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The Circular Hill Problem Regarding Arbitrary Disturbing Forces: The Periodic Solutions that are Emerging from the Equilibria

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Abstract: In this work, sufficient conditions for computing periodic solutions have been obtained in the circular Hill Problem with regard to arbitrary disturbing forces. This problem will be solved by means of using the averaging theory for dynamical systems as the main mathematical tool that has been applied in this work.

Keywords: Periodic solutions; circular Hill problem; averaging theory

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1. Introduction

The three-body problem, as it is known, is a special case of the $n-$body problem where the motion of three point masses under their mutual gravitational interactions is described. This classical problem represents a large range of astronomical situations. Because a general solution of the problem exists but it is unusable in practice, some simplifications to it have been made. The Hill problem is an example of this. There are many studies on the Hill problem [1–11], and many modifications of this classic problem have appeared in the mathematical, astronomical, and mechanical literature. For example, in [6], Markellos and Roy introduced a photogravitational Hill problem as a simplified, non-trivial form of the photogravitational restricted three-body problem. In [8], Papadakis made a mainly numerical study of this problem on the planar case. On the other hand, in [2], the authors focused their attention on the case of a perturbed spatial Hill lunar problem using the averaging theory, and in [3], Chavineau and Mignard studied the trajectories of the Hill problem to describe the effect of solar perturbations on the relative motion of a binary asteroid. In a central force field, the binary objects case can be approached from potential [4].

Our main goal in this study was to find, using the averaging theory, sufficient conditions for the existence of periodic orbits that come from the Lagrangian points of the circular Hill problem with arbitrary disturbing forces.

Firstly, we assumed that the first two masses were small, compared to the third one (i.e., we assumed m_1 and m_2 were small as compared to M), and that the two small masses were close together, as compared to their separation from the third mass.

In the case of the circular Hill problem, the equations of the relative motion are [3,4]:

$$
\begin{aligned}
&\ddot{q}_1 - 2\omega\dot{q}_2 - 3\omega^2 q_1 + \tfrac{Gmq_1}{(q_1^2+q_2^2+q_3^2)^{3/2}} = 0,\\
&\ddot{q}_2 + 2\omega\dot{q}_1 + \tfrac{Gmq_2}{(q_1^2+q_2^2+q_3^2)^{3/2}} = 0,\\
&\ddot{q}_3 + \omega^2 q_3 + \tfrac{Gmq_3}{(q_1^2+q_2^2+q_3^2)^{3/2}} = 0,
\end{aligned} \tag{1}
$$

where G is the gravitational constant, $m = m_1 + m_2$, and ω is the angular velocity of the motion of the barycenter of the m_1-m_2 system around M.

The equilibrium solutions of the differential system (1) are given by

$$
\mathcal{L}_1 = \left(\left(\frac{Gm}{3\omega^2}\right)^{1/3}, 0, 0\right), \qquad \mathcal{L}_2 = \left(-\left(\frac{Gm}{3\omega^2}\right)^{1/3}, 0, 0\right).
$$

In this work, we provide sufficient conditions for the existence of periodic solutions coming from the equilibrium point, $\mathcal{L}_1$. The same conditions are valid for the equilibrium, $\mathcal{L}_2$.

Making the change of variables $x = q_1 - \left(\frac{Gm}{3\omega^2}\right)^{1/3}$, $y = q_2$ and $z = q_3$, and linearizing, we can rewrite the unperturbed system of differential Equations (1) in the form:

$$
\begin{aligned}
&\ddot{x} - 9\omega^2 x - 2\omega\dot{y} = 0,\\
&\ddot{y} + 2\omega\dot{x} + 3\omega^2 y = 0,\\
&\ddot{z} + 4\omega^2 z = 0.
\end{aligned} \tag{2}
$$

Using the averaging theory, we will provide a system of nonlinear equations whose simple zeros provide periodic solutions of the perturbed system:

$$
\begin{aligned}
&\ddot{x} - 9\omega^2 x - 2\omega\dot{y} = \varepsilon F_1(t, x, \dot{x}, y, \dot{y}, z, \dot{z}),\\
&\ddot{y} + 2\omega\dot{x} + 3\omega^2 y = \varepsilon F_2(t, x, \dot{x}, y, \dot{y}, z, \dot{z}),\\
&\ddot{z} + 4\omega^2 z = \varepsilon F_3(t, x, \dot{x}, y, \dot{y}, z, \dot{z}),
\end{aligned} \tag{3}
$$

where the parameter ε is small, and F_1, F_2 and F_3 are the smooth and periodic functions in the variable t in resonance $p : q$, $(p, q$ positive integers relatively prime) with some of the periodic solutions for $\varepsilon = 0$.

The unperturbed system (2) has a unique singular point, the origin, with eigenvalues $\pm\omega_1 i$, $\pm\omega_2 i$, $\pm\omega_3$ with

$$
\omega_1 = 2\omega, \quad \omega_2 = \omega\sqrt{2\sqrt{7}-1} \quad \text{and} \quad \omega_3 = \omega\sqrt{2\sqrt{7}+1},
$$

and thus, the phase space $(x, \dot{x}, y, \dot{y}, z, \dot{z})$ has two planes (excluding the origin) covered by periodic solutions, with periods

$$
T_1 = \frac{2\pi}{\omega_1} \quad \text{or} \quad T_2 = \frac{2\pi}{\omega_2}.
$$

Accordingly, they belong to the plane associated to the eigenvectors with eigenvalues $\pm\omega_1\, i$ or $\pm\omega_2\, i$, respectively. Now, we are going to study if and when the parameter ε is sufficiently small, and when the perturbed functions F_i for $i \in \{1, 2, 3\}$ have a period of either pT_1/q or pT_2/q, whether these solutions still exist for the perturbed system (3).

Let $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2)$ be the Malkin bifurcation function (see [12]) for the system (3) and $X^0 = (X_1^0, X_2^0)$. Then:

$$\begin{aligned} \mathcal{G}_1(X^0) &= \frac{1}{pT_1}\int_0^{pT_1} \cos(\omega_1 t)F_1^*(t)dt, \\ \mathcal{G}_2(X^0) &= \frac{1}{pT_1}\int_0^{pT_1} \sin(\omega_1 t)F_1^*(t)dt, \end{aligned} \tag{4}$$

where

$$F_1^*(t) = F_3$$

with $F_3 = F_3(\sigma_1^1(t), \cdots, \sigma_6^1(t))$, and $\sigma_j^1(t) = 0, j = 1, \cdots 4$,

$$\begin{aligned} \sigma_5^1(t) &= \frac{X_1 \sin(2t\omega) - X_2\cos(2t\omega)}{2\omega} \\ \sigma_6^1(t) &= X_1\cos(2t\omega) + X_2\sin(2t\omega). \end{aligned}$$

As usual, we are going to call a *simple zero* of the system $\mathcal{G}(X^0) = 0$ to any solution $X^{0*} = (X_1^{0*}, X_2^{0*})$, such that $\det\left(\frac{\partial\mathcal{G}}{\partial X^0}|_{X^0=X^{0*}}\right) \neq 0$.

If X^{0*} is a simple zero, then the T_1-periodic solution of the unperturbed system with X^{0*} as the initial value gives a T_1-periodic solution of the perturbed system (see Appendix A).

Given an initial condition, we can use this result in order to find sufficient conditions for the existence of families of periodic orbits.

Theorem 1. *Suppose that the functions of the equations of motion of (3), F_1, F_2, and F_3 are smooth and periodic functions of the pT_1/q period in the variable t, with p and q being positive integers that are relatively prime, and where the ratio of the frequencies ω_2/ω_1 are not resonant with π. If $\varepsilon \neq 0$ is sufficiently small, then for every simple zero that is not null, X^{0*}, of the nonlinear system $\mathcal{G}(X^0) = 0$, the system (3) has a periodic solution $(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon))$ such that $(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon)) \to (x(t), y(t), z(t)) = (\sigma_1^1(t), \sigma_3^1(t), \sigma_5^1(t))|_{X^0=X^{0*}}$, when $\epsilon \to 0$—that is, when the periodic solution for the unperturbed system (2) travelled p times.*

In Section 2, Theorem 1 will be proved using the averaging theory (see [13–18]).

In the next corollary, an application of Theorem 1 is shown, whose proof is implemented in Section 3.

Corollary 1. *Assume that $f_i(t, x, \dot{x}, y, \dot{y}, z, \dot{z}), i = 1, 2, 3$, are arbitrary, smooth functions, and*

$$\begin{aligned} F_1(t, x, \dot{x}, y, \dot{y}, z, \dot{z}) &= f_1(t, x, \dot{x}, y, \dot{y}, z, \dot{z}), \\ F_2(t, x, \dot{x}, y, \dot{y}, z, \dot{z}) &= f_2(t, x, \dot{x}, y, \dot{y}, z, \dot{z}), \\ F_3(t, x, \dot{x}, y, \dot{y}, z, \dot{z}) &= \sin(\omega_1 t) + \omega\dot{z} + f_3(t, x, \dot{x}, y, \dot{y}, z, \dot{z})xyz + 1. \end{aligned}$$

If $\varepsilon \neq 0$ is sufficiently small, the perturbed system (3) has one periodic solution $(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon))$ such that, whenever $\varepsilon \to 0$, $(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon))$ tends to the periodic solution of the unperturbed system $(x(t), y(t), z(t)) = (\sigma_1^1(t), \sigma_3^1(t), \sigma_5^1(t))|_{X^0=\left(0, \sqrt[3]{\frac{4}{3\omega}}\right)}$.

This corollary will be proved in Section 3.

Figure 1 shows the solution of the system (3) for the smooth function F_1, F_2 and F_3.

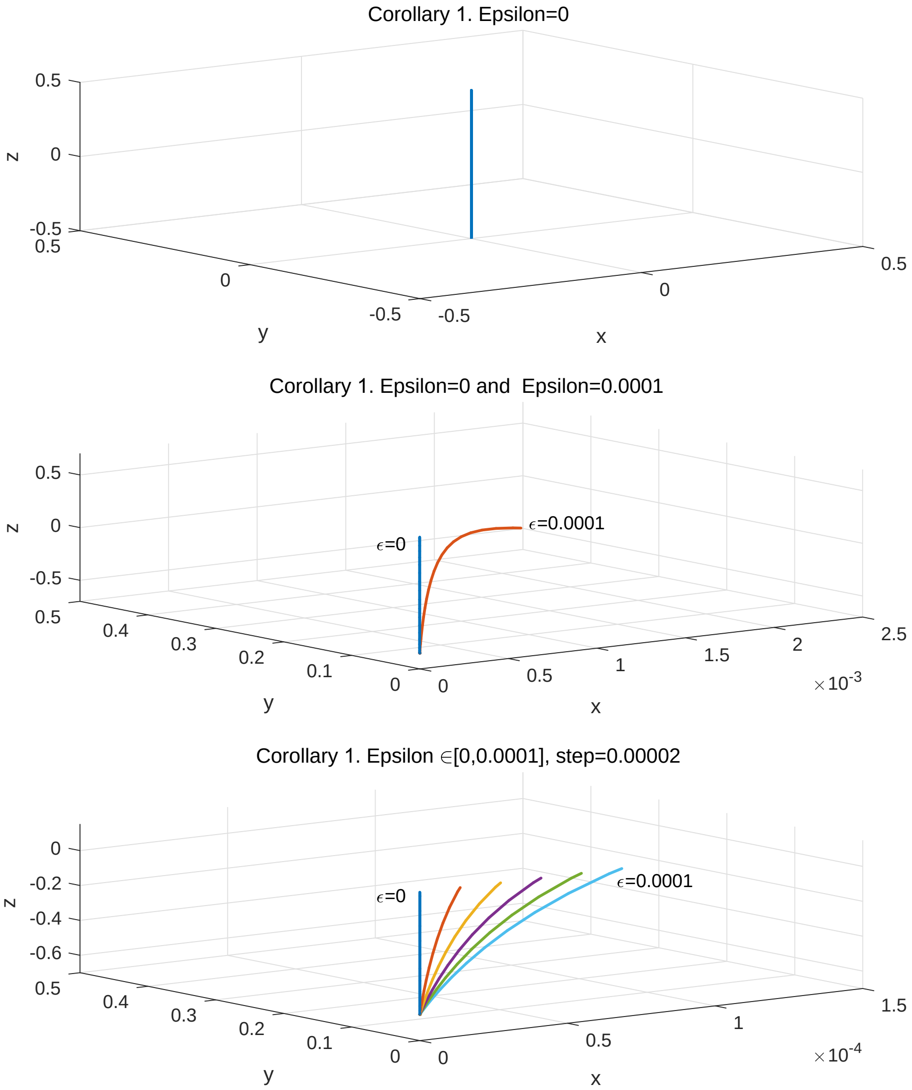

Figure 1. Solutions of the differential system (3) for the smooth functions $F_1 = 1 - x\dot{x}^2 - \sin(\omega_1 t)y$, $F_2 = \sin(\omega_1 t)\dot{x}^2 - yz + z^2$, and $F_3 = \sin(\omega_1 t) + \omega\dot{z} + \dot{x}xyz + 1$ for Corollary 1 with $\varepsilon > 0$, $w = 1$, $\omega_1 = 2\omega$, $t = [0, 2\pi/\omega_1]$, and $X^{0*} = \left(0, \sqrt[3]{\frac{4}{3\omega}}\right)$.

Let $\overline{\mathcal{G}} = (\mathcal{G}_3, \mathcal{G}_4)$ be the Malkin bifurcation function (see [12]) for system (3) and $Y^0 = (Y_1^0, Y_2^0)$. Then:

$$\mathcal{G}_3(Y^0) = \frac{1}{pT_2}\int_0^{pT_2}[\cos(\omega_2 t)F_3^*(t) - \sin(\omega_2 t)F_4^*(t)]dt,$$
$$\mathcal{G}_4(Y^0) = \frac{1}{pT_2}\int_0^{pT_2}[\cos(\omega_2 t)F_4^*(t) + \sin(\omega_2 t)F_3^*(t)]dt,$$

where

$$F_3^*(t) = \left(\frac{1}{2} + \frac{2}{\sqrt{7}}\right)F_2 \quad \text{and} \quad F_4^*(t) = -\frac{1}{2}\sqrt{\frac{2}{\sqrt{7}} - \frac{1}{7}F_1}$$

with $F_i = F_i(\sigma_1^2(t), \cdots, \sigma_6^2(t)), i = 1, 2$, and

$$
\begin{aligned}
\sigma_1^2(t) &= -\frac{1}{9\omega}\left(4-\sqrt{7}\right)\left(Y_1\cos\left(\sqrt{2\sqrt{7}-1}\,\omega t\right)+Y_2\sin\left(\sqrt{2\sqrt{7}-1}\,\omega t\right)\right)\\
\sigma_2^2(t) &= -\frac{\left(\sqrt{7}-2\right)\left(Y_2\cos\left(\sqrt{2\sqrt{7}-1}\,\omega t\right)-Y_1\sin\left(\sqrt{2\sqrt{7}-1}\,\omega t\right)\right)}{\sqrt{2\sqrt{7}-1}}\\
\sigma_3^2(t) &= -\frac{Y_2\cos\left(\sqrt{2\sqrt{7}-1}\,\omega t\right)-Y_1\sin\left(\sqrt{2\sqrt{7}-1}\,\omega t\right)}{\sqrt{2\sqrt{7}-1}\,\omega}\\
\sigma_4^2(t) &= Y_1\cos\left(\sqrt{2\sqrt{7}-1}\,\omega t\right)+Y_2\sin\left(\sqrt{2\sqrt{7}-1}\,\omega t\right)
\end{aligned}
$$

and $\sigma_j^2(t)=0, j=5,6$.

As has been stated before, if Y^{0*} is a simple zero of $\overline{\mathcal{G}}(Y^{0*})=0$, then the T_2-periodic solution of the unperturbed system with Y^{0*} as the initial value gives a T_2-periodic solution of the perturbed system (see Appendix A).

Theorem 2. *Suppose that the functions of the equations of motion of (3), F_1, F_2, and F_3, are smooth and periodic functions of the pT_2/q period in the variable t, with p and q positive integers relatively prime, and the ratio of the frequencies ω_2/ω_1 not resonant with π. If $\varepsilon \neq 0$ is sufficiently small, then for every simple and not null zero, Y^{0*} of the nonlinear system $\overline{\mathcal{G}}(Y^0)=0$, the system (3) has a periodic solution $(x(t,\varepsilon),y(t,\varepsilon),z(t,\varepsilon))$ such that $(x(t,\varepsilon),y(t,\varepsilon),z(t,\varepsilon))\to(x(t),y(t),z(t))=(\sigma_1^2(t),\sigma_3^2(t),\sigma_5^2(t))|_{Y^0=Y^{0*}}$, when $\epsilon\to 0$—that is, when the periodic solution for the unperturbed system (2) travelled p times.*

The proof of Theorem 2 is given in Section 2.

In the next corollary, an application of Theorem 2 is shown, whose proof is implemented in Section 3.

Corollary 2. *Let $g(t)$ be the arbitrary and smooth function, and thus consider:*

$$
\begin{aligned}
F_1(t,x,\dot{x},y,\dot{y},z,\dot{z}) &= (1+x\dot{z})\sin(\omega_2 t),\\
F_2(t,x,\dot{x},y,\dot{y},z,\dot{z}) &= x+\dot{y}+2z,\\
F_3(t,x,\dot{x},y,\dot{y},z,\dot{z}) &= x^3+\dot{y}^2+g(t).
\end{aligned}
$$

If $\varepsilon\neq 0$ is sufficiently small, the perturbed system (3) has one periodic solution, $(x(t,\varepsilon),y(t,\varepsilon),z(t,\varepsilon))$ such that, whenever $\varepsilon\to 0$, $(x(t,\varepsilon),y(t,\varepsilon),z(t,\varepsilon))$ tends to the periodic solution of the unperturbed system $(x(t),y(t),z(t))=(\sigma_1^2(t),\sigma_3^2(t),\sigma_5^2(t))|_{Y^0=\left(\frac{\sqrt{14\sqrt{7}-7}\,w}{\sqrt{7}-(7+4\sqrt{7})w},0\right)}$, $\omega\neq\frac{1}{9}\left(4-\sqrt{7}\right)$.

Corollary 2 will be proved in Section 3.

Figure 2 shows the solution of the system (3) for the smooth function, F_1, F_2 and F_3.

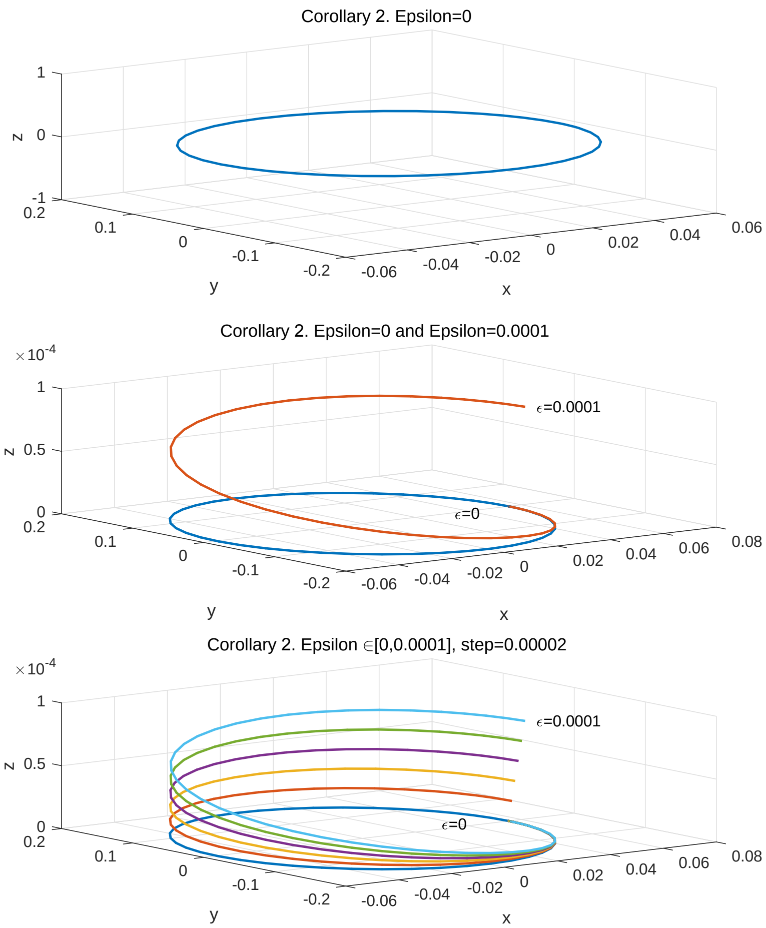

Figure 2. Solutions of the differential system (3) for the smooth functions $F_1 = (1 + x\dot{z})\sin(\omega_2 t)$, $F_2 = x + \dot{y} + 2z$ and $F_3 = x^3 + \dot{y}^2 + t$ for Corollary 2 with $\varepsilon > 0$, $w = 1$, $\omega_2 = \omega\sqrt{2\sqrt{7} - 1}$, $t = [0, 2\pi/\omega_2]$ and $Y^{0*} = \left(\frac{\sqrt{14\sqrt{7}-7w}}{\sqrt{7}-(7+4\sqrt{7})w}, 0\right)$.

2. Proof of Theorems 1 and 3

The differential system (3) must be written as a first-order differential system defined in $\mathbb{R}^6$. For that, the variables $(x_1, x_2, y_1, y_2, z_1, z_2) = (x, \dot{x}, y, \dot{y}, z, \dot{z})$ have been introduced in the following form, this being:

$$\begin{aligned}
\frac{dx_1}{dt} &= x_2 \\
\frac{dx_2}{dt} &= 9\omega^2 x_1 + 2\omega y_2 + \varepsilon F_1(x_1, x_2, y_1, y_2, z_1, z_2) \\
\frac{dy_1}{dt} &= y_2 \\
\frac{dy_2}{dt} &= -2\omega x_2 - 3\omega^2 y_1 + \varepsilon F_2(x_1, x_2, y_1, y_2, z_1, z_2) \\
\frac{dz_1}{dt} &= z_2 \\
\frac{dz_2}{dt} &= -4\omega^2 z_1 + \varepsilon F_3(x_1, x_2, y_1, y_2, z_1, z_2).
\end{aligned} \tag{5}$$

It must be remarked that if $\varepsilon = 0$, the differential system (5) is equivalent to the differential system (2), well-known as the *unperturbed system*. Otherwise, when $\varepsilon \neq 0$, this is known as the *perturbed system*

The change of variables $(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2, \bar{z}_1, \bar{z}_2) \to (x_1, x_2, y_1, y_2, z_1, z_2)$ given by

$$\begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{\sqrt{7}-4}{9w} & 0 & -\frac{4+\sqrt{7}}{9w} & -\frac{4+\sqrt{7}}{9w} \\ 0 & 0 & 0 & \frac{2-\sqrt{7}}{\sqrt{-1+2\sqrt{7}}} & -\frac{2+\sqrt{7}}{\sqrt{1+2\sqrt{7}}} & \frac{2+\sqrt{7}}{\sqrt{1+2\sqrt{7}}} \\ 0 & 0 & 0 & -\frac{1}{\sqrt{-1+2\sqrt{7}}\,w} & \frac{1}{\sqrt{1+2\sqrt{7}}\,w} & -\frac{1}{\sqrt{1+2\sqrt{7}}\,w} \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & -\frac{1}{2w} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{y}_1 \\ \bar{y}_2 \\ \bar{z}_1 \\ \bar{z}_2 \end{pmatrix} \tag{6}$$

writes the linear part of the differential system (5) in its real Jordan normal form, and this system in the new variables $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2, \bar{z}_1, \bar{z}_2)$ becomes

$$\begin{aligned} \dot{\bar{x}}_1 &= \omega_1 \bar{x}_2 + \varepsilon F_1^* \\ \dot{\bar{x}}_2 &= -\omega_1 \bar{x}_1 + \varepsilon F_2^* \\ \dot{\bar{y}}_1 &= \omega_2 \bar{y}_2 + \varepsilon F_3^* \\ \dot{\bar{y}}_2 &= -\omega_2 \bar{y}_1 + \varepsilon F_4^* \\ \dot{\bar{z}}_1 &= \omega_3 \bar{z}_1 + \varepsilon F_5^* \\ \dot{\bar{z}}_2 &= -\omega_3 \bar{z}_2 + \varepsilon F_6^*, \end{aligned} \tag{7}$$

where

$$\begin{aligned} F_1^* &= F_3 \\ F_2^* &= 0 \\ F_3^* &= \left(\frac{1}{2} + \frac{2}{\sqrt{7}}\right) F_2 \\ F_4^* &= -\frac{1}{2}\sqrt{\frac{2}{\sqrt{7}} - \frac{1}{7}}\, F_1 \\ F_5^* &= \left(\frac{1}{4} - \frac{1}{\sqrt{7}}\right) F_2 - \frac{1}{4}\sqrt{\frac{1}{7} + \frac{2}{\sqrt{7}}}\, F_1 \\ F_6^* &= \left(\frac{1}{4} - \frac{1}{\sqrt{7}}\right) F_2 + \frac{1}{4}\sqrt{\frac{1}{7} + \frac{2}{\sqrt{7}}}\, F_1 \end{aligned}$$

with $F_i = F_i(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6)$, and

$$\begin{aligned} \sigma_1 &= \frac{\left(\sqrt{7}-4\right)\bar{y}_1 - \left(4+\sqrt{7}\right)(\bar{z}_1 + \bar{z}_2)}{9w} \\ \sigma_2 &= \frac{\sqrt{5+2\sqrt{7}}(\bar{z}_2 - \bar{z}_1) - \sqrt{2\sqrt{7}-5}\,\bar{y}_2}{\sqrt{3}} \\ \sigma_3 &= \frac{\sqrt{2\sqrt{7}-1}(\bar{z}_1 - \bar{z}_2) - \sqrt{1+2\sqrt{7}}\,\bar{y}_2}{3\sqrt{3}w} \\ \sigma_4 &= \bar{y}_1 + \bar{z}_1 + \bar{z}_2 \\ \sigma_5 &= -\frac{\bar{x}_2}{2\omega} \\ \sigma_6 &= \bar{x}_1. \end{aligned}$$

In order to prove Theorems 1 and 2, in the following lemma we are going to obtain the general expressions of the periodic orbits of the unperturbed system.

Lemma 1. *If $\epsilon = 0$, the T_1-periodic solutions of the system of the differential Equations (7) are*

$$\begin{aligned} \bar{x}_1(t) &= X_1^0 \cos(\omega_1 t) + X_2^0 \sin(\omega_1 t), \\ \bar{x}_2(t) &= X_2^0 \cos(\omega_1 t) - X_1^0 \sin(\omega_1 t), \\ \bar{y}_1(t) &= 0, \\ \bar{y}_2(t) &= 0, \\ \bar{z}_1(t) &= 0, \\ \bar{z}_2(t) &= 0, \end{aligned} \tag{8}$$

and the T_2-periodic solutions are

$$\begin{aligned} \bar{x}_1(t) &= 0, \\ \bar{x}_2(t) &= 0, \\ \bar{y}_1(t) &= Y_1^0 \cos(\omega_2 t) + Y_2^0 \sin(\omega_2 t), \\ \bar{y}_2(t) &= Y_2^0 \cos(\omega_2 t) - Y_1^0 \sin(\omega_2 t), \\ \bar{z}_1(t) &= 0, \\ \bar{z}_2(t) &= 0. \end{aligned}$$

Proof of Lemma 1. If $\varepsilon = 0$, the system (7) is a system of linear differential equations, and the proof follows easily. □

Proof of Theorem 1. Since the positive integers p and q are relatively prime, and the functions $F_i, i = 1,2,3$, of the system (3) are $pT_1/q-$periodic in the variable t, we can consider that the system (7) and the solutions (8) are pT_1-periodic.

Now, we shall study what periodic solutions of the unperturbed system (7) with $\varepsilon = 0$ of the type (8) persist as periodic solutions, with the perturbed one for $\varepsilon \neq 0$ being sufficiently small.

If we write the system (7) in the form

$$\dot{\mathrm{x}}(t) = G_0(t, \bar{\mathrm{x}}) + \epsilon G_1(t, \bar{\mathrm{x}}) + \epsilon^2 G_2(t, \bar{\mathrm{x}}, \epsilon),$$

where

$$\bar{\mathrm{x}} = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{y}_1 \\ \bar{y}_2 \\ \bar{z}_1 \\ \bar{z}_2 \end{pmatrix}, G_0(t, \bar{\mathrm{x}}) = \begin{pmatrix} \omega_1 \bar{x}_2 \\ -\omega_1 \bar{x}_1 \\ \omega_2 \bar{y}_2 \\ -\omega_2 \bar{y}_1 \\ \omega_3 \bar{z}_1 \\ -\omega_3 \bar{z}_2 \end{pmatrix}, G_1(t, \bar{\mathrm{x}}) = \begin{pmatrix} F_1^* \\ F_2^* \\ F_3^* \\ F_4^* \\ F_5^* \\ F_6^* \end{pmatrix}, G_2(t, \bar{\mathrm{x}}, \varepsilon) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

we can apply Theorem A1 of the Appendix A. Our starting point is the description of the different elements which appear in this theorem for the particular case of the differential system (7). In this case, $\Omega = \mathbb{R}^6$, $k = 2$, and $n = 6$. Now, let $r_1 > 0$ be arbitrarily small, and let $r_2 > 0$ be arbitrarily large. If V is the open and bounded subset of the plane $\bar{x}_i = 0, i = 3, \ldots, 6$, of the form $V = \{(X_1^0, X_2^0, 0, 0, 0, 0) \in \mathbb{R}^6 \ : r_1 < \sqrt{\sum_{i=1}^{2} (X_i^0)^2} < r_2\}$. If $\alpha = (X_1^0, X_2^0)$, then we identify V with the set $\{\alpha \in \mathbb{R}^2 \ : \ r_1 < ||\alpha|| < r_2\}$, where $|| \cdot ||$ is the Euclidean norm in $\mathbb{R}^2$. If we denote by $\mathrm{Cl}(V)$, the closure of V, the function $\beta \colon \mathrm{Cl(V)} \to \mathbb{R}^4$ is $\beta(\alpha) = (0,0,0,0)$. Using all these elements in our system, we have

$$\begin{aligned} \mathcal{Z} = \{ \mathrm{z}_\alpha = (\alpha,\ \beta(\alpha)),\ \alpha \in \mathrm{Cl}(V) \} &= \\ = \{ (X_1^0, X_2^0, 0, 0, 0, 0) \in \mathbb{R}^6 \ : \ r_1 \leq \sqrt{\sum_{i=1}^{2} (X_i^0)^2} \leq r_2 \} \end{aligned}$$

Now, for each $z_\alpha \in \mathcal{Z}$, we consider the pT_1-periodic solution

$$\mathrm{x}(t, z_\alpha) = (X_1(t), X_2(t), 0, 0, 0, 0)$$

given by (8).

The fundamental matrix $M_{z_\alpha}(t)$ of the linear differential system (7) with $\varepsilon = 0$, that is, associated to the pT_1-periodic solution $z_\alpha = (X_1^0, X_2^0, 0, 0, 0, 0)$ and such that $M_{z_\alpha}(0) = Id$ in $\mathbb{R}^6$, is

$$M_{z_\alpha}(t) = M(t) =$$

$$= \begin{pmatrix} \cos(\omega_1 t) & \sin(\omega_1 t) & 0 & 0 & 0 & 0 \\ -\sin(\omega_1 t) & \cos(\omega_1 t) & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(\omega_2 t) & \sin(\omega_2 t) & 0 & 0 \\ 0 & 0 & -\sin(\omega_2 t) & \cos(\omega_2 t) & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\omega_3 t} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-\omega_3 t} \end{pmatrix},$$

where $M(t)$ does not depend on the particular periodic solution $\mathrm{x}(t, z_\alpha,\ 0)$.

Computing the matrix $M = M^{-1}(0) - M^{-1}(pT_1)$, we get

$$M = \begin{pmatrix} 2\sin^2(p\pi) & \sin(2p\pi) & 0 & 0 & 0 & 0 \\ -\sin(2p\pi) & 2\sin^2(p\pi) & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\sin^2\left(\frac{p\pi\omega_2}{\omega_1}\right) & \sin\left(\frac{2p\pi\omega_2}{\omega_1}\right) & 0 & 0 \\ 0 & 0 & -\sin\left(\frac{2p\pi\omega_2}{\omega_1}\right) & 2\sin^2\left(\frac{p\pi\omega_2}{\omega_1}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 - e^{-\frac{2p\pi\omega_3}{\omega_1}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 - e^{\frac{2p\pi\omega_3}{\omega_1}}, \end{pmatrix}$$

and M satisfies the assumptions of statement (ii) of Theorem A1, since the determinant

$$\begin{vmatrix} 2\sin^2\left(\frac{p\pi\omega_2}{\omega_1}\right) & \sin\left(\frac{2p\pi\omega_2}{\omega_1}\right) & 0 & 0 \\ -\sin\left(\frac{2p\pi\omega_2}{\omega_1}\right) & 2\sin^2\left(\frac{p\pi\omega_2}{\omega_1}\right) & 0 & 0 \\ 0 & 0 & 1 - e^{-\frac{2p\pi\omega_3}{\omega_1}} & \\ 0 & 0 & 0 & 1 - e^{\frac{2p\pi\omega_3}{\omega_1}} \end{vmatrix} =$$

$$= -16\sinh^2\left(\frac{p\pi\omega_3}{\omega_1}\right)\sin^2\left(\frac{p\pi\omega_2}{\omega_1}\right) \neq 0,$$

owing to the ratio of the frequencies, is non-resonant with π. In other words, all the assumptions of Theorem A1 are satisfied by the system (7).

In our particular system, if $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2, \bar{z}_1, \bar{z}_2)$, the map $\xi : \mathbb{R}^6 \to \mathbb{R}^2$ is given by $\xi(\bar{x}) = (\bar{x}_1, \bar{x}_2)$, and if we calculate the function $\mathcal{G}(X_1^0, X_2^0)$

$$\mathcal{G}(X_1^0, X_2^0) = \mathcal{G}(\alpha) = \xi\left(\frac{1}{pT_1}\int_0^{pT_1} M_{z_\alpha}^{-1}(t) G_1(t, \mathrm{x}(t, z_\alpha, 0))dt\right),$$

we obtain $\mathcal{G}(X^0) = (\mathcal{G}_1(X^0), \mathcal{G}_2(X^0))$, where the functions $\mathcal{G}_1, \mathcal{G}_2$ are given in (4). Then, by means of Theorem A1 of the Appendix A, for every simple zero $X^{0*} \in V$ of the system of nonlinear functions $\mathcal{G}(X^0) = 0$, there is a periodic solution $(\bar{x}_1(t,\varepsilon), \bar{x}_2(t,\varepsilon), \bar{y}_1(t,\varepsilon), \bar{y}_2(t,\varepsilon), \bar{z}_1(t,\varepsilon), \bar{z}_2(t,\varepsilon))$ of system (7), so that

$$(\bar{x}_1(0,\varepsilon), \bar{x}_2(0,\varepsilon), \bar{y}_1(0,\varepsilon), \bar{y}_2(0,\varepsilon), \bar{z}_1(0,\varepsilon), \bar{z}_2(0,\varepsilon)) \overset{\varepsilon \to 0}{\to} (X_1^{0*}, X_2^{0*}, 0, 0, 0, 0).$$

Going back through the change of variables (6), the periodic solution $(x_1(t,\varepsilon), x_2(t,\varepsilon), y_1(t,\varepsilon), y_2(t,\varepsilon), z_1(t,\varepsilon), z_2(t,\varepsilon))$ of system (7) has been obtained, so that

$$\begin{pmatrix} x_1(t,\varepsilon) \\ x_2(t,\varepsilon) \\ y_1(t,\varepsilon) \\ y_2(t,\varepsilon) \\ z_1(t,\varepsilon) \\ z_2(t,\varepsilon) \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{X_1 \sin(2t\omega) - X_2 \cos(2t\omega)}{2\omega} \\ X_1 \cos(2t\omega) + X_2 \sin(2t\omega) \end{pmatrix} \text{ when } \varepsilon \to 0.$$

Consequently, a periodic solution $(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon))$ of system (3) has been achieved, so that

$$(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon)) \to \begin{pmatrix} 0 \\ 0 \\ \frac{X_1 \sin(2t\omega) - X_2 \cos(2t\omega)}{2\omega} \end{pmatrix} \text{ when } \varepsilon \to 0.$$

This completes the proof of Theorem 1. □

Proof of Theorem 2. The proof is analogous to the proof of Theorem 1 changing the roles of T_1 for T_2, and we obtain the periodic solution

$$\begin{pmatrix} -\frac{(4-\sqrt{7})\left(Y_1 \cos\left(\sqrt{2\sqrt{7}-1}\, t\omega\right) + Y_2 \sin\left(\sqrt{2\sqrt{7}-1}\, t\omega\right)\right)}{9\omega} \\ -\frac{(\sqrt{7}-2)\left(Y_2 \cos\left(\sqrt{2\sqrt{7}-1}\, t\omega\right) - Y_1 \sin\left(\sqrt{2\sqrt{7}-1}\, t\omega\right)\right)}{\sqrt{2\sqrt{7}-1}} \\ -\frac{Y_2 \cos\left(\sqrt{2\sqrt{7}-1}\, t\omega\right) - Y_1 \sin\left(\sqrt{2\sqrt{7}-1}\, t\omega\right)}{\sqrt{2\sqrt{7}-1}\,\omega} \\ Y_1 \cos\left(\sqrt{2\sqrt{7}-1}\, t\omega\right) + Y_2 \sin\left(\sqrt{2\sqrt{7}-1}\, t\omega\right) \\ 0 \\ 0 \end{pmatrix} \text{ when } \varepsilon \to 0.$$

In consequence, a periodic solution $(x,y,z)(t,\varepsilon)$ of system (3) has been obtained, so that

$$(x,y,z)(t,\varepsilon) \to \begin{pmatrix} -\frac{(4-\sqrt{7})\left(Y_1 \cos\left(\sqrt{2\sqrt{7}-1}\, t\omega\right) + Y_2 \sin\left(\sqrt{2\sqrt{7}-1}\, t\omega\right)\right)}{9\omega} \\ -\frac{Y_2 \cos\left(\sqrt{2\sqrt{7}-1}\, t\omega\right) - Y_1 \sin\left(\sqrt{2\sqrt{7}-1}\, t\omega\right)}{\sqrt{2\sqrt{7}-1}\,\omega} \\ 0 \end{pmatrix} \text{ when } \varepsilon \to 0.$$

□

3. Proof of Corollaries 1 and 2

Proof of Corollary 1. Under the assumptions of Corollary 1, we have the function

$$\mathcal{G}_1(X^0) = \frac{1}{pT_1}\int_0^{pT_1} \cos(\omega_1 t) F_1^*(t)dt = \frac{3}{8} X_1^0 \left[(X_1^0)^2 + (X_2^0)^2\right]\omega,$$
$$\mathcal{G}_2(X^0) = \frac{1}{pT_1}\int_0^{pT_1} \sin(\omega_1 t) F_1^*(t)dt = \frac{1}{8}[4 + 3(X_1^0)^2 X_2^0 \omega + 3(X_2^0)^3\omega]$$

and

$$\det\left(\frac{\partial \mathcal{G}}{\partial X^0}\right)\Big|_{X^{0*}=(X_1^0, X_2^0)} = \frac{3}{8}\sqrt[3]{\frac{9\omega^2}{2}} \neq 0, \quad \forall X_1^0,\ X_2^0 \in \mathbb{R}, \omega \neq 0$$

Therefore, the solution $X^{0*} = \left(0, \sqrt[3]{\frac{4}{3\omega}}\right)$ of the nonlinear system $\mathcal{G}(X^0) = 0$ is simple, and by Theorem 1 we only have one periodic solution for the system of this corollary. $\square$

Proof of Corollary 2. Under the assumptions of Corollary 2, we have the function

$$\begin{aligned}
\mathcal{G}_3(Y^0) &= \frac{1}{pT_2}\int_0^{pT_2} [\cos(\omega_2 t)F_3^*(t) - \sin(\omega_2 t)F_4^*(t)]\,dt = \\
&= \frac{w\left(\left(7+4\sqrt{7}\right)Y_1^0 + \sqrt{14\sqrt{7}-7}\right) - \sqrt{7}Y_1^0}{28w} \\
\mathcal{G}_4(Y^0) &= \frac{1}{pT_2}\int_0^{pT_2} [\cos(\omega_2 t)F_4^*(t) + \sin(\omega_2 t)F_3^*(t)]\,dt = \\
&= \frac{1}{28}Y_2^0\left(-\frac{\sqrt{7}}{w} + 4\sqrt{7} + 7\right),
\end{aligned}$$

and, if $\omega \neq \frac{1}{9}(4-\sqrt{7})$,

$$\det\left(\frac{\partial\overline{\mathcal{G}}}{\partial Y^0}\right)\Bigg|_{Y^{0*}=(Y_1^0,Y_2^0)} = \frac{1}{112}\left(\frac{1}{w^2} - \frac{2\left(4+\sqrt{7}\right)}{w} + 8\sqrt{7} + 23\right) \neq 0.$$

Therefore, the solution $Y^{0*} = \left(\frac{\sqrt{14\sqrt{7}-7}\,w}{\sqrt{7}-(7+4\sqrt{7})w}, 0\right)$ of the nonlinear system $\overline{\mathcal{G}}(Y^0) = 0$ is simple. By means of Theorem 1, the system of this Corollary has only this periodic solution. $\square$

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Appendix A. Basic Results on Averaging Theory

In this appendix we present the basic result from the averaging theory that we shall need for proving the main results of this paper.

We consider the problem of the bifurcation of $T-$periodic solutions from a differential system of the form

$$\dot{\mathrm{x}}(t) = G_0(t,\mathrm{x}) + \epsilon G_1(t,\mathrm{x}) + \epsilon^2 G_2(t,\mathrm{x},\epsilon), \tag{A1}$$

with $\epsilon \neq 0$ sufficiently small. Here the functions $G_0, G_1\colon \mathbb{R}\times\Omega \to \mathbb{R}^n$ and $G_2\colon \mathbb{R}\times\Omega\times(-\epsilon_0,\epsilon_0) \to \mathbb{R}^n$ are $\mathcal{C}^2$ functions, $T-$periodic in the first variable, and Ω is an open subset of $\mathbb{R}^n$. The main assumption is that the unperturbed system

$$\dot{\mathrm{x}}(t) = G_0(t,\mathrm{x}), \tag{A2}$$

has a submanifold of periodic solutions. A solution of this problem is given using the averaging theory.

Malkin (see [13] and references therein) studied the bifurcation of T–periodic solutions in the T–periodic system $\dot{x} = G_0(t,x) + \varepsilon G_1(t,x,\varepsilon)$, whose unperturbed system has a family of T–periodic solutions with initial conditions given by a smooth function $\beta\colon \mathbb{R}^k \to \mathbb{R}^n$, and proved that if the bifurcation function

$$\mathcal{G}(\alpha) = \int_0^T \begin{pmatrix} < u_1(t,\alpha), g(t,x(t,\beta(\alpha)),0) > \\ \cdots \\ < u_k(t,\alpha), g(t,x(t,\beta(\alpha)),0) > \end{pmatrix} dt,$$

where u_i, $i = 1, \ldots, k$, are k linearly independent T-periodic solutions of the adjoint linearized differential system, has a simple zero α_0 such that $\det(\mathcal{G})|_{\alpha=\alpha_0} \neq 0$, then for any $\varepsilon > 0$ sufficiently small, the system $\dot{x} = G_0(t,x) + \varepsilon G_1(t,x,\varepsilon)$ has an unique T–periodic solution x_ε such that $x_\varepsilon(0) \to \beta(\alpha_0)$ as $\varepsilon \to 0$.

This can be rephrased as follows: Let $\mathrm{x}(t,\mathrm{z},\epsilon)$ be the solution of the system (A2) such that $\mathrm{x}(0,\mathrm{z},\epsilon) = \mathrm{z}$. We write the linearization of the unperturbed system along a periodic solution $\mathrm{x}(t,\mathrm{z},0)$ as

$$\dot{\mathrm{y}} = D_{\mathrm{x}} G_0(t, \mathrm{x}(t,\mathrm{z},0))\mathrm{y}. \tag{A3}$$

In what follows we denote by $M_{\mathrm{z}}(t)$ some fundamental matrix of the linear differential system (A3), and by $\xi\colon \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ the projection of $\mathbb{R}^n$ onto its first k coordinates; i.e., $\xi(x_1, \ldots, x_n) = (x_1, \ldots, x_k)$.

We assume that there exists a k-dimensional submanifold $\mathcal{Z}$ of Ω filled with $T-$periodic solutions of (A2). Then an answer to the problem of bifurcation of $T-$periodic solutions from the periodic solutions contained in $\mathcal{Z}$ for system (A1) is given in the following result.

Theorem A1. *Let V be an open and bounded subset of $\mathbb{R}^k$, and let $\beta\colon \mathrm{Cl}(V) \to \mathbb{R}^{n-k}$ be a $\mathcal{C}^2$ function. We assume that*

(i) $\mathcal{Z} = \{\mathrm{z}_\alpha = (\alpha, \beta(\alpha)), \alpha \in \mathrm{Cl}(V)\} \subset \Omega$ *and that for each* $\mathrm{z}_\alpha \in \mathcal{Z}$ *the solution* $\mathrm{x}(t,\mathrm{z}_\alpha)$ *of (A2) is T-periodic;*

(ii) *for each* $\mathrm{z}_\alpha \in \mathcal{Z}$ *there is a fundamental matrix* $M_{\mathrm{z}_\alpha}(t)$ *of (A3) such that the matrix* $M_{\mathrm{z}_\alpha}^{-1}(0) - M_{\mathrm{z}_\alpha}^{-1}(T)$ *has in the upper right corner the* $k \times (n-k)$ *zero matrix, and in the lower right corner a* $(n-k) \times (n-k)$ *matrix* Δ_α *with* $\det(\Delta_\alpha) \neq 0$.

We consider the function $\mathcal{G}\colon \mathrm{Cl}(V) \to \mathbb{R}^k$

$$\mathcal{G}(\alpha) = \xi\Big(\frac{1}{T}\int_0^T M_{\mathrm{z}_\alpha}^{-1}(t) G_1(t, \mathrm{x}(t,\mathrm{z}_\alpha,0))dt\Big).$$

If there exists $a \in V$ *with* $\mathcal{G}(a) = 0$ *and* $\det((d\mathcal{G}/d\alpha)(a)) \neq 0$, *then there is a* $T-$*periodic solution* $\mathrm{x}(t,\epsilon)$ *of system (A1) such that* $\mathrm{x}(0,\epsilon) \to \mathrm{z}_a$ *as* $\epsilon \to 0$.

For a proof of Theorem A1 see [12,19,20] for shorter proof.

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