# On *p*-Adic Fermionic Integrals of *q*-Bernstein Polynomials Associated with *q*-Euler Numbers and Polynomials †

**Lee-Chae Jang [1], Taekyun Kim [2,*], Dae San Kim [3] and Dmitry Victorovich Dolgy [4,5]**

[1]    Graduate School of Education, Konkuk University, Seoul 143-701, Korea; Lcjang@konkuk.ac.kr
[2]    Department of Mathematics, Kwangwoon University, Seoul 139-701, Korea
[3]    Department of Mathematics, Sogang University, Seoul 121-742, Korea; dskim@sogang.ac.kr
[4]    Kwangwoon Institute for Advanced Studies, Kwangwoon University, Seoul 139-701, Korea; d_dol@mail.ru
[5]    Institute of National Sciences, Far Eastern Federal University, 690950 Vladivostok, Russia
*    Correspondence: tkkim@kw.ac.kr
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**Abstract:** We study a *q*-analogue of Euler numbers and polynomials naturally arising from the *p*-adic fermionic integrals on $\mathbb{Z}_p$ and investigate some properties for these numbers and polynomials. Then we will consider *p*-adic fermionic integrals on $\mathbb{Z}_p$ of the two variable *q*-Bernstein polynomials, recently introduced by Kim, and demonstrate that they can be written in terms of the *q*-analogues of Euler numbers. Further, from such *p*-adic integrals we will derive some identities for the *q*-analogues of Euler numbers.

**Keywords:** two variable *q*-Berstein polynomial; two variable *q*-Berstein operator; *q*-Euler number; *q*-Euler polynomial

## 1. Introduction

As is well known, the classical Bernstein polynomial of order *n* for $f \in C[0, 1]$ is defined by (see [1–3]),

$$\mathbb{B}_n(f|x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x), \quad 0 \leq x \leq 1, \tag{1}$$

where $\mathbb{B}_n$ is called the Bernstein operater of order *n*, and (see [4–30]),

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad n, k \geq 0, \tag{2}$$

are called the Bernstein basis polynomials (or Bernstein polynomials of degree *n*).

The Weierstrass approximation theorem states that every continuous function defined on $[0, 1]$ can be uniformly approximated as closely as desired by a polynomial function. In 1912, S. N. Bernstein explicitly constructed a sequence of polynomials that uniformly approximates any given continuous function *f* on $[0, 1]$. Namely, he showed that $\mathbb{B}_n(f|x)$ tends uniformly to $f(x)$ as $n \to \infty$ on $[0, 1]$ (see [3]). For $q \in \mathbb{C}$, with $0 < |q| < 1$, and $n, k \in \mathbb{Z}_{\geq 0}$, with $n \geq k$, the *q*-Bernstein polynomials of degree *n* are defined by Kim as (see [8])

$$B_{k,n}(x, q) = \binom{n}{k} [x]_q^k [1-x]_{\frac{1}{q}}^{n-k}, \tag{3}$$

where $[x]_q = \frac{1-q^x}{1-q}$. For any $f \in C[0,1]$, the $q$-Bernstein operator of order $n$ is defined as

$$\mathbb{B}_{n,q}(f|x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x,q) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} [x]_q^k [1-x]_{\frac{1}{q}}^{n-k}, \tag{4}$$

where $0 \leq x \leq 1$, and $n \in \mathbb{Z}_{\geq 0}$, (see [8,13]).

Here we note that a different version of $q$-Bernstein polynomials from Kim's was introduced earlier in 1997 by Phillips (see [22]). His $q$-Bernstein polynomial of order $n$ for $f$ is defined by

$$\mathbb{B}_n(f,q;x) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-1-k} (1 - q^s x),$$

where $f$ is a function defined on $[0,1]$, $q$ is any positive real number, and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \quad [n]_q! = [1]_q[2]_q \ldots [n]_q, (n \geq 1), \quad [0]_q! = 1.$$

The properties of Phillips' $q$-Bernstein polynomilas for $q \in (0,1)$ were treated for example in [6,15,16,22–24], while those for $q > 1$ were developed for instance in [17–20].

A Bernoulli trial is an experiment where only two outcomes, whether a particular event $A$ occurs or not, are possible. Flipping of coin is an example of Bernoulli trial, where only two outcomes, namely head and tail, are possible. Conventionally, it is said that the outcome of Bernoulli trial is a "success" if $A$ occurs and a "failure" otherwise. Let $P_n(k)$ denote the probability of $k$ successes in $n$ independent Bernoulli trials with the probability of success $r$. Then it is given by the binomial probability law

$$P_n(k) = \binom{n}{k} r^k (1-r)^{n-k}, \text{ for } k = 0, 1, 2, \cdots, n. \tag{5}$$

We remark here that the Bernstein basis is the probability mass function of the binomial distribution from the definition of Bernstein polynomials. Let $p$ be a fixed odd prime number. Throughout this paper, we will use the notations $\mathbb{Z}_p, \mathbb{Q}_p$, and $\mathbb{C}_p$ to denote respectively the ring of $p$-adic integers, the field of $p$-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. The $p$-adic norm in $\mathbb{C}_p$ is normalized in such a way that $|p|_p = \frac{1}{p}$. It is known that in terms of the recurrence relation the Euler numbers are given as follows (see [10,11]):

$$E_0 = 1, \ (E+1)^n + E_n = 2\delta_{0,n}, \tag{6}$$

where $\delta_{n,k}$ is the Kronecker's symbol. Then the Euler polynomials can be given as (see [10])

$$E_n(x) = \sum_{l=0}^{n} \binom{n}{l} E_l x^{n-l}, \tag{7}$$

The $q$-Euler polynomials, considered by L. Carlitz, are given by

$$\mathcal{E}_{0,q} = 1, \ q(q\mathcal{E}_q + 1)^n + \mathcal{E}_{n,q} = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases} \tag{8}$$

with the understanding that $\mathcal{E}_q^n$ is to be replaced by $\mathcal{E}_{n,q}$ (see [5]). Note that $\lim_{q \to 1} \mathcal{E}_{n,q} = E_n$, $(n \geq 0)$.

Let $f(x)$ be a continuous function on $\mathbb{Z}_p$. Then the $p$-adic fermionic integral on $\mathbb{Z}_p$ is defined by Kim as (see [12])

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \tag{9}$$

where we notice that $\mu_{-1}(x + p^N\mathbb{Z}_p) = (-1)^x$ is a measure.

From (9), we note that (see [12])

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0), \tag{10}$$

where $f_1(x) = f(x+1)$. By (10), we easily get (see [25])

$$\int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y) = E_n(x), \ (n \geq 0), \tag{11}$$

When $x = 0$, we note that $\int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = E_n$, $(n \geq 0)$. Let $q$ be an indeterminate in $\mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$. Taking (11) into consideration, we may investigate a $q$-analogue of Euler polynomials which are given by (see [12,26])

$$\int_{\mathbb{Z}_p} [x+y]_q^n d\mu_{-1}(y) = E_{n,q}(x), \ (n \geq 0), \tag{12}$$

When $x = 0$, $E_{n,q} = E_{n,q}(0)$, $(n \geq 0)$ are said to be the $q$-Euler numbers. Using (9), we can easily see that

$$\int_{\mathbb{Z}_p} [x]_q^n d\mu_{-1}(x) = \frac{2}{(1-q)^n} \sum_{l=0}^n \binom{n}{l}(-1)^l \frac{1}{1+q^l}$$
$$= 2\sum_{m=0}^\infty (-1)^m [m]_q^n, \ (n \geq 0). \tag{13}$$

Thus, by (13), we get

$$E_{n,q} = 2\sum_{m=0}^\infty (-1)^m [m]_q^n = \frac{2}{(1-q)^n} \sum_{l=0}^n \binom{n}{l}(-1)^l \frac{1}{1+q^l}. \tag{14}$$

For $n, k \geq 0$, with $n \geq k$, and $q \in \mathbb{C}_p$, with $|1-q|_p < p^{-\frac{1}{p-1}}$, we define the $p$-adic $q$-Bernstein polynomials as follows:

$$B_{k,n}(x,q) = \binom{n}{k}[x]_q^k[1-x]_{\frac{1}{q}}^{n-k}. \tag{15}$$

Then we consider the $p$-adic $q$-Bernstein operator defined for continuous functions $f$ on $\mathbb{Z}_p$ and given by

$$\mathbb{B}_{n,q}(f|x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) B_{k,n}(x,q), \ (x \in \mathbb{Z}_p). \tag{16}$$

We study a $q$-analogue of Euler numbers and polynomials naturally arising from the $p$-adic fermionic integrals on $\mathbb{Z}_p$ and investigate some properties for these numbers and polynomials. Then we will consider $p$-adic fermionic integrals on $\mathbb{Z}_p$ of the two variable $q$-Bernstein polynomials, recently introduced by Kim in [8], and demonstrate that they can be written in terms of the $q$-analogues of Euler numbers. Further, from such $p$-adic integrals we will derive some identities for the $q$-analogues of Euler numbers.

## 2. *q*-Bernstein Polynomials Associated with *q*-Euler Numbers and Polynomials

We assume that $q \in \mathbb{C}_p$, with $|1 - q|_p < p^{-\frac{1}{p-1}}$, throughout this section. From (12), we notice that

$$\sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!} = \sum_{m=0}^{\infty} (-1)^m e^{[m+x]_q t}.$$ (17)

By (10), we get

$$\int_{\mathbb{Z}_p} [x+1]_q^n d\mu_{-1}(x) + \int_{\mathbb{Z}_p} [x]_q^n d\mu_{-1}(x) = 2\delta_{0,n}, \ (n \geq 0).$$ (18)

Thus, from (12), we have

$$E_{n,q}(1) + E_{n,q} = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n > 0. \end{cases}$$ (19)

On the other hand,

$$\begin{aligned} E_{n,q}(x) &= \int_{\mathbb{Z}_p} [x+y]_q^n d\mu_{-1}(y) \\ &= \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{lx} \int_{\mathbb{Z}_p} [y]_q^l d\mu_{-1}(y) \\ &= \sum_{l=0}^{n} \binom{n}{l} q^{lx} E_{l,q} [x]_q^{n-l} = (q^x E_q + [x]_q)^n, \end{aligned}$$ (20)

with the understanding that $E_q^n$ is to be replaced by $E_{n,q}$. From (19) and (20), we note that

$$E_{0,q} = 1, \ (qE_q + 1)^n + E_{n,q} = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n > 0. \end{cases}$$ (21)

Now, we observe that

$$\begin{aligned} E_{n,q}(2) &= (q^2 E_q + 1 + q)^n = (q(qE_q + 1) + 1)^n \\ &= \sum_{l=0}^{n} q^l (qE_q + 1)^l \binom{n}{l} = 2 - E_{0,q} - \sum_{l=1}^{n} q^l E_{l,q} \binom{n}{l} \\ &= 2 - \sum_{l=0}^{n} q^l E_{l,q} \binom{n}{l} = 2 - (qE_q + 1)^n. \end{aligned}$$ (22)

Now, by combining (21) with (22), we have the following theorem.

**Theorem 1.** *For any $n \geq 0$, we have*

$$E_{n,q}(2) = 2 + E_{n,q}, \ (n > 0), \ E_{0,q}(2) = 1.$$ (23)

Invoking (9), we can derive the following equation

$$\int_{\mathbb{Z}_p} [1 - x + y]_{q^{-1}}^n d\mu_{-1}(y) = (-1)^n q^n \int_{\mathbb{Z}_p} [x+y]_q^n d\mu_{-1}(y),$$ (24)

where $n$ is a nonnegative integer. By (12) and (24), we get

$$E_{n,q^{-1}}(1-x) = (-1)^n q^n E_{n,q}(x), \ (n > 0).$$ (25)

On the other hand, we have

$$\int_{\mathbb{Z}_p} [1-x]_{q^{-1}}^n d\mu_{-1}(x) = (-1)^n q^n \int_{\mathbb{Z}_p} [x-1]_q^n d\mu_{-1}(x) \tag{26}$$
$$= (-1)^n q^n E_{n,q}(-1),$$

as $[-x]_{q^{-1}} = -q[x]_q$. By (25) and (26), we get

$$\int_{\mathbb{Z}_p} [1-x]_{q^{-1}}^n d\mu_{-1}(x) = (-1)^n q^n E_{n,q}(-1) = E_{n,q^{-1}}(2). \tag{27}$$

Therefore, by (23) and (27), we have

**Theorem 2.** *For any $n > 0$, we have*

$$\int_{\mathbb{Z}_p} [1-x]_{q^{-1}}^n d\mu_{-1}(x) = 2 + E_{n,q^{-1}}. \tag{28}$$

For $q \in \mathbb{C}_p$, with $|1-q|_p < p^{-\frac{1}{p-1}}$, and $x_1, x_1 \in \mathbb{Z}_p$, the two variable $q$-Bernstein polynomials are defined by

$$B_{k,n}(x_1, x_2|q) = \begin{cases} \binom{n}{k} [x_1]_q^k [1-x_2]_{q^{-1}}^{n-k}, & \text{if } n \geq k, \\ 0, & \text{if } n < k, \end{cases} \tag{29}$$

where $n, k \geq 0$. From (29), we note that

$$B_{n-k,n}(1-x_2, 1-x_1|q^{-1}) = B_{k,n}(x_1, x_2|q), \quad B_{k,n}(x, x|q) = B_{k,n}(x, q), \tag{30}$$

where $n, k \geq 0$ and $x_1, x_2 \in \mathbb{Z}_p$. For continuous functions $f$ on $\mathbb{Z}_p$, the two variable $q$-Bernstein operator of order $n$ is defined by

$$\mathbb{B}_{n,q}(f|x_1, x_2) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} [x_1]_q^k [1-x_2]_{q^{-1}}^{n-k} \tag{31}$$
$$= \sum_{k=0}^n f\left(\frac{k}{n}\right) B_{k,n}(x_1, x_2|q),$$

where $n, k \in \mathbb{Z}_{\geq 0}$, and $x_1, x_2 \in \mathbb{Z}_p$. In particular, if $f = 1$, then we have

$$\mathbb{B}_{n,q}(1|x_1, x_2) = \sum_{k=0}^n \binom{n}{k} [x_1]_q^k [1-x_2]_{q^{-1}}^{n-k} \tag{32}$$
$$= (1 + [x_1]_q - [x_2]_q)^n,$$

where we used the fact

$$[1-x]_{q^{-1}} = 1 - [x]_q. \tag{33}$$

Taking the double $p$-adic fermionic integral on $\mathbb{Z}_p$ as in the following, we have

$$\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} B_{k,n}(x_1, x_2|q) d\mu_{-1}(x_1) d\mu_{-1}(x_2)$$

$$= \binom{n}{k} \int_{\mathbb{Z}_p} [x_1]_q^k d\mu_{-1}(x_1) \int_{\mathbb{Z}_p} [1 - x_2]_{q^{-1}}^{n-k} d\mu_{-1}(x_2) \tag{34}$$

$$= \begin{cases} \binom{n}{k} E_{k,q}(2 + E_{n-k,q^{-1}}), & \text{if } n > k, \\ E_{k,q}, & \text{if } n = k. \end{cases}$$

Therefore, from (34) we obtain the next theorem.

**Theorem 3.** *For any* $n, k \in \mathbb{Z}_{\geq 0}$, *with* $n \geq k$, *we have*

$$\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} B_{k,n}(x_1, x_2|q) d\mu_{-1}(x_1) d\mu_{-1}(x_2)$$

$$= \begin{cases} \binom{n}{k} E_{n,q}(2 + E_{n,q^{-1}}), & \text{if } n > k, \\ E_{k,q}, & \text{if } n = k. \end{cases} \tag{35}$$

Making the use of the definition of the two variable $q$-Bernstein polynomials and from (33), we notice that

$$\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} B_{k,n}(x_1, x_2|q) d\mu_{-1}(x_1) d\mu_{-1}(x_2)$$

$$= \sum_{l=0}^{k} \binom{n}{n-k} \binom{k}{l} (-1)^{k+l} \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} [1 - x_1]_{q^{-1}}^{k-l} [1 - x_2]_{q^{-1}}^{n-k} d\mu_{-1}(x_1) d\mu_{-1}(x_2)$$

$$= \binom{n}{k} \int_{\mathbb{Z}_p} [1 - x_2]_{q^{-1}}^{n-k} d\mu_{-1}(x_2) \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} [1 - x_1]_{q^{-1}}^{k-l} d\mu_{-1}(x_1) \tag{36}$$

$$= \binom{n}{k} \int_{\mathbb{Z}_p} [1 - x_2]_{q^{-1}}^{n-k} d\mu_{-1}(x_2) \left\{ 1 + \sum_{l=0}^{k-1} \binom{k}{l} (2 + E_{k-l}, q^{-1}) \right\}.$$

Therefore, from (34) and (36) we deduce the following theorem.

**Theorem 4.** *For any* $k \geq 0$, *we have*

$$E_{k,q} = 2(2^k - 1) + \sum_{l=0}^{k} \binom{k}{l} E_{k-l,q^{-1}}. \tag{37}$$

For $m, n, k \in \mathbb{Z}_{\geq 0}$, we observe that

$$\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} B_{k,n}(x_1, x_2|q) B_{k,m}(x_1, x_2|q) d\mu_{-1}(x_1) d\mu_{-1}(x_2)$$

$$= \binom{n}{k} \binom{m}{k} \int_{\mathbb{Z}_p} [x_1]_q^{2k} d\mu_{-1}(x_1) \int_{\mathbb{Z}_p} [1 - x_2]_{q^{-1}}^{n+m-2k} d\mu_{-1}(x_2) \tag{38}$$

$$= \binom{n}{k} \binom{m}{k} E_{2k,q} \int_{\mathbb{Z}_p} [1 - x_2]_{q^{-1}}^{n+m-2k} d\mu_{-1}(x_2).$$

On the other hand,

$$\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} B_{k,n}(x_1, x_2|q) B_{k,m}(x_1, x_2|q) d\mu_{-1}(x_1) d\mu_{-1}(x_2)$$

$$= \sum_{l=0}^{2k} \binom{n}{k} \binom{m}{k} \binom{2k}{l} (-1)^{2k-l}$$

$$\times \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} [1-x_1]_{q^{-1}}^{2k-l} [1-x_2]_{q^{-1}}^{n+m-2k} d\mu_{-1}(x_1) d\mu_{-1}(x_2) \tag{39}$$

$$= \binom{n}{k} \binom{m}{k} \int_{\mathbb{Z}_p} [1-x_2]_{q^{-1}}^{n+m-2k} d\mu_{-1}(x_2)$$

$$\times \left\{ 1 + \sum_{l=0}^{2k-1} \binom{2k}{l} (-1)^{2k-l} \int_{\mathbb{Z}_p} [1-x_1]_{q^{-1}}^{2k-l} d\mu_{-1}(x_1) \right\}.$$

Hence, by (28), (38) and (39), we arrive at the following theorem.

**Theorem 5.** *For any $k \in \mathbb{N}$, we have*

$$E_{2k,q} = -2 + \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k-l} E_{2k-l,q^{-1}}. \tag{40}$$

Let $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_{\geq 0}$, with $s \in \mathbb{N}$. Then we clearly have

$$\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \prod_{i=1}^{s} B_{k,n_i}(x_1, x_2|q) d\mu_{-1}(x_1) d\mu_{-1}(x_2)$$

$$= \prod_{i=1}^{s} \binom{n_i}{k} \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} [x_1]_q^{sk} [1-x_2]_{q^{-1}}^{n_1+\cdots+n_s-sk} (x_1) d\mu_{-1}(x_2) \tag{41}$$

$$= \prod_{i=1}^{s} \binom{n_i}{k} E_{sk,q} \int_{\mathbb{Z}_p} [1-x_2]_{q^{-1}}^{n_1+\cdots+n_s-sk} d\mu_{-1}(x_2).$$

On the other hand,

$$\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \prod_{i=1}^{s} B_{k,n_i}(x_1, x_2|q) d\mu_{-1}(x_1) d\mu_{-1}(x_2)$$

$$= \sum_{l=0}^{sk} \prod_{i=1}^{s} \binom{n_i}{k} \binom{sk}{l} (-1)^{sk-l} \tag{42}$$

$$\times \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} [1-x_1]_{q^{-1}}^{sk-l} [1-x_2]_{q^{-1}}^{n_1+\cdots+n_s-sk} d\mu_{-1}(x_1) d\mu_{-1}(x_2).$$

By (41) and (42), we get

$$E_{sk,q} = \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk-l} \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} [1-x_1]_{q^{-1}}^{sk-l} d\mu_{-1}(x_1)$$

$$= 1 + \sum_{l=0}^{sk-1} \binom{sk}{l} (-1)^{sk-l} \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} [1-x_1]_{q^{-1}}^{sk-l} d\mu_{-1}(x_1). \tag{43}$$

Hence (28) and (43) together yield the next theorem.

**Theorem 6.** *For any $s \in \mathbb{N}$, we have*

$$E_{sk,q} = -2 + \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk-l} E_{sk-l,q^{-1}}. \tag{44}$$

## 3. Conclusions

In the previous paper [8], the $q$-Bernstein polynomials were introduced as a generalization of the classical Bernstein polynomials. Here we studied some properties of a $q$-analogue of Euler numbers and polynomials arising from the $p$-adic fermionic integrals on $\mathbb{Z}_p$. Then we considered $p$-adic fermionic integrals on $\mathbb{Z}_p$ of the two variable $q$-Bernstein polynomials, recently introduced by Kim, and show that they can be expressed in terms of the $q$-analogues of Euler numbers. Along the same line, we can introduce a new $q$-Bernoulli numbers and polynomials, different from the classical Carlitz $q$-Bernoulli numbers $\beta_{n,q}$ and polynomials $\beta_{n,q}(x)$, by considering the Volkenborn integrals in lieu of the $p$-adic fermionic integrals on $\mathbb{Z}_p$. Then we may investigate Volkenborn integrals on $\mathbb{Z}_p$ of the $q$-Bernstein polynomials and unveil their connections with those new $q$-Bernoulli numbers which is our ongoing project.

## References

1.  Açikgöz, M.; Erdal, D.; Araci, S. A new approach to $q$-Bernoulli numbers and $q$-Bernoulli polynomials related to $q$-Bernstein polynomials. *Adv. Differ. Equ.* **2010**, 2015, 272. [CrossRef]
2.  Araci, S.; Acikgoz, M. A note on the Frobenius-Euler numbers and polynomials associated with Bernstein polynomials. *Adv. Stud. Contemp. Math. (Kyungshang)* **2012**, *22*, 399–406.
3.  Bernstein, S. Démonstration du théorème de Weierstrass fondée sur le calcul des probabilities. *Commun. Soc. Math. Kharkov* **1912**, *2*, 1–2.
4.  Bayad, A.; Kim, T. Identities involving values of Bernstein, $q$-Bernoulli, and $q$-Euler polynomials. *Russ. J. Math. Phys.* **2011**, *18*, 133–143. [CrossRef]
5.  Carlitz, L. Degenerate Stirling, Bernoulli and Eulerian numbers. *Utilitas Math.* **1979**, *15*, 51–88.
6.  Goodman, T.N.T.; Oruç, H.; Phillips, G.M. Convexity and generalized Bernstein polynomials. *Proc. Edinb. Math. Soc.* **1999**, *42*, 179–190. [CrossRef]
7.  Kim, T.; Choi, J.; Kim, Y.-H. On the k-dimensional generalization of $q$-Bernstein polynomials. *Proc. Jangjeon Math. Soc.* **2011**, *14*, 199–207.
8.  Kim, T. A note on $q$-Bernstein polynomials. *Russ. J. Math. Phys.* **2011**, *18*, 73–82. [CrossRef]
9.  Kim, T.; Choi, J.; Kim, Y.-H. Some identities on the $q$-Bernstein polynomials, $q$-Stirling numbers and $q$-Bernoulli numbers. *Adv. Stud. Contemp. Math. (Kyungshang)* **2010**, *20*, 335–341.
10. Kim, T.; Lee, B.; Choi, J.; Kim, Y.-H.; Rim, S.H. On the $q$-Euler numbers and weighted $q$-Bernstein polynomials. *Adv. Stud. Contemp. Math. (Kyungshang)* **2011**, *21*, 13–18.
11. Kim, T. Identities on the weighted $q$-Euler numbers and $q$-Bernstein polynomials. *Adv. Stud. Contemp. Math. (Kyungshang)* **2012**, *22*, 7–12. [CrossRef]
12. Kim, T. A study on the $q$-Euler numbers and the fermionic $q$-integral of the product of several type $q$-Bernstein polynomials on $\mathbb{Z}_p$. *Adv. Stud. Contemp. Math. (Kyungshang)* **2013**, *23*, 5–11.
13. Kim, T.; Ryoo, C. S.; Yi, H. A note on $q$-Bernoulli numbers and $q$-Bernstein polynomials. *Ars Comb.* **2012**, *104*, 437–447.
14. Kurt, V. Some relation between the Bernstein polynomials and second kind Bernoulli polynomials. *Adv. Stud. Contemp. Math. (Kyungshang)* **2013**, *23*, 43–48.
15. Oruç, H.; Phillips, G.M. A generalization of Bernstein polynomials. *Proc. Edinb. Math. Soc.* **1999**, *42*, 403–413. [CrossRef]
16. Oruç, H.; Tuncer, N. On the convergence and iterates of $q$-Bernstein polynomials. *J. Approx. Theory* **2002**, *117*, 301–313. [CrossRef]
17. Ostrovska, S. $q$-Bernstein polynomials and their iterates *J. Approx. Theory* **2003** *123*, 232–255. [CrossRef]
18. Ostrovska, S. On the $q$-Bernstein polynomials. *Adv. Stud. Contemp. Math. (Kyungshang)* **2005**, *11*, 193–204.

19. Ostrovska, S. On the *q*-Bernstein polynomials of the logarithmic function in the case *q* > 1. *Math. Slovaca* **2016**, *66*, 73–78. [CrossRef]

20. Ostrovska, S. On the approximation of analytic functions by the *q*-Bernstein polynomials in the case *q* > 1. *Electron. Trans. Numer. Anal.* **2010**, *37*, 105–112.

21. Park, J.-W.; Pak, H.K.; Rim, S.-H.; Kim, T.; Lee, S.-H. A note on the *q*-Bernoulli numbers and *q*-Bernstein polynomials. *J. Comput. Anal. Appl.* **2013**, *15*, 722–729.

22. Phillips, G.M. Bernstein polynomials based on the *q*-integers. *Ann. Numer. Math.* **1997**, *4*, 511–518.

23. Phillips, G.M. A de Casteljau algorithm for generalized Bernstein polynomials. *BIT* **1997**, *37*, 232–236. [CrossRef]

24. Phillips, G.M. A generalization of the Bernstein polynomials based on the *q*-integers. *ANZIAM J.* **2000**, *42*, 79–86. [CrossRef]

25. Rim, S.-H.; Joung, J.; Jin, J.-H.; Lee, S.-J. A note on the weighted Carlitz's type *q*-Euler numbers and *q*-Bernstein polynomials. *Proc. Jangjeon Math. Soc.* **2012**, *15*, 195–201.

26. Ryoo, C.S. Some identities of the twisted *q*-Euler numbers and polynomials associated with *q*-Bernstein polynomials. *Proc. Jangjeon Math. Soc.* **2011**, *14*, 239–248.

27. Simsek, Y.; Gunay, M. On Bernstein type polynomials and their applications. *Adv. Differ. Equ.* **2015**, *2015*, 79. [CrossRef]

28. Siddiqui, M.A.; Agrawal, R.R.; Gupta, N. On a class of modified new Bernstein operators. *Adv. Stud. Contemp. Math. (Kyungshang)* **2014**, *24*, 97–107.

29. Srivastava, H.M.; Mursaleen, M.; Alotaibi, A.M.; Nasiruzzaman, M.; Al-Abied, A.A.H. Some approximation results involving the *q*-Szász-Mirakyan-Kantorovich type operators via Dunkl's generalization. *Math. Methods Appl. Sci.* **2017**, *40*, 5437–5452. [CrossRef]

30. Tunc, T.; Simsek, E. Some approximation properties of Szász-Mirakjan-Bernstein operators. *Eur. J. Pure Appl. Math.* **2014**, *7*, 419–428.