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On the Catalan Numbers and Some of Their Identities

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Received: 5 December 2018 ; Accepted: 4 January 2019; Published: 8 January 2019

Abstract: The main purpose of this paper is using the elementary and combinatorial methods to study the properties of the Catalan numbers, and give two new identities for them. In order to do this, we first introduce two new recursive sequences, then with the help of these sequences, we obtained the identities for the convolution involving the Catalan numbers.

Keywords: catalan numbers; elementary and combinatorial methods; recursive sequence; convolution sums

JEL Classification: 11B83; 11B75

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1. Introduction

For any non-negative integer n, the famous Catalan numbers C_n are defined as $C_n = \frac{1}{n+1} \cdot \binom{2n}{n}$. For example, the first several Catalan numbers are $C_0 = 1$, $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, $C_4 = 14$, $C_5 = 42$, $C_6 = 132$, $C_7 = 429$, $C_8 = 1430$, $\cdots$. The Catalan numbers C_n satisfy the recursive formula

$$C_n = \sum_{i=1}^{n} C_{i-1} \cdot C_i.$$

The generating function of the Catalan numbers C_n is

$$\frac{2}{1+\sqrt{1-4x}} = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{n+1} \cdot x^n = \sum_{n=0}^{\infty} C_n \cdot x^n. \tag{1}$$

These numbers occupy a pivotal position in combinatorial mathematics, as many counting problems are closely related to Catalan numbers, and some famous examples can be found in R. P. Stanley [1]. Many papers related to the Catalan numbers and other special sequences can also be found in references [1–20], especially the works of T. Kim et al. give a series of new identities for the Catalan numbers, see [9–14], these are important results in the related field.

The main purpose of this paper is to consider the calculating problem of the following convolution sums involving the Catalan numbers:

$$\sum_{a_1+a_2+\cdots+a_h=n} C_{a_1} \cdot C_{a_2} \cdot C_{a_3} \cdots C_{a_h}, \tag{2}$$

where the summation is taken over all h-dimension non-negative integer coordinates $(a_1, a_2, \cdots, a_h)$ such that the equation $a_1 + a_2 + \cdots + a_h = n$.

About the convolution sums (2), it seems that none had studied it yet, at least we have not seen any related results before. We think this problem is meaningful. The reason is based on the following two aspects: First, it can reveal the profound properties of the Catalan numbers themselves. Second, for the other sequences, such as Fibonacci numbers, Fubini numbers, and Euler numbers, etc. (see [21–23]), there are corresponding results, so the Catalan numbers should have a corresponding identity. In this paper, we use the elementary and combinatorial methods to answer this question. That is, we shall prove the following:

Theorem 1. *For any positive integer h, we have the identity*

$$\sum_{a_1+a_2+\cdots+a_{2h+1}=n} C_{a_1}\cdot C_{a_2}\cdot C_{a_3}\cdots C_{a_{2h+1}}$$
$$=\frac{1}{(2h)!}\sum_{i=0}^{h} C(h,i)\sum_{j=0}^{\min(n,i)}\frac{(n-j+h+i)!\cdot C_{n-j+h+i}}{(n-j)!}\cdot\binom{i}{j}\cdot(-4)^j,$$

where $C(h,i)$ are defined as $C(1,0)=-2$, $C(h,h)=1$, $C(h+1,h)=C(h,h-1)-(8h+2)\cdot C(h,h)$, $C(h+1,0)=8\cdot C(h,1)-2\cdot C(h,0)$, and for all integers $1\le i\le h-1$, we have the recursive formula

$$C(h+1,i)=C(h,i-1)-(8i+2)\cdot C(h,i)+(4i+4)(4i+2)\cdot C(h,i+1).$$

Theorem 2. *For any positive integer h and non-negative n, we have*

$$\sum_{a_1+a_2+\cdots+a_{2h}=n} C_{a_1}\cdot C_{a_2}\cdot C_{a_3}\cdots C_{a_{2h}}$$
$$=\frac{1}{(2h-1)!}\sum_{i=0}^{h-1}\sum_{j=0}^{n} D(h,i+1)\cdot\binom{i+\frac{1}{2}}{j}\cdot(-4)^j\cdot\frac{(n-j+h+i)!\cdot C_{n-j+h+i}}{(n-j)!},$$

where $\binom{n+\frac{1}{2}}{i}=\left(n+\frac{1}{2}\right)\cdot\left(n-1+\frac{1}{2}\right)\cdots\left(n-i+1+\frac{1}{2}\right)/i!$, $D(k,i)$ are defined as $D(k,0)=0$, $D(k,k)=1$, $D(k+1,k)=D(k,k-1)-(8k-2)$, $D(k+1,1)=24D(k,2)-6D(k,1)$, and for all integers $1\le i\le k-1$,

$$D(k+1,i)=D(k,i-1)-(8i-2)\cdot D(k,i)+4i(4i+2)\cdot D(k,i+1).$$

To better illustrate the sequence $\{C(k,i)\}$ and $D(h,i)$, we compute them using mathematical software and list some values in the following Tables 1 and 2.

Table 1. Values of $C(k,i)$.

$C(k,i)$	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
$k=1$	−2	1					
$k=2$	12	−12	1				
$k=3$	−120	180	−30	1			
$k=4$	1680	−3360	840	−56	1		
$k=5$	−30,240	75,600	−25,200	2520	−90	1	
$k=6$	665,280	−1,995,840	831,600	−110,880	5940	−132	1

Table 2. Values of $D(k,i)$.

$D(k,i)$	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
$k=1$	0	1					
$k=2$	0	−6	1				
$k=3$	0	60	−20	1			
$k=4$	0	−840	420	−42	1		
$k=5$	0	15,120	−10,080	1512	−72	1	
$k=6$	0	−332,640	277,200	−55,440	3960	−110	1

Observing these two tables, we can easily find that if $2k-1=p$ is a prime, then for all integers $0 \leq i < k$, we have the congruences $C(k,i) \equiv 0 \bmod (2k-1)(2k)$ and $D(k,i) \equiv 0 \bmod (2k-1)(2k-2)$. So we propose the following two conjectures:

Conjecture 1. *Let p be a prime. Then for any integer $0 \leq i < \frac{p+1}{2}$, we have the congruence*

$$C\left(\frac{p+1}{2}, i\right) \equiv 0 \bmod p(p+1).$$

Conjecture 2. *Let p be a prime. Then for any integer $0 \leq i < \frac{p+1}{2}$, we have the congruence*

$$D\left(\frac{p+1}{2}, i\right) \equiv 0 \bmod p(p-1).$$

For some special integers n and h, from Theorem 1 and Theorem 2 we can also deduce several interesting corollaries. In fact if we take $n=0$ and $h=1$ in the theorems respectively, then we have the following four corollaries:

Corollary 1. *For any positive integer h, we have the identity*

$$\sum_{i=0}^{h} C(h,i) \cdot (h+i)! \cdot C_{h+i} = (2h)!.$$

Corollary 2. *For any positive integer h, we have the identity*

$$\sum_{i=1}^{h} D(h,i) \cdot (h+i-1)! \cdot C_{h+i-1} = (2h-1)!.$$

Corollary 3. *For any integer $n \geq 0$, we have the identity*

$$\sum_{a+b+d=n} C_a \cdot C_b \cdot C_d = (n+1) \cdot \left[\frac{1}{2} \cdot (n+2) \cdot C_{n+2} - (2n+1) \cdot C_{n+1}\right].$$

Corollary 4. *For any integer $n \geq 0$, we have the identity*

$$\sum_{u+v+w+x+y=n} C_u \cdot C_v \cdot C_w \cdot C_x \cdot C_y = \frac{(n+1)(n+2)(4n^2+8n+3)}{6} \cdot C_{n+2}$$
$$- \frac{(n+3)(n+2)(n+1)(2n+3)}{6} \cdot C_{n+3} + \frac{(n+4)(n+3)(n+2)(n+1)}{24} \cdot C_{n+4}.$$

2. Several Simple Lemmas

To prove our theorems, we need following four simple lemmas. First we have:

Lemma 1. *Let function $f(x) = \frac{2}{1+\sqrt{1-4x}}$. Then for any positive integer h, we have the identity*

$$(2h)! \cdot f^{2h+1}(x) = \sum_{i=0}^{h} C(h,i) \cdot (1-4x)^i \cdot f^{(h+i)}(x),$$

where $f^{(i)}(x)$ denotes the i-order derivative of $f(x)$ for x, and $\{C(h,i)\}$ are defined as the same as in Theorem 1.

Proof. In fact, this identity and its generalization had appeared in D. S. Kim and T. Kim's important work [9] (see Theorem 3.1), but only in different forms. For the completeness of our results, here we give a different proof by mathematical induction. First from the properties of the derivative we have

$$f'(x) \quad = \quad \frac{4}{(1+\sqrt{1-4x})^2} \cdot \frac{1}{\sqrt{1-4x}} = \frac{f^2(x)}{\sqrt{1-4x}}$$

or identity

$$f^2(x) = (1-4x)^{\frac{1}{2}} \cdot f'(x). \tag{3}$$

From (3) and note that $C(1,0) = -2$ and $C(1,1) = 1$ we have

$$2f(x) \cdot f'(x) = -2\,(1-4x)^{-\frac{1}{2}} \cdot f'(x) + (1-4x)^{\frac{1}{2}} \cdot f''(x)$$

and

$$2!f^3(x) = -2f'(x) + (1-4x) \cdot f''(x) = \sum_{i=0}^{1} C(1,i) \cdot (1-4x)^i \cdot f^{(1+i)}(x).$$

That is, Lemma 1 is true for $h = 1$.

Assume that Lemma 1 is true for $h = k \geq 1$. That is,

$$(2k)! \cdot f^{2k+1}(x) = \sum_{i=0}^{k} C(k,i) \cdot (1-4x)^i \cdot f^{(k+i)}(x). \tag{4}$$

Then from (3), (4), the definition of $C(k,i)$, and the properties of the derivative we can deduce that

$$\begin{aligned} (2k+1)! \cdot f^{2k}(x) \cdot f'(x) \quad &= \quad \sum_{i=0}^{k} C(k,i) \cdot (1-4x)^i \cdot f^{(k+i+1)}(x) \\ &\quad - \sum_{i=1}^{k} 4i \cdot C(k,i) \cdot (1-4x)^{i-1} \cdot f^{(k+i)}(x) \end{aligned}$$

or

$$\begin{aligned} rrl(2k+1)! \cdot f^{2k+2}(x) = \qquad & \sum_{i=0}^{k} C(k,i) \cdot (1-4x)^{i+\frac{1}{2}} \cdot f^{(k+i+1)}(x) \\ & - \sum_{i=1}^{k} 4i \cdot C(k,i) \cdot (1-4x)^{i-\frac{1}{2}} \cdot f^{(k+i)}(x). \end{aligned} \tag{5}$$

Applying (5) and the properties of the derivative we also have

$$\begin{aligned} &(2k+2)! \cdot f^{2k+1}(x) \cdot f'(x) = \sum_{i=0}^{k} C(k,i) \cdot (1-4x)^{i+\frac{1}{2}} \cdot f^{(k+i+2)}(x) \\ &- \sum_{i=0}^{k} (4i+2) \cdot C(k,i) \cdot (1-4x)^{i-\frac{1}{2}} \cdot f^{(k+i+1)}(x) \\ &- \sum_{i=1}^{k} 4i \cdot C(k,i) \cdot (1-4x)^{i-\frac{1}{2}} \cdot f^{(k+i+1)}(x) \\ &+ \sum_{i=1}^{k} (4i) \cdot (4i-2) \cdot C(k,i) \cdot (1-4x)^{i-\frac{3}{2}} \cdot f^{(k+i)}(x) \end{aligned}$$

or note that identity (3) we have

$$
\begin{aligned}
& (2k+2)! \cdot f^{2k+3}(x) = \sum_{i=0}^{k} C(k,i) \cdot (1-4x)^{i+1} \cdot f^{(k+i+2)}(x) \\
& - \sum_{i=0}^{k} (4i+2) \cdot C(k,i) \cdot (1-4x)^{i} \cdot f^{(k+i+1)}(x) \\
& - \sum_{i=1}^{k} 4i \cdot C(k,i) \cdot (1-4x)^{i} \cdot f^{(k+i+1)}(x) \\
& + \sum_{i=1}^{k} (4i) \cdot (4i-2) \cdot C(k,i) \cdot (1-4x)^{i-1} \cdot f^{(k+i)}(x) \\
= \; & C(k,k) \cdot (1-4x)^{k+1} \cdot f^{(2k+2)}(x) + \sum_{i=1}^{k} C(k,i-1) \cdot (1-4x)^{i} \cdot f^{(k+i+1)}(x) \\
& -2C(k,0) \cdot f^{(k+1)}(x) - \sum_{i=1}^{k} (4i+2) \cdot C(k,i) \cdot (1-4x)^{i} \cdot f^{(k+i+1)}(x) \qquad (6) \\
& - \sum_{i=1}^{k} 4i \cdot C(k,i) \cdot (1-4x)^{i} \cdot f^{(k+i+1)}(x) + 8 \cdot C(k,1) \cdot f^{(k+1)}(x) \\
& + \sum_{i=1}^{k-1} (4i+4) \cdot (4i+2) \cdot C(k,i+1) \cdot (1-4x)^{i} \cdot f^{(k+i+1)}(x) \\
= \; & (1-4x)^{k+1} \cdot f^{(2k+2)}(x) + (8 \cdot C(k,1) - 2 \cdot C(k,0)) \cdot f^{(k+1)}(x) \\
& + (C(k,k-1) - (8k+2) \cdot C(k,k)) \cdot (1-4x)^{k} \cdot f^{(2k+1)}(x) \\
& + \sum_{i=1}^{k-1} (C(k,i-1) - (8i+2) \cdot C(k,i) + (4i+4)(4i+2) \cdot C(k,i+1)) \\
& \times (1-4x)^{i} \cdot f^{(k+i+1)}(x) \\
= \; & \sum_{i=0}^{k+1} C(k+1,i) \cdot (1-4x)^{i} \cdot f^{(k+i+1)}(x),
\end{aligned}
$$

where we have used the identities $C(k+1,k) = C(k,k-1) - (8k+2) \cdot C(k,k)$, $C(k,k) = 1$, $C(k+1,0) = 8 \cdot C(k,1) - 2 \cdot C(k,0)$ and for all integers $1 \le i \le k-1$,

$$C(k+1,i) = C(k,i-1) - (8i+2) \cdot C(k,i) + (4i+4)(4i+2) \cdot C(k,i+1).$$

It is clear that (6) implies Lemma 1 is true for $h = k+1$.

This proves Lemma 1 by mathematical induction. □

Lemma 2. *For any positive integer h, we have the identity*

$$(2h-1)! \cdot f^{2h}(x) = \sum_{i=0}^{h-1} D(h,i+1) \cdot (1-4x)^{i+\frac{1}{2}} \cdot f^{(h+i)}(x),$$

where $D(h,i)$ are defined as the same as in Theorem 2.

Proof. It is clear that using the methods of proving Lemma 1 we can easily deduce Lemma 2. □

Lemma 3. *Let h be any positive integer. Then for any integer $k \ge 0$, we have the identity*

$$(1-4x)^{k} \cdot f^{(h+k)}(x) = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{\min(n,k)} \frac{C_{n-i+h+k}}{(n-i)!} \binom{k}{i} \cdot (-4)^{i} \right) \cdot x^{n}.$$

Proof. From the binomial theorem we have

$$(1-4x)^k = \sum_{i=0}^{k} \binom{k}{i} \cdot (-4x)^i. \tag{7}$$

On the other hand, from (1) we also have

$$f^{(h+k)}(x) = \sum_{n=0}^{\infty} \frac{(n+h+k)! \cdot C_{n+h+k}}{n!} \cdot x^n. \tag{8}$$

Combining (7) and (8) we have

$$\begin{aligned}
& (1-4x)^k \cdot f^{(h+k)}(x) \\
= & \left(\sum_{i=0}^{k} \binom{k}{i} \cdot (-4x)^i \right) \left(\sum_{n=0}^{\infty} \frac{(n+h+k)! \cdot C_{n+h+k}}{n!} \cdot x^n \right) \\
= & \sum_{n=0}^{\infty} \sum_{i=0}^{k} \frac{(n+h+k)! \cdot C_{n+h+k}}{n!} \cdot \binom{k}{i} \cdot (-4)^i \cdot x^{n+i} \\
= & \sum_{n=0}^{\infty} \left(\sum_{i=0}^{\min(n,k)} \frac{(n-i+h+k)! \cdot C_{n-i+h+k}}{(n-i)!} \binom{k}{i} \cdot (-4)^i \right) \cdot x^n.
\end{aligned}$$

This proves Lemma 3. □

Lemma 4. *Let h be any positive integer. Then for any integer $k \geq 0$, we have the identity*

$$(1-4x)^{k+\frac{1}{2}} \cdot f^{(h+k)}(x) = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \binom{k+\frac{1}{2}}{i} \cdot (-4)^i \cdot \frac{C_{n-i+h+k}}{(n-i)!} \right) \cdot x^n.$$

Proof. From the power series expansion of the function we know that

$$(1-4x)^{k+\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{k+\frac{1}{2}}{n} \cdot (-4)^n \cdot x^n. \tag{9}$$

Applying (8) and (9) we have

$$\begin{aligned}
& (1-4x)^{k+\frac{1}{2}} \cdot f^{(h+k)}(x) \\
= & \left(\sum_{n=0}^{\infty} \binom{k+\frac{1}{2}}{n} \cdot (-4)^n \cdot x^n \right) \left(\sum_{n=0}^{\infty} \frac{(n+h+k)! \cdot C_{n+h+k}}{n!} \cdot x^n \right) \\
= & \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \binom{k+\frac{1}{2}}{i} \cdot (-4)^i \cdot \frac{(n-i+h+k)! \cdot C_{n-i+h+k}}{(n-i)!} \right) \cdot x^n.
\end{aligned}$$

This proves Lemma 4. □

3. Proofs of the Theorems

In this section, we shall complete the proofs of our theorems. First we prove Theorem 1. From (1) and the multiplicative properties of the power series we have

$$(2h)! \cdot f^{2h+1}(x) = (2h)! \sum_{n=0}^{\infty} \left(\sum_{a_1+a_2+\cdots+a_{2h+1}=n} C_{a_1} \cdot C_{a_2} \cdots C_{a_{2h+1}} \right) \cdot x^n. \tag{10}$$

On the other hand, from Lemma 1 and Lemma 3 we also have

$$
\begin{aligned}
&(2h)! \cdot f^{2h+1}(x) = \sum_{i=0}^{h} C(h,i) \cdot (1-4x)^i \cdot f^{(h+i)}(x) \\
&= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{h} C(h,i) \sum_{j=0}^{\min(n,i)} \frac{(n-j+h+i)!C_{n-j+h+i}}{(n-j)!} \binom{i}{j} (-4)^j \right) x^n.
\end{aligned} \tag{11}
$$

Combining (10) and (11) we may immediately deduce the identity

$$
\begin{aligned}
&\sum_{a_1+a_2+\cdots+a_{2h+1}=n} C_{a_1} \cdot C_{a_2} \cdot C_{a_3} \cdots C_{a_{2h+1}} \\
= \quad & \frac{1}{(2h)!} \sum_{i=0}^{h} C(h,i) \sum_{j=0}^{\min(n,i)} \frac{(n-j+h+i)! \cdot C_{n-j+h+i}}{(n-j)!} \cdot \binom{i}{j} \cdot (-4)^j.
\end{aligned}
$$

This proves Theorem 1.

Now we prove Theorem 2. For any positive integer h, from (1) we have

$$
f^{2h}(x) = \sum_{n=0}^{\infty} \left(\sum_{a_1+a_2+\cdots+a_{2h}=n} C_{a_1} \cdot C_{a_2} \cdots C_{a_{2h}} \right) \cdot x^n. \tag{12}
$$

On the other hand, from Lemma 2 and Lemma 4 we also have

$$
\begin{aligned}
&(2h-1)! \cdot f^{2h}(x) = \sum_{i=0}^{h-1} D(h,i+1) \cdot (1-4x)^{i+\frac{1}{2}} \cdot f^{(h+i)}(x) \\
= \quad & \sum_{i=0}^{h-1} D(h,i+1) \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \binom{i+\frac{1}{2}}{j} (-4)^j \frac{(n-j+h+i)! \cdot C_{n-j+h+i}}{(n-j)!} \right) x^n \\
= \quad & \sum_{n=0}^{\infty} \sum_{i=0}^{h-1} \sum_{j=0}^{n} D(h,i+1) \binom{i+\frac{1}{2}}{j} (-4)^j \frac{(n-j+h+i)!C_{n-j+h+i}}{(n-j)!} x^n.
\end{aligned} \tag{13}
$$

From (12), (13), and Lemma 2 we may immediately deduce the identity

$$
\begin{aligned}
&\sum_{a_1+a_2+\cdots+a_{2h}=n} C_{a_1} \cdot C_{a_2} \cdot C_{a_3} \cdots C_{a_{2h}} \\
= \quad & \frac{1}{(2h-1)!} \sum_{i=0}^{h-1} \sum_{j=0}^{n} D(h,i+1) \cdot \binom{i+\frac{1}{2}}{j} \cdot (-4)^j \cdot \frac{(n-j+h+i)! \cdot C_{n-j+h+i}}{(n-j)!}.
\end{aligned}
$$

This completes the proof of Theorem 2.

4. Conclusions

The main results of this paper are Theorem 1 and Theorem 2. They gave two special expressions for convolution (2). In addition, Corollary 1 gives a close relationship between $C(h,i)$ and C_{h+i}. Corollary 2 gives a close relationship between $D(h,i)$ and D_{h+i-1}. Corollary 3 and Corollary 4 give two exact representations for the special cases of Theorem 1 with $h=1$ and $h=2$.

About the new sequences $C(h,i)$ and $D(h,i)$, we proposed two interesting conjectures related to congruence mod p, where p is an odd prime. We believe that these conjectures are correct, but at the moment we cannot prove them. We also believe that these two conjectures will certainly attract the interest of many readers, thus further promoting the study of the properties of $C(h,i)$ and C_{h+i}.

Author Contributions: All authors have equally contributed to this work. All authors read and approved the final manuscript.

Funding: This work is supported by the National Natural Science Foundation (N. S. F.) (11771351) and (11826205) of China.

Acknowledgments: The authors would like to thank the Editor and referees for their very helpful and detailed comments, which have significantly improved the presentation of this paper.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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