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Representation by Chebyshev Polynomials for Sums of Finite Products of Chebyshev Polynomials

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Abstract: In this paper, we consider sums of finite products of Chebyshev polynomials of the first, third, and fourth kinds, which are different from the previously-studied ones. We represent each of them as linear combinations of Chebyshev polynomials of all kinds whose coefficients involve some terminating hypergeometric functions ${}_2F_1$. The results may be viewed as a generalization of the linearization problem, which is concerned with determining the coefficients in the expansion of the product of two polynomials in terms of any given sequence of polynomials. These representations are obtained by explicit computations.

Keywords: Chebyshev polynomials of the first, second, third, and fourth kinds; sums of finite products; representation

1. Introduction and Preliminaries

We first fix some notations that will be used throughout this paper. For any nonnegative integer n, the falling factorial sequence $(x)_n$ and the rising factorial sequence $< x >_n$ are respectively given by:

$$(x)_n = x(x-1)\cdots(x-n+1), \quad (n \geq 1), \ \ (x)_0 = 1, \tag{1}$$

$$< x >_n = x(x+1)\cdots(x+n-1), \quad (n \geq 1), \ \ < x >_0 = 1. \tag{2}$$

Then, we easily see that the two factorial sequences are related by:

$$(-1)^n (x)_n =< -x >_n . \tag{3}$$

The Gauss hypergeometric function ${}_2F_1(a,b;c;x)$ is defined by:

$${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{< a >_n < b >_n}{< c >_n} \frac{x^n}{n!}, \ \ (|x| < 1). \tag{4}$$

In this paper, we only need very basic facts about Chebyshev polynomials of the first, second, third, and fourth kinds, which we recall briefly in the following. The Chebyshev polynomials belong to the family of orthogonal polynomials. We let the interested reader refer to [1–4] for more details on these.

In terms of generating functions, the Chebyshev polynomials of the first, second, third, and fourth kinds are respectively given by:

$$F_1(t,x) = \frac{1-xt}{1-2xt+t^2} = \sum_{n=0}^{\infty} T_n(x)t^n, \tag{5}$$

$$F_2(t,x) = \frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty} U_n(x)t^n, \tag{6}$$

$$F_3(t,x) = \frac{1-t}{1-2xt+t^2} = \sum_{n=0}^{\infty} V_n(x)t^n, \tag{7}$$

$$F_4(t,x) = \frac{1+t}{1-2xt+t^2} = \sum_{n=0}^{\infty} W_n(x)t^n. \tag{8}$$

They are also explicitly given by the following expressions:

$$\begin{aligned} T_n(x) &= {}_2F_1\left(-n, n; \frac{1}{2}; \frac{1-x}{2}\right) \\ &= \frac{n}{2}\sum_{l=0}^{\left[\frac{n}{2}\right]} (-1)^l \frac{1}{n-l}\binom{n-l}{l}(2x)^{n-2l}, \quad (n \geq 1), \end{aligned} \tag{9}$$

$$\begin{aligned} U_n(x) &= (n+1)\,{}_2F_1\left(-n, n+2; \frac{3}{2}; \frac{1-x}{2}\right) \\ &= \sum_{l=0}^{\left[\frac{n}{2}\right]} (-1)^l \binom{n-l}{l}(2x)^{n-2l}, \quad (n \geq 0), \end{aligned} \tag{10}$$

$$\begin{aligned} V_n(x) &= {}_2F_1\left(-n, n+1; \frac{1}{2}; \frac{1-x}{2}\right) \\ &= \sum_{l=0}^{n} \binom{n+l}{2l} 2^l (x-1)^l, \quad (n \geq 0), \end{aligned} \tag{11}$$

$$\begin{aligned} W_n(x) &= (2n+1)\,{}_2F_1\left(-n, n+1; \frac{3}{2}; \frac{1-x}{2}\right) \\ &= (2n+1)\sum_{l=0}^{n} \frac{2^l}{2l+1}\binom{n+l}{2l}(x-1)^l, \quad (n \geq 0), \end{aligned} \tag{12}$$

The Chebyshev polynomials of all four kinds are also expressed by the Rodrigues formulas, which are given by:

$$T_n(x) = \frac{(-1)^n 2^n n!}{(2n)!}(1-x^2)^{\frac{1}{2}} \frac{d^n}{dx^n}(1-x^2)^{n-\frac{1}{2}}, \tag{13}$$

$$U_n(x) = \frac{(-1)^n 2^n (n+1)!}{(2n+1)!}(1-x^2)^{-\frac{1}{2}} \frac{d^n}{dx^n}(1-x^2)^{n+\frac{1}{2}}, \tag{14}$$

$$\begin{aligned} &(1-x)^{-\frac{1}{2}}(1+x)^{\frac{1}{2}} V_n(x) \\ &= \frac{(-1)^n 2^n n!}{(2n)!} \frac{d^n}{dx^n}(1-x)^{n-\frac{1}{2}}(1+x)^{n+\frac{1}{2}}, \end{aligned} \tag{15}$$

$$\begin{aligned} &(1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}} W_n(x) \\ &= \frac{(-1)^n 2^n n!}{(2n)!} \frac{d^n}{dx^n}(1-x)^{n+\frac{1}{2}}(1-x)^{n-\frac{1}{2}}. \end{aligned} \tag{16}$$

They satisfy orthogonalities with respect to various weight functions as given in the following:

$$\int_{-1}^{1} (1-x^2)^{-\frac{1}{2}} T_n(x) T_m(x)\,dx = \frac{\pi}{\varepsilon_n}\delta_{n,m}, \tag{17}$$

where:

$$\varepsilon_n = \begin{cases} 1, & \text{if} \quad n = 0, \\ 2, & \text{if} \quad n \geq 1, \end{cases} \tag{18}$$

$$\delta_{n,m} = \begin{cases} 0, & \text{if} \quad n \neq m, \\ 1, & \text{if} \quad n = m. \end{cases} \tag{19}$$

$$\int_{-1}^{1} (1-x^2)^{\frac{1}{2}} U_n(x) U_m(x) dx = \frac{\pi}{2} \delta_{n,m}, \tag{20}$$

$$\int_{-1}^{1} \left(\frac{1+x}{1-x} \right)^{\frac{1}{2}} V_n(x) V_m(x) dx = \pi \delta_{n,m}, \tag{21}$$

$$\int_{-1}^{1} \left(\frac{1-x}{1+x} \right)^{\frac{1}{2}} W_n(x) W_m(x) dx = \pi \delta_{n,m}. \tag{22}$$

For convenience, we let:

$$\alpha_{m,r}(x) = \sum_{i_1+\cdots+i_{r+1}=m} T_{i_1}(x) \cdots T_{i_{r+1}}(x), \quad (m, r \geq 0), \tag{23}$$

$$\beta_{m,r}(x) = \sum_{i_1+\cdots+i_{r+1}=m} V_{i_1}(x) \cdots V_{i_{r+1}}(x), \quad (m, r \geq 0), \tag{24}$$

$$\gamma_{m,r}(x) = \sum_{i_1+\cdots+i_{r+1}=m} W_{i_1}(x) \cdots W_{i_{r+1}}(x), \quad (m, r \geq 0), \tag{25}$$

Here, all the sums in (23)–(25) are over all nonnegative integers $i_1, \cdots, i_{r+1}$, with $i_1 + i_2 + \cdots + i_{r+1} = m$. Furthermore, note here that $\alpha_{m,r}(x)$, $\beta_{m,r}(x)$, $\gamma_{m,r}(x)$ all have degree m.

Further, let us put:

$$\begin{aligned} &\sum_{l=0}^{m} \sum_{i_1+\cdots+i_{r+1}=m-l} \binom{r+l}{r} x^l T_{i_1}(x) \cdots T_{i_{r+1}}(x) \\ &- \sum_{l=0}^{m-2} \sum_{i_1+\cdots+i_{r+1}=m-l-2} \binom{r+l}{r} x^l T_{i_1}(x) \cdots T_{i_{r+1}}(x), \quad (m \geq 2, r \geq 1), \end{aligned} \tag{26}$$

$$\sum_{l=0}^{m} \sum_{i_1+\cdots+i_{r+1}=l} \binom{r-1+m-l}{r-1} V_{i_1}(x) \cdots V_{i_{r+1}}(x), \quad (m \geq 0, r \geq 1), \tag{27}$$

$$\sum_{l=0}^{m} \sum_{i_1+\cdots+i_{r+1}=l} (-1)^{m-l} \binom{r-1+m-l}{r-1} W_{i_1}(x) \cdots W_{i_{r+1}}(x), \quad (m \geq 0, r \geq 1). \tag{28}$$

We considered the expression (26) in [5] and (27) and (28) in [6] and were able to express each of them in terms of the Chebyshev polynomials of all four kinds. It is amusing to note that in such expressions, some terminating hypergeometric functions ${}_2F_1$ and ${}_3F_2$ appear respectively for (26)–(28). We came up with studying the sums in (26)–(28) by observing that they are respectively equal to $\frac{1}{2^{r-1}r!} T^{(r)}_{m+r}(x)$, $\frac{1}{2^r r!} V^{(r)}_{m+r}(x)$, and $\frac{1}{2^r r!} W^{(r)}_{m+r}(x)$. Actually, these easily follow by differentiating the generating functions in (5), (7), and (8).

In this paper, we consider the expressions $\alpha_{m,r}(x)$, $\beta_{m,r}(x)$, and $\gamma_{m,r}(x)$ in (23)–(25), which are sums of finite products of Chebyshev polynomials of the first, third, and fourth kinds, respectively. Then, we express each of them as linear combinations of $T_n(x)$, $U_n(x)$, $V_n(x)$, and $W_n(x)$. Here, we remark that $\alpha_{m,r}(x)$, $\beta_{m,r}(x)$, and $\gamma_{m,r}(x)$ are expressed in terms of $U^{(r)}_{m-j+r}(x)$, $(j = 0, 1, \cdots, m)$ (see Lemmas 2 and 3) by making use of the generating function in (6). This is unlike the previous works for (26)–(28) (see [5,6]), where we showed they are respectively equal to $\frac{1}{2^{r-1}r!}T_{m+r}^{(r)}(x)$, $\frac{1}{2^r r!}V_{m+r}^{(r)}(x)$, and $\frac{1}{2^r r!}W_{m+r}^{(r)}(x)$ by exploiting the generating functions in (5), (7) and (8). Then, our results for $\alpha_{m,r}(x)$, $\beta_{m,r}(x)$, and $\gamma_{m,r}(x)$ will be found by making use of Lemmas 1 and 2, the general formulas in Propositions 1 and 2, and integration by parts. As we can notice here, generating functions play important roles in the present and the previous works in [5,6]. We would like to remark here that the technique of generating functions has been widely used not only in mathematics, but also in physics and biology. For this matter, we recommend the reader to refer to [7–9]. The next three theorems are our main results.

Theorem 1. *For any nonnegative integers m, r, the following identities hold true.*

$$\sum_{i_1+\cdots+i_{r+1}=m} T_{i_1}(x)\cdots T_{i_{r+1}}(x)$$
$$= \frac{1}{r!}\sum_{s=0}^{\left[\frac{m}{2}\right]}\sum_{l=0}^{s}\frac{\varepsilon_{m-2s}(-1)^l(m+r-l)!}{l!(m-s-l)!(s-l)!}\,{}_2F_1\left(2l-m,-r-1;l-m-r;\frac{1}{2}\right)T_{m-2s}(x) \tag{29}$$

$$= \frac{1}{r!}\sum_{s=0}^{\left[\frac{m}{2}\right]}\sum_{l=0}^{s}\frac{(-1)^l(m-2s+1)(m+r-l)!}{l!(m-s+1-l)!(s-l)!}\,{}_2F_1\left(2l-m,-r-1;l-m-r;\frac{1}{2}\right)U_{m-2s}(x) \tag{30}$$

$$= \frac{1}{r!}\sum_{s=0}^{m}\sum_{l=0}^{\left[\frac{s}{2}\right]}\frac{(-1)^l(m+r-l)!}{l!\left(m-\left[\frac{s}{2}\right]-l\right)!\left(\left[\frac{s}{2}\right]-l\right)!}\,{}_2F_1\left(2l-m,-r-1;l-m-r;\frac{1}{2}\right)V_{m-s}(x) \tag{31}$$

$$= \frac{1}{r!}\sum_{s=0}^{m}\sum_{l=0}^{\left[\frac{s}{2}\right]}\frac{(-1)^{s+l}(m+r-l)!}{l!\left(m-\left[\frac{s}{2}\right]-l\right)!\left(\left[\frac{s}{2}\right]-l\right)!}\,{}_2F_1\left(2l-m,-r-1;l-m-r;\frac{1}{2}\right)W_{m-s}(x). \tag{32}$$

Theorem 2. *For any nonnegative integers m, r, we have the following identities.*

$$\sum_{i_1+\cdots+i_{r+1}=m} V_{i_1}(x)\cdots V_{i_{r+1}}(x)$$
$$= \frac{1}{r!}\sum_{k=0}^{m}\sum_{l=0}^{\left[\frac{m-k}{2}\right]}\frac{(-1)^{m-k}\varepsilon_k(k+2s+r)!}{(s+k)!s!}\binom{r+1}{m-k-2s}{}_2F_1(-s,-s-k;-k-2s-r;1)\,T_k(x) \tag{33}$$

$$= \frac{1}{r!}\sum_{k=0}^{m}\sum_{s=0}^{\left[\frac{m-k}{2}\right]}\frac{(-1)^{m-k}(k+1)(k+2s+r)!}{(s+k+1)!s!}\binom{r+1}{m-k-2s}{}_2F_1(-s,-s-k-1;-k-2s-r;1)\,U_k(x) \tag{34}$$

$$= \frac{1}{r!}\sum_{k=0}^{m}\sum_{s=0}^{m-k}\frac{(-1)^{m-k-s}(k+r+s)!}{\left(k+\left[\frac{s+1}{2}\right]\right)!\left[\frac{s}{2}\right]!}\binom{r+1}{m-k-s}{}_2F_1\left(-\left[\frac{s}{2}\right],-\left[\frac{s+1}{2}\right]-k;-k-s-r;1\right)V_k(x) \tag{35}$$

$$= \frac{1}{r!}\sum_{k=0}^{m}\sum_{s=0}^{m-k}\frac{(-1)^{m-k}(k+r+s)!}{\left(k+\left[\frac{s+1}{2}\right]\right)!\left[\frac{s}{2}\right]!}\binom{r+1}{m-k-s}{}_2F_1\left(-\left[\frac{s}{2}\right],-\left[\frac{s+1}{2}\right]-k;-k-s-r;1\right)W_k(x) \tag{36}$$

Theorem 3. *For any nonnegative integers m, r, the following identities are valid.*

$$\sum_{i_1+\cdots+i_{r+1}=m} W_{i_1}(x)\cdots W_{i_{r+1}}(x)$$
$$= \frac{1}{r!}\sum_{k=0}^{m}\sum_{l=0}^{\left[\frac{m-k}{2}\right]}\frac{\varepsilon_k(k+2s+r)!}{(s+k)!s!}\binom{r+1}{m-k-2s}{}_2F_1(-s,-s-k;-k-2s-r;1)\,T_k(x) \tag{37}$$

$$= \frac{1}{r!}\sum_{k=0}^{m}\sum_{s=0}^{\left[\frac{m-k}{2}\right]}\frac{(k+1)(k+2s+r)!}{(s+k+1)!s!}\binom{r+1}{m-k-2s}{}_2F_1(-s,-s-k-1;-k-2s-r;1)\,U_k(x) \tag{38}$$

$$= \frac{1}{r!}\sum_{k=0}^{m}\sum_{s=0}^{m-k}\frac{(-1)^s(k+r+s)!}{\left(k+\left[\frac{s+1}{2}\right]\right)!\left[\frac{s}{2}\right]!}\binom{r+1}{m-k-s}{}_2F_1\left(-\left[\frac{s}{2}\right],-\left[\frac{s+1}{2}\right]-k;-k-s-r;1\right)V_k(x) \quad (39)$$

$$= \frac{1}{r!}\sum_{k=0}^{m}\sum_{s=0}^{m-k}\frac{(k+r+s)!}{\left(k+\left[\frac{s+1}{2}\right]\right)!\left[\frac{s}{2}\right]!}\binom{r+1}{m-k-s}{}_2F_1\left(-\left[\frac{s}{2}\right],-\left[\frac{s+1}{2}\right]-k;-k-s-r;1\right)W_k(x) \quad (40)$$

Before moving on to the next section, we would like to say a few words on the previous works that are associated with the results in the present paper. In terms of Bernoulli polynomials, quite a few sums of finite products of some special polynomials are expressed. They include Chebyshev polynomials of all four kinds, and Bernoulli, Euler, Genocchi, Legendre, Laguerre, Fibonacci, and Lucas polynomials (see [10–16]). All of these expressions in terms of Bernoulli polynomials have been derived from the Fourier series expansions of the functions closely related to each such polynomials. Further, as for Chebyshev polynomials of all four kinds and Legendre, Laguerre, Fibonacci, and Lucas polynomials, certain sums of finite products of such polynomials are also expressed in terms of all four kinds of Chebyshev polynomials in [5,6,17,18]. Finally, the reader may want to look at [19–21] for some applications of Chebyshev polynomials.

2. Proof of Theorem 1

In this section, we will prove Theorem 1. In order to do this, we first state Propositions 1 and 2 that are needed in proving Theorems 1–3. Here, we note that the facts (a), (b), (c), and (d) in Proposition 1 are stated respectively in the Equations (24) of [22], (36) of [22], (23) of [23], and (38) of [23]. All of them follow easily from the orthogonality relations in (17) and (20)–(22), Rodrigues' formulas in (13)–(16), and integration by parts.

Proposition 1. *For any polynomial $q(x) \in \mathbb{R}[x]$ of degree n, we have the following formulas.*

(a) $q(x) = \sum_{k=0}^{n} C_{k,1}T_k(x)$, *where:*

$$C_{k,1} = \frac{(-1)^k 2^k k! \epsilon_k}{(2k)!\pi}\int_{-1}^{1} q(x)\frac{d^k}{dx^k}(1-x^2)^{k-\frac{1}{2}}dx,$$

(b) $q(x) = \sum_{k=0}^{n} C_{k,2}U_k(x)$, *where:*

$$C_{k,2} = \frac{(-1)^k 2^{k+1}(k+1)!}{(2k+1)!\pi}\int_{-1}^{1} q(x)\frac{d^k}{dx^k}(1-x^2)^{k+\frac{1}{2}}dx,$$

(c) $q(x) = \sum_{k=0}^{n} C_{k,3}V_k(x)$, *where:*

$$C_{k,3} = \frac{(-1)^k k! 2^k}{(2k)!\pi}\int_{-1}^{1} q(x)\frac{d^k}{dx^k}(1-x)^{k-\frac{1}{2}}(1+x)^{k+\frac{1}{2}}dx,$$

(d) $q(x) = \sum_{k=0}^{n} C_{k,4}W_k(x)$, *where,*

$$C_{k,4} = \frac{(-1)^k k! 2^k}{(2k)!\pi}\int_{-1}^{1} q(x)\frac{d^k}{dx^k}(1-x)^{k+\frac{1}{2}}(1+x)^{k+\frac{1}{2}}dx,$$

The next proposition is stated and proven in [17].

Proposition 2. *For any nonnegative integers m, k, we have the following formulas:*

(a)

$$\int_{-1}^{1}(1-x^2)^{k-\frac{1}{2}}x^m dx = \begin{cases} 0, & \text{if } m \equiv 1 \pmod 2, \\ \frac{m!(2k)!\pi}{2^{m+2k}\left(\frac{m}{2}+k\right)!\left(\frac{m}{2}\right)!k!}, & \text{if } m \equiv 0 \pmod 2. \end{cases}$$

(b)

$$\int_{-1}^{1}(1-x^2)^{k+\frac{1}{2}}x^m dx = \begin{cases} 0, & \text{if } m \equiv 1 \pmod 2, \\ \frac{m!(2k+2)!\pi}{2^{m+2k+2}\left(\frac{m}{2}+k+1\right)!\left(\frac{m}{2}\right)!(k+1)!}, & \text{if } m \equiv 0 \pmod 2. \end{cases}$$

(c)

$$\int_{-1}^{1}(1-x)^{k-\frac{1}{2}}(1+x)^{k+\frac{1}{2}}x^m dx = \begin{cases} \frac{(m+1)!(2k)!\pi}{2^{m+2k+1}\left(\frac{m+1}{2}+k\right)!\left(\frac{m+1}{2}\right)!k!}, & \text{if } m \equiv 1 \pmod 2, \\ \frac{m!(2k)!\pi}{2^{m+2k}\left(\frac{m}{2}+k\right)!\left(\frac{m}{2}\right)!k!}, & \text{if } m \equiv 0 \pmod 2. \end{cases}$$

(d)

$$\int_{-1}^{1}(1-x)^{k+\frac{1}{2}}(1+x)^{k-\frac{1}{2}}x^m dx = \begin{cases} -\frac{(m+1)!(2k)!\pi}{2^{m+2k+1}\left(\frac{m+1}{2}+k\right)!\left(\frac{m+1}{2}\right)!k!}, & \text{if } m \equiv 1 \pmod 2, \\ \frac{m!(2k)!\pi}{2^{m+2k}\left(\frac{m}{2}+k\right)!\left(\frac{m}{2}\right)!k!}, & \text{if } m \equiv 0 \pmod 2. \end{cases}$$

The following lemma was shown in [24] and can be derived by differentiating [23].

Lemma 1. *For any nonnegative integers n, r, the following identity holds:*

$$\sum_{i_1+\cdots+i_{r+1}=n} U_{i_1}(x)\cdots U_{i_{r+1}}(x) = \frac{1}{2^r r!}U_{n+k}^{(r)}(x), \tag{41}$$

where the sum is over all nonnegative integers $i_1, \cdots, i_{r+1}$, with $i_1+\cdots+i_{r+1}=n$.

Further, Equation (41) is equivalent to:

$$\left(\frac{1}{1-2xt+t^2}\right)^{r+1} = \frac{1}{2^r r!}\sum_{n=0}^{\infty} U_{n+r}^{(r)}(x)t^n. \tag{42}$$

In reference [24], the following lemma is stated for $m \geq r+1$. However, it holds for any nonnegative integer m, under the usual convention $\binom{r+1}{j}=0$, for $j > r+1$. Therefore, we are going to give a proof for the next lemma.

Lemma 2. *Let m, r be any nonnegative integers. Then, the following identity holds.*

$$\begin{aligned} &\sum_{i_1+\cdots+i_{r+1}=m} T_{i_1}(x)\cdots T_{i_{r+1}}(x) \\ &= \frac{1}{2^r r!}\sum_{j=0}^{m}(-1)^j\binom{r+1}{j}x^j U_{m-j+r}^{(r)}(x), \end{aligned} \tag{43}$$

where $\binom{r+1}{j}=0$, for $j > r+1$.

Proof. By making use of (42), we have:

$$\begin{aligned}
&\sum_{m=0}^{\infty}\left(\sum_{i_1+\cdots+i_{r+1}=m} T_{i_1}(x)\cdots T_{i_{r+1}}(x)\right)t^m \\
&=\left(\frac{1}{1-2xt+t^2}\right)^{r+1}(1-xt)^{r+1} \\
&=\frac{1}{2^r r!}\sum_{n=0}^{\infty}U_{n+r}^{(r)}(x)t^n\sum_{j=0}^{r+1}\binom{r+1}{j}(-x)^j t^j \\
&=\frac{1}{2^r r!}\sum_{m=0}^{\infty}\left(\sum_{j=0}^{\min\{m,r+1\}}(-1)^j\binom{r+1}{j}x^j U_{m-j+r}^{(r)}(x)\right)t^m \\
&=\frac{1}{2^r r!}\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}(-1)^j\binom{r+1}{j}x^j U_{m-j+r}^{(r)}(x)\right)t^m.
\end{aligned} \tag{44}$$

Now, by comparing both sides of (44), we have the desired result. □

From (10), we see that the rth derivative of $U_n(x)$ is given by:

$$U_n^{(r)}(x)=\sum_{l=0}^{\left[\frac{n-r}{2}\right]}(-1)^l\binom{n-l}{l}(n-2l)_r 2^{n-2l}x^{n-2l-r}. \tag{45}$$

Especially, we have:

$$x^j U_{m-j+r}^{(r)}(x)=\sum_{l=0}^{\left[\frac{m-j}{2}\right]}(-1)^l\binom{m-j+r-l}{l}(m-j+r-2l)_r 2^{m-j+r-2l}x^{m-2l}. \tag{46}$$

In this section, we will show (29) and (31) of Theorem 1 and leave similar proofs for (30) and (32) as exercises to the reader. As in (23), let us put:

$$\alpha_{m,r}(x)=\sum_{i_1+\cdots+i_{r+1}=m} T_{i_1}(x)\cdots T_{i_{r+1}}(x),$$

and set:

$$\alpha_{m,r}(x)=\sum_{k=0}^{m} C_{k,1}T_k(x). \tag{47}$$

Then, we can now proceed as follows by using (a) of Proposition 1, (43) and (46), and integration by parts k times.

$$\begin{aligned}
C_{k,1} &= \frac{(-1)^k 2^k k! \epsilon_k}{(2k)!\pi} \int_{-1}^{1} \alpha_{m,r}(x) \frac{d^k}{dx^k}(1-x^2)^{k-\frac{1}{2}} dx \\
&= \frac{(-1)^k 2^k k! \epsilon_k}{(2k)!\pi 2^r r!} \sum_{j=0}^{m} (-1)^j \binom{r+1}{j} \int_{-1}^{1} x^j U_{m-j+r}^{(r)}(x) \frac{d^k}{dx^k}(1-x^2)^{k-\frac{1}{2}} dx \\
&= \frac{(-1)^k 2^k k! \epsilon_k}{(2k)!\pi 2^r r!} \sum_{j=0}^{m} (-1)^j \binom{r+1}{j} \sum_{l=0}^{\left[\frac{m-j}{2}\right]} (-1)^l \binom{m-j+r-l}{l} \\
&\quad \times (m-j+r-2l)_r 2^{m-j+r-2l} \int_{-1}^{1} x^{m-2l} \frac{d^k}{dx^k}(1-x^2)^{k-\frac{1}{2}} dx. \\
&= \frac{2^k k! \epsilon_k}{(2k)!\pi 2^r r!} \sum_{j=0}^{m} (-1)^j \binom{r+1}{j} \sum_{l=0}^{\left[\frac{m-j}{2}\right]} (-1)^l \binom{m-j+r-l}{l} \\
&\quad \times (m-j+r-2l)_r 2^{m-j+r-2l} (m-2l)_k \int_{-1}^{1} x^{m-k-2l} (1-x^2)^{k-\frac{1}{2}} dx \\
&= \frac{2^k k! \epsilon_k}{(2k)!\pi 2^r r!} \sum_{l=0}^{\left[\frac{m-k}{2}\right]} \sum_{j=0}^{m-2l} (-1)^j \binom{r+1}{j} (-1)^l \binom{m-j+r-l}{l} \\
&\quad \times (m-j+r-2l)_r 2^{m-j+r-2l} (m-2l)_k \int_{-1}^{1} x^{m-k-2l} (1-x^2)^{k-\frac{1}{2}} dx.
\end{aligned} \tag{48}$$

Now, from (a) of Proposition 2 and after some simplifications, we see that:

$$\begin{aligned}
\alpha_{m,r}(x) &= \frac{1}{r!} \sum_{0 \le k \le m,\, k \equiv m (\mathrm{mod} 2)} \sum_{l=0}^{\left[\frac{m-k}{2}\right]} \sum_{j=0}^{m-2l} \epsilon_k (-1)^j \binom{r+1}{j} 2^{-j} \\
&\quad \times \frac{(-1)^l (m-j+r-l)!(m-2l)!}{l!(m-j-2l)! \left(\frac{m+k}{2}-l\right)! \left(\frac{m-k}{2}-l\right)!} T_k(x) \\
&= \frac{1}{r!} \sum_{s=0}^{\left[\frac{m}{2}\right]} \sum_{l=0}^{s} \frac{\epsilon_{m-2s}(-1)^l (m-2l)!}{l!(m-s-l)!(s-l)!} \\
&\quad \times \sum_{j=0}^{m-2l} \frac{2^{-j}(-1)^j (m+r-l-j)!(r+1)_j}{j!(m-2l-j)!} T_{m-2s}(x) \\
&= \frac{1}{r!} \sum_{s=0}^{\left[\frac{m}{2}\right]} \sum_{l=0}^{s} \frac{\epsilon_{m-2s}(-1)^l (m+r-l)!}{l!(m-s-l)!(s-l)!} \\
&\quad \times \sum_{j=0}^{m-2l} \frac{2^{-j}(-1)^j (m-2l)_j (r+1)_j}{j!(m+r-l)_j} T_{m-2s}(x) \\
&= \frac{1}{r!} \sum_{s=0}^{\left[\frac{m}{2}\right]} \sum_{l=0}^{s} \frac{\epsilon_{m-2s}(-1)^l (m+r-l)!}{l!(m-s-l)!(s-l)!} \\
&\quad \times \sum_{j=0}^{m-2l} \frac{2^{-j} < 2l-m >_j < -r-1 >_j}{j! < l-m-r >_j} T_{m-2s}(x) \\
&= \frac{1}{r!} \sum_{s=0}^{\left[\frac{m}{2}\right]} \sum_{l=0}^{s} \frac{\epsilon_{m-2s}(-1)^l (m+r-l)!}{l!(m-s-l)!(s-l)!} \\
&\quad \times_2 F_1 \left(2l-m, -r-1; l-m-r; \frac{1}{2}\right) T_{m-2s}(x),
\end{aligned} \tag{49}$$

where we note that we made the change of variables $m-k=2s$.

This completes the proof for (29). Next, we let:

$$\alpha_{m,r}(x) = \sum_{k=0}^{m} C_{k,3} V_k(x). \tag{50}$$

Then, we can obtain the following by making use of (c) of Proposition 1, (43) and (46), and integration by parts k times.

$$\begin{aligned} C_{k,3} = \frac{k!2^k}{(2k)!\pi 2^r r!} \sum_{l=0}^{\left[\frac{m-k}{2}\right]} \sum_{j=0}^{m-2l} (-1)^j \binom{r+1}{j} (-1)^l \binom{m-j+r-l}{l} (m-j+r-2l)_r 2^{m-j+r-2l} \\ \times (m-2l)_k \int_{-1}^{1} x^{m-2l-k} (1-x)^{k-\frac{1}{2}} (1+x)^{k+\frac{1}{2}} dx. \end{aligned} \tag{51}$$

where we note from (c) of Proposition 2 that:

$$\begin{aligned} &\int_{-1}^{1} x^{m-2l-k} (1-x)^{k-\frac{1}{2}} (1+x)^{k+\frac{1}{2}} dx \\ &= \begin{cases} \frac{(m-2l-k+1)!(2k)!\pi}{2^{m+k-2l+1}\left(\frac{m+k+1}{2}-l\right)!\left(\frac{m-k+1}{2}-l\right)!k!}, & \text{if } k \not\equiv m \pmod 2, \\ \frac{(m-2l-k)!(2k)!\pi}{2^{m+k-2l}\left(\frac{m+k}{2}-l\right)!\left(\frac{m-k}{2}-l\right)!k!}, & \text{if } k \equiv m \pmod 2. \end{cases} \end{aligned} \tag{52}$$

From (50)–(52), and after some simplifications, we get:

$$\alpha_{m,r}(x) = \sum\nolimits_1 + \sum\nolimits_2, \tag{53}$$

where:

$$\begin{aligned} \sum\nolimits_1 &= \frac{1}{r!} \sum_{0 \le k \le m, k \not\equiv m (\mathrm{mod} 2)} \sum_{l=0}^{\left[\frac{m-k}{2}\right]} \sum_{j=0}^{m-2l} (-1)^j \binom{r+1}{j} 2^{-j-1} \\ &\quad \times \frac{(-1)^l (m-j+r-l)!(m-2l)!(m-2l-k+1)}{l!(m-j-2l)!\left(\frac{m+k+1}{2}-l\right)!\left(\frac{m-k+1}{2}-l\right)!} V_k(x), \\ \sum\nolimits_2 &= \frac{1}{r!} \sum_{0 \le k \le m, k \equiv m (\mathrm{mod} 2)} \sum_{l=0}^{\left[\frac{m-k}{2}\right]} \sum_{j=0}^{m-2l} (-1)^j \binom{r+1}{j} 2^{-j} \\ &\quad \times \frac{(-1)^l (m-j+r-l)!(m-2l)!}{l!(m-j-2l)!\left(\frac{m+k}{2}-l\right)!\left(\frac{m-k}{2}-l\right)!} V_k(x). \end{aligned} \tag{54}$$

Proceeding analogously to the case of (29), we observe from (54) that:

$$\begin{aligned} \sum\nolimits_1 &= \frac{1}{r!} \sum_{s=0}^{\left[\frac{m-1}{2}\right]} \sum_{l=0}^{s} \frac{(-1)^l (m+r-l)!}{l!(m-s-l)!(s-l)!} \\ &\quad \times \sum_{j=0}^{m-2l} \frac{2^{-j}(-1)^j (r+1)_j (m-2l)_j}{j!(m+r-l)_j} V_{m-2s-1}(x) \\ &= \frac{1}{r!} \sum_{s=0}^{\left[\frac{m-1}{2}\right]} \sum_{l=0}^{s} \frac{(-1)^l (m+r-l)!}{l!(m-s-l)!(s-l)!} \\ &\quad \times {}_2F_1\left(2l-m, -r-1; l-m-r; \frac{1}{2}\right) V_{m-2s-1}(x), \end{aligned} \tag{55}$$

$$\begin{aligned}\sum\nolimits_2 =& \frac{1}{r!}\sum_{s=0}^{\left[\frac{m}{2}\right]}\sum_{l=0}^{s}\frac{(-1)^l(m+r-l)!}{l!(m-s-l)!(s-l)!}\\&\times\sum_{j=0}^{m-2l}\frac{2^{-j}(-1)^j(r+1)_j(m-2l)_j}{j!(m+r-l)_j}V_{m-2s}(x)\\=&\frac{1}{r!}\sum_{s=0}^{\left[\frac{m}{2}\right]}\sum_{l=0}^{s}\frac{(-1)^l(m+r-l)!}{l!(m-s-l)!(s-l)!}\\&\times{}_2F_1\left(2l-m,-r-1;l-m-r;\frac{1}{2}\right)V_{m-2s}(x).\end{aligned} \tag{56}$$

We now obtain the result in (31) from (53), (55) and (56).

3. Proofs of Theorems 2 and 3

In this section, we will show (34) and (36) for Theorem 2, leaving (33) and (35) as exercises to the reader, and note that Theorem 3 follows from (33)–(36) by simple observation. The next lemma can be shown analogously to Lemma 1.

Lemma 3. *For any nonnegative integers m, r, the following identities are valid.*

$$\begin{aligned}&\sum_{i_1+\cdots+i_{r+1}=m}V_{i_1}(x)\cdots V_{i_{r+1}}(x)\\&=\frac{1}{2^r r!}\sum_{j=0}^{m}(-1)^j\binom{r+1}{j}U_{m-j+r}^{(r)}(x),\end{aligned} \tag{57}$$

$$\begin{aligned}&\sum_{i_1+\cdots+i_{r+1}=m}W_{i_1}(x)\cdots W_{i_{r+1}}(x)\\&=\frac{1}{2^r r!}\sum_{j=0}^{m}\binom{r+1}{j}U_{m-j+r}^{(r)}(x),\end{aligned} \tag{58}$$

where $\binom{r+1}{j}=0$, for $j>r+1$.

As in (24), let us set:

$$\beta_{m,r}(x)=\sum_{i_1+\cdots+i_{r+1}=m}V_{i_1}(x)\cdots V_{i_{r+1}}(x),$$

and put:

$$\beta_{m,r}(x)=\sum_{k=0}^{m}C_{k,2}U_k(x). \tag{59}$$

First, we note:

$$U_{m-j+r}^{(r+k)}(x)=\sum_{l=0}^{\left[\frac{m-j-k}{2}\right]}(-1)^l\binom{m-j+r-l}{l}(m-j+r-2l)_{r+k}2^{m-j+r-2l}x^{m-j-k-2l}. \tag{60}$$

Then, we have the following by exploiting (b) of Proposition 1, (57) and (60), and integration by parts k times.

$$
\begin{aligned}
C_{k,2} &= \frac{(-1)^k 2^{k+1}(k+1)!}{(2k+1)!\pi} \int_{-1}^{1} \beta_{m,r}(x) \frac{d^k}{dx^k}(1-x^2)^{k+\frac{1}{2}} dx \\
&= \frac{(-1)^k 2^{k+1}(k+1)!}{(2k+1)!\pi 2^r r!} \sum_{j=0}^{m} (-1)^j \binom{r+1}{j} \int_{-1}^{1} U_{m-j+r}^{(r)}(x) \frac{d^k}{dx^k}(1-x^2)^{k+\frac{1}{2}} dx \\
&= \frac{2^{k+1}(k+1)!}{(2k+1)!\pi 2^r r!} \sum_{j=0}^{m-k} (-1)^j \binom{r+1}{j} \int_{-1}^{1} U_{m-j+r}^{(r+k)}(x)(1-x^2)^{k+\frac{1}{2}} dx \\
&= \frac{2^{k+1-r}(k+1)!}{(2k+1)!\pi r!} \sum_{j=0}^{m-k} (-1)^j \binom{r+1}{j} \sum_{l=0}^{\left[\frac{m-j-k}{2}\right]} (-1)^l \binom{m-j+r-l}{l} \\
&\quad \times (m-j+r-2l)_{r+k} 2^{m-j+r-2l} \int_{-1}^{1} x^{m-j-k-2l}(1-x^2)^{k+\frac{1}{2}} dx
\end{aligned} \tag{61}
$$

where we note from (b) of Proposition 2 that:

$$
\begin{aligned}
&\int_{-1}^{1} x^{m-j-k-2l}(1-x^2)^{k+\frac{1}{2}} dx \\
&= \begin{cases} 0, & \text{if } j \not\equiv m-k \pmod 2, \\ \frac{(m-j-k-2l)!(2k+2)!\pi}{2^{m-j+k-2l+2}\left(\frac{m-j+k}{2}+1-l\right)!\left(\frac{m-j-k}{2}-l\right)!(k+1)!}, & \text{if } j \equiv m-k \pmod 2. \end{cases}
\end{aligned} \tag{62}
$$

From (59), (61) and (62), and after some simplifications, we obtain:

$$
\begin{aligned}
\beta_{m,r}(x) &= \frac{1}{r!} \sum_{k=0}^{m} \sum_{0 \le j \le m-k, j \equiv m-k (\mathrm{mod} 2)} \sum_{l=0}^{\left[\frac{m-k-j}{2}\right]} (-1)^j \binom{r+1}{j} (k+1) \\
&\quad \times \frac{(-1)^l (m-j+r-l)!}{l! \left(\frac{m-j+k}{2}+1-l\right)! \left(\frac{m-j-k}{2}-l\right)!} U_k(x) \\
&= \frac{1}{r!} \sum_{k=0}^{m} \sum_{s=0}^{\left[\frac{m-k}{2}\right]} \frac{(-1)^{m-k}(k+1)(k+2s+r)!}{(s+k+1)!s!} \binom{r+1}{m-k-2s} \\
&\quad \times \sum_{l=0}^{s} \frac{(-1)^l (s+k+1)_l (s)_l}{l!(k+2s+r)_l} U_k(x) \\
&= \frac{1}{r!} \sum_{k=0}^{m} \sum_{s=0}^{\left[\frac{m-k}{2}\right]} \frac{(-1)^{m-k}(k+1)(k+2s+r)!}{(s+k+1)!s!} \binom{r+1}{m-k-2s} \\
&\quad \times \sum_{l=0}^{s} \frac{< -s >_l < -s-k-1 >_l}{l! < -k-2s-r >_l} U_k(x) \\
&= \frac{1}{r!} \sum_{k=0}^{m} \sum_{s=0}^{\left[\frac{m-k}{2}\right]} \frac{(-1)^{m-k}(k+1)(k+2s+r)!}{(s+k+1)!s!} \binom{r+1}{m-k-2s} \\
&\quad \times {}_2F_1(-s,-s-k-1;-k-2s-r;1)\, U_k(x).
\end{aligned} \tag{63}
$$

This completes the proof for (34). Next, we let:

$$
\beta_{m,r}(x) = \sum_{k=0}^{m} C_{k,4} W_k(x). \tag{64}
$$

Then, from (d) of Proposition 1, (57) and (60), and integration by parts k times, we have:

$$C_{k,4}=\frac{k!2^{k-r}}{(2k)!\pi r!}\sum_{j=0}^{m-k}(-1)^j\binom{r+1}{j}\sum_{l=0}^{\left[\frac{m-j-k}{2}\right]}(-1)^l\binom{m-j+r-l}{l}$$
$$\times(m-j+r-2l)_{r+k}2^{m-j+r-2l}\int_{-1}^{1}x^{m-j-k-2l}(1-x)^{k+\frac{1}{2}}(1-x)^{k-\frac{1}{2}}dx \tag{65}$$

From (d) of Proposition 2, we observe that:

$$\begin{aligned}&\int_{-1}^{1}x^{m-j-k-2l}(1-x)^{k+\frac{1}{2}}(1-x)^{k-\frac{1}{2}}dx\\&=\begin{cases}-\frac{(m-j-k-2l+1)!(2k)!\pi}{2^{m-j+k-2l+1}\left(\frac{m-j+k+1}{2}-l\right)!\left(\frac{m-j-k+1}{2}-l\right)!k!}, & \text{if } j\not\equiv m-k \pmod 2,\\ \frac{(m-j-k-2l)!(2k)!\pi}{2^{m-j+k-2l}\left(\frac{m-j+k}{2}-l\right)!\left(\frac{m-j-k}{2}-l\right)!k!}, & \text{if } j\equiv m-k \pmod 2.\end{cases}\end{aligned} \tag{66}$$

By (64)–(66), and after some simplifications, we get:

$$\begin{aligned}\beta_{m,r}(x)&=-\frac{1}{2r!}\sum_{k=0}^{m}\sum_{0\le j\le m-k,j\not\equiv m-k(\mathrm{mod}2)}(-1)^j\binom{r+1}{j}\sum_{l=0}^{\left[\frac{m-j-k}{2}\right]}\\&\quad\times\frac{(-1)^l(m-j+r-l)!(m-j-k-2l+1)}{l!\left(\frac{m-j+k+1}{2}-l\right)!\left(\frac{m-j-k+1}{2}-l\right)!}W_k(x)\\&\quad+\frac{1}{r!}\sum_{k=0}^{m}\sum_{0\le j\le m-k,j\equiv m-k(\mathrm{mod}2)}(-1)^j\binom{r+1}{j}\sum_{l=0}^{\left[\frac{m-j-k}{2}\right]}\\&\quad\times\frac{(-1)^l(m-j+r-l)!}{l!\left(\frac{m-j+k}{2}-l\right)!\left(\frac{m-j-k}{2}-l\right)!}W_k(x)\\&=\frac{1}{r!}\sum_{k=0}^{m}\sum_{s=0}^{\left[\frac{m-k-1}{2}\right]}(-1)^{m-k}\binom{r+1}{m-k-2s-1}\frac{(k+2s+r+1)!}{(s+k+1)!s!}\\&\quad\times\sum_{l=0}^{s}\frac{(-1)^l(s+k+1)_l(s)_l}{l!(k+2s+r+1)_l}W_k(x)\\&\quad+\frac{1}{r!}\sum_{k=0}^{m}\sum_{s=0}^{\left[\frac{m-k}{2}\right]}(-1)^{m-k}\binom{r+1}{m-k-2s}\frac{(k+2s+r)!}{(s+k)!s!}\\&\quad\times\sum_{l=0}^{s}\frac{(-1)^l(s+k)_l(s)_l}{l!(k+2s+r)_l}W_k(x).\end{aligned} \tag{67}$$

Further modification of (67) gives us:

$$\begin{aligned}
\beta_{m,r}(x) &= \frac{1}{r!}\sum_{k=0}^{m}\sum_{s=0}^{\left[\frac{m-k-1}{2}\right]}(-1)^{m-k}\frac{(k+2s+r+1)!}{(s+k+1)!s!}\binom{r+1}{m-k-2s-1} \\
&\times {}_2F_1(-s,-s-k-1;-k-2s-r-1;1)W_k(x) \\
&+\frac{1}{r!}\sum_{k=0}^{m}\sum_{l=0}^{\left[\frac{m-k}{2}\right]}(-1)^{m-k}\frac{(k+2s+r)!}{(s+k)!s!}\binom{r+1}{m-k-2s} \\
&\times {}_2F_1(-s,-s-k;-k-2s-r;1)W_k(x) \\
&=\frac{1}{r!}\sum_{k=0}^{m}\sum_{s=0}^{m-k}\frac{(-1)^{m-k}(k+r+s)!}{\left(k+\left[\frac{s+1}{2}\right]\right)!\left[\frac{s}{2}\right]!}\binom{r+1}{m-k-s} \\
&\times {}_2F_1\left(-\left[\frac{s}{2}\right],-\left[\frac{s+1}{2}\right]-k;-k-s-r;1\right)W_k(x).
\end{aligned} \tag{68}$$

This finishes up the proof for (36).

Remark 1. *We note from (57) and (58) that the only difference between $\beta_{m,r}(x)$ and $\gamma_{m,r}(x)$ (see (24) and (25)) is the alternating sign $(-1)^j$ in their sums, which corresponds to $(-1)^{m-k}$ in (33)–(36). This remark gives the results in (37)–(40) of Theorem 3.*

4. Conclusions

Our paper can be viewed as a generalization of the linearization problem, which is concerned with determining the coefficients in the expansion $a_n(x)b_m(x) = \sum_{k=0}^{n+m} c_k(nm)p_k(x)$ of the product $a_n(x)b_m(x)$ of two polynomials $a_n(x)$ and $b_m(x)$ in terms of an arbitrary polynomial sequence $\{p_k(x)\}_{k\geq 0}$. Our pursuit of this line of research can also be justified from another fact; namely, the famous Faber–Pandharipande–Zagier and Miki identities follow by expressing the sum $\sum_{k=1}^{m-1}\frac{1}{k(m-k)}B_k(x)B_{m-k}(x)$ as a linear combination of Bernoulli polynomials. For some details on this, we let the reader refer to the Introduction of [15]. Here, we considered sums of finite products of the Chebyshev polynomials of the first, third, and fourth kinds and represented each of those sums of finite products as linear combinations of $T_n(x)$, $U_n(x)$, $V_n(x)$, and $W_n(x)$, which involve some terminating hypergeometric function ${}_2F_1$. Here, we remark that $\alpha_{m,r}(x)$, $\beta_{m,r}(x)$, and $\gamma_{m,r}(x)$ are expressed in terms of $U^{(r)}_{m-j+r}(x)$, $(j=0,1,\cdots,m)$ (see Lemmas 2 and 3) by making use of the generating function in (6). This is unlike the previous works for (26)–(28) (see [5,6]), where we showed they are respectively equal to $\frac{1}{2^{r-1}r!}T^{(r)}_{m+r}(x)$, $\frac{1}{2^r r!}V^{(r)}_{m+r}(x)$, and $\frac{1}{2^r r!}W^{(r)}_{m+r}(x)$ by exploiting the generating functions in (5), (7) and (8). Then, our results for $\alpha_{m,r}(x)$, $\beta_{m,r}(x)$, and $\gamma_{m,r}(x)$ were found by making use of Lemmas 1 and 2, the general formulas in Propositions 1 and 2, and integration by parts. It is certainly possible to represent such sums of finite products by other orthogonal polynomials, which is one of our ongoing projects. More generally, along the same line as the present paper, we are planning to consider some sums of finite products of many special polynomials and want to find their applications.

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