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Some Symmetric Identities Involving Fubini Polynomials and Euler Numbers

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Abstract: The aim of this paper is to use elementary methods and the recursive properties of a special sequence to study the computational problem of one kind symmetric sums involving Fubini polynomials and Euler numbers, and give an interesting computational formula for it. At the same time, we also give a recursive calculation method for the general case.

Keywords: Fubini polynomials; Euler numbers; symmetric identities; elementary method; computational formula

MSC: 11B83; 11B37

1. Introduction

For any integer $n \geq 0$, the Fubini polynomials $\{F_n(y)\}$ are defined by the coefficients of the generating function

$$\frac{1}{1 - y(e^t - 1)} = \sum_{n=0}^{\infty} \frac{F_n(y)}{n!} \cdot t^n, \quad (1)$$

where $F_0(y) = 1$, $F_1(y) = y$, and so on. $F_n(1) = F_n$ are called Fubini numbers. These polynomials and numbers are closely connected with the Stirling numbers. Some contents and properties of Stirling numbers can be found in reference [1]. T. Kim et al. [2] proved the identity

$$F_n(y) = \sum_{k=0}^n S_2(n, k) k! y^k, \quad (n \geq 0),$$

where $S_2(n, k)$ are the Stirling numbers of the second kind. It not only associated Fubini polynomials with Stirling numbers, but also stressed the importance of researching Fubini polynomials.

Please note that the identity (see [3,4])

$$\frac{2e^{tx}}{1 + e^t} = \sum_{n=0}^{\infty} \frac{E_n(x)}{n!} \cdot t^n, \quad (2)$$

where $E_n(x)$ signifies the Euler polynomials.

It is distinct that if taking $y = -\frac{1}{2}$ in (1) and $x = 0$ in (2), then from (1) and (2) we can get the identity

$$E_n(0) = F_n\left(-\frac{1}{2}\right), \quad n \in \mathbb{N}^* \quad (3)$$

where $E_n(0) = E_n$ is the Euler number (see [5] for related contents).

On the other hand, two variable Fubini polynomials are defined by means of the following (see [2,6])

$$\frac{e^{xt}}{1-y(e^t-1)} = \sum_{n=0}^{\infty} \frac{F_n(x,y)}{n!} \cdot t^n,$$

and $F_n(y) = F_n(0, y)$ for all integers $n \geq 0$. About the properties of $F_n(x, y)$, several scholars have also researched it, especially T. Kim and others have done a large amount of vital works. For instance, they proved a series of identities linked to $F_n(x, y)$ (see [2,7]), one of which is

$$F_n(x, y) = \sum_{l=0}^n \binom{n}{l} x^l \cdot F_{n-l}(y), n \in \mathbb{N}^*0.$$

These polynomials occupy indispensable positions in the theory and application of mathematics. In particular, they are widely used in combinatorial mathematics. Therefore, several scholars have researched their various properties, and acquired a series of vital results. Some involved contents can be found in references [5,7–17].

The goal of this paper is to use elementary methods and recursive properties of a special sequence to research the computational problem of the sums

$$\sum_{a_1+a_2+\dots+a_k=n} \frac{F_{a_1}(y)}{(a_1)!} \cdot \frac{F_{a_2}(y)}{(a_2)!} \dots \frac{F_{a_k}(y)}{(a_k)!}, \quad (4)$$

where the summation is over all k -tuples with non-negative integer coordinates (a_1, a_2, \dots, a_k) such that $a_1 + a_2 + \dots + a_k = n$.

About this content, it seems there is no valid method to solve the computational problem of (4). However, this problem is significant, it can reveal the structure of Fubini polynomials itself and its internal relations, at least it can reflect the combination properties of Fubini polynomials.

In this paper, we will take elementary methods and the properties of $F_n(y)$ to obtain a fascinating computational formula for (4). Simultaneously, we can also acquire a recursive calculation method for the general case. That is, we are going to prove the following major result:

Theorem 1. For any positive integers n and k , we have the identity

$$\begin{aligned} & \sum_{a_1+a_2+\dots+a_k=n} \frac{F_{a_1}(y)}{(a_1)!} \cdot \frac{F_{a_2}(y)}{(a_2)!} \dots \frac{F_{a_k}(y)}{(a_k)!} \\ &= \frac{1}{(k-1)!(y+1)^{k-1}} \cdot \frac{1}{n!} \sum_{i=0}^{k-1} C(k-1, i) F_{n+k-1-i}(y), \end{aligned}$$

where the sequence $\{C(k, i)\}$ is defined as follows: For any positive integer k and integers $0 \leq i \leq k$, we define $C(k, 0) = 1$, $C(k, k) = k!$ and

$$C(k+1, i+1) = C(k, i+1) + (k+1)C(k, i), \text{ for all } 0 \leq i < k,$$

providing $C(k, i) = 0$, if $i > k$.

The characteristic of this theorem is to represent a complex sum of Fubini polynomials as a linear combination of a single Fubini polynomial. Of course, our method can also be further generalized, provided a corresponding results for $F_n(x, y)$. It is just that its form is not so pretty, so we are not listing it here. If taking $k = 3, 4$ and 5 , then from our theorem we may instantly deduce the following several corollaries:

Corollary 1. For any positive integer n , we have the identity

$$\sum_{a+b+c=n} \frac{F_a(y)}{a!} \cdot \frac{F_b(y)}{b!} \cdot \frac{F_c(y)}{c!} = \frac{1}{2 \cdot n! \cdot (y+1)^2} (F_{n+2}(y) + 3F_{n+1}(y) + 2F_n(y)).$$

Corollary 2. For any positive integer n , we have the identity

$$\begin{aligned} & \sum_{a+b+c+d=n} \frac{F_a(y)}{a!} \cdot \frac{F_b(y)}{b!} \cdot \frac{F_c(y)}{c!} \cdot \frac{F_d(y)}{d!} \\ &= \frac{1}{6 \cdot n! \cdot (y+1)^3} (F_{n+3}(y) + 6F_{n+2}(y) + 11F_{n+1}(y) + 6F_n(y)). \end{aligned}$$

Corollary 3. For any positive integer n , we have the identity

$$\begin{aligned} & \sum_{a+b+c+d+e=n} \frac{F_a(y)}{a!} \cdot \frac{F_b(y)}{b!} \cdot \frac{F_c(y)}{c!} \cdot \frac{F_d(y)}{d!} \cdot \frac{F_e(y)}{e!} \\ &= \frac{1}{24 \cdot n! \cdot (y+1)^4} (F_{n+4}(y) + 10F_{n+3}(y) + 35F_{n+2}(y) + 50F_{n+1}(y) + 24F_n(y)). \end{aligned}$$

If taking $y = -\frac{1}{2}$ in our theorem, then from (3) we can also infer the following:

Corollary 4. For any positive integers n and $k \geq 2$, we have the identity

$$\sum_{a_1+a_2+\dots+a_k=n} \frac{E_{a_1}}{(a_1)!} \cdot \frac{E_{a_2}}{(a_2)!} \cdots \frac{E_{a_k}}{(a_k)!} = \frac{2^{k-1}}{(k-1)!} \cdot \frac{1}{n!} \sum_{i=0}^{k-1} C(k-1, i) E_{n+k-1-i}.$$

If $n = p$ is an odd prime, then taking $y = 1$ in Corollaries 1 and 2, we also have the following congruences.

Corollary 5. For any odd prime p , we have the congruence

$$22F_p \equiv F_{p+2} + 3F_{p+1} \pmod{p}.$$

Corollary 6. For any odd prime p , we have the congruence

$$186F_p \equiv F_{p+3} + 6F_{p+2} + 11F_{p+1} \pmod{p}.$$

2. A Simple Lemma

For purpose of proving our theorem, we need a uncomplicated lemma. As a matter of convenience, we first present a new sequence $\{C(k, i)\}$ as follows. For any positive integer k and integers $0 \leq i \leq k$, we define $C(k, 0) = 1$, $C(k, k) = k!$ and

$$C(k+1, i+1) = C(k, i+1) + (k+1)C(k, i), \quad 1 \leq i \leq k, \quad C(k, i) = 0, \text{ if } i > k.$$

For clarity, for $1 \leq k \leq 9$, we list values of $C(k, i)$ in the Table 1.

Table 1. Values of $C(k, i)$.

$C(k, i)$	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$	$i=8$	$i=9$
$k=1$	1	1								
$k=2$	1	3	2							
$k=3$	1	6	11	6						
$k=4$	1	10	35	50	24					
$k=5$	1	15	85	225	274	120				
$k=6$	1	21	175	735	1624	1764	720			
$k=7$	1	28	322	1960	6769	13,132	13,068	5040		
$k=8$	1	36	546	4536	22,449	67,284	118,124	109,584	40,320	
$k=9$	1	45	870	9450	63,273	269,325	723,680	1,172,700	1,026,576	362,880

Obviously, the values of $C(k, i)$ can be easily calculated by using a computer program. Hence, for any positive integer k , the computational problem of (4) can be solved fully.

In this table of numerical values, we also find that for prime $p = 3, 5$ and 7 , we have the congruence

$$C(p-1, i) \equiv 0 \pmod{p} \text{ for all } 1 \leq i \leq p-2.$$

For all prime $p > 7$ is true? This is an enjoyable open problem.

If this congruence is true, then we can also deduce that for any positive integer n and odd prime p , one has the congruence

$$F_{n+p-1}(y) + F_n(y) \equiv 0 \pmod{p}.$$

Now let function $f(t) = \frac{1}{1-y(e^t-1)}$. Then we have the following

Lemma 1. For any positive integer k , we have the identity

$$\sum_{i=0}^k C(k, i) f^{(k-i)}(t) = k!(y+1)^k f^{k+1}(t),$$

where $f^{(0)}(t) = f(t)$, $f^{(r)}(t)$ denotes the r -order derivative of $f(t)$ for variable t .

Proof. Now we prove this lemma by induction. From the definition of the derivative we acquire

$$f'(t) = \frac{ye^t}{(1-y(e^t-1))^2} = -f(t) + (y+1)f^2(t) \quad (5)$$

or

$$f'(t) + f(t) = (y+1)f^2(t). \quad (6)$$

Please note that $C(1, 0) = 1$ and $C(1, 1) = 1$, so the lemma is true for $k = 1$.

Suppose that the lemma is true for all integer $k \geq 1$. That is,

$$\sum_{i=0}^k C(k, i) f^{(k-i)}(t) = k!(y+1)^k f^{k+1}(t). \quad (7)$$

Then take the derivative for t in (7) and applying (5) and (7) we obtain

$$\begin{aligned} & \sum_{i=0}^k C(k, i) f^{(k+1-i)}(t) = (k+1)!(y+1)^k f^k(t) \cdot f'(t) \\ &= (k+1)!(y+1)^k f^k(t) \cdot (-f(t) + (y+1)f^2(t)) \\ &= (k+1)!(y+1)^{k+1} f^{k+2}(t) - (k+1)!(y+1)^k f^{k+1}(t) \\ &= (k+1)!(y+1)^{k+1} f^{k+2}(t) - (k+1) \left(\sum_{i=0}^k C(k, i) f^{(k-i)}(t) \right). \end{aligned} \quad (8)$$

It is evident that (8) implies

$$\begin{aligned}
 & (k+1)!(y+1)^{k+1}f^{k+2}(t) \\
 = & C(k,0)f^{(k+1)}(t) + \sum_{i=0}^{k-1} (C(k,i+1) + (k+1)C(k,i))f^{(k-i)}(t) + (k+1)!f(t) \\
 = & C(k,0)f^{(k+1)}(t) + \sum_{i=0}^{k-1} C(k+1,i+1)f^{(k-i)}(t) + (k+1)!f(t) \\
 = & \sum_{i=0}^{k+1} C(k+1,i)f^{(k+1-i)}(t),
 \end{aligned} \tag{9}$$

where we have used the identities $C(k,0) = 1$ and $C(k,k) = k!$. Now the lemma follows from (9) and mathematical induction. \square

3. Proof of the Theorem

In this section, the proof of our theorem will be completed. Firstly, for any positive integer k , from the definition of $f(t)$ and the properties of the power series we obtain

$$f^{(k)}(t) = \sum_{n=0}^{\infty} \frac{F_{n+k}(y)}{n!} \cdot t^n \tag{10}$$

and

$$\begin{aligned}
 f^k(t) &= \left(\sum_{a_1=0}^{\infty} \frac{F_{a_1}(y)}{a_1!} \cdot t^{a_1} \right) \left(\sum_{a_2=0}^{\infty} \frac{F_{a_2}(y)}{a_2!} \cdot t^{a_2} \right) \cdots \left(\sum_{a_k=0}^{\infty} \frac{F_{a_k}(y)}{a_k!} \cdot t^{a_k} \right) \\
 = & \left(\sum_{a_1=0}^{\infty} \sum_{a_2=0}^{\infty} \cdots \sum_{a_k=0}^{\infty} \frac{F_{a_1}(y)}{(a_1)!} \cdot \frac{F_{a_2}(y)}{(a_2)!} \cdots \frac{F_{a_k}(y)}{(a_k)!} \cdot t^{a_1+a_2+\cdots+a_k} \right) \\
 = & \sum_{n=0}^{\infty} \left(\sum_{a_1+a_2+\cdots+a_k=n} \frac{F_{a_1}(y)}{(a_1)!} \cdot \frac{F_{a_2}(y)}{(a_2)!} \cdots \frac{F_{a_k}(y)}{(a_k)!} \right) \cdot t^n.
 \end{aligned} \tag{11}$$

From (10), (11) and Lemma we acquire

$$\begin{aligned}
 & \frac{1}{(k-1)!(y+1)^{k-1}} \cdot \sum_{i=0}^{k-1} C(k-1,i) \sum_{n=0}^{\infty} \frac{F_{n+k-1-i}(y)}{n!} \cdot t^n \\
 = & \sum_{n=0}^{\infty} \left(\sum_{a_1+a_2+\cdots+a_k=n} \frac{F_{a_1}(y)}{(a_1)!} \cdot \frac{F_{a_2}(y)}{(a_2)!} \cdots \frac{F_{a_k}(y)}{(a_k)!} \right) \cdot t^n.
 \end{aligned} \tag{12}$$

Comparing the coefficients of t^n in (12) we have the identity

$$\begin{aligned}
 & \sum_{a_1+a_2+\cdots+a_k=n} \frac{F_{a_1}(y)}{(a_1)!} \cdot \frac{F_{a_2}(y)}{(a_2)!} \cdots \frac{F_{a_k}(y)}{(a_k)!} \\
 = & \frac{1}{(k-1)!(y+1)^{k-1}} \cdot \frac{1}{n!} \sum_{i=0}^{k-1} C(k-1,i) F_{n+k-1-i}(y).
 \end{aligned}$$

This completes the proof of our Theorem.

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