infty$ be given. Let $t_0 \neq t_k$, $k = 1, 2, \dots$ be the given initial time and $r > 0$. Without loss of generality we can assume $t_0 \in [0, t_1)$. Let $E = C([-r, 0], \mathbb{R}^n)$ with $||\phi||_0 = \max_{s\in[-r,0]} ||\phi(s)||$ for $\phi \in E$, and $||.||$ is a norm in $\mathbb{R}^n$.

In many applications in science and engineering, the fractional order q is often less than 1, so we restrict $q \in (0, 1)$ everywhere in the paper.

1: *The Riemann–Liouville (RL) fractional derivative* of order $q \in (0, 1)$ of $m(t)$ is given by (see, for example, [13–15])

$$ {}_{t_0}^{RL}D_t^q m(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_{t_0}^{t} (t-s)^{-q}\, m(s) ds, \quad t \ge t_0, $$

where $\Gamma(.)$ denotes the Gamma function.

2: *The Caputo fractional derivative* of order $q \in (0, 1)$ is defined by (see, for example, [13–15])

$$ {}_{t_0}^{C}D_t^q m(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q}\, m'(s) ds, \quad t \ge t_0. $$

3: *The Grünwald–Letnikov fractional derivative* is given by (see, for example, [13–15]))

$$ {}_{t_0}^{GL}D_t^q m(t) = \lim_{h\to 0+} \frac{1}{h^q} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r\, (qCr)\, m(t - rh), \quad t \ge t_0. $$

The Mittag–Leffler function with one parameter is defined as

$$E_\alpha(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(1+\alpha\, i)}, \quad \alpha > 0, \; z \in C.$$

Definition 1. ([16]) The function $m(t) \in C^q([t_0, T], \mathbb{R}^n)$ if $m(t)$ is differentiable on $[t_0, T]$ (i.e., $m'(t)$ exists) and the Caputo derivative ${}_{t_0}^{C}D^q m(t)$ exists for $t \in [t_0, T]$.

Consider the initial value problem (IVP) for the nonlinear *impulsive Caputo fractional delay differential equation* (IFrDDE)

$$\begin{aligned} &{}_{t_0}^{C}D^q x(t) = F(t, x_t) \text{ for } t \geq t_0,\; t \neq t_k, k = 1, \ldots, \\ &\Delta x(t)|_{t=t_k} = I_k(x(t_k - 0)) \quad \text{for } k = 1, 2, \ldots, \\ &x(t_0 + s) = \phi(s), \;\; s \in [-r, 0], \end{aligned} \tag{1}$$

where $0 < q < 1$, $\Delta x(t)|_{t=t_k} = x(t_k + 0) - x(t_k)$, $x_t = x(t+s), s \in [-r, 0]$, $F : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$, $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $(k = 1, 2, 3, \ldots)$, $\phi \in E$, where $x(t_k + 0) = \lim_{t \to t_k,\, t > t_k} x(t) < \infty$, $x(t_k - 0) = \lim_{t \to t_k,\, t < t_k} x(t) = x(t_k)$.

Denote $\Phi_k(x) = x + I_k(x), k = 1, 2, \ldots,\; x \in \mathbb{R}^n$.

We will denote the solution of the IVP for IFrDDE (1) by $x(t; t_0, \phi)$ for $t \geq t_0$. The solution of IFrDDE (1) is a piecewise continuous function. In connection with this, we introduce the following sets of functions:

$$\begin{aligned} PC(a, b) = \Big\{ &x : [a, b] \to \mathbb{R}^n \text{ such that } \; x(t) \in C([a, b]/\{t_k\}), \\ &x(t_k) = x(t_k - 0) = \lim_{t \to t_k - 0} x(t),\; x(t_k + 0) = \lim_{t \to t_k + 0} x(t) < \infty \Big\}, \\ PC^1(a, b) = \Big\{ &x \in PC(a, b) \text{ such that } \; x(t) \in C^1([a, b]/\{t_k\}), \\ &x'(t_k) = x'(t_k - 0) = \lim_{t \to t_k - 0} x'(t),\; x'(t_k + 0) = \lim_{t \to t_k + 0} x'(t) < \infty \Big\}. \end{aligned}$$

The fractional derivatives depend significantly on their lower limit and it allows different interpretations of piecewise continuous solutions of impulsive differential equations. This phenomena is not characteristic for ordinary derivatives. In the literature, there are two main approaches to interpret the solutions of impulsive fractional delay differential equations:

First approach to the solutions of (1) (A1 for IFrDDE).

The solution of the IVP for IFrDDE (1) satisfies the equalities (integral)

$$x(t) = x(t; t_0, \phi) = \\ \begin{cases} \phi(t - t_0), & t \in [t_0 - r, t_0], \\ \phi(0) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} F(s, x_s) ds & \\ \quad + \sum_{i=1}^{k} I_i(x(t_i - 0)), & t \in (t_k, t_{k+1}],\; k = 0, 1, 2, \ldots. \end{cases} \tag{2}$$

Formula (2) is given and used in [17]. It is a generalization to the formula proved in [18] for the solution of impulsive fractional differential equations without delays.

Second approach to the solutions of (1) (A2 for IFrDDE).

The idea of this approach is based on the dependence of the Caputo fractional derivative on the initial time point of the interval of differential equation, i.e., the lower limit of the Caputo fractional derivative is changing at each moment of impulse of the differential equation. Sometimes, Equation (1) in this case is written by

$$
\begin{aligned}
&{}_{t_k}^{C}D^q x(t) = F(t, x_t) \text{ for } t \in (t_k, t_{k+1}],\ k = 0, 1, \dots, \\
&\Delta x(t)|_{t=t_k} = I_k(x(t_k - 0)) \quad \text{for } k = 1, 2, \dots, \\
&x(t_0 + s) = \phi(s),\ \ s \in [-r, 0].
\end{aligned} \tag{3}
$$

Then, the solution of the IVP for IFrDDE (1), respectively (3), is given by

$$
x(t) = \begin{cases} \phi(t - t_0), & t \in [t_0 - r, t_0], \\ \phi(0) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} F(s, x_s) ds \text{ for } \ t \in (t_0, t_1], & \\ \Phi_k(x(t_k - 0)) + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} F(s, x_s) ds & \\ \qquad \text{for } \ t \in (t_k, t_{k+1}],\ k = 1, 2, \dots . & \end{cases} \tag{4}
$$

Remark 1. *Both Formulas (2) and (4) differ for fractional differential equations and they are generalizations to impulsive ordinary differential equations. Both formulas coincide in the case of the ordinary derivative ($q = 1$) because in this case we have*

$$
\begin{aligned}
&\Phi_k(x(t_k - 0)) + \int_{t_k}^{t} F(s, x_s) ds = x(t_k - 0) + I_k(x(t_k - 0)) + \int_{t_k}^{t} f(s, x_s) ds \\
&= \phi(0) + \int_{t_0}^{t_k} F(s, x_s) ds + \sum_{i=1}^{k-1} I_i(x(t_i - 0)) + I_k(x(t_k - 0)) + \int_{t_k}^{t} f(s, x_s) ds \\
&= \phi(0) + \int_{t_0}^{t} F(s, x_s) ds + \sum_{i=1}^{k} I_i(x(t_i - 0)),
\end{aligned} \tag{5}
$$

but

$$
\begin{aligned}
&\Phi_k(x(t_k - 0)) + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} F(s, x_s) ds \\
&= x(t_k - 0) + I_k(x(t_k - 0)) + \frac{1}{\Gamma(q)} \int_{t_k}^{t} F(s, x_s) ds \\
&= \Big(\phi(0) + \frac{1}{\Gamma(q)} \int_{t_0}^{t_k} (t_k - s)^{q-1} F(s, x_s) ds + \sum_{i=1}^{k-1} I_i(x(t_i - 0))\Big) \\
&\quad + I_k(x(t_k - 0)) + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} F(s, x_s) ds \\
&\neq \phi(0) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} F(s, x_s) ds + \sum_{i=1}^{k} I_i(x(t_i - 0)).
\end{aligned} \tag{6}
$$

Remark 2. *In the case $q = 1$, the solution of the impulsive ordinary differential equation on each interval of continuity could be considered as a solution of the same differential equation with a new initial condition, defined by the impulsive function. It allows the application of induction w.r.t. the interval of continuity.*

This is different for fractional differential equation.

If A1 for IFrDDE is applied, then ${}_{t_0}^{C}D^q x(t) \neq {}_{t_k}^{C}D^q x(t)$ for $t \in (t_k, t_{k+1})$ and induction w.r.t. the interval of continuity is not useful.

If A2 for IFrDDE is applied, then induction w.r.t. the interval of continuity could be used.

A detailed explanation of advantages/disadvantages of both the above approaches for equations without delays is given in [19,20]. The definition of the solution $x(t; t_0, \phi)$ of the IVP for IFrDDE (1) depends on your point of view.

In this paper, we will use approach A2 for IFrDDE.

3. Lyapunov Functions and Their Fractional Derivatives

In this paper, we will use piecewise continuous Lyapunov functions [21]):

Definition 1. *Let $\alpha < \beta \leq \infty$ be given numbers and $\mathcal{D} \subset \mathbb{R}^n$, $0 \in \mathcal{D}$ be a given set. We will say that the function $V(t,x) : [\alpha - r, \beta) \times \mathcal{D} \to \mathbb{R}_+$ belongs to the class $\Lambda([\alpha - r, \beta), \mathcal{D})$ if*

1. *The function $V(t,x)$ is continuous on $[\alpha, \beta)/\{t_k\} \times \mathcal{D}$ and it is locally Lipschitz with respect to its second argument;*
2. *For each $t_k \in (\alpha, \beta)$ and $x \in \mathcal{D}$, there exist finite limits*

$$V(t_k, x) = V(t_k - 0, x) = \lim_{t \uparrow t_k} V(t,x), \quad \textit{and} \;\; V(t_k + 0, x) = \lim_{t \downarrow t_k} V(t,x).$$

In connection with the Caputo fractional derivative, it is necessary to define in an appropriate way the derivative of the Lyapunov functions among the studied equation. The choice $\frac{dV(t,x)}{dt}$ is adapted from the case of ordinary differential equations, but it is not applicable since it does not depend on the initial time point (such as the Caputo fractional derivative).

We will give a brief overview of the three main types derivatives of Lyapunov functions $V(t,x) \in \Lambda([t_0 - r, T), \mathcal{D})$ among solutions of fractional differential equations in the literature:

- <u>*first type*</u>– Let $x(t) \in \mathcal{D}$, $t \in [t_0 - r, T)$, be a solution of the IVP for the IFrDDE (3) (according to A2 for IFrDDE). Then, we can consider **the Caputo fractional derivative** of the function $V(t,x) \in \Lambda([t_0 - r, T), \mathcal{D})$ defined by

$$
\begin{aligned}
&{}_{t_k}^{\;c}D^q V(t, x(t)) = \frac{1}{\Gamma(1-q)} \int_{t_k}^{t} (t-s)^{-q} \frac{d}{ds}\Big(V(s, x(s))\Big) ds, \\
&t \in J_k = (t_k, t_{k+1}), \quad k = 1, 2, \cdots : \; t_k \in (t_0, T).
\end{aligned} \tag{7}
$$

  This type of derivative is applicable for continuously differentiable Lyapunov functions.
- <u>*second type*</u>– Let $\psi \in C([-\tau, 0], \mathcal{D})$. Then, **the Dini fractional derivative** of the Lyapunov function $V(t,x) \in \Lambda([t_0 - r, T), \mathcal{D})$ is defined by

$$
\begin{aligned}
&{}_{t_k}D^q_{(3)} V(t, \psi(0), \psi) \\
&= \limsup_{h \to 0} \frac{1}{h^q} \Big[V(t, \psi(0)) - \sum_{r=1}^{[\frac{t-t_k}{h}]} (-1)^{r+1} \, {}_qC_r V(t - rh, \psi(0) - h^q F(t, \psi_0)) \Big] \\
&\quad t \in J_k, \quad k = 1, 2, \cdots : \; t_k \in (t_0, T),
\end{aligned} \tag{8}
$$

  where $\psi_0 = \psi(s)$, $s \in [-r, 0]$.

  The Dini fractional derivative (8) keeps the concept of fractional derivatives because it has a memory.

  Note that Dini fractional derivative, defined by (8), is based on the notation

$$D^+ V(t, \psi(0), \psi) = \limsup_{h \to 0} \frac{1}{h^q} \Big[V(t, \psi(0)) - V(t - h, \psi(0) - h^q F(t, \psi_0)) \Big]. \tag{9}$$

  In [17], the notation (9) is used directly. However, the notation (9) does not depend on the order q of the fractional derivative nor on the initial time t_0, which is typical for the Caputo fractional derivative. The operator defined by (9) has no memory. In addition, if $x(t)$ is a solution of (3), then $D^+ V(t, x(t), x) \neq {}_{t_k}^{\;c}D^q V(t, x(t))$.

For fractional differential equations without any impulses, notation similar to (9) is defined and $V(t-h, x-h^q F(t,x)) = \sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}\,{}_qC_r V(t-rh, x-h^q F(t,x))$ is applied [16].

- <u>*third type*</u>—let the initial data $(t_0,\phi_0) \in \mathbb{R}_+ \times C([-\tau,0],\mathcal{D})),\ \mathcal{D}) \subset \mathbb{R}^n$, of IVP for IFrDDE (3) and $\psi \in C([-\tau,0],\mathcal{D})$ be given. Then, **the Caputo fractional Dini derivative** of the Lyapunov function $V(t,x) \in \Lambda([t_0-r,T),\mathcal{D})$ is defined by:

$$\begin{aligned} &{}_{t_k}^{c}D^q_{(3)}V(t,\psi;t_0,\phi_0(0)) \\ &= \limsup_{h\to 0^+}\frac{1}{h^q}\Big\{V(t,\psi(0)) - V(t_0,\phi_0(0)) \\ &- \sum_{r=1}^{[\frac{t-t_k}{h}]}(-1)^{r+1}\,{}_qC_r\Big(V(t-rh,\psi(0)-h^qF(t,\psi_0)) - V(t_0,\phi_0(0))\Big)\Big\}, \\ &\text{for } t\in J_k, \end{aligned} \tag{10}$$

where $\psi_0 = \psi(s),\ s\in[-r,0]$,

or its equivalent

$$\begin{aligned} &{}_{t_k}^{c}D^q_{(3)}V(t,\psi;t_k,\phi_0(0)) \\ &= \limsup_{h\to 0^+}\frac{1}{h^q}\Big\{V(t,\psi(0)) + \sum_{r=1}^{[\frac{t-t_k}{h}]}(-1)^r\,{}_qC_r V(t-rh,\psi(0)-h^q f(t,\psi_0))\Big\} \\ &- \frac{V(t_0,\phi_0(0))}{(t-t_k)^q\Gamma(1-q)}, \qquad \text{for } t\in J_k. \end{aligned} \tag{11}$$

The derivative ${}_{t_k}^{c}D^q_{(3)}V(t,\psi;t_0,\phi_0(0))$ given by (11) depends significantly on both the fractional order q and the initial data (t_0,ϕ_0) of IVP for IFrDDE (3) and it makes this type of derivative close to the idea of the Caputo fractional derivative of a function.

Remark 3. *For any initial data $(t_0,\phi_0)\in\mathbb{R}_+\times C([-\tau,0],\mathcal{D})$ of the IVP for IFrDDE (3), the relation between the Dini fractional derivative defined by (8) and for any $t\in J_k,\ \psi\in C([-\tau,0],\mathcal{D})$ and the Caputo fractional Dini derivative defined by (11) is given by*

$${}_{t_0}^{c}D^q_{(3)}V(t,\psi;t_0,\phi_0(0)) = {}_{t_k}D^q_{(3)}V(t,\psi(0),\psi) - \frac{V(t_0,\phi_0(0))}{(t-t_k)^q\Gamma(1-q)}$$

or

$${}_{t_k}^{c}D^q_{(3)}V(t,\psi;t_0,\phi_0(0)) = {}_{t_k}D^q_{(3)}V(t,\psi(0),\psi) - {}_{t_0}^{RL}D^q\Big(V(t_0,\phi_0(0))\Big).$$

In the next example, to simplify the calculations and to emphasize the derivatives and their properties, we will consider the scalar case, i.e., $n=1$.

Example 1. *(Quadratic Lyapunov function). Let $V(t,x)=x^2$, with $x\in\mathbb{R}$.*

Case 1. Caputo fractional derivative: Let $x(t)=x(t;t_0,\phi_0), t\in[t_0-\tau,T)$ be a solution of the IVP for IFrDDE (3) with $n=1$ and $\phi_0\in C([-\tau,0],\mathbb{R})$ and we get

$${}_{t_k}^{c}D^q V(t,x(t)) = \frac{2}{\Gamma(1-q)}\int_{t_k}^{t}(t-s)^{-q}\,x(s)\frac{d}{ds}\Big(x(s)\Big)ds \le 2x(t)F(t,x_t),\ \ t\in J_k. \tag{12}$$

Case 2. Dini fractional derivative. Consider IVP for IFrDDE (3) with given initial data $(t_0,\phi_0)\in\mathbb{R}_+\times C([-\tau,0],\mathbb{R})$. Let $\psi\in C([-\tau,0],\mathbb{R})$ be any function. Apply (8) and we obtain

$$ {}_{t_k}D^q_{(3)}V(t,\psi(0),\psi) = 2\psi(0)F(t,\psi_0) + \frac{(\psi(0))^2}{(t-t_k)^q\Gamma(1-q)}, \quad t \in J_k, \tag{13} $$

where $\psi_0 = \psi(s), s \in [-r, 0]$.

Note, if we apply (9) directly, we obtain $D^+V(t,\psi(0),\psi) = 2\psi(0)F(t,\psi_0)$.

Case 3. Caputo fractional Dini derivative. Use (11), Case 2 and Remark 3 and we obtain

$$ \begin{aligned} &{}^{c}_{t_k}D^q_{(3)}V(t,\psi;t_0,\phi_0) = 2\psi(0)F(t,\psi_0) + ((\psi(0))^2 - (\phi_0(0))^2)\frac{(t-t_k)^{-q}}{\Gamma(1-q)} \\ &= {}_{t_k}D^q_{(3)}V(t,\psi(0),\psi) - {}^{RL}_{t_k}D^q\Big(V(t_0,\phi_0(0))\Big), \ \textit{for } t \in J_k, \ \psi \in C([-\tau,0],\mathbb{R}). \end{aligned} \tag{14} $$

Example 2. *(Lyapunov function depending directly on the time variable). Let* $V(t,x) = m(t)\ x^2$ *where* $m \in C^1(\mathbb{R}_+,\mathbb{R}_+)$.

Case 1. Caputo fractional derivative. Let $x(t) = x(t;t_0,\phi_0)$ *be a solution of the IVP for IFrDDE (3). The fractional derivative*

$$ {}^{c}_{t_k}D^qV(t,x(t)) = {}^{c}_{t_k}D^q\Big(m(t)\ x^2(t)\Big) = \frac{1}{\Gamma(1-q)}\int_{t_k}^{t}\frac{m'(s)x^2(s)+2m(s)x(s)x'(s)}{(t-s)^q}ds $$

is difficult to obtain in the general case for any solution of (3). In addition, its upper bound is difficult to find.

Case 2. Dini fractional derivative. Let $\psi \in C([-\tau,0],\mathcal{D})$ *be given. Applying (8), we obtain*

$$ {}_{t_k}D^q_{(3)}V(t,\psi(0),\psi) = \psi(0)\ m(t)F(t,\psi_0) + (\psi(0))^2\ {}^{RL}_{t_k}D^q\Big(m(t)\Big). \tag{15} $$

Note that if we use (9) directly, then $D^+V(t,\psi(0),\psi) = 2\psi(0)\ m(t)F(t,\psi_0)$, *which unusually is missing the derivative of the function* $m(t)$ *(compare with the case of ordinary derivatives).*

Case 3. Caputo fractional Dini derivative. Use (11) and we obtain

$$ \begin{aligned} &{}^{c}_{t_k}D^q_{(3)}V(t,\psi;t_0,\phi_0(0)) \\ &= 2\psi(0)m(t)F(t,\psi_0) + (\psi(0))^2\ {}^{RL}_{t_k}D^q\Big(m(t)\Big) - (\phi_0(0))^2m(t_0)\frac{(t-t_k)^{-q}}{\Gamma(1-q)} \\ &= {}_{t_k}D^q_{(3)}V(t,\psi(0),\psi) - V(t_0,\phi_0(0))\frac{(t-t_k)^{-q}}{\Gamma(1-q)} \\ &= {}_{t_k}D^q_{(3)}V(t,\psi(0),\psi) - {}^{RL}_{t_k}D^q\Big(V(t_0,\phi_0(0))\Big). \end{aligned} \tag{16} $$

4. Some Comparison Results for Lyapunov Functions

4.1. Comparison Results for Delay Fractional Differential Equations

First, we will prove several comparison results for fractional delay differential equation without any impulses. We will use Lyapunov function with the Razumikhin condition $V(t+\Theta,\psi(\Theta)) \le p(V(t,\psi(0)))$, $\Theta \in [-r,0]$ for $\psi \in C([-\tau,0],\mathbb{R}^n)$, where $p \in C([0,\infty),\mathbb{R}_+)$, $p(s) > s$ for $s > 0$.

Remark 4. *A comparison result is given in Theorem 4.5 [22] by applying definition (9) for the derivative of V and incorrectly replacing it with the Caputo derivative in the proof. Some comparison results applying A1 for IFrDDE are obtained in [17], but induction w.r.t. the interval of continuity is incorrectly used (see Remark 2).*

Consider the IVP for the following delay fractional differential equation (FrDDE)

$$ \begin{aligned} &{}^{C}_{\tau_0}D^qx(t) = F(t,x_t) \text{ for } t \in (\tau_0,\Theta], \\ &x(\tau_0+s) = \phi(s), \ \ s \in [-r,0], \end{aligned} \tag{17} $$

where $x \in R^n$, $\phi \in C([-r,0],\mathcal{D})$, $\mathcal{D} \subset \mathbb{R}^n$.

Lemma 1. *(Comparison result with the Caputo fractional derivative) Assume:*

1. *The function $x(t) = x(t;\tau_0,\phi) \in C^q([\tau_0,\Theta],\mathcal{D})$ is a solution of the IVP for FrDDE (17).*
2. *The function $V \in \Lambda([\tau_0 - r,\Theta],\mathcal{D})$, $\mathcal{D} \subset \mathbb{R}^n$ is such that there exist positive numbers $p,\alpha :\ p > \frac{1}{E_q(-\alpha r^q)}$ such that, for any point $t \in [\tau_0,\Theta]\ :\ V(t+s,x(t+s)) < pV(t,x(t))$, $s \in [-r,0)$, the fractional derivative ${}^{c}_{\tau_0}D^q V(t,x(t))$ exists and the inequality*

$$ {}^{c}_{\tau_0}D^q V(t,x(t)) \le -\alpha V(t,x(t)) \tag{18}$$

holds.

Then, $V(t,x(t;\tau_0,\phi)) \le \max_{s\in[-r,0]} V(t_0+s,\phi(s))E_q(-\alpha(t-t_0)^q)$ for $t \in [\tau_0,\Theta]$.

Proof. Denote $B = \max_{s\in[-r,0]} V(\tau_0+s,\phi_0(s))$. Let $\varepsilon > 0$ be an arbitrary number. Define the functions $m(t) = V(t,x(t))$ for $t \in [\tau_0 - r,\Theta]$. We will prove that

$$ m(t) < BE_q(-\alpha(t-\tau_0)^q) + \varepsilon,\ \ t \ge \tau_0. \tag{19}$$

For $t = \tau_0$, the inequality $m(\tau_0) \le B < B + \varepsilon$ holds. Assume (19) is not true and, therefore, there exists a point $t^* \in (\tau_0,\Theta)$ such that

$$ m(t) < BE_q(-\alpha(t-\tau_0)^q) + \varepsilon,\ \ t \in [\tau_0,t^*)\ \ \text{and}\ m(t^*) = BE_q(-\alpha(t^*-\tau_0)^q) + \varepsilon. \tag{20}$$

Let $s \in [-r,0]$.
Case 1. Let $t^* + s \in [\tau_0,t^*]$.
Then, $t^* + s \ge \tau_0 - r$, $t^* - \tau_0 > 0$ and $0 \le t^* - \tau_0 - r \le t^* - \tau_0 + s$. Using the inequality $E_q(-\alpha a^q)E_q(-\alpha(t-a)^q) \le E_q(-\alpha t^q)$ for $t \ge a$ with $\alpha > 0, a \ge 0$ and the choice of p, we obtain

$$\begin{aligned} p(m(t^*)) &= pBE_q(-\alpha(t^*-\tau_0)^q) + p\varepsilon > B\frac{E_q(-\alpha(t^*-\tau_0)^q)}{E_q(-\alpha r^q)} + \varepsilon \\ &\ge BE_q(-\alpha(t^*-\tau_0-r)^q) + \varepsilon \ge BE_q(-\alpha(t^*+s-\tau_0)^q) + \varepsilon > m(t^*+s). \end{aligned} \tag{21}$$

Case 2. Let $t^* - r \le t^* + s < \tau_0$.
Then, $t^* - \tau_0 < -s \le r$ and we get

$$ p(m(t^*)) > B\frac{E_q(-\alpha(t^*-\tau_0)^q)}{E_q(-\alpha r^q)} + \varepsilon \ge B + \varepsilon > B \ge m(t^*+s). \tag{22}$$

From (21), (22) and condition 2 of Lemma 1, it follows that the fractional derivative ${}^{c}_{\tau_0}D^q m(t^*)$ exists.
From (20), it follows that

$$ {}^{c}_{\tau_0}D^q\Big(m(t^*) - BE_q(-\alpha(t^*-\tau_0)^q) - \varepsilon\Big) > 0. \tag{23}$$

Then, using ${}^{c}_{\tau_0}D^q E_q(-\alpha(t-\tau_0)^q) = -aE_q(-\alpha(t-\tau_0)^q)$, we get ${}^{c}_{\tau_0}D^q m(t^*) > -B\alpha E_q(-\alpha(t^*-\tau_0)^q)$.
From inequality (18), we get ${}^{c}_{\tau_0}D^q V(t^*,x(t^*)) \le -\alpha V(t^*,x(t^*))$. Therefore, the inequality $-B\alpha E_q(-\alpha(t^*-\tau_0)^q) < {}^{c}_{\tau_0}D^q m(t^*) \le -\alpha m(t^*)$ holds or $BE_q(-\alpha(t^*-\tau_0)^q) > m(t^*)$, which contradicts (20). Therefore, inequality (19) holds for an arbitrary $\varepsilon > 0$. Thus, the claim in our Lemma is true. □

Remark 5. *If $p = 1$ in condition 2 of Lemma 1 then since $E_q(-\alpha a^q)E_q(-\alpha(t-a)^q) \ne E_q(-\alpha t^q)$ for $t \ge a$ (see [23]), the inequality (21) is not true and the claim of Lemma 1 is not true.*

Note the comparison result for Lyapunov functions is true if the Caputo fractional derivative in Lemma 1 is replaced by any of the other two derivatives. In the proof, we will use the following result.

Lemma 2. *[22] Let $m \in C([\tau_0, \Theta], \mathbb{R})$ and there exists $\xi \in (\tau_0, \Theta]$ such that $m(\xi) = 0$ and $m(t) < 0$ for $t \in [\tau_0, \xi)$. Then, ${}^{GL}_{\tau_0}D^q_+ m(\xi) > 0$.*

Lemma 3. *(Comparison result with the Dini fractional derivative) Assume the function $V \in \Lambda([\tau_0 - r, \Theta], \mathcal{D})$, $\mathcal{D} \subset \mathbb{R}^n$, is such that there exist positive numbers $p, \alpha :\ p > \frac{1}{E_q(-\alpha r^q)}$ such that for any point $t \in [\tau_0, \Theta]$ and any function $\psi \in C([-r, 0], \mathbb{R}^n) :\ V(t+s, \psi(s)) < pV(t, \psi(0)), s \in [-r, 0)$ the inequality*

$$
{}_{\tau_0}D^q_{(17)} V(t, \psi(0), \psi) \leq -\alpha V(t, \psi(0)) \tag{24}
$$

holds.

Then, $V(t, x(t; \tau_0, \phi)) \leq \max_{s\in[-r,0]} V(t_0 + s, \phi(s)) E_q(-\alpha(t - t_0)^q)$ for $t \in [\tau_0, \Theta]$ where $x(t; \tau_0, \phi) \in C^q([\tau_0, \Theta], \mathcal{D}$ is a solution of the IVP for FrDDE (17).

Proof. The proof is similar to that in Lemma 1. The main difference is in connection with inequality (23). Follow the proof in Lemma 1 and in this case we use Lemma 2 and obtain

$$
{}^{GL}_{\tau_0}D^q_+ \Big(m(t^*) - BE_q(-\alpha(t^* - \tau_0)^q) - \varepsilon \Big) > 0. \tag{25}
$$

Now, using ${}^{c}_{\tau_0}D^q u(t) = {}^{GL}_{\tau_0}D^q_+ (u(t) - u(\tau_0)) = {}^{GL}_{\tau_0}D^q_+ u(t) - \frac{u(\tau_0)}{(t-\tau_0)^q \Gamma(1-q)}$, we get

$$
\begin{aligned}
{}^{GL}_{\tau_0}D^q_+ m(t^*) &> {}^{GL}_{\tau_0}D^q_+ \Big(BE_q(-\alpha(t^* - \tau_0)^q) - B \Big) + \frac{B + \varepsilon}{(t - \tau_0)^q \Gamma(1-q)} \\
&> {}^{c}_{\tau_0}D^q \Big(BE_q(-\alpha(t^* - \tau_0)^q) = -B\alpha E_q(-\alpha(t^* - \tau_0)^q).
\end{aligned} \tag{26}
$$

It remains to show that we have a contradiction. To see this for any $t \in [\tau_0, t^*]$ and $h > 0$, we let

$$
S(x(t), h) = \sum_{r=1}^{[\frac{t-\tau_0}{h}]} (-1)^{r+1} {}_qC_r \Big[x(t - rh) - \phi(0) \Big].
$$

Now, ${}^{C}_{\tau_0}D^q_+ x(t) = {}^{GL}_{\tau_0} D^q_+ (x(t) - x(\tau_0)) = \limsup_{h\to 0+} \frac{1}{h^q} \Big[x(t) - x(\tau_0) - S(x(t), h) \Big] = F(t, x_t)$. Therefore, $S\,(x(t), h) = x(t) - \phi(0) - h^q F(t, x_t) - \Lambda(h^q)$ or

$$
x(t) - h^q F(t, x_t) = S\,(x(t), h) + \phi(0) + \Lambda(h^q) \tag{27}
$$

with $\frac{||\Lambda(h^q)||}{h^q} \to 0$ as $h \to 0$. Then, for any $t \in [t_0, t^*]$, we obtain

$$
\begin{aligned}
&{}^{GL}_{\tau_0}D^q_+ m(t) = m(t) - \sum_{r=1}^{[\frac{t-\tau_0}{h}]} (-1)^{r+1} {}_qC_r m(t - rh) \\
&= \Big\{ V(t, x(t)) - \sum_{r=1}^{[\frac{t-\tau_0}{h}]} (-1)^{r+1} {}_qC_r V(t - rh, x(t) - h^q F(t, x_t) \Big\} \\
&\quad + \sum_{r=1}^{[\frac{t-\tau_0}{h}]} (-1)^{r+1} {}_qC_r \Big\{ V(t - rh, S\,(x(t), h) + \phi(0) + \Lambda(h^q)) \\
&\quad - V(t - rh, x(t - rh)) \Big\}.
\end{aligned} \tag{28}
$$

Since V is locally Lipschitzian in its second argument with a Lipschitz constant $L > 0$, we obtain

$$\begin{aligned}
&\sum_{r=1}^{[\frac{t-\tau_0}{h}]} (-1)^{r+1}\, {}_qC_r \Big\{ V(t-rh, S(x(t),h) + \phi(0) + \Lambda(h^q)) - V(t-rh, x(t-rh)) \Big\} \\
&\leq L \Big|\Big| \sum_{r=1}^{[\frac{t-\tau_0}{h}]} (-1)^{r+1}\, {}_qC_r \sum_{j=1}^{[\frac{t-\tau_0}{h}]} (-1)^{j+1}\, {}_qC_j \Big(x(t-jh) - \phi(0) \Big) \\
&\quad - \sum_{r=1}^{[\frac{t-\tau_0}{h}]} (-1)^{r+1}\, {}_qC_r \Big((x(t-rh) - \phi(0)) \Big) \Big|\Big| + L||\Lambda(h^q)|| \, \Big| \sum_{r=1}^{[\frac{t-\tau_0}{h}]} (-1)^{r+1}\, {}_qC_r \Big| \\
&= L \Big|\Big| \Big(\sum_{r=0}^{[\frac{t-\tau_0}{h}]} (-1)^{r+1}\, {}_qC_r \Big) \Big(\sum_{j=1}^{[\frac{t-\tau_0}{h}]} (-1)^{j+1}\, {}_qC_j \Big(x(t-jh) - \phi(0) \Big) \Big) \Big|\Big| \\
&\quad + L||\Lambda(h^q)|| \, \Big| \sum_{r=1}^{[\frac{t-\tau_0}{h}]} (-1)^{r+1}\, {}_qC_r \Big|.
\end{aligned} \tag{29}$$

Substitute (29) in (28), divide both sides by h^q, take the limit as $h \to 0^+$, use $\sum_{r=0}^{\infty} {}_qC_r z^r = (1+z)^q$ if $|z| \leq 1$, and we obtain for any $t \in [\tau_0, t^*]$ the inequality

$$\begin{aligned}
{}^{GL}_{\tau_0}D^q_+ m(t) &\leq \lim_{h\to 0+} \frac{1}{h^q} \Big\{ V(t, x(t)) \\
&\quad - \sum_{r=1}^{[\frac{t-\tau_0}{h}]} (-1)^{r+1}\, {}_qC_r V(t-rh, x(t) - h^q f(t, x^*(t)) \Big\} \\
&\quad + L \lim_{h\to 0+} \frac{||\Lambda(h^q)||}{h^q} \lim_{h\to 0+} \Big| \sum_{r=1}^{[\frac{t-\tau_0}{h}]} (-1)^{r+1}\, {}_qC_r \Big| \\
&\quad + L \lim_{h\to 0^+} \sup \Big| \sum_{r=1}^{[\frac{t-\tau_0}{h}]} (-1)^{r+1}\, {}_qC_r \Big| \, \Big|\Big| \frac{1}{h^q} \sum_{j=1}^{[\frac{t-\tau_0}{h}]} (-1)^{j+1}\, {}_qC_j \Big(x(t-jh) - \phi(0) \Big) \Big|\Big| \\
&\leq \lim_{h\to 0+} \frac{1}{h^q} \Big\{ V(t, x(t)) - \sum_{r=1}^{[\frac{t-\tau_0}{h}]} (-1)^{r+1}\, {}_qC_r V(t-rh, x(t) - h^q f(t, x^*(t)) \Big\}.
\end{aligned} \tag{30}$$

Let $t = t^*$. Define the function $\psi(\Theta) = x(t^* + \Theta)$, $\theta \in [-\tau, 0]$. Applying condition 2 to (30) for $t = t^*$, we get

$$\begin{aligned}
&{}^{GL}_{t_0}D^q_+ m(t^*) \leq \lim_{h\to 0+} \frac{1}{h^q} \Big\{ V(t^*, \psi(0)) - \sum_{r=1}^{[\frac{t^*-\tau_0}{h}]} (-1)^{r+1}\, {}_qC_r V(t^* - rh, \psi(0) - h^q F(t^*, \psi) \Big\} \\
&= {}_{\tau_0}D^q_{(17)} V(t^*, \psi(0), \psi_0) \leq -\alpha V(t^*, u(0)) = -\alpha m(t^*) = -\alpha B E_q(-\alpha(t^* - \tau_0)).
\end{aligned} \tag{31}$$

Inequality (31) contradicts (26). □

Remark 6. *(Comparison result with the Caputo fractional Dini derivative) Lemma 3 remains true if the Dini fractional derivative in inequality (24) is replaced by the Caputo fractional Dini derivative* ${}^{c}_{\tau_0}D^q_{(17)} V(t, \psi; \tau_0, \phi_0(0))$.

4.2. Comparison Results for Impulsive Delay Fractional Differential Equations

Now, we will prove some comparison result for IFrDDE (3) using approach A2 for IFrDDE.

Lemma 4. *(Comparison result with the Caputo fractional derivative) Assume:*

1. *The function $x(t) = x(t; t_0, \phi_0) \in PC^q([t_0, \Theta], \mathcal{D})$ is a solution of the IVP for FrDDE (3) with $\phi_0 \in C([-r, 0], \mathcal{D})$.*
2. *The function $V \in \Lambda([t_0 - r, \Theta], \mathcal{D})$, $\mathcal{D} \subset \mathbb{R}^n$ is such that there exist positive numbers p, α : $p > \frac{1}{E_q(-\alpha r^q)}$:*
   - (i) *for any $k = 0, 1, \ldots$ and any point $t \in [t_k, t_{k+1}] \bigcap (t_0, \Theta]$ such that $V(t+s, x(t+s)) < pV(t, x(t))$, $s \in [-r, 0)$ the fractional derivative ${}_{t_k}^{c}D^q V(t, x(t))$ exists and the inequality*

$$ {}_{t_k}^{c}D^q V(t, x(t)) \le -\alpha V(t, x(t)) \tag{32} $$

     holds;
   - (ii) *for all $k = 1, 2, \ldots$: $t_k \in (t_0, \Theta)$ and $x \in \mathcal{D}$, the inequality $V(t_k, x + I_k(x)) \le V(t_k, x)$.*

   Then,

$$ V(t, x(t; t_0, \phi_0)) \le \max_{s \in [-r, 0]} V(t_0 + s, \phi_0(s)) \Big(\prod_{i=1}^{k} E_q(-\alpha(t_i - t_{i-1})^q) \Big) E_q(-\alpha(t - t_k)^q), $$
$$ t \in (t_k, t_{k+1}] \bigcap [t_0, \Theta]. $$

Proof. Let $\varepsilon > 0$ be an arbitrary number. Define the functions $m(t) = V(t, x(t))$ for $t \in [\tau_0 - r, \Theta]$.

We use Lemma 1 and induction w.r.t. to the interval (see Remark 2) to prove the claim. For $t \in [t_0, t_1]$, the claim follows directly from Lemma 1, i.e.,

$$ m(t) = V(t, x(t)) \le \max_{s \in [-r, 0]} V(t_0 + s, \phi_0(s)) E_q(-\alpha(t - t_0)^q), \quad t \in [t_0, t_1]. \tag{33} $$

Let $t \in (t_1, t_2]$. Denote $B_1 = m(t_1)$. Let $\varepsilon > 0$ be an arbitrary number. According to (33), the inequality $B_1 \le \max_{s \in [-r, 0]} V(t_0 + s, \phi_0(s)) E_q(-\alpha(t_1 - t_0)^q)$ holds.

We will prove

$$ m(t) < B_1 E_q(-\alpha(t - t_1)^q) + \varepsilon, \ \ t \in (t_1 + 0, t_2]. \tag{34} $$

From condition 2(ii), we get $m(t_1 + 0) \le m(t_1) < m(t_1) + \varepsilon$. Assume (34) is not true on $(t_1, t_2]$, i.e., there exists a point $t^* \in (t_1, t_2)$ such that

$$ m(t) < B_1 E_q(-\alpha(t - t_1)^q) + \varepsilon, \ \ t \in [t_1, t^*) \ \text{ and } m(t^*) = B_1 E_q(-\alpha(t^* - t_1)^q) + \varepsilon. \tag{35} $$

Let $s \in [-r, 0]$.

Case 1. Let $t^* + s \in [t_1, t^*]$.

Then, $t^* + s \ge t_1 - r$. Then, $t^* - t_1 > 0$ and $0 \le t^* - t_1 - r \le t^* - t_1 + s$. Using the inequality $E_q(-\alpha a^q) E_q(-\alpha(t - a)^q) \le E_q(-\alpha t^q)$ for $t \ge a$ with $\alpha > 0, a \ge 0$ and (35), we get

$$ \begin{aligned} p\, m(t^*) &= pB_1 E_q(-\alpha(t^* - t_1)^q) + p\varepsilon > B_1 \frac{E_q(-\alpha(t^* - t_1)^q)}{E_q(-\alpha r^q)} + \varepsilon \\ &\ge B_1 E_q(-\alpha(t^* - t_1 - r)^q) + \varepsilon \ge B_1 E_q(-\alpha(t^* + s - t_1)^q) + \varepsilon > m(t^* + s). \end{aligned} \tag{36} $$

Case 2. Let $t^* - r \le t^* + s < t_1$.

Then, $t^* - t_1 < -s \le r$ and we get

$$ p\, m(t^*) > B_1 \frac{E_q(-\alpha(t^* - t_1)^q)}{E_q(-\alpha r^q)} + \varepsilon \ge B_1 + \varepsilon > B_1 \ge m(t^* + s), \ s \in [-r, 0]. \tag{37} $$

From (36), (22) and condition 2 (i) of Lemma 4, it follows that the fractional derivative ${}_{t_1}^{c}D^q m(t^*)$ exists.

Similar to the proof in Lemma 1 with $\tau_0 = t_1$ and $B = B_1$, we obtain a contradiction.

Therefore, inequality (34) holds for an arbitrary $\varepsilon > 0$. Thus, for $t \in (t_1, t_2]$,

$$m(t) \le B_1 E_q(-\alpha(t-t_1)^q) \le \max_{s\in[-r,0]} V(t_0+s, \phi_0(s)) E_q(-\alpha(t_1-t_0)^q) E_q(-\alpha(t-t_1)^q).$$

Using induction, we prove the claim. □

In the case when the Dini fractional derivative, defined by (8), is applied instead of the Caputo fractional derivative by using Lemma 3, we obtain the following result:

Lemma 5. *(Comparison result with the Dini fractional derivative) Assume the function $V \in \Lambda([\tau_0 - r, \Theta], \mathcal{D})$, $\mathcal{D} \subset \mathbb{R}^n$ is such that there exist positive numbers p, α : $p > \frac{1}{E_q(-\alpha r^q)}$:*

(i) for any $k = 0, 1, \ldots$ and any point $t \in [t_k, t_{k+1}] \bigcap (t_0, \Theta]$ and any function $\psi \in C([-r, 0], \mathbb{R}^n)$: $V(t+s, \psi(s)) < pV(t, \psi(0))$, $s \in [-r, 0)$ the inequality

$$ {}_{t_k}D^q_{(3)} V(t, \psi(0), \psi) \le -\alpha V(t, \psi(0)) \tag{38}$$

holds;

(ii) for all $k = 1, 2, \ldots$: $t_k \in (t_0, \Theta)$ and $x \in \mathcal{D}$ the inequality $V(t_k, x + I_k(x)) \le V(t_k, x)$.

Then,

$$\begin{aligned} V(t, x(t; t_0, \phi_0)) \le \max_{s\in[-r,0]} V(t_0+s, \phi_0(s)) \Big(\prod_{i=1}^{k} E_q(-\alpha(t_i - t_{i-1})^q) \Big) E_q(-\alpha(t-t_k)^q), \\ t \in (t_k, t_{k+1}] \bigcap [t_0, \Theta], \end{aligned} \tag{39}$$

where $x(t; \tau_0, \phi_0) \in PC^q([\tau_0, \Theta], \mathcal{D})$ is a solution of the IVP for FrDDE (3) with $\phi_0 \in C([-r, 0], \mathcal{D})$.

Remark 7. *(Comparison result with the Caputo fractional Dini derivative) Lemma 5 remains true if the Dini fractional derivative in inequality (38) is replaced by Caputo fractional Dini derivative ${}_{t_k}^{c}D^q_{(3)} V(t, \psi; \tau_0, \phi_0(0))$.*

5. Application to Neural Networks

5.1. Problem Formulation

We will study neural networks modeled by impulsive Caputo fractional differential equations with bounded time dependent delays and distributed delays. We will consider the case when the lower limit of the fractional derivative is changed after each impulse, i.e., we will use approach A2 for IFrDDE. Following the notations in (3), we consider the general model of Hopfield's graded response neural networks with impulses and bounded delays and distributed delays (INND)

$$\begin{aligned} {}_{t_k}^{C}D_t^q x_i(t) &= -c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t) f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}(t) g_j(x_j(t - \tau_j(t))) \\ &\quad + \sum_{j=1}^{n} d_{ij}(t) \int_{-r}^{0} K_{ij}(s) h_j(x_j(t+s))ds + I_i(t) \\ &\qquad \text{for } t \in (t_k, t_{k+1}], k = 0, 1, 2, \ldots, \\ \Delta x_i(t)|_{t=t_k} &= I_{k,i}(x_i(t_k - 0)), \qquad k = 1, 2, \ldots, \\ x_i(t) &= \phi_i^0(t - t_0), \ \ t \in [t_0 - r, t_0], \ \ i = 1, 2, \ldots n, \end{aligned} \tag{40}$$

where n represents the number of neurons in the network, $x_i(t)$ is the pseudostate variable denoting the average membrane potential of the i-th neuron at time t, $x(t) = (x_1(t), x_2(t), \dots, x_n(t)) \in \mathbb{R}^n$, $q \in (0,1)$, $c_i(t) > 0, i = 1,2,\dots,n$, is the self-regulating parameter of the i-th unit, they correspond to the rate with which the i-th neuron rests its potential in the resting state in isolation, $a_{ij}(t), b_{ij}(t)$, $i, j = 1,2,\dots,n$, correspond to the synaptic connection strength of the i-th neuron to the j-th neuron at time t and $t - \tau_j(t)$, respectively, $f_j(x), g_j(x), h_j(x)$ are nonlinear activation functions such that $f(x) = (f_1(x_1), f_2(x_2), \dots, f_n(x_n))$, $h(x) = (h_1(x_1), h_2(x_2), \dots, h_n(x_n))$, $g(x) = (g_1(x_1), g_2(x_2), \dots, g_n(x_n))$; $I = (I_1, I_2, \dots, I_n)$ is an external bias vector, $\tau_j(t)$ represents the transmission delay along the axis of the j-th unit and satisfies $0 \le \tau_j(t) \le r$, the $t_k, k = 1,2,\dots$, are points of acting the state displacements, the functions $\Phi_{k,i}(t,u,v)$, $k = 1,2,\dots$ are the impulsive functions giving the impulsive perturbation of the i-th neuron on the point t_k, the numbers $x_i(t_k - 0) = x_i(t_k)$ and $x_i(t_k + 0)$ are the state of the i-th neuron before and after impulsive perturbation at time t_k; $K_{ij}(.)$ is the delay kernel with $\int_{-r}^{0} |K_{ij}(s)| ds = 1$, $\phi_i^0 \in C([-r,0], \mathbb{R})$, $i = 1,2,\dots,n$ are the initial functions.

The slave system is given by

$$\begin{aligned}
{}_{t_k}^{C}D_t^q y_i(t) &= -c_i(t) y_i(t) + \sum_{j=1}^{n} a_{ij}(t) f_j(y_j(t)) + \sum_{j=1}^{n} b_{ij}(t) g_j(y_j(t - \tau_j(t))) \\
&\quad + \sum_{j=1}^{n} d_{ij}(t) \int_{-r}^{0} K_{ij}(s) h_j(y_j(t+s)) ds - u_i(t) + I_i(t) \\
&\qquad \text{for } t \in (t_k, t_{k+1}], k = 0,1,2,\dots, \\
\Delta y_i(t)|_{t=t_k} &= I_{k,i}(y_i(t_k - 0)), \qquad k = 1,2,\dots, \\
y_i(t) &= \varphi_i^0(t - t_0), \;\; t \in [t_0 - r, t_0], \;\; i = 1,2,\dots n,
\end{aligned} \tag{41}$$

where $u_i(t)$, $i = 1,2,\dots,n$ are the suitable controllers at time t, $\varphi_i^0 \in C([-r,0], \mathbb{R})$, $i = 1,2,\dots,n$.

5.2. Mittag–Leffler Synchronization

Definition 2. *The master impulsive Caputo fractional system (40) and the slave impulsive Caputo fractional system (41) are globally Mittag–Leffler synchronized if for any initial functions $\phi_i^0, \varphi_i^0 \in C([-r,0], \mathbb{R})$ there exist constants $C, K > 0$ such that*

$$\begin{aligned}
&||x(t; t_0, \phi^0) - y(t; t_0, \varphi^0)|| \le \{ m(\phi^0 - \varphi^0) E_q(-C(t - t_k)^q) \prod_{j=0}^{k-1} E_q(-C(t_{j+1} - t_j)^q) \}^K, \\
&\qquad t \in J_k = (t_k, t_{k=1}], \; k = 0,1,2,\dots,
\end{aligned}$$

where $m \in C(\mathbb{R}_+^n, \mathbb{R}_+)$ (with $m(0) = 0$) is Lipschitz.

Remark 8. *The synchronization of the problem (40) is studied in [11] and the authors consider the case of constant strengths between the neurons and linear impulsive functions. They used approach A1 for IFrDDE. The main result is based on incorrectly citing and using the Lemma from [17] where they use the derivative (9), which is different than the Caputo fractional derivative (see Remarks 2 and 4).*

The main goal of the paper is to implement appropriate controllers $u_i(t)$, $i = 1,2,\dots,n$ for the response system, such that the controlled response system (41) could be synchronized with the drive system (40).

5.2.1. Output Coupling Controller

Inspired by the ideas in [24], the control inputs in the response system are taken as output coupling $u_j(t) = \sum_{j=1}^{n} m_{ij}(f_j(y_j(t)) - f_j(x_j(t))),\ i = 1,2,\dots,n$. The synchronization via output coupling is important because, in many real systems, only output signals can be measured.

Define the synchronization error $e_i(t) = y_i(t) - x_i(t)$. Therefore, the error dynamics between (40) and (41) can be expressed by

$$\begin{aligned} {}_{t_k}^{C}D_t^q e_i(t) &= -c_i(t)e_i(t) + \sum_{j=1}^{n}(a_{ij}(t) - m_{ij})F_j(e_j(t)) + \sum_{j=1}^{n} b_{ij}(t)G_j(e_j(t-\tau_j(t))) \\ &\quad + \sum_{j=1}^{n} d_{ij}(t)\int_{-r}^{0} K_{ij}(s)H_j(e_j(t+s))ds \text{ for } t \in (t_k, t_{k+1}], k = 0,1,2,\dots, \\ \Delta e_i(t)|_{t=t_k} &= L_{k,i}(e_i(t_k - 0)), \qquad k = 1,2,\dots, \\ e_i(t) &= \Phi_i^0(t - t_0),\ \ t \in [t_0 - r, t_0],\ \ i = 1,2,\dots n, \end{aligned} \tag{42}$$

where $F_j(e_j(t)) = f_j(y_j(t)) - f_j(x_j(t)),\ G_j(e_j(t)) = g_j(y_j(t)) - g_j(x_j(t)),\ H_j(e_j(t)) = h_j(y_j(t)) - h_j(x_j(t))$ for $t \in (t_k, t_{k+1}], k = 0,1,2,\dots,\ L_{k,i}(e_j(t_k - 0)) = L_{k,i}(y_j(t_k - 0)) - L_{k,i}(x_j(t_k - 0))$, $i = 1,2,\dots,n, k = 1,2,\dots,$ and $\Phi_i^0(s) = \varphi_i^0(s) - \phi_i^0(s),\ s \in [-r, 0]$.

We assume the following:

Assumption A1. The neuron activation functions are Lipschitz, i.e., there exist positive numbers $\lambda_i, \mu_i, \nu_i\ \ i = 1,2,\dots,n$ such that $|f_i(u) - f_i(v)| \le \lambda_i|u - v|$, $|g_i(u) - g_i(v)| \le \mu_i|u - v|$ and $|h_i(u) - h_i(v)| \le \nu_i|u - v|, i = 1,2,\dots,n$ for $u, v \in \mathbb{R}$.

Assumption A2. There exist positive numbers $M_{i,j},\ C_{i,j}, D_{ij}\ \ i,j = 1,2,\dots,n$ such that $|a_{i,j}(t)| \le M_{i,j}$, $|b_{i,j}(t)| \le C_{i,j}$, $|d_{i,j}(t)| \le D_{i,j}$ for $t > 0$.

Assumption A3. There exists a constant $\eta > 0$ such that $c_i(t) \ge \eta,\ i = 1,2,\dots,n,\ t \ge 0$.

Assumption A4. The impulsive functions $\Phi_{k,i}(u) = u + I_{k,i}(u)$ are Lipschitz with constants $A_{k,i} \in (0,1]$, i.e., $|\Phi_{k,i}(u) - \Phi_{k,i}(v)| \le A_{k,i}|u - v|, i = 1,2,\dots,n, k = 1,2,\dots$ for $u, v \in \mathbb{R}$.

Assumption A5. The inequality

$$\begin{aligned} 2\eta > &\sum_{i=1}^{n}\Big(\max_{j=1,2,\dots,n}(M_{ij} + |m_{ij}|)\lambda_j + \sum_{j=1}^{n}(D_{ij}\nu_j + C_{ij}\mu_j)\Big) \\ &+ \max_{i=1,2,\dots,n}\sum_{j=1}^{n}\Big[(M_{ij} + |m_{ij}|)\lambda_j + C_{ij}\mu_j + D_{ij}\nu_j\Big] \end{aligned} \tag{43}$$

holds.

Remark 9. *If assumption A1 is satisfied, then the functions F, G, H in (42) satisfy $|F_j(u)| \le \lambda_j|u|$, $|G_j(u)| \le \mu_j|u|$, $|H_j(u)| \le \nu_j|u|$, $j = 1,2,\dots,n$ for any $u \in \mathbb{R}$. If assumption A4 is satisfied, then the impulsive functions $L_{k,i}$ in (42) satisfy $|u + L_{k,i}| \le A_{k,i}|u|$ $i = 1,2,\dots,n,\ k = 1,2,\dots$ for any $u \in \mathbb{R}$.*

The case of multiple time constant delays (no distributed delays) and the constant synaptic connection strength between neurons is studied in [22] by using quadratic Lyapunov function. We will study the case of variable bounded synaptic connection strength and nonlinear impulsive functions.

Theorem 1. *Let assumptions A1–A5 be satisfied.*

Then, the master impulsive Caputo fractional system (40) and the slave impulsive Caputo fractional system (41) are globally Mittag–Leffler synchronized.

Proof. According to condition A5, there exists a positive constant α such that $\alpha \leq \frac{2\eta - B_1 - B_2}{B_3}$, where

$$\begin{aligned}
B_1 &= \sum_{i=1}^{n} \Big(\max_{j=1,2,\ldots,n} (M_{ij} + |m_{ij}|)\lambda_j + \sum_{j=1}^{n} (D_{ij}\nu_j + C_{ij}\mu_j) \Big), \\
B_2 &= \max_{i=1,2,\ldots,n} \sum_{j=1}^{n} \Big[C_{ij}\mu_j + D_{ij}\nu_j + (M_{ij} + |m_{ij}|)\lambda_j \Big], \\
B_3 &= 1 + \frac{r^q}{\Gamma(1+q)} \sum_{i=1}^{n} \sum_{j=1}^{n} (D_{ij}\nu_j + C_{ij}\mu_j).
\end{aligned}$$

Choose the positive constants $p = 1 + \frac{\alpha r^q}{\Gamma(1+q)}$. Then, $\frac{1}{E_q(-\alpha r^q)} \leq 1 + \frac{\alpha r^q}{\Gamma(1+q)} = p$ (see (3.8) and (3.11) [23]).

Consider the quadratic functions $V(t,x) = x^T x$. Let the point $t > 0 : \ t \in (t_m, t_{m+1}]$, $m \geq 0$ being an integer, be such that $\sup_{s\in[-r,0]} V(t+s, e(t+s)) = \sup_{s\in[-r,0]} \sum_{j=1}^{n} e_j^2(t+s) = p\sum_{j=1}^{n} e_j^2(t) = pV(t, e(t))$ where $e(t)$ is a solution of (42). Then, since $\tau_j(t) \in [0,r]$, $t \geq 0$, we have $pV(t,e(t)) = p\sum_{j=1}^{n}(e_j(t))^2 \geq \sum_{j=1}^{n}(e_j(t-\tau_j(t)))^2 \geq (e_i(t-\tau_j(t)))^2, i = 1,2,\ldots,n$. In addition, $(e_j(t+s))^2 \leq \sup_{s\in[-r,0]} V(t+s, e(t+s)) = pV(t,e(t))$, $s \in [-r,0]$.

From conditions A1,A2, A3, A5, the choice of the constants α, p, and Remark 9, we get for the chosen above point t:

$$\begin{aligned}
&{}^{C}_{t_m}D^q_t V(t,e(t)) = {}^{C}_{t_m}D^q_t e^T(t)e(t) \leq 2\sum_{i=1}^{n} e_i(t)\, {}^{C}_{t_m}D^q_t e_i(t) \\
&\leq -2\sum_{i=1}^{n} c_i e_i^2(t) + 2\sum_{i=1}^{n}\sum_{j=1}^{n} (|a_{ij}(t) + |m_{ij}|)|F_j(e_j(t))||e_i(t)| \\
&\quad + 2\sum_{i=1}^{n}\sum_{j=1}^{n} |b_{ij}(t)||G_j(e_j(t-\tau(t)))|e_i(t)| \\
&\quad + 2\sum_{i=1}^{n}\sum_{j=1}^{n} |d_{ij}(t)| \int_{-r}^{0} |K_{ij}(s)||H_j(e_j(t+s))|ds|e_i(t)| \\
&\leq -2\sum_{i=1}^{n} c_i e_i^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{n} (M_{ij} + |m_{ij}|)\lambda_j((e_j(t))^2 + (e_i(t))^2) \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}\mu_j((e_j(t-\tau(t)))^2 + (e_i(t))^2) \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} |d_{ij}(t)| \int_{-r}^{0} |K_{ij}(s)|\nu_j((e_j(t+s)))^2 + (e_i(t))^2)ds \\
&\leq \Big\{ -2\eta + \sum_{i=1}^{n} \max_{j=1,2,\ldots,n} (M_{ij} + |m_{ij}|)\lambda_j \\
&\quad + \max_{i=1,2,\ldots,n} \sum_{j=1}^{n} \Big[(M_{ij} + |m_{ij}|)\lambda_j + C_{ij}\mu_j + D_{ij}\nu_j \Big] \\
&\quad + p(\sum_{i=1}^{n}\sum_{j=1}^{n} (D_{ij}\nu_j + C_{ij}\mu_j) \Big\} V(t,e(t)) \leq -\alpha V(t,e(t)).
\end{aligned} \tag{44}$$

Let $t = t_k$, k be a natural number and $x \in \mathbb{R}^n$, $x - (x_1, x_2, \ldots, x_n)$. Then, according to condition A4 and Remark 9, the inequalities

$$V(t, x + L_k(x)) = \sum_{i=1}^{n} (x_i + L_{k,i}(x))^2 \leq \sum_{i=1}^{n} A_{k,i}^2 x_i^2 \leq V(t,x) \tag{45}$$

hold.

Inequalities (44), (45) and Lemma 4 prove the claim. □

In the case when the conditions A3 and A5 are not satisfied (i.e., the bounds of the functions $c_i(t)$ are not small enough), we introduce:

Assumption A6. There exists a continuous positive function $m(t) \in C([0,\infty),(0,\infty))$ such that $0 < \beta \leq m(t) \leq \gamma$, β, γ are constants, the fractional derivative ${}_{t_k}^{RL}D_t^q m(t)$ exists for $t \in (t_k, t_{k+1})$, $k = 0,1,2,\dots,$

$$2 \min_{i=1,2,\dots,n} c_i(t) - \frac{{}_{t_k}^{RL}D^q\Big(m(t)\Big)}{m(t)} \geq \xi > 0 \;\text{ for } t \in (t_k, t_{k+1}),\; k = 0,1,2,\dots,$$

and

$$\xi > \sum_{i=1}^{n}\Big(\max_{j=1,2,\dots,n}(M_{ij} + |m_{ij}|)\lambda_j + \frac{1}{\beta}\sum_{j=1}^{n}(D_{ij}\nu_j + C_{ij}\mu_j)\Big)$$
$$+ \max_{i=1,2,\dots,n}\sum_{j=1}^{n}\Big[(M_{ij} + |m_{ij}|)\lambda_j + C_{ij}\mu_j + D_{ij}\nu_j\Big].$$

Theorem 2. *Let ssumptions A1, A2, A4 and A6 be satisfied.*

Then, the master impulsive Caputo fractional system (40) and the slave impulsive Caputo fractional system (41) are globally Mittag–Leffler synchronized.

Proof. According to condition A6, there exists a positive constant α such that $\alpha \leq \frac{\xi - B_4 - B_2}{B_5}$ where B_2 is defined in Theorem 1 and

$$B_4 = \sum_{i=1}^{n}\Big(\max_{j=1,2,\dots,n}(M_{ij} + |m_{ij}|)\lambda_j + \frac{1}{\beta}\sum_{j=1}^{n}(D_{ij}\nu_j + C_{ij}\mu_j)\Big),$$

$$B_5 = 1 + \frac{r^q}{\beta\Gamma(1+q)}\sum_{i=1}^{n}\sum_{j=1}^{n}(D_{ij}\nu_j + C_{ij}\mu_j).$$

Choose the positive constants $p = 1 + \frac{\alpha r^q}{\Gamma(1+q)}$ and consider the Lyapunov function $V(t,x) = m(t)\sum_{i=1}^{n} x_i^2$, $x = (x_1, x_2, \dots, x_n)$ where the function $m(t)$ is defined in Assumption A6.

Let k be any nonegative integer, the point $t \in (t_k, t_{k+1})$ and the function $\psi \in C([-r,0],\mathbb{R}^n)$ be such that $V(t+s,\psi(s)) = m(t+s)\sum_{i=1}^{n}(\psi_i(s))^2 < pm(t)\sum_{i=1}^{n}(\psi_i(0))^2 = pV(t,\psi(0)), s \in [-r,0)$. Then, we have $(\psi_i(s))^2 \leq \frac{1}{\beta}\beta(\psi_i(s))^2 \leq \frac{1}{\beta}m(t+s)(\psi_i(s))^2 \leq \frac{1}{\beta}m(t+s)\sum_{i=1}^{n}(\psi_i(s))^2 = \frac{1}{\beta}V(t+s,\psi(s)) < p\frac{1}{\beta}V(t,\psi(0)) = p\frac{1}{\beta}m(t)\sum_{i=1}^{n}(\psi_i(0))^2$, $s \in [-r,0]$.

From conditions A1, A2, A6, the choice of the constants α, p, Example 2, Case 2 and Equation (15), we get for the chosen above point t and function ψ:

$$
\begin{aligned}
& {}_{t_k}D^q_{(3)}V(t,\psi(0),\psi) \\
& \le\ m(t)\sum_{i=1}^{n}\Big\{-2\sum_{i=1}^{n}c_i(t)|\psi_i(0)|^2+2\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}(t)+m_{ij}||F_j(\psi_j(0))|\psi_i(0)| \\
& \quad +2\sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}(t)||G_j(\psi_j(\tau(0)))|\psi_i(0)|+\ {}^{RL}_{t_k}D^q\Big(m(t)\Big)\sum_{i=1}^{n}(\psi_(0))^2 \\
& \quad +2\sum_{i=1}^{n}\sum_{j=1}^{n}|d_{ij}(t)|\int_{-r}^{0}|K_{ij}(s)||H_j(\psi_j(s))|ds|\psi_i(0)|\Big\} \\
& \le\Big\{-2\min_{i=1,2,\dots,n}c_i(t)+\sum_{i=1}^{n}\max_{j=1,2,\dots,n}(M_{ij}+|m_{ij}|)\lambda_j+\frac{{}^{RL}_{t_k}D^q\Big(m(t)\Big)}{m(t)} \\
& \quad +\max_{i=1,2,\dots,n}\sum_{j=1}^{n}\Big[(M_{ij}+|m_{ij}|)\lambda_j+C_{ij}\mu_j+D_{ij}\nu_j\Big] \\
& \quad +p\frac{1}{\beta}(\sum_{i=1}^{n}\sum_{j=1}^{n}(D_{ij}\nu_j+C_{ij}\mu_j)\Big\}V(t,\psi(0))<-\alpha V(t,\psi(0)).
\end{aligned}
\tag{46}
$$

For any natural number k and $x\in\mathbb{R}^n, x=(x_1,x_2,\dots,x_n)$ according to condition A4 and Remark 9, we have $V(t_k,x+L_k(x))\le\sum_{i=1}^{n}A^2_{k,i}x_i^2\le V(t,x)$.

According to Lemma 5, the claim of Theorem 2 follows. □

Example 3. *Consider the master impulsive Caputo fractional system (40) with $n=3$, $c_i(t)\equiv c_i$, with the activation functions $f_j(s)=g_j(s)=h_j(s)=0.5\,tanh(s)$, the delays $\tau_i(t)=|sin(t)|\le 1$, i.e., $r=1$ and $|a_{ij}(t)|\le M_{ij}$, $|b_{ij}(t)|\le C_{ij}$, $|d_{ij}(t)|\le D_{ij}$ $i,j=1,2,3$, $t\ge 0$, where $M=\{M_{ij}\}, C=\{C_{ij}\}$ are given by*

$$
M=\begin{pmatrix}0.1 & 0.5 & 0.3\\ 0.2 & 0.3 & 0.2\\ 0.4 & 0.2 & 0.1\end{pmatrix},\ C=\begin{pmatrix}0.1 & 0.1 & 0.2\\ 0.3 & 0.2 & 0.1\\ 0.2 & 0.5 & 0.3\end{pmatrix},\qquad D=\begin{pmatrix}0.2 & 0.1 & 0.1\\ 0.1 & 0.2 & 0.1\\ 0.2 & 0.3 & 0.1\end{pmatrix}.
$$

Let the output coupling controller be $u_j(t)=(tanh(y_j(t))-tanh(x_j(t))\sum_{j=1}^{3}m_{ij},\ i=1,2,3$ with

$$
m=\begin{pmatrix}0.1 & 0.2 & 0.1\\ 0.2 & 0.1 & 0.1\\ 0.1 & 0.2 & 0.1\end{pmatrix}.
$$

Then, $\lambda_i=\mu_i=\nu_i=0.5$ and $\sum_{i=1}^{3}\Big(\max_{j=1,2,3}(M_{ij}+m_{ij})\lambda_j+\sum_{j=1}^{3}(D_{ij}\nu_j+C_{ij}\mu_j)\Big)=2.5$ and $\max_{i=1,2,3}\sum_{j=1}^{3}\Big[(M_{ij}+m_{ij})\lambda_j+C_{ij}\mu_j+D_{ij}\nu_j\Big]=0.5\max_{i=1,2,3}\sum_{j=1}^{3}\Big[(M_{ij}+m_{ij})\lambda_j+C_{ij}\mu_j+D_{ij}\nu_j\Big]=1.35$.

Therefore, if $c_i>\frac{3.85}{2}=1.925,\ i=1,2,3$ then, according to Theorem 1, the master impulsive Caputo fractional system (40) and the slave impulsive Caputo fractional system (41) are globally Mittag–Leffler synchronized.

Example 4. *Consider the master impulsive Caputo fractional system (40) with $n=2$, $q=0.3$, $t_k=k, k=0,1,2,\dots$, $c_i(t)=\frac{0.55}{(t-k)^{0.3}\Gamma(0.7)}$ for $t\in(k,k+1]$, with the activation functions $f_j(s)=g_j(s)=h_j(s)=\frac{1}{1+e^{-s}}$, the delays $\tau_i(t)=|sin(t)|\le 1$, i.e., $r=1$ and $|a_{ij}(t)|\le M_{ij}$, $|b_{ij}(t)|\le C_{ij}$, $|d_{ij}(t)|\le D_{ij}$ $i,j=1,2$, $t\ge 0$ where $M=\{M_{ij}\}, C=\{C_{ij}\}$ are given by*

$$
M=\begin{pmatrix}0.1 & 0.03\\ 0.02 & 0.3\end{pmatrix},\ C=\begin{pmatrix}0.001 & 0.002\\ 0.003 & 0.001\end{pmatrix},\qquad D=\begin{pmatrix}0.002 & 0.001\\ 0.001 & 0.002\end{pmatrix}.
$$

Let the output coupling controller be $u_j(t) = (\frac{1}{1+e^{-y_j(t)}} - \frac{1}{1+e^{-x_j(t)}})\sum_{j=1}^{3} m_{ij}$, $i = 1,2$ *with*

$$m = \begin{pmatrix} 0.1 & 0.02 \\ 0.02 & 0.1 \end{pmatrix} M + m = \begin{pmatrix} 0.2 & 0.05 \\ 0.04 & 0.4 \end{pmatrix}.$$

Let $m(t) = E_{0.3}(-(t-k)^{0.3}) + 0.1$ *with* $\beta = 0.1$. *Then,* ${}_{t_k}^{RL}D^q\Big(m(t)\Big) = -E_{0.3}(-(t-k)^{0.3}) + \frac{1.1}{(t-k)^{0.3}\Gamma(0.7)}$ *and* $2\min_{i=1,2,\dots,n} c_i(t) - \frac{{}_{t_k}^{RL}D^q\Big(m(t)\Big)}{m(t)} = E_{0.3}(-(t-k)^{0.3}) > E_{0.3}(-1) = \xi = 0.456594 > 0$.

Then, $\lambda_i = \mu_i = \nu_i = 0.25$ *and* $0.25\sum_{i=1}^{2}\Big(\max_{j=1,2}(M_{ij} + |m_{ij}|) + 10\sum_{j=1}^{2}(D_{ij} + C_{ij})\Big) = 0.1775$, $0.25\max_{i=1,2}\sum_{j=1}^{2}\Big[(M_{ij} + |m_{ij}|) + C_{ij} + D_{ij}\Big] = 0.114$. *Then,* $E_{0.3}(-1) > 0.1775 + 0.114$ *and according to Theorem 2 the master impulsive Caputo fractional system (40) and the slave impulsive Caputo fractional system (41) are globally Mittag–Leffler synchronized.*

5.2.2. State Coupling Controllers

Note that in [25] the state coupling was used to achieve the exponential lag synchronization of chaotic neural networks with impulsive effects. Now, we will consider the case when the control inputs are $u_j(t) = N_j(y_j(t)) - x_j(t))$, $j = 1,2,\dots,n$. Then, the synchronization error $e_i(t) = y_i(t) - x_i(t)$ will satisfy

$$\begin{aligned}
&{}_{t_k}^{C}D_t^q e_i(t) = -c_i(t)e_i(t) + \sum_{j=1}^{n} a_{ij}(t)F_j(e_j(t)) + \sum_{j=1}^{n} b_{ij}(t)G_j(e_j(t-\tau_j(t))) \\
&\quad + \sum_{j=1}^{n} d_{ij}(t)\int_{-r}^{0} K_{ij}(s)H_j(e_j(t+s))ds - N_i e_i(t) \text{ for } t \in (t_k, t_{k+1}], k = 0,1,2,\dots, \\
&\Delta e_i(t)|_{t=t_k} = L_{k,i}(e_i(t_k - 0)), \qquad k = 1,2,\dots \\
&e_i(t) = \Phi_i^0(t - t_0), \ \ t \in [t_0 - r, t_0], \ \ i = 1,2,\dots n.
\end{aligned} \tag{47}$$

In this case, we can derive the following result (its proof is similar to the one in Theorem 1 and we omit it). We assume the following:

Assumption A7. The inequality

$$\begin{aligned}
2(\eta + \min_{i=1,2,\dots,n} N_i) > &\sum_{i=1}^{n}\Big(\max_{j=1,2,\dots,n} M_{ij}\lambda_j + \sum_{j=1}^{n}(D_{ij}\nu_j + C_{ij}\mu_j)\Big) \\
&+ \max_{i=1,2,\dots,n}\sum_{j=1}^{n}\Big[M_{ij}\lambda_j + C_{ij}\mu_j + D_{ij}\nu_j\Big]
\end{aligned} \tag{48}$$

holds.

Theorem 3. *Let assumptions A1–A4 and A7 be satisfied.*

Then, the master impulsive Caputo fractional system (40) and the slave impulsive Caputo fractional system (41) are globally Mittag–Leffler synchronized.

In the case when assumptions A3 and A7 are not satisfied, we introduce the following:

Assumption A8. There exists a continuous positive function $m(t) \in C([0,\infty),(0,\infty))$ such that $0 < \beta \le m(t) \le \gamma$, β,γ are constants, the fractional derivative ${}_{t_k}^{RL}D_t^q m(t)$ exists for $t \in (t_k, t_{k+1})$, $k = 0,1,2,\dots,$

$$2 \min_{i=1,2,\ldots,n}(c_i(t)+N_i) - \frac{{}^{RL}_{t_k}D^q\Big(m(t)\Big)}{m(t)} \geq \xi > 0 \ \text{ for } t \in (t_k, t_{k+1}),\ k = 0,1,2,\ldots,$$

and

$$\xi > \sum_{i=1}^{n}\Big(\max_{j=1,2,\ldots,n} M_{ij}\lambda_j + \frac{1}{\beta}\sum_{j=1}^{n}(D_{ij}\nu_j + C_{ij}\mu_j)\Big) + \max_{i=1,2,\ldots,n}\sum_{j=1}^{n}\Big[M_{ij}\lambda_j + C_{ij}\mu_j + D_{ij}\nu_j\Big].$$

Theorem 4. *Let assumptions A1, A2, A4 and A8 be satisfied.*

Then, the master impulsive Caputo fractional system (40) and the slave impulsive Caputo fractional system (41) are globally Mittag–Leffler synchronized.

6. Conclusions

The paper presents sufficient conditions for the global Mittag–Leffler synchronization of a fractional-order neural network with time-varying and distributed delay and with impulsive effects. We consider the case of two types of controllers, output coupling controller and state coupling controller. The study is based on the application of the fractional generalization of the Lyapunov–Razumikhin technique. We study the case of the time varying rate with which the i-th neuron resets its potential to the resting state in isolation and time varying synaptic connection strength of the i-th neuron to the j-th neuron. The case when the lower bound of the Caputo fractional derivative is changeable at each point of impulse is investigated. Consequently, our results are significant for various applications in engineering and technology.

It would be interesting to extend our results to the case of non-Lipschitz discontinuous activation functions applying both approaches for the interpretation of solutions of fractional equations with impulses. This would lead to wider possibilities for appropriate modeling of the connections between neurons in the networks. This topic goes beyond the scope of this paper and will be a challenging issue for future research.

Author Contributions: Conceptualization, R.A., S.H. and D.O.; Methodology, R.A., S.H. and D.O.; Formal Analysis, R.A., S.H. and D.O.; Investigation, R.A., S.H. and D.O.;Writing—Original Draft Preparation, R.A., S.H. and D.O.; Writing—Review and Editing, R.A., S.H. and D.O.; Supervision, R.A., S.H. and D.O.; Funding Acquisition, S.H.

Funding: This research was partially supported by Fund of Plovdiv University FP17FMI008.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Nouh, M.I.; Abed El Salam, E.A.B. Analytical solution to the fractional polytropic gas spheres. *Eur. Phys. J. Plus* **2018**, *149*, 133–149. [CrossRef]
2. Srivastava, T.; Singh, A.P.; Agarwal, H. Modeling the under-Aactuated mechanical system with fractional order derivative. *Progr. Fract. Differ. Appl.* **2015**, *1*, 57–64.
3. Longhi, S. Fractional Schrödinger equation in optics. *Opt. Lett.* **2015**, *40*, 1117–1120. [CrossRef] [PubMed]
4. Sadati, S.J.; Ghaderi, R.; Ranjbar, A. Some fractional comparison results and stability theorem for fractional time delay systems. *Romanian Rep. Phys.* **2013**, *65*, 94–102.
5. Wang, J.R.; Zhang, Y. Ulam-Hyers-Mittag–Leffler stability of fractional-order delay differential equations. *Optim. J. Math. Program. Oper. Res.* **2014**, *63*, 1181–1190. [CrossRef]
6. Hristova, S.; Stamova, I. On the Mittag–Leffler stability of impulsive fractional neural networks with finite delays. *Int. J. Pure Appl. Math.* **2016**, *109*, 105–117. [CrossRef]
7. Li, Y.; Chen, Y.; Podlubny, I. Mittag—Leffler stability of fractional order nonlinear dynamic systems. *Automatica* **2009**, *45*, 1965–1969. [CrossRef]

8. Sadati, S.J.; Baleanu, D.; Ranjbar, A.; Ghaderi, R.; Abdeljawad (Maraaba), T. Mittag–Leffler stability theorem for fractional nonlinear systems with delay. *Abstr. Appl. Anal.* **2010**, *2010*, 108651. [CrossRef]
9. Yang, X.; Li, C.; Huang, T.; Song, Q. Mittag–Leffler stability analysis of nonlinear fractional-order systems with impulses. *Appl. Math. Comput.* **2017**, *293*, 416–422. [CrossRef]
10. Peng, X.; Wu, H.; Song, K.; Shi, J. Global synchronization in finite time for fractional-order neural networks with discontinuous activations and time delays. *Neural Net.* **2017**, *94*, 46–54. [CrossRef] [PubMed]
11. Rifhat, R.; Muhammadhaji, A.; Teng, Z. Global Mittag–Leffler synchronization for impulsive fractional-order neural networks with delays. *Int. J. Nonlinear Sci. Numer. Simul.* **2018**, *19*, 205–213. [CrossRef]
12. Zhang, W.; Wu, R.; Cao, J.; Alsaedid, A.; Hayat, T. Synchronization of a class of fractional-order neural networks with multiple time delays by comparison principles. *Nonliinear Anal. Model. Control* **2017**, *22*, 636–645. [CrossRef]
13. Das, S. *Functional Fractional Calculus*; Springer: Berlin/Heidelberg, Germany, 2011.
14. Diethelm, K. *The Analysis of Fractional Differential Equations*; Springer: Berlin/Heidelberg, Germany, 2010.
15. Podlubny, I. *Fractional Differential Equations*; Academic Press: San Diego, CA, USA, 1999.
16. Devi, J.V.; Rae, F.A.M.; Drici, Z. Variational Lyapunov method for fractional differential equations. *Comput. Math. Appl.* **2012**, *64*, 2982–2989. [CrossRef]
17. Stamova, I.; Stamov, G. Stability analysis of impulsive functional systems of fractional order. *Commun. Nonlinear Sci. Numer. Simul.* **2014**, *19*, 702–709. [CrossRef]
18. Feckan, M.; Zhou, Y.; Wang, J.R. On the concept and existence of solution for impulsive fractional differential equations. *Commun. Nonlinear Sci. Numer. Simul.* **2012**, *17*, 3050–3060. [CrossRef]
19. Agarwal, R.; Hristova, S.; O'Regan, D. A survey of Lyapunov functions, stability and impulsive Caputo fractional differential equations. *Fract. Calc. Appl. Anal.* **2016**, *19*, 290–318. [CrossRef]
20. Wang, J.R.; Feckan, M.; Zhou, Y. A survey on impulsive fractional differential equations. *Fract. Calc. Appl. Anal.* **2016**, **19**, 806–831. [CrossRef]
21. Lakshmikantham, V.; Bainov, D.D.; Simeonov, P.S. *Theory of Impulsive Differential Equations*; World Scientific: Singapore, 1989.
22. Laskin, N. Principles of fractional quantum mechanics. *Fract. Dyn.* **2011**, 393–427. [CrossRef]
23. Mainardi, F. On some properties of the Mittag–Leffler function $E_\alpha(-t^\alpha)$, completely monotone for $t > 0$ with $0 < \alpha < 1$. *Discrt. Contin. Dyn. Syst. Ser. B* **2014**, *19*, 2267–2278.
24. Kaslik, E.; Sivasundaram, S. Nonlinear dynamics and chaos in fractional-order neural networks. *Neural Netw.* **2012**, *32*, 245–256. [CrossRef] [PubMed]
25. Yang, Y.; Cao, J. Exponential lag synchronization of a class of chaotic delayed neural networks with impulsive effects. *Phys. A* **2007**, *386*, 492–502. [CrossRef]