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Symmetric Identities of Hermite-Bernoulli Polynomials and Hermite-Bernoulli Numbers Attached to a Dirichlet Character χ

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Abstract: We aim to introduce arbitrary complex order Hermite-Bernoulli polynomials and Hermite-Bernoulli numbers attached to a Dirichlet character χ and investigate certain symmetric identities involving the polynomials, by mainly using the theory of p-adic integral on $\mathbb{Z}_p$. The results presented here, being very general, are shown to reduce to yield symmetric identities for many relatively simple polynomials and numbers and some corresponding known symmetric identities.

Keywords: q-Volkenborn integral on $\mathbb{Z}_p$; Bernoulli numbers and polynomials; generalized Bernoulli polynomials and numbers of arbitrary complex order; generalized Bernoulli polynomials and numbers attached to a Dirichlet character χ

1. Introduction and Preliminaries

For a fixed prime number p, throughout this paper, let $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ be the ring of p-adic integers, the field of p-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. In addition, let $\mathbb{C}$, $\mathbb{Z}$, and $\mathbb{N}$ be the field of complex numbers, the ring of rational integers and the set of positive integers, respectively, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let $\mathrm{UD}(\mathbb{Z}_p)$ be the space of all uniformly differentiable functions on $\mathbb{Z}_p$. The notation $[z]_q$ is defined by

$$[z]_q := \frac{1-q^z}{1-q} \quad (z \in \mathbb{C};\, q \in \mathbb{C} \setminus \{1\};\, q^z \neq 1)\,.$$

Let ν_p be the normalized exponential valuation on $\mathbb{C}_p$ with $|p|_p = p^{\nu_p(p)} = p^{-1}$. For $f \in \mathrm{UD}(\mathbb{Z}_p)$ and $q \in \mathbb{C}_p$ with $|1-q|_p < 1$, q-Volkenborn integral on $\mathbb{Z}_p$ is defined by Kim [1]

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)\, q^x. \tag{1}$$

For recent works including q-Volkenborn integration see References [1–10].

The ordinary p-adic invariant integral on $\mathbb{Z}_p$ is given by [7,8]

$$I_1(f) = \lim_{q\to 1} I_q(f) = \int_{\mathbb{Z}_p} f(x)\, dx. \tag{2}$$

It follows from Equation (2) that

$$I_1(f_1) = I_1(f) + f'(0), \tag{3}$$

where $f_n(x) := f(x+n)$ $(n \in \mathbb{N})$ and $f'(0)$ is the usual derivative. From Equation (3), one has

$$\int_{\mathbb{Z}_p} e^{xt}\, dx = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \tag{4}$$

where B_n are the nth Bernoulli numbers (see References [11–14]; see also Reference [15] (Section 1.7)). From Equation (2) and (3), one gets

$$\begin{aligned} \frac{n\int_{\mathbb{Z}_p} e^{xt}dx}{\int_{\mathbb{Z}_p} e^{nxt}dx} &= \frac{1}{t}\left(\int_{\mathbb{Z}_p} e^{(x+n)t}dx - \int_{\mathbb{Z}_p} e^{xt}dx\right) \\ = \sum_{j=0}^{n-1} e^{jt} &= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{n-1} j^k\right)\frac{t^k}{k!} = \sum_{k=0}^{\infty} S_k(n-1)\frac{t^k}{k!}, \end{aligned} \tag{5}$$

where

$$S_k(n) = 1^k + \cdots + n^k \quad (k \in \mathbb{N},\, n \in \mathbb{N}_0)\,. \tag{6}$$

From Equation (4), the generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ are defined by the following p-adic integral (see Reference [15] (Section 1.7))

$$\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{\alpha \text{ times}} e^{(x+y_1+y_2+\cdots+y_\alpha)t} dy_1 dy_2 \cdots dy_\alpha = \left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!} \tag{7}$$

in which $B_n^{(1)}(x) := B_n(x)$ are classical Bernoulli numbers (see, e.g., [1–10]).

Let $d,\ p \in \mathbb{N}$ be fixed with $(d,p) = 1$. For $N \in \mathbb{N}$, we set

$$\begin{aligned} X = X_d &= \varprojlim_{N} \left(\mathbb{Z}/dp^N\mathbb{Z}\right); \\ a + dp^N\mathbb{Z}_p &= \left\{x \in X \,|\, x \equiv a \left(\mathrm{mod}\ dp^N\right)\right\} \\ &\quad \left(a \in \mathbb{Z} \text{ with } 0 \le a < dp^N\right); \\ X^* &= \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p)\,, \quad X_1 = \mathbb{Z}_p. \end{aligned} \tag{8}$$

Let χ be a Dirichlet character with conductor $d \in \mathbb{N}$. The generalized Bernoulli polynomials attached to χ are defined by means of the generating function (see, e.g., [16])

$$\int_X \chi(y) e^{(x+y)t} dy = \frac{t\sum\limits_{j=0}^{d-1}\chi(j)\,e^{jt}}{e^{dt}-1} e^{xt} = \sum_{n=0}^{\infty} B_{n,\chi}(x)\frac{t^n}{n!}. \tag{9}$$

Here $B_{n,\chi} := B_{n,\chi}(0)$ are the generalized Bernoulli numbers attached to χ. From Equation (9), we have (see, e.g., [16])

$$\int_X \chi(x) x^n dx = B_{n,\chi} \quad \text{and} \quad \int_X \chi(y)(x+y)^n\, dy = B_{n,\chi}(x). \tag{10}$$

Define the p-adic functional $T_k(\chi, n)$ by (see, e.g., [16])

$$T_k(\chi, n) = \sum_{\ell=0}^{n} \chi(\ell)\, \ell^k \quad (k \in \mathbb{N}). \tag{11}$$

Then one has (see, e.g., [16])

$$B_{k,\chi}(nd) - B_{k,\chi} = kT_{k-1}(\chi, nd-1) \quad (k,\ n,\ d \in \mathbb{N}). \tag{12}$$

Kim et al. [16] (Equation (2.14)) presented the following interesting identity

$$\frac{dn \int_X \chi(x)\, e^{xt}\, dx}{\int_X e^{dnxt}\, dx} = \sum_{\ell=0}^{nd-1} \chi(\ell)\, e^{\ell t} = \sum_{k=0}^{\infty} T_k(\chi, nd-1)\, \frac{t^k}{k!} \quad (n \in \mathbb{N}). \tag{13}$$

Very recently, Khan [17] (Equation (2.1)) (see also Reference [11]) introduced and investigated λ-Hermite-Bernoulli polynomials of the second kind ${}_HB_n(x, y|\lambda)$ defined by the following generating function

$$\begin{aligned} &\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x+u}{\lambda}}\, (1+\lambda t^2)^{\frac{y}{\lambda}}\, \mathrm{d}\mu_0(u) \\ &= \frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} (1+\lambda t)^{\frac{x}{\lambda}} (1+\lambda t^2)^{\frac{y}{\lambda}} = \sum_{m=0}^{\infty} {}_HB_m(x, y|\lambda) \frac{t^m}{m!} \end{aligned} \tag{14}$$

$$\left(\lambda,\ t \in \mathbb{C}_p \ \text{ with } \ \lambda \neq 0,\ |\lambda t| < p^{-\frac{1}{p-1}}\right).$$

Hermite-Bernoulli polynomials ${}_HB_k^{(\alpha)}(x, y)$ of order α are defined by the following generating function

$$\left(\frac{t}{e^t - 1}\right)^{\alpha} e^{xt + yt^2} = \sum_{k=0}^{\infty} {}_HB_k^{(\alpha)}(x, y) \frac{t^k}{k!} \quad (\alpha,\ x,\ y \in \mathbb{C};\ |t| < 2\pi) \tag{15}$$

where ${}_HB_k^{(1)}(x, y) := {}_HB_k(x, y)$ are Hermite-Bernoulli polynomials, cf. [18,19]. For more information related to systematic works of some special functions and polynomials, see References [20–29].

We aim to introduce arbitrary complex order Hermite-Bernoulli polynomials attached to a Dirichlet character χ and investigate certain symmetric identities involving the polynomials (15) and (31), by mainly using the theory of p-adic integral on $\mathbb{Z}_p$. The results presented here, being very general, are shown to reduce to yield symmetric identities for many relatively simple polynomials and numbers and some corresponding known symmetric identities.

2. Symmetry Identities of Hermite-Bernoulli Polynomials of Arbitrary Complex Number Order

Here, by mainly using Kim's method in References [30,31], we establish certain symmetry identities of Hermite-Bernoulli polynomials of arbitrary complex number order.

Theorem 1. *Let α, x, y, $z \in \mathbb{C}$, η_1, $\eta_2 \in \mathbb{N}$, and $n \in \mathbb{N}_0$. Then,*

$$\begin{aligned} &\sum_{m=0}^{n} \sum_{\ell=0}^{m} \binom{n}{m} \binom{m}{\ell}\, {}_HB_{n-m}^{(\alpha)}(\eta_2 x, \eta_2^2 z)\, S_{m-\ell}(\eta_1 - 1)\, B_\ell^{(\alpha-1)}(\eta_1 y)\, \eta_1^{n-m-1}\, \eta_2^m \\ &= \sum_{m=0}^{n} \sum_{\ell=0}^{m} \binom{n}{m} \binom{m}{\ell}\, {}_HB_{n-m}^{(\alpha)}(\eta_1 x, \eta_1^2 z)\, S_{m-\ell}(\eta_2 - 1)\, B_\ell^{(\alpha-1)}(\eta_2 y)\, \eta_2^{n-m-1}\, \eta_1^m \end{aligned} \tag{16}$$

and

$$\begin{aligned}
&\sum_{m=0}^{n}\sum_{j=0}^{\eta_1-1}\binom{n}{m}\eta_1^{m-1}\,\eta_2^{n-m}B_{n-m}^{(\alpha-1)}(\eta_1 y)\,{}_HB_m^{(\alpha)}\left(\eta_2 x+\frac{\eta_2}{\eta_1}j,\eta_2^2 z\right)\\
&=\sum_{m=0}^{n}\sum_{j=0}^{\eta_1-1}\binom{n}{m}\eta_2^{m-1}\,\eta_1^{n-m}\,B_{n-m}^{(\alpha-1)}(\eta_2 y)\,{}_HB_m^{(\alpha)}\left(\eta_1 x+\frac{\eta_1}{\eta_2}j,\eta_1^2 z\right).
\end{aligned} \tag{17}$$

Proof. Let

$$F(\alpha;\eta_1,\eta_2)(t):=\frac{e^{\eta_1\eta_2 t}-1}{\eta_1\eta_2 t}\left(\frac{\eta_1 t}{e^{\eta_1 t}-1}\right)^{\alpha}e^{\eta_1\eta_2 xt+\eta_1^2\eta_2^2 zt^2}\left(\frac{\eta_2 t}{e^{\eta_2 t}-1}\right)^{\alpha}e^{\eta_1\eta_2 yt} \tag{18}$$

$$(\alpha,\,x,\,y,\,z\in\mathbb{C};\,t\in\mathbb{C}\setminus\{0\};\,\eta_1,\,\eta_2\in\mathbb{N};\,1^{\alpha}:=1).$$

Since $\lim_{t\to 0}\eta t/(e^{\eta t}-1)=1=\lim_{t\to 0}(e^{\eta t}-1)/(\eta t)$ $(\eta\in\mathbb{N})$, $F(\alpha;\eta_1,\eta_2)(t)$ may be assumed to be analytic in $|t|<2\pi/(\eta_1\eta_2)$. Obviously $F(\alpha;\eta_1,\eta_2)(t)$ is symmetric with respect to the parameters η_1 and η_2.

Using Equation (4), we have

$$F(\alpha;\eta_1,\eta_2)(t):=\left(\frac{\eta_1 t}{e^{\eta_1 t}-1}\right)^{\alpha}e^{\eta_1\eta_2 xt+\eta_1^2\eta_2^2 zt^2}\frac{\int_{\mathbb{Z}_p}e^{\eta_2 t\,u}du}{\int_{\mathbb{Z}_p}e^{\eta_1\eta_2 t\,u}du}\left(\frac{\eta_2 t}{e^{\eta_2 t}-1}\right)^{\alpha-1}e^{\eta_1\eta_2 yt}. \tag{19}$$

Using Equations (5) and (15), we find

$$\begin{aligned}
F(\alpha;\eta_1,\eta_2)(t)=\sum_{n=0}^{\infty}{}_HB_n^{(\alpha)}(\eta_2 x,\eta_2^2 z)\frac{(\eta_1 t)^n}{n!}\cdot\frac{1}{\eta_1}\sum_{m=0}^{\infty}S_m(\eta_1-1)\frac{(\eta_2 t)^m}{m!}\\
\cdot\sum_{\ell=0}^{\infty}B_{\ell}^{(\alpha-1)}(\eta_1 y)\frac{(\eta_2 t)^{\ell}}{\ell!}.
\end{aligned} \tag{20}$$

Employing a formal manipulation of double series (see, e.g., [32] (Equation (1.1)))

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A_{k,n}=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}A_{k,n-pk}\quad(p\in\mathbb{N}) \tag{21}$$

with $p=1$ in the last two series in Equation (20), and again, the resulting series and the first series in Equation (20), we obtain

$$\begin{aligned}
F(\alpha;\eta_1,\eta_2)(t)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{\ell=0}^{m}\frac{{}_HB_{n-m}^{(\alpha)}(\eta_2 x,\eta_2^2 z)\,S_{m-\ell}(\eta_1-1)\,B_{\ell}^{(\alpha-1)}(\eta_1 y)}{(n-m)!\,(m-\ell)!\,\ell!}\\
\times\eta_1^{n-m-1}\,\eta_2^{m}\,t^n.
\end{aligned} \tag{22}$$

Noting the symmetry of $F(\alpha;\eta_1,\eta_2)(t)$ with respect to the parameters η_1 and η_2, we also get

$$\begin{aligned}
F(\alpha;\eta_1,\eta_2)(t)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{\ell=0}^{m}\frac{{}_HB_{n-m}^{(\alpha)}(\eta_1 x,\eta_1^2 z)\,S_{m-\ell}(\eta_2-1)\,B_{\ell}^{(\alpha-1)}(\eta_2 y)}{(n-m)!\,(m-\ell)!\,\ell!}\\
\times\eta_2^{n-m-1}\,\eta_1^{m}\,t^n.
\end{aligned} \tag{23}$$

Equating the coefficients of t^n in the right sides of Equations (22) and (23), we obtain the first equality of Equation (16).

For (17), we write

$$F(\alpha;\eta_1,\eta_2)(t)=\frac{1}{\eta_1}\left(\frac{\eta_1 t}{e^{\eta_1 t}-1}\right)^{\alpha} e^{\eta_1\eta_2 xt+\eta_1^2\eta_2^2 zt^2}\,\frac{e^{\eta_1\eta_2 t}-1}{e^{\eta_2 t}-1}\left(\frac{\eta_2 t}{e^{\eta_2 t}-1}\right)^{\alpha-1} e^{\eta_1\eta_2 yt}. \tag{24}$$

Noting

$$\frac{e^{\eta_1\eta_2 t}-1}{e^{\eta_2 t}-1}=\sum_{j=0}^{\eta_1-1} e^{\eta_2 jt}=\sum_{j=0}^{\eta_1-1} e^{\eta_1\frac{\eta_2}{\eta_1}jt},$$

we have

$$F(\alpha;\eta_1,\eta_2)(t)=\frac{1}{\eta_1}\sum_{j=0}^{\eta_1-1}\left(\frac{\eta_1 t}{e^{\eta_1 t}-1}\right)^{\alpha} e^{\eta_1\left(\eta_2 x+\frac{\eta_2}{\eta_1}j\right)t+\eta_1^2\eta_2^2 zt^2}\left(\frac{\eta_2 t}{e^{\eta_2 t}-1}\right)^{\alpha-1} e^{\eta_1\eta_2 yt}. \tag{25}$$

Using Equation (15), we obtain

$$\begin{aligned}F(\alpha;\eta_1,\eta_2)(t)&=\frac{1}{\eta_1}\sum_{n=0}^{\infty} B_n^{(\alpha-1)}(\eta_1 y)\,\frac{(\eta_2 t)^n}{n!}\\&\times\sum_{m=0}^{\infty}\sum_{j=0}^{\eta_1-1} {}_HB_m^{(\alpha)}\left(\eta_2 x+\frac{\eta_2}{\eta_1}j,\eta_2^2 z\right)\frac{(\eta_1 t)^m}{m!}.\end{aligned} \tag{26}$$

Applying Equation (21) with $p=1$ to the right side of Equation (26), we get

$$\begin{aligned}F(\alpha;\eta_1,\eta_2)(t)&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{j=0}^{\eta_1-1} B_{n-m}^{(\alpha-1)}(\eta_1 y)\\&\times {}_HB_m^{(\alpha)}\left(\eta_2 x+\frac{\eta_2}{\eta_1}j,\eta_2^2 z\right)\frac{\eta_1^{m-1}\,\eta_2^{n-m}}{m!(n-m)!}\,t^n.\end{aligned} \tag{27}$$

In view of symmetry of $F(\alpha;\eta_1,\eta_2)(t)$ with respect to the parameters η_1 and η_2, we also obtain

$$\begin{aligned}F(\alpha;\eta_1,\eta_2)(t)&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{j=0}^{\eta_1-1} B_{n-m}^{(\alpha-1)}(\eta_2 y)\\&\times {}_HB_m^{(\alpha)}\left(\eta_1 x+\frac{\eta_1}{\eta_2}j,\eta_1^2 z\right)\frac{\eta_2^{m-1}\,\eta_1^{n-m}}{m!(n-m)!}\,t^n.\end{aligned} \tag{28}$$

Equating the coefficients of t^n in the right sides of Equation (27) and Equation (28), we have Equation (17). □

Corollary 1. *By substituting $\alpha=1$ in Theorem 1, we have*

$$\begin{aligned}&\sum_{m=0}^{n}\sum_{\ell=0}^{m}\binom{n}{m}\binom{m}{\ell}\,{}_HB_{n-m}(\eta_2 x,\eta_2^2 z)\,S_{m-\ell}(\eta_1-1)\,(\eta_1 y)^{\ell}\,\eta_1^{n-m-1}\,\eta_2^{m}\\&=\sum_{m=0}^{n}\sum_{\ell=0}^{m}\binom{n}{m}\binom{m}{\ell}\,B_{n-m}(\eta_1 x,\eta_1^2 z)\,S_{m-\ell}(\eta_2-1)\,(\eta_2 y)^{\ell}\,\eta_2^{n-m-1}\,\eta_1^{m}\end{aligned}$$

and

$$\begin{aligned}&\sum_{m=0}^{n}\sum_{j=0}^{\eta_1-1}\binom{n}{m}\eta_1^{m-1}\eta_2^{n-m}(\eta_1 y)^{n-m}\,{}_HB_m\left(\eta_2 x+\frac{\eta_2}{\eta_1}j,\eta_2^2 z\right)\\&=\sum_{m=0}^{n}\sum_{j=0}^{\eta_1-1}\binom{n}{m}\eta_2^{m-1}\eta_1^{n-m}(\eta_2 y)^{n-m}\,{}_HB_m\left(\eta_1 x+\frac{\eta_1}{\eta_2}j,\eta_1^2 z\right).\end{aligned}\tag{29}$$

Corollary 2. *Taking $\alpha=1$ and $z=0$ in Theorem 1, we have*

$$\begin{aligned}&\sum_{m=0}^{n}\sum_{\ell=0}^{m}\binom{n}{m}\binom{m}{\ell}B_{n-m}(\eta_2 x)\,S_{m-\ell}(\eta_1-1)\,(\eta_1 y)^{\ell}\,\eta_1^{n-m-1}\,\eta_2^{m}\\&=\sum_{m=0}^{n}\sum_{\ell=0}^{m}\binom{n}{m}\binom{m}{\ell}B_{n-m}(\eta_1 x)\,S_{m-\ell}(\eta_2-1)\,(\eta_2 y)^{\ell}\,\eta_2^{n-m-1}\,\eta_1^{m}\end{aligned}$$

and

$$\begin{aligned}&\sum_{m=0}^{n}\sum_{j=0}^{\eta_1-1}\binom{n}{m}\eta_1^{m-1}\eta_2^{n-m}(\eta_1 y)^{n-m}B_m\left(\eta_2 x+\frac{\eta_2}{\eta_1}j\right)\\&=\sum_{m=0}^{n}\sum_{j=0}^{\eta_1-1}\binom{n}{m}\eta_2^{m-1}\eta_1^{n-m}(\eta_2 y)^{n-m}B_m\left(\eta_1 x+\frac{\eta_1}{\eta_2}j\right).\end{aligned}\tag{30}$$

3. Symmetry Identities of Arbitrary Order Hermite-Bernoulli Polynomials Attached to a Dirichlet Character χ

We begin by introducing generalized Hermite-Bernoulli polynomials attached to a Dirichlet character χ of order $\alpha\in\mathbb{C}$ defined by means of the following generating function:

$$\left(\frac{t\sum_{j=0}^{d-1}\chi(j)\,e^{jt}}{e^{dt}-1}\right)^{\alpha}e^{xt+yt^2}=\sum_{n=0}^{\infty}{}_HB_{n,\chi}^{(\alpha)}(x,y)\frac{t^n}{n!}\tag{31}$$

$$(\alpha,\;x,\;y\in\mathbb{C}),$$

where χ is a Dirichlet character with conductor d.

Here, $B_{n,\chi}^{(\alpha)}(x) := {}_HB_{n,\chi}^{(\alpha)}(x,0)$, $B_{n,\chi}^{(\alpha)} := {}_HB_{n,\chi}^{(\alpha)}(0,0)$, and $B_{n,\chi} := {}_HB_{n,\chi}^{(1)}(0,0)$ are called the generalized Hermite-Bernoulli polynomials and numbers attached to χ of order α and Hermite-Bernoulli numbers attached to χ, respectively.

Remark 1. *Taking $y=0$ in Equation (31) gives ${}_HB_{n,\chi}^{(\alpha)}(x,0) :={}_H B_{n,\chi}^{(\alpha)}(x)$, cf. [33].*

Remark 2. *Equation (15) is obtained when $\chi:=1$ in Equation (31).*

Remark 3. *The Hermite-Bernoulli polynomials ${}_HB_n(x,y)$ are obtained when $\chi:=1$ and $\alpha=1$ in Equation (31).*

Remark 4. *The generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ is obtained when $\chi:=1$ and $y=0$ in Equation (31).*

Remark 5. *The classical Bernoulli polynomials attached to χ is obtained when $\alpha=1$ and $y=0$ in Equation (31).*

Theorem 2. *Let α, x, y, $z \in \mathbb{C}$, η_1, $\eta_2 \in \mathbb{N}$, and $n \in \mathbb{N}_0$. Then,*

$$\begin{aligned} &\sum_{m=0}^{n}\sum_{\ell=0}^{m}\binom{n}{m}\binom{m}{\ell}\eta_1^{n-m-1}\eta_2^{m}\,{}_HB_{n-m,\chi}^{(\alpha)}\left(\eta_2 x,\eta_2^2 z\right)\,B_{m-\ell,\chi}^{(\alpha-1)}\left(\eta_1 y\right)\,T_\ell(\chi,d\eta_1-1) \\ &=\sum_{m=0}^{n}\sum_{\ell=0}^{m}\binom{n}{m}\binom{m}{\ell}\eta_2^{n-m-1}\eta_1^{m}\,{}_HB_{n-m,\chi}^{(\alpha)}\left(\eta_1 x,\eta_1^2 z\right)\,B_{m-\ell,\chi}^{(\alpha-1)}\left(\eta_2 y\right)\,T_\ell(\chi,d\eta_2-1) \end{aligned} \tag{32}$$

and

$$\begin{aligned} &\sum_{m=0}^{n}\sum_{\ell=0}^{d\eta_1-1}\chi(\ell)\binom{n}{m}\eta_1^{n-m-1}\,\eta_2^{m}\,{}_HB_{n-m,\chi}^{(\alpha)}\left(\eta_2 x+\frac{\ell\eta_2}{\eta_1},\eta_2^2 z\right)B_{m,\chi}^{(\alpha-1)}\left(\eta_1 y\right) \\ &=\sum_{m=0}^{n}\sum_{\ell=0}^{d\eta_2-1}\chi(\ell)\binom{n}{m}\eta_2^{n-m-1}\,\eta_1^{m}\,{}_HB_{n-m,\chi}^{(\alpha)}\left(\eta_1 x+\frac{\ell\eta_1}{\eta_2},\eta_1^2 z\right)B_{m,\chi}^{(\alpha-1)}\left(\eta_2 y\right), \end{aligned} \tag{33}$$

where χ is a Dirichlet character with conductor d.

Proof. Let

$$\begin{aligned} G(\alpha;\eta_1,\eta_2)(t) := \frac{d}{\int_X e^{d\eta_1\eta_2 ut}du}\left(\frac{\eta_1 t\sum_{j=0}^{d-1}\chi(j)\,e^{j\eta_1 t}}{e^{d\eta_1 t}-1}\right)^{\alpha} e^{\eta_1\eta_2 xt+\eta_1^2\eta_2^2 zt^2} \\ \times\left(\frac{\eta_2 t\sum_{j=0}^{d-1}\chi(j)\,e^{j\eta_2 t}}{e^{d\eta_2 t}-1}\right)^{\alpha} e^{\eta_1\eta_2 yt} \end{aligned} \tag{34}$$

$$\left(\alpha,\ x,\ y,\ z\in\mathbb{C};\ t\in\mathbb{C}\setminus\{0\};\ \eta_1,\ \eta_2\in\mathbb{N};\ 1^\alpha:=1\right).$$

Obviously $G(\alpha;\eta_1,\eta_2)(t)$ is symmetric with respect to the parameters η_1 and η_2. As in the function $F(\alpha;\eta_1,\eta_2)(t)$ in Equation (18), $G(\alpha;\eta_1,\eta_2)(t)$ can be considered to be analytic in a neighborhood of $t=0$. Using Equation (9), we have

$$\begin{aligned} G(\alpha;\eta_1,\eta_2)(t) = \frac{d\int_X \chi(u)e^{\eta_2 ut}du}{\int_X e^{d\eta_1\eta_2 ut}du}\left(\frac{\eta_1 t\sum_{j=0}^{d-1}\chi(j)\,e^{j\eta_1 t}}{e^{d\eta_1 t}-1}\right)^{\alpha} e^{\eta_1\eta_2 xt+\eta_1^2\eta_2^2 zt^2} \\ \times\left(\frac{\eta_2 t\sum_{j=0}^{d-1}\chi(j)\,e^{j\eta_2 t}}{e^{d\eta_2 t}-1}\right)^{\alpha-1} e^{\eta_1\eta_2 yt}. \end{aligned} \tag{35}$$

Applying Equations (13) and (31) to Equation (35), we obtain

$$\begin{aligned} G(\alpha;\eta_1,\eta_2)(t) := \frac{1}{\eta_1}\sum_{n=0}^{\infty}{}_HB_{n,\chi}^{(\alpha)}\left(\eta_2 x,\eta_2^2 z\right)\frac{(\eta_1 t)^n}{n!}\sum_{m=0}^{\infty}B_{m,\chi}^{(\alpha-1)}\left(\eta_1 y\right)\frac{(\eta_2 t)^m}{m!} \\ \times\sum_{\ell=0}^{\infty}T_\ell(\chi,d\eta_1-1)\,\frac{(\eta_2 t)^\ell}{\ell!}. \end{aligned} \tag{36}$$

Similarly as in the proof of Theorem 1, we find

$$\begin{aligned} G(\alpha;\eta_1,\eta_2)(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{\ell=0}^{m}\frac{\eta_1^{n-m-1}\eta_2^{m}}{(n-m)!(m-\ell)!\ell!} \\ \times{}_HB_{n-m,\chi}^{(\alpha)}\left(\eta_2 x,\eta_2^2 z\right)\,B_{m-\ell,\chi}^{(\alpha-1)}\left(\eta_1 y\right)\,T_\ell(\chi,d\eta_1-1)\,t^n. \end{aligned} \tag{37}$$

In view of the symmetry of $G(\alpha;\eta_1,\eta_2)(t)$ with respect to the parameters η_1 and η_2, we also get

$$\begin{aligned} G(\alpha;\eta_1,\eta_2)(t) &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{\ell=0}^{m} \frac{\eta_2^{n-m-1}\eta_1^{m}}{(n-m)!(m-\ell)!\ell!} \\ &\times {}_HB_{n-m,\chi}^{(\alpha)}\left(\eta_1 x,\eta_1^2 z\right) B_{m-\ell,\chi}^{(\alpha-1)}(\eta_2 y)\; T_\ell(\chi,d\eta_2-1)\, t^n. \end{aligned} \tag{38}$$

Equating the coefficients of t^n of the right sides of Equations (37) and (38), we obtain Equation (32).

From Equation (13), we have

$$\frac{d\int_X \chi(u) e^{\eta_2 u t}du}{\int_X e^{d\eta_1\eta_2 u t}du} = \frac{1}{\eta_1}\sum_{\ell=0}^{d\eta_1-1}\chi(\ell)\, e^{\ell\eta_2 t}. \tag{39}$$

Using Equation (39) in Equation (35), we get

$$\begin{aligned} G(\alpha;\eta_1,\eta_2)(t) &= \frac{1}{\eta_1}\sum_{\ell=0}^{d\eta_1-1}\chi(\ell)\left(\frac{\eta_1 t\sum\limits_{j=0}^{d-1}\chi(j)\,e^{j\eta_1 t}}{e^{d\eta_1 t}-1}\right)^{\alpha} e^{\left(\eta_2 x+\frac{\ell\eta_2}{\eta_1}\right)\eta_1 t+\eta_1^2\eta_2^2 z t^2} \\ &\times\left(\frac{\eta_2 t\sum\limits_{j=0}^{d-1}\chi(j)\,e^{j\eta_2 t}}{e^{d\eta_2 t}-1}\right)^{\alpha-1} e^{\eta_1\eta_2 y t}. \end{aligned} \tag{40}$$

Using Equation (31), similarly as above, we obtain

$$\begin{aligned} G(\alpha;\eta_1,\eta_2)(t) &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{\ell=0}^{d\eta_1-1}\chi(\ell)\,{}_HB_{n-m,\chi}^{(\alpha)}\left(\eta_2 x+\frac{\ell\eta_2}{\eta_1},\eta_2^2 z\right) \\ &\times B_{m,\chi}^{(\alpha-1)}(\eta_1 y)\,\frac{\eta_1^{n-m-1}\eta_2^{m}}{(n-m)!m!}\,t^n. \end{aligned} \tag{41}$$

Since $G(\alpha;\eta_1,\eta_2)(t)$ is symmetric with respect to the parameters η_1 and η_2, we also have

$$\begin{aligned} G(\alpha;\eta_1,\eta_2)(t) &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{\ell=0}^{d\eta_2-1}\chi(\ell)\,{}_HB_{n-m,\chi}^{(\alpha)}\left(\eta_1 x+\frac{\ell\eta_1}{\eta_2},\eta_1^2 z\right) \\ &\times B_{m,\chi}^{(\alpha-1)}(\eta_2 y)\,\frac{\eta_2^{n-m-1}\eta_1^{m}}{(n-m)!m!}\,t^n. \end{aligned} \tag{42}$$

Equating the coefficients of t^n of the right sides in Equation (41) and Equation (42), we get Equation (33). □

4. Conclusions

The results in Theorems 1 and 2, being very general, can reduce to yield many symmetry identities associated with relatively simple polynomials and numbers using Remarks 1–5. Setting $z = 0$ and $\alpha \in \mathbb{N}$ in the results in Theorem 1 and Theorem 2 yields the corresponding known identities in References [33,34], respectively.

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