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Bounds for the Coefficient of Faber Polynomial of Meromorphic Starlike and Convex Functions

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Abstract: Let Σ be the class of meromorphic functions f of the form $f(\zeta) = \zeta + \sum_{n=0}^{\infty} a_n \zeta^{-n}$ which are analytic in $\Delta := \{\zeta \in \mathbb{C} : |\zeta| > 1\}$. For $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, the nth Faber polynomial $\Phi_n(w)$ of $f \in \Sigma$ is a monic polynomial of degree n that is generated by a function $\zeta f'(\zeta)/(f(\zeta) - w)$. For given $f \in \Sigma$, by $F_{n,i}(f)$, we denote the ith coefficient of $\Phi_n(w)$. For given $0 \le \alpha < 1$ and $0 < \beta \le 1$, let us consider domains $\mathbb{H}_\alpha$ and $S_\beta \subset \mathbb{C}$ defined by $\mathbb{H}_\alpha = \{w \in \mathbb{C} : \mathrm{Re}(w) > \alpha\}$ and $S_\beta = \{w \in \mathbb{C} : |\arg(w)| < \beta\}$, which are symmetric with respect to the real axis. A function $f \in \Sigma$ is called meromorphic starlike of order α if $\zeta f'(\zeta)/f(\zeta) \in \mathbb{H}_\alpha$ for all $\zeta \in \Delta$. Another function $f \in \Sigma$ is called meromorphic strongly starlike of order β if $\zeta f'(\zeta)/f(\zeta) \in S_\beta$ for all $\zeta \in \Delta$. In this paper we investigate the sharp bounds of $F_{n,n-i}(f)$, $n \in \mathbb{N}_0$, $i \in \{2, 3, 4\}$, for meromorphic starlike functions of order α and meromorphic strongly starlike of order β. Similar estimates for meromorphic convex functions of order α $(0 \le \alpha < 1)$ and meromorphic strongly convex of order β $(0 < \beta \le 1)$ are also discussed.

Keywords: meromorphic functions; starlike functions; convex functions; Faber polynomials; coefficient problems

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1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in $\mathbb{C}$. Let $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$ and $\Delta = \{\zeta \in \mathbb{C} : |\zeta| > 1\}$ be the punctured unit disk and the exterior of $\mathbb{D}$.

Let Σ by the class of meromorphic functions

$$f(\zeta) = \zeta + \sum_{n=0}^{\infty} a_n \zeta^{-n}, \quad \zeta \in \Delta, \tag{1}$$

that are univalent in Δ. Let $\tilde{\Sigma}$ be class of functions in Σ which have the form (1) with $a_0 = 0$.

Let $\alpha \in [0, 1)$ be given and consider a domain $\mathbb{H}_\alpha := \{w \in \mathbb{C} : \mathrm{Re}(w) > \alpha\}$ which is symmetric with respect to the real axis. A meromorphic function $f \in \Sigma$ is called starlike of order α if f satisfies $\zeta f'(\zeta)/f(\zeta) \in \mathbb{H}_\alpha$ for all $\zeta \in \Delta$. A meromorphic function $f \in \Sigma$ is called convex of order α if f satisfies $1 + \zeta f''(\zeta)/f'(\zeta) \in \mathbb{H}_\alpha$ for all $\zeta \in \Delta$. By $\mathcal{S}^*_{\tilde{\Sigma}}(\alpha)$ and $\mathcal{K}_{\tilde{\Sigma}}(\alpha)$ we denote the classes of starlike and convex functions of order α. That is, $f \in \mathcal{S}^*_{\tilde{\Sigma}}(\alpha)$ if and only if $f \in \Sigma$ and f satisfies

$$\mathrm{Re}\left\{\frac{\zeta f'(\zeta)}{f(\zeta)}\right\} > \alpha, \quad \zeta \in \Delta.$$

Furthermore, $f \in \mathcal{K}_{\tilde{\Sigma}}(\alpha)$ if and only if $f \in \Sigma$ and f satisfies

$$\operatorname{Re}\left\{1 + \frac{\zeta f''(\zeta)}{f'(\zeta)}\right\} > \alpha, \quad \zeta \in \Delta.$$

For given $\beta \in (0,1]$, consider a domain $S_\beta = \{w \in \mathbb{C} : |\arg(w)| < \beta\}$ which is symmetric with respect to the real axis. A meromorphic function $f \in \Sigma$ is called strongly starlike of order β if f satisfies $\zeta f'(\zeta)/f(\zeta) \in S_\beta$ for all $\zeta \in \Delta$. A meromorphic function $f \in \Sigma$ is called strongly convex of order β if f satisfies $1 + \zeta f''(\zeta)/f'(\zeta) \in S_\beta$ for all $\zeta \in \Delta$. By $\mathcal{SS}^*_{\tilde{\Sigma}}(\beta)$ and $\mathcal{SK}_{\tilde{\Sigma}}(\beta)$ we denote the classes of strongly starlike and strongly convex functions of order β. That is, $f \in \mathcal{SS}^*_{\tilde{\Sigma}}(\beta)$ if and only if $f \in \Sigma$ and f satisfies

$$\left|\arg\left\{\frac{\zeta f'(\zeta)}{f(\zeta)}\right\}\right| < \frac{\pi}{2}\beta, \quad \zeta \in \Delta.$$

In addition, $f \in \mathcal{SK}_{\tilde{\Sigma}}(\beta)$ if and only if $f \in \Sigma$ and f satisfies

$$\left|\arg\left\{1 + \frac{\zeta f''(\zeta)}{f'(\zeta)}\right\}\right| < \frac{\pi}{2}\beta, \quad \zeta \in \Delta.$$

Note that $\mathcal{S}^*_{\tilde{\Sigma}} := \mathcal{S}^*_{\tilde{\Sigma}}(0) = \mathcal{SS}^*_{\tilde{\Sigma}}(1)$ and $\mathcal{K}_{\tilde{\Sigma}} = \mathcal{K}_{\tilde{\Sigma}}(0) = \mathcal{SK}_{\tilde{\Sigma}}(1)$ are the classes of starlike and convex functions which are frequently studied classes in the area of univalent function theory.

Computing the bounds of coefficients is an interesting problem to study. In particular, the bound of the nth coefficient of functions in $\mathcal{S}^*_{\tilde{\Sigma}}(\alpha)$ and $\mathcal{SS}^*_{\tilde{\Sigma}}(\beta)$ was found by Pommerenke [1] and Brannan et al. [2]. Another interesting problem is to find the bound of $\Lambda_\gamma(f) := a_1 - \gamma a_0^2$, $\gamma \in \mathbb{C}$, which is known as Fekete–Szegö functional for meromorphic functions. Many authors examined the functional $\Lambda_\gamma(f)$ over subclasses of Σ (see [3–5]). The object of this paper is to investigate bounds of new functionals over the classes $\mathcal{S}^*_{\tilde{\Sigma}}(\alpha)$, $\mathcal{K}_{\tilde{\Sigma}}(\alpha)$, $\mathcal{SS}^*_{\tilde{\Sigma}}(\beta)$ and $\mathcal{SK}_{\tilde{\Sigma}}(\beta)$, generated by polynomials.

For the $f \in \Sigma$ consider the expansion

$$\frac{\zeta f'(\zeta)}{f(\zeta) - w} = \sum_{n=0}^{\infty} \Phi_n(w)\zeta^{-n}, \quad \zeta \in \Delta. \tag{2}$$

The nth Faber polynomial Φ_n of the function $f \in \Sigma$ is a monic polynomial of degree n given by the formula

$$\Phi_n(w) = \sum_{k=0}^{n} F_{n,k}(f) w^k. \tag{3}$$

Since Φ_n is monic, there must be $F_{n,n}(f) = 1$. If f has the form (1), by dividing the expression $\zeta f'(\zeta)$ by $(f(\zeta) - w)$, the formulas Φ_i are of w as follows:

$$\Phi_0(w) = 1, \quad \Phi_1(w) = w - a_0, \quad \Phi_2(w) = w^2 - 2a_0 w + (a_0^2 - 2a_1), \tag{4}$$

$$\Phi_3(w) = w^3 - 3a_0 w^2 + (3a_0^2 - 3a_1)w + (-a_0^3 + 3a_1 a_0 - 3a_2) \tag{5}$$

and

$$\begin{aligned}\Phi_4(w) = w^4 - 4a_0 w^3 + (6a_0^2 - 4a_1)w^2 - 4(a_0^3 - 2a_0 a_1 + a_2)w \\ + (a_0^4 - 4a_0^2 a_1 + 2a_1^2 + 4a_0 a_2 - 4a_3).\end{aligned} \tag{6}$$

Moreover, if $f \in \tilde{\Sigma}$, then $a_0 = 0$ and we have

$$\Phi_0(w) = 1, \quad \Phi_1(w) = w, \quad \Phi_2(w) = w^2 - 2a_1, \quad \Phi_3(w) = w^3 - 3a_1 w - 3a_2$$

and

$$\Phi_4(w) = w^4 - 4a_1 w^2 - 4a_2 w + (2a_1^2 - 4a_3).$$

In this paper, we investigate the bounds of coefficients in $\Phi_n(w)$ for given functions in the classes $\mathcal{S}^*_{\tilde{\Sigma}}(\alpha)$, $\mathcal{SS}^*_{\tilde{\Sigma}}(\beta)$, $\mathcal{K}_{\tilde{\Sigma}}(\alpha)$ and $\mathcal{SK}_{\tilde{\Sigma}}(\beta)$. In Section 2, we will formulate the functional $F_{n,n-i}(f)$, $i \in \{1,2,3,4\}$ in terms of coefficients that appear in $f \in \Sigma$. Then sharp bounds $F_{n,n-i}(f)$, $i \in \{2,3,4\}$, for given f in $\mathcal{S}^*_{\tilde{\Sigma}}(\alpha)$ and $\mathcal{SS}^*_{\tilde{\Sigma}}(\beta)$ will be examined in Section 3. In Section 4, the sharp bounds $F_{n,n-i}(f)$, $i \in \{2,3,4\}$ over the classes $\mathcal{K}_{\tilde{\Sigma}}(\alpha)$ and $\mathcal{SK}_{\tilde{\Sigma}}(\beta)$ will be discussed.

Let $\mathcal{P}$ be a class of functions p:

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D} \tag{7}$$

such that $p(0) = 1$ and $p(z)$ is into the right-half plane $\mathbb{H} := \mathbb{H}_0 = \{w \in \mathbb{C} : \mathrm{Re}(w) > 0\}$. The following property for functions in $\mathcal{P}$ is well-known (e.g., [6], p. 41) and will be used for our considerations.

Lemma 1. *If $p \in \mathcal{P}$ and has the form (7), then the sharp inequality $|c_n| \le 2$ holds for $n \in \mathbb{N}$.*

Also, the following lemma for functions in $\mathcal{P}$ will be used for our proofs. It contains the well-known formula for c_2 (e.g., [6], p. 166), the formula for c_3 due to Libera and Zlotkiewicz [7,8] and the formula for c_4 found by the authors [9].

Lemma 2. *If $p \in \mathcal{P}$ is of the form (7) with $c_1 \in \mathbb{R}$ and $c_1 \ge 0$, then*

$$2c_2 = c_1^2 + \tau(4 - c_1^2), \tag{8}$$

$$4c_3 = c_1^3 + 2c_1(4 - c_1^2)\tau - c_1(4 - c_1^2)\tau^2 + 2(4 - c_1^2)(1 - |\tau|^2)\eta \tag{9}$$

and

$$\begin{aligned} 8c_4 = & c_1^4 + (4 - c_1^2)\tau\left[c_1^2(\tau^2 - 3\tau + 3) + 4\tau\right] \\ & - 4(4 - c_1^2)(1 - |\tau|^2)\left[c_1(\tau - 1)\eta + \overline{\tau}\eta^2 - \left(1 - |\eta|^2\right)\xi\right] \end{aligned} \tag{10}$$

for some $\tau, \eta, \xi \in \overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \le 1\}$.

2. Some Identities for Coefficients of Faber Polynomials

Let $f \in \Sigma$. Since $\Phi_n(w)$ is a monic polynomial of degree n, $F_{n,n}(f) = 1$ ($n \in \mathbb{N}_0$). Some initial coefficients of $\Phi_n(w)$ for early n can be obtained by the formulas in (4)–(6). For example, $F_{1,0}(f) = -a_0$, $F_{2,0}(f) = a_0^2 - 2a_1$ and $F_{2,1}(f) = -2a_0$. In fact, the functionals $F_{n,n-i}(f)$, $i \in \{1,2,3,4\}$, are obtained by (2) and (3), and are represented as follows.

$$F_{n,n-1}(f) = -na_0 \ (n \ge 1), \tag{11}$$

$$F_{n,n-2}(f) = \frac{1}{2}n(n-1)a_0^2 - na_1 \quad (n \ge 2), \tag{12}$$

$$F_{n,n-3}(f) = -\frac{1}{6}n(n-1)(n-2)a_0^3 + n(n-2)a_0a_1 - na_2 \quad (n \ge 3) \tag{13}$$

and

$$\begin{aligned} F_{n,n-4}(f) = & \frac{1}{24}n(n-1)(n-2)(n-3)a_0^4 - \frac{1}{2}n(n-2)(n-3)a_0^2a_1 \\ & + \frac{1}{2}n(n-3)a_1^2 + n(n-3)a_0a_2 - na_3 \quad (n \ge 4). \end{aligned} \tag{14}$$

Indeed, from (2) and (3), we get the following identity (see also [6], p. 57):

$$\begin{aligned}\Phi_n(w) &= (w-a_0)^n - na_1(w-a_0)^{n-2} - na_2(w-a_0)^{n-3} + \cdots \\ &= w^n - na_0 w^{n-1} + \cdots .\end{aligned} \tag{15}$$

Hence, the Formula (11) follows from (15).

Next we will show that the formula for $F_{n,n-4}(f)$, $n \geq 4$, is given by (14). For this, we assume that the expressions (12) and (13) are true. When $n = 4$, the assertion is clear by (6). Suppose now that (14) holds for $4 \leq n \leq k$ and recall the following recurrence formula from (2) and (3) (see also [6], p. 57):

$$\Phi_{k+1}(w) = (w-a_0)\Phi_k(w) - \sum_{\nu=1}^{k-1} a_{k-\nu}\Phi_\nu(w) - (k+1)a_k. \tag{16}$$

By differentiating the both sides of (16), since $\Phi_\nu^{(k-3)}(w) = 0$ for $\nu \leq k-4$, we get

$$\Phi_{k+1}^{(k-3)}(0) = (k-3)\Phi_k^{(k-4)}(0) - \sum_{i=0}^{3} a_i \Phi_{k-i}^{(k-3)}(0). \tag{17}$$

By dividing the both sides of (17) by $(k-3)!$ and using $\Phi_n^{(k)}(0)/k! = F_{n,k}(f)$, we obtain

$$F_{k+1,k-3}(f) = F_{k,k-4}(f) - \sum_{i=0}^{3} a_i F_{k-i,k-3}(f).$$

Therefore, by using the equalities (11)–(13), we get

$$\begin{aligned}F_{k+1,k-3}(f) = &\frac{1}{24}k(k+1)(k-1)(k-2)a_0^4 - \frac{1}{2}(k+1)(k-1)(k-2)a_0^2a_1 \\ &+ \frac{1}{2}(k+1)(k-2)a_1^2 + (k+1)(k-2)a_0a_2 - (k+1)a_3,\end{aligned}$$

which means that (14) holds for $n = k+1$. Thus, it follows by induction that (14) holds for all $n \in \mathbb{N}$ with $n \geq 4$.

It now remains to be checked that the formulas for $F_{n,n-2}(f)$ and $F_{n,n-3}(f)$ are true. By a similar process with the above we can obtain the identities (12) and (13), and the detailed proofs of them are omitted.

3. Bounds for the Coefficient of Faber Polynomial of Meromorphic Starlike Functions

In this section we find the sharp bounds for $F_{n,n-i}(f)$, $i \in \{1,2,3,4\}$, where f is in $\mathcal{S}_\Sigma^*(\alpha)$ and $\mathcal{SS}_\Sigma^*(\beta)$.

From (11), we see that $|F_{n,n-1}(f)| \leq n|a_0|$ for $f \in \Sigma$. Then, for $f \in \mathcal{S}_\Sigma^*(\alpha)$, the inequality $|F_{n,n-1}(f)| \leq 2(1-\alpha)n$ follows from $|a_0| \leq 2(1-\alpha)$ [10], p. 232. Similarly, for $f \in \mathcal{SS}_\Sigma^*(\beta)$, by the inequality $|a_0| \leq 2\beta$ [10], p. 233, we have $|F_{n,n-1}(f)| \leq 2\beta n$.

Next, the following result gives the sharp bounds for $F_{n,n-i}(f)$, $i \in \{2,3,4\}$, of $f \in \mathcal{S}_\Sigma^*(\alpha)$.

Theorem 1. *Let $\alpha \in [0,1)$ and $f \in \mathcal{S}_\Sigma^*(\alpha)$ be of the form (1). Then the following inequalities hold:*

$$|F_{n,n-2}(f)| \leq (1-\alpha)(2\rho_2+1)n, \quad n \in \mathbb{N}\setminus\{1\}; \tag{18}$$

$$|F_{n,n-3}(f)| \leq \frac{2}{3}(1-\alpha)(1+\rho_3)(1+2\rho_3)n, \quad n \in \mathbb{N}\setminus\{1,2\}; \tag{19}$$

$$|F_{n,n-4}(f)| \leq \frac{1}{6}(1-\alpha)(1+\rho_4)(1+2\rho_4)(3+2\rho_4)n, \quad n \in \mathbb{N}\setminus\{1,2,3\}, \tag{20}$$

where $\rho_k = (1-\alpha)(n-k)$, $k \in \{2,3,4\}$. *All the results are sharp and the equalities hold for the function* f_1 *given with*

$$f_1(\zeta) = \zeta(1-\zeta^{-1})^{2(1-\alpha)}, \quad \zeta \in \Delta. \tag{21}$$

Before proving the above result, let us recall the notion of the subordination. For analytic functions f and g we say that f is subordinate to g and write $f \prec g$, if there is an analytic function $\omega : \mathbb{D} \to \mathbb{D}$ with $\omega(0) = 0$ such that $f = g \circ \omega$ on $\mathbb{D}$. If g is univalent, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$.

The following lemma is a special case of more general results due to ([3], Theorem 1) and will be used to obtain our results in this section.

Lemma 3. *Let* $\varphi(z) = 1 + \sum_{n=1}^{\infty} B_n z^n$ *belong to* $\mathcal{P}$. *If* f *has the form* (1) *and satisfies* $-zg'(z)/g(z) \prec \varphi(z)$, *where* $g(z) = f(1/z)$, *then*

$$|a_1 - \gamma a_0^2| \le \frac{1}{2}|B_1| \cdot \max\left\{1, \left|\frac{B_2}{B_1} - (1-2\gamma)B_1\right|\right\}.$$

This result is sharp.

Here, note that the condition $-zg'(z)/g(z) \prec \varphi(z)$ in Lemma 3 is well-defined since the function $-zg'(z)/g(z)$ has a removable singularity at $z = 0$ and

$$\lim_{z\to 0}\left(-\frac{zg'(z)}{g(z)}\right) = 1 = \varphi(0).$$

Now we prove Theorem 1.

Proof of Theorem 1. Let $f \in \mathcal{S}_\Sigma^*(\alpha)$ be of the form (1) and $g(z) = f(1/z)$, $z \in \mathbb{D}^*$.

Since $F_{n,n-2}(f) = -n[a_1 - ((n-1)/2)a_0^2]$ and $-zg'(z)/g(z) \prec \varphi(z)$, where $\varphi \in \mathcal{P}$ is the function defined by

$$\varphi(z) = \frac{1+(1-2\alpha)z}{1-z} = 1 + 2(1-\alpha)\sum_{n=1}^{\infty} z^n,$$

by applying Lemma 3 with $B_1 = 2(1-\alpha) = B_2$ and $\gamma = (n-1)/2$, we have the inequality (18).

By dividing the expands in numerator and denominator, we note that

$$\begin{aligned}\frac{\zeta f'(\zeta)}{f(\zeta)} &= 1 - a_0\zeta^{-1} + (a_0^2 - 2a_1)\zeta^{-2} + (-a_0^3 + 3a_0a_1 - 3a_2)\zeta^{-3} \\ &\quad + (a_0^4 - 4a_0^2a_1 + 2a_1^2 + 4a_0a_2 - 4a_3)\zeta^{-4} + \cdots, \quad \zeta \in \Delta.\end{aligned} \tag{22}$$

Since $f \in \mathcal{S}_\Sigma^*(\alpha)$ and $g(z) = f(\zeta)$, where $z = 1/\zeta \in \mathbb{D}^*$, we have

$$\mathrm{Re}\left\{\frac{1}{1-\alpha}\left(-\frac{zg'(z)}{g(z)} - \alpha\right)\right\} > 0, \quad z \in \mathbb{D}^*. \tag{23}$$

Recall that the function $-zg'(z)/g(z)$ has a removable singularity at $z = 0$ and

$$\lim_{z\to 0}\frac{1}{1-\alpha}\left(-\frac{zg'(z)}{g(z)} - \alpha\right) = 1.$$

Therefore, the inequality (23) holds for all $z \in \mathbb{D}$ and there exists a function $p \in \mathcal{P}$ such that

$$\frac{1}{1-\alpha}\left(-\frac{zg'(z)}{g(z)} - \alpha\right) = p(z), \quad z \in \mathbb{D}. \tag{24}$$

Since $\zeta f'(\zeta)/f(\zeta) = -zg'(z)/g(z)$, where $\zeta = 1/z$, if p has the form given by (7), then (24) implies that

$$\frac{\zeta f'(\zeta)}{f(\zeta)} = 1 + (1-\alpha)\sum_{n=1}^{\infty} c_n \zeta^{-n}, \quad \zeta \in \Delta. \tag{25}$$

Equating the coefficients in (22) and (25), we get

$$a_0 = -(1-\alpha)c_1, \quad a_1 = \frac{1}{2}(1-\alpha)[(1-\alpha)c_1^2 - c_2], \tag{26}$$

$$a_2 = \frac{1}{6}(1-\alpha)[-(1-\alpha)^2 c_1^3 + 3(1-\alpha)c_1c_2 - 2c_3] \tag{27}$$

and

$$a_3 = \frac{1}{24}(1-\alpha)[(1-\alpha)^3c_1^4 - 6(1-\alpha)^2c_1^2c_2 + 3(1-\alpha)c_2^2 + 8(1-\alpha)c_1c_3 - 6c_4]. \tag{28}$$

Let $n \in \mathbb{N}$ with $n \geq 3$. By substituting the expressions (26) and (27) into (13), we obtain

$$F_{n,n-3}(f) = \frac{1}{6}(1-\alpha)n[(1-\alpha)^2(n-3)^2c_1^3 + 3(1-\alpha)(n-3)c_1c_2 + 2c_3].$$

Therefore, it follows from the triangle inequality and Lemma 1 that the inequality (19) holds.

Next, let $n \in \mathbb{N}$ with $n \geq 4$. By using the Equations (26)–(28) and (14), we have

$$F_{n,n-4}(f) = \frac{1}{24}(1-\alpha)n[\lambda_5c_1^4 + \lambda_4c_1^2c_2 + \lambda_3c_2^2 + \lambda_2c_1c_3 + \lambda_1c_4],$$

where $\lambda_5 = \rho_4^3$, $\lambda_4 = 6\rho_4^2$, $\lambda_3 = 3\rho_4$, $\lambda_2 = 8\rho_4$ and $\lambda_1 = 6$. Since $\lambda_i \geq 0$ for all $i \in \{1,2,3,4,5\}$, the inequality (20) follows from the triangle inequality and Lemma 1.

The function f_1 defined by (21) has the form (1) with

$$a_0 = -2(1-\alpha), \quad a_1 = 1 - 3\alpha + 2\alpha^2, \quad a_2 = \frac{2}{3}\alpha(1 - 3\alpha + 2\alpha^2)$$

and

$$a_3 = \frac{1}{6}\alpha(1 - \alpha - 4\alpha^2 + 4\alpha^3).$$

Putting these quantities into (12)–(14), we get

$$F_{n,n-2}(f_1) = (1-\alpha)(2n + 4\alpha - 2\alpha n - 3)n,$$

$$F_{n,n-3}(f_1) = \frac{2}{3}(1-\alpha)n[2(1-\alpha)^2(n-3)^2 + 3(1-\alpha)(n-3) + 1]$$

and

$$F_{n,n-4}(f_1) = \frac{1}{6}(1-\alpha)n[4(1-\alpha)^3(n-4)^3 + 12(1-\alpha)^2(n-4)^2 + 11(1-\alpha)(n-4) + 3],$$

respectively, which show that the inequalities (18)–(20) are sharp. The proof of Theorem 1 is now completed. □

The sharp bounds for $F_{n,n-i}(f)$, $i \in \{2,3,4\}$, where $f \in \mathcal{SS}^*_\Sigma(\beta)$, are given as in the following theorem.

Theorem 2. *Let $\beta \in (0,1]$ and $f \in \mathcal{SS}^*_\Sigma(\beta)$. Then*

$$|F_{n,n-2}(f)| \leq \beta n \cdot \max\{1, \beta(2n-3)\}, \quad n \in \mathbb{N} \setminus \{1\}. \tag{29}$$

If β and n satisfy one of the following conditions:

(i) $3 \leq n \leq (14\beta+1)/(6\beta)$;
(ii) $(14\beta+1)/(6\beta) \leq n \leq (7\beta+2)/(3\beta)$ *and* $\beta^2(6n^2-27n+29) \leq 2$,

then we have

$$|F_{n,n-3}(f)| \leq \frac{2}{3}\beta n, \quad n \in \mathbb{N} \setminus \{1,2\}. \tag{30}$$

If β and n are satisfying one of the following conditions:

(iii) $n \geq (7\beta+2)/(3\beta)$;
(iv) $(14\beta+1)/(6\beta) \leq n \leq (7\beta+2)/(3\beta)$ *and* $\beta^2(6n^2-27n+29) \geq 2$,

then we have

$$|F_{n,n-3}(f)| \leq \frac{2}{9}\beta n[1+\beta^2(29-27n+6n^2)], \quad n \in \mathbb{N} \setminus \{1,2\}. \tag{31}$$

The inequalities (29)–(31) are sharp.

Let $\mathcal{B}_0$ be a class of Schwarz functions ω:

$$\omega(z) = \sum_{n=1}^{\infty} d_n z^n, \quad z \in \mathbb{D}, \tag{32}$$

such that $\omega(0) = 0$ and $\omega(z) \in \mathbb{D}$. Then $\omega \in \mathcal{B}_0$ if and only if $p(z) := (1+\omega(z))/(1-\omega(z)) \in \mathcal{P}$. The following property for the Schwarz functions will be used for our proof of Theorem 2.

Lemma 4 ([11], Prokhorov and Szynal)**.** *If $\omega \in \mathcal{B}_0$ has the form (32), then for any real numbers μ and ν the following sharp estimate holds:*

$$\Psi(\mu,\nu) := |d_3 + \mu d_1 d_2 + \nu d_1^3| \leq \hat{\Psi}(\mu,\nu), \tag{33}$$

where

$$\hat{\Psi}(\mu,\nu) := \begin{cases} 1, & (\mu,\nu) \in D_1 \cup D_2 \cup \{(2,1)\}, \\ |\nu|, & (\mu,\nu) \in \bigcup_{k=3}^{7} D_k, \\ \frac{2}{3}(|\mu|+1)\left(\frac{|\mu|+1}{3(|\mu|+1+\nu)}\right)^{1/2}, & (\mu,\nu) \in D_8 \cup D_9, \\ \frac{1}{3}\nu\left(\frac{\mu^2-4}{\mu^2-4\nu}\right)\left(\frac{\mu^2-4}{3(\nu-1)}\right)^{1/2}, & (\mu,\nu) \in D_{10} \cup D_{11} \setminus \{(2,1)\}, \\ \frac{2}{3}(|\mu|-1)\left(\frac{|\mu|-1}{3(|\mu|-1-\nu)}\right)^{1/2}, & (\mu,\nu) \in D_{12}. \end{cases} \tag{34}$$

Here, the sets $D_i \subset \mathbb{R}^2$, $i \in \{1,2,\cdots,12\}$, are defined as follows.

$$D_1 = \left\{(\mu,\nu) \in \mathbb{R}^2 : |\mu| \leq \frac{1}{2}, |\nu| \leq 1\right\},$$

$$D_2 = \left\{(\mu,\nu) \in \mathbb{R}^2 : \frac{1}{2} \leq |\mu| \leq 2, \frac{4}{27}(|\mu|+1)^3 - (|\mu|+1) \leq \nu \leq 1\right\},$$

$$D_3 = \left\{(\mu,\nu) \in \mathbb{R}^2 : |\mu| \leq \frac{1}{2}, \nu \leq -1\right\},$$

$$D_4 = \left\{(\mu,\nu) \in \mathbb{R}^2 : |\mu| \geq \frac{1}{2}, \nu \leq -\frac{2}{3}(|\mu|+1)\right\},$$

$$D_5 = \left\{(\mu,\nu) \in \mathbb{R}^2 : |\mu| \leq 2, \nu \geq 1\right\},$$

$$D_6 = \left\{(\mu,\nu) \in \mathbb{R}^2 : 2 \leq |\mu| \leq 4, \nu \geq \frac{1}{12}(\mu^2+8)\right\},$$

$$D_7 = \left\{ (\mu, \nu) \in \mathbb{R}^2 : |\mu| \geq 4, \nu \geq \frac{2}{3}(|\mu| - 1) \right\},$$

$$D_8 = \left\{ (\mu, \nu) \in \mathbb{R}^2 : \frac{1}{2} \leq |\mu| \leq 2, -\frac{2}{3}(|\mu| + 1) \leq \nu \leq \frac{4}{27}(|\mu| + 1)^3 - (|\mu| + 1) \right\},$$

$$D_9 = \left\{ (\mu, \nu) \in \mathbb{R}^2 : |\mu| \geq 2, -\frac{2}{3}(|\mu| + 1) \leq \nu \leq \frac{2|\mu|(|\mu| + 1)}{\mu^2 + 2|\mu| + 4} \right\},$$

$$D_{10} = \left\{ (\mu, \nu) \in \mathbb{R}^2 : 2 \leq |\mu| \leq 4, \frac{2|\mu|(|\mu| + 1)}{\mu^2 + 2|\mu| + 4} \leq \nu \leq \frac{1}{12}(\mu^2 + 8) \right\},$$

$$D_{11} = \left\{ (\mu, \nu) \in \mathbb{R}^2 : |\mu| \geq 4, \frac{2|\mu|(|\mu| + 1)}{\mu^2 + 2|\mu| + 4} \leq \nu \leq \frac{2|\mu|(|\mu| - 1)}{\mu^2 - 2|\mu| + 4} \right\},$$

$$D_{12} = \left\{ (\mu, \nu) \in \mathbb{R}^2 : |\mu| \geq 4, \frac{2|\mu|(|\mu| - 1)}{\mu^2 - 2|\mu| + 4} \leq \nu \leq \frac{2}{3}(|\mu| - 1) \right\}.$$

Now we prove Theorem 2.

Proof of Theorem 2. Let $\beta \in (0, 1]$ and $f \in \mathcal{SS}^*_\Sigma(\beta)$. Further, $g(z) = f(1/z), z \in \mathbb{D}^*$. Since $-zg'(z)/g(z) \prec \varphi(z)$, where $\varphi \in \mathcal{P}$ is the function defined by

$$\varphi(z) = \left(\frac{1+z}{1-z} \right)^\beta = 1 + 2\beta z + 2\beta^2 z^2 + \cdots,$$

the inequality (29) follows from (12) and Lemma 3 with $B_1 = 2\beta$, $B_2 = 2\beta^2$ and $\gamma = (n-1)/2$.

Since $f \in \mathcal{SS}^*_\Sigma(\beta)$, we have

$$\mathrm{Re} \left\{ \left(\frac{\zeta f'(\zeta)}{f(\zeta)} \right)^{1/\beta} \right\} > 0, \quad \zeta \in \Delta.$$

By a similar argument with the proof of Theorem 1, there exists a function $p \in \mathcal{P}$ such that

$$\frac{\zeta f'(\zeta)}{f(\zeta)} = (p(1/\zeta))^\beta, \quad \zeta \in \Delta. \tag{35}$$

Here, we choose the branch of functions $z \mapsto (p(z))^\beta$ for $z \in \mathbb{D}$, so that $p(0)^\beta = 1$.

Let p have the form given by (7). Then, by the Laurent queue for $(p(z))^\beta$ and by equating the coefficients in (35), we obtain

$$a_0 = -\beta c_1, \quad a_1 = \frac{1}{4}\beta[(1+\beta)c_1^2 - 2c_2] \tag{36}$$

and

$$a_2 = \frac{1}{36}\beta[(-4 - 3\beta + \beta^2)c_1^3 + 6(2+\beta)c_1 c_2 - 12c_3]. \tag{37}$$

Let $n \in \mathbb{N}$ with $n \geq 3$. By using the equalities (13), (36) and (37) we have

$$F_{n,n-3}(f) = \frac{1}{36}\beta n \cdot [12c_3 + \kappa_1 c_1 c_2 + \kappa_2 c_1^3], \tag{38}$$

where

$$\kappa_1 = 6[-2 + \beta(-7 + 3n)]$$

and

$$\kappa_2 = 4 + \beta(21 - 9n) + \beta^2(29 - 27n + 6n^2).$$

Note that $\kappa_2 \geq 0$ for $n \geq 3$.

When the condition (iii) is satisfied, we have $\kappa_1 \geq 0$. Therefore, the inequality (31) follows from the triangle inequality and Lemma 1.

Now, let $n < (7\beta+2)/(3\beta)$. Let $\omega(z) = (p(z)-1)/(p(z)+1)$ and suppose ω has the form given by (32). Using the relations

$$c_1 = 2d_1, \quad c_2 = 2(d_1^2 + d_2) \quad \text{and} \quad c_3 = 2(d_1^3 + 2d_1d_2 + d_3),$$

together with (38), we obtain

$$F_{n,n-3}(f) = \frac{2}{3}\beta n \Psi(\mu, \nu), \tag{39}$$

where Ψ is defined by (33) with

$$\mu = \beta(3n-7) \quad \text{and} \quad \nu = \frac{1}{3}[1 + \beta^2(6n^2 - 27n + 29)]. \tag{40}$$

Suppose that (i) is satisfied. Then it holds that $0 < \mu \leq 1/2$ and $0 < \nu < 1$. Indeed, let $I_\beta = [3, (14\beta+1)/(6\beta)]$ and consider a function $k : I_\beta \to \mathbb{R}$ defined by

$$k(x) = \frac{1}{3}[1 + \beta^2(6x^2 - 27x + 29)].$$

Then $k(x)$ increases on I_β. Thus, we have

$$0 < \frac{43}{123} = k(3) \leq k(x) \leq k\left(\frac{14\beta+1}{6\beta}\right) = \frac{1}{18}(7 + 8 - 8\beta^2) \leq \frac{25}{64} < 1$$

for $x \in I_\beta$, which leads us to get $0 < \nu < 1$. Therefore, we have $(\mu, \nu) \in D_1$, and it follows from (39) and Lemma 4 that the inequality (30) holds.

Now consider the case $(14\beta+1)/(6\beta) \leq n \leq (7\beta+2)/(3\beta)$. In this case, we have $1/2 \leq \mu < 2$. Therefore, we get

$$-4\mu^3 + 15\mu + 32 \geq 30. \tag{41}$$

Moreover it is observed that

$$-3\beta^2(18n^2 - 87n + 109) \geq -6(4 - \beta + 2\beta^2) \geq -30. \tag{42}$$

By combining (40), (41) and (42), we have

$$\begin{aligned} &27\nu - 4(\mu+1)^3 + 27(\mu+1) \\ &= -4\mu^3 + 15\mu + 32 - 3\beta^2(18n^2 - 87n + 109) \geq 0, \end{aligned}$$

which implies that $\nu \geq (4/27)(\mu+1)^3 - (\mu+1)$. Now, if $\beta^2(6n^2 - 27n + 29) \leq 2$, then $\nu \leq 1$ and $(\mu, \nu) \in D_2$. Thus, it follows from (39) and Lemma 4 that the inequality (30) holds. If $\beta^2(6n^2 - 27n + 29) \geq 2$, then $\nu \geq 1$ and $(\mu, \nu) \in D_5$. Therefore, by Lemma 4, we obtain the inequality (31).

Finally, let us consider the sharpness of this result. For given $m \in \mathbb{N}$, define a function $g_m : \mathbb{D}^* \to \mathbb{C}$ by

$$g_m(z) = \frac{1}{z}\exp\left[-\int_0^z \frac{1}{t}\left(\left(\frac{1-t^m}{1+t^m}\right)^\beta - 1\right)\mathrm{d}t\right] \tag{43}$$

and let $\hat{f}_m(\zeta) = g_m(1/\zeta)$, $\zeta \in \Delta$. Then we get

$$\hat{f}_1(\zeta) = \zeta - 2\beta + \beta^2\zeta^{-1} - \frac{2}{9}\beta(1-\beta^2)\zeta^{-2} + \frac{1}{9}\beta^2(1-\beta^2)\zeta^{-3} + \cdots, \quad z \in \Delta,$$

$$\hat{f}_2(\zeta) = \zeta - \beta\zeta^{-1} - \frac{1}{9}\beta(1-\beta^2)\zeta^{-5} + \cdots, \quad z \in \Delta$$

and

$$\hat{f}_3(\zeta) = \zeta - \frac{2}{3}\beta\zeta^{-2} - \frac{1}{9}\beta^2\zeta^{-5} + \cdots, \quad z \in \Delta.$$

Hence, from (11)–(14), we have

$$F_{n,n-2}(\hat{f}_1) = \beta^2(2n-3)n, \quad F_{n,n-2}(\hat{f}_2) = \beta n, \quad F_{n,n-3}(\hat{f}_3) = 2\beta n/3$$

and

$$F_{n,n-3}(\hat{f}_1) = \frac{2}{9}\beta n[1 + \beta^2(29 - 27n + 6n^2)].$$

The inequality (29) is sharp for the function $\hat{f}_2$ when $1 \geq \beta(2n-3)$ and for the function $\hat{f}_1$ when $1 \leq \beta(2n-3)$. When β and n satisfy the condition (i) or (ii), the equality in (30) holds for $\hat{f}_3$. In addition, the equality in (31) holds for $\hat{f}_1$, when β and n satisfy the condition (iii) or (iv). The proof of Theorem 2 is completed. □

4. Bounds for the Coefficient of Faber Polynomial of Meromorphic Convex Functions

In this section we find the sharp bounds for $F_{n,n-i}(f)$, $i \in \{2,3,4\}$, of f in $\mathcal{K}_{\tilde{\Sigma}}(\alpha)$ and $\mathcal{SK}_{\tilde{\Sigma}}(\beta)$. We find the sharp bounds for the functional $a_3 - \gamma a_1^2$ of f in $\mathcal{K}_{\tilde{\Sigma}}(\alpha)$ and $\mathcal{SK}_{\tilde{\Sigma}}(\beta)$ for our investigations.

Proposition 1. *Let $\alpha \in [0,1)$ and $\gamma \in \mathbb{R}$. If $f \in \mathcal{K}_{\tilde{\Sigma}}(\alpha)$, then*

$$|a_3 - \gamma a_1^2| \leq \frac{1}{6}(1-\alpha)\max\{1, |\alpha - 6\gamma + 6\alpha\gamma|\}. \tag{44}$$

This result is sharp.

Proof. Suppose $f \in \mathcal{K}_{\tilde{\Sigma}}(\alpha)$. Then we have

$$1 + \frac{\zeta f''(\zeta)}{f'(\zeta)} = 1 + 2a_1\zeta^{-2} + 6a_2\zeta^{-3} + 2(a_1^2 + 6a_3)\zeta^{-4} + \cdots, \quad \zeta \in \Delta. \tag{45}$$

Since $f \in \mathcal{K}_{\tilde{\Sigma}}(\alpha)$, a similar argument of the proof of Theorem 1 implies that there exists a function $p \in \mathcal{P}$ such that

$$\frac{1}{1-\alpha}\left(1 + \frac{\zeta f''(\zeta)}{f'(\zeta)} - \alpha\right) = p(1/\zeta), \quad \zeta \in \Delta.$$

Let p have the form given by (7). Then

$$(1-a)p(1/\zeta) + \alpha = 1 + (1-\alpha)\sum_{n=1}^{\infty} c_n\zeta^{-n}, \quad \zeta \in \Delta. \tag{46}$$

Therefore, by equating the coefficients in (45) and (46) we get $c_1 = 0$,

$$a_1 = \frac{1}{2}(1-\alpha)c_2, \quad a_2 = \frac{1}{6}(1-\alpha)c_3 \quad \text{and} \quad a_3 = -\frac{1}{24}(1-\alpha)^2c_2^2 + \frac{1}{12}(1-\alpha)c_4. \tag{47}$$

Since $c_1 = 0$, by Lemma 2, we have

$$c_2 = 2\tau \quad \text{and} \quad c_4 = 2\tau^2 - 2(1-|\tau|^2)\left(\overline{\tau}\eta^2 - (1-|\eta|^2)\xi\right), \tag{48}$$

where $\tau, \eta, \xi \in \overline{\mathbb{D}}$. Substituting (48) into (47) we obtain

$$\frac{6}{1-\alpha}(a_3 - \gamma a_1^2) = (\alpha - 6\gamma + 6\alpha\gamma)\tau^2 - (1-|\tau|^2)\overline{\tau}\eta^2 + (1-|\tau|^2)(1-|\eta|^2)\xi. \tag{49}$$

Taking the absolute values of the both sides in (49) and the triangle inequality together with $|\xi| \leq 1$ yield that

$$|a_3 - \gamma a_1^2| \leq \frac{1}{6}(1-\alpha)H_1(|\tau|, |\eta|), \tag{50}$$

where $H_1 : [0,1] \times [0,1] \to \mathbb{R}$ is a function defined by

$$H_1(x,y) = |\alpha - 6\gamma + 6\alpha\gamma|x^2 + (1-x^2)xy^2 + (1-x^2)(1-y^2).$$

A simple computation gives us to get

$$\begin{aligned} H_1(x,y) \leq H_1(x,0) &= (|\alpha - 6\gamma + 6\alpha\gamma| - 1)\, x^2 + 1 \\ &= \max\{1, |\alpha - 6\gamma + 6\alpha\gamma|\}, \quad (x,y) \in [0,1] \times [0,1]. \end{aligned} \tag{51}$$

Since $\tau, \eta \in \overline{\mathbb{D}}$, it follows from (50) and (51) that the inequality (44) holds.

Now, consider a function $\tilde{f}_1 : \Delta \to \mathbb{C}$ such that $\tilde{f}_1'(\zeta) = (1-\zeta^{-4})^{(1-\alpha)/2}$. Then we have $\tilde{f}_1 \in \mathcal{K}_{\tilde{\Sigma}}(\alpha)$ and

$$\tilde{f}_1(\zeta) = \zeta + \frac{1}{6}(1-\alpha)\zeta^{-3} + \cdots, \quad \zeta \in \Delta,$$

which implies that $a_3 - \gamma a_1^2 = (1-\alpha)/6$. This shows that the inequality (44) is sharp for $\tilde{f}_1$ when $|\alpha - 6\gamma + 6\alpha\gamma| \leq 1$. Next we consider a function $\tilde{f}_2 : \Delta \to \mathbb{C}$ such that $\tilde{f}_2'(\zeta) = (1-\zeta^{-2})^{1-\alpha}$. Then we have $a_1 = 1-\alpha$ and $a_3 = \alpha(1-\alpha)/6$, which implies that

$$a_3 - \gamma a_1^2 = \frac{1}{6}(1-\alpha)(\alpha - 6\gamma + 6\alpha\gamma).$$

Thus, when $|\alpha - 6\gamma + 6\alpha\gamma| \geq 1$, the inequality (44) is sharp with the extremal function $\tilde{f}_2$ and it completes the proof of Proposition 1. □

Proposition 2. *Let $\beta \in (0,1]$ and $\gamma \in \mathbb{R}$. If $f \in \mathcal{SK}_{\tilde{\Sigma}}(\beta)$ has the form given by (1), then*

$$|a_3 - \gamma a_1^2| \leq \frac{\beta}{6} \cdot \max\{1, 6\beta|\gamma|\}. \tag{52}$$

This result is sharp.

Proof. Let $f \in \mathcal{SK}_{\tilde{\Sigma}}(\beta)$. Then, by a similar argument as in the proof of Theorem 1, we have

$$1 + \frac{\zeta f''(\zeta)}{f'(\zeta)} = (p(1/\zeta))^{\beta}, \quad \zeta \in \Delta, \tag{53}$$

for some $p \in \mathcal{P}$. If p is of the form (7), then we get $c_1 = 0$ from (53) and

$$a_1 = \frac{1}{2}\beta c_2, \quad a_2 = \frac{1}{6}\beta c_3 \quad \text{and} \quad a_3 = -\frac{1}{24}\beta c_2^2 + \frac{1}{12}\beta c_4.$$

Therefore, we have

$$a_3 - \gamma a_1^2 = \beta\left[-\left(\frac{1}{24} + \frac{1}{4}\beta\gamma\right)c_2^2 + \frac{1}{12}c_4\right].$$

Using the relations in (48), we have

$$(6/\beta)(a_3 - \gamma a_1^2) = -6\beta\gamma\tau^2 - (1-|\tau|^2)\overline{\tau}\eta^2 + (1-|\tau|^2)(1-|\eta|^2)\xi$$

with $\tau, \eta, \xi \in \overline{\mathbb{D}}$. Therefore, we get

$$|a_3 - \gamma a_1^2| \leq \frac{\beta}{6} \cdot H_2(|\tau|, |\eta|), \tag{54}$$

where $H_2 : [0,1] \times [0,1] \to \mathbb{R}$ is a function defined by

$$H_2(x, y) = 6\beta|\gamma|x^2 + (1-x^2)xy^2 + (1-x^2)(1-y^2).$$

Since

$$H_2(x, y) \leq \max\{1, 6\beta|\gamma|\}, \quad (x, y) \in [0,1] \times [0,1],$$

the inequality (52) follows from (54).

Finally, we will show that this result is sharp. Consider a function $\tilde{f}_3 \in \mathcal{SK}_{\tilde{\Sigma}}(\beta)$ such that $\zeta\tilde{f}_3'(\zeta) = g_2(1/\zeta), \zeta \in \Delta$, where g_2 is the function defined by (43) with $m = 2$. Then $\tilde{f}_3$ is represented by

$$\tilde{f}_3(\zeta) = \zeta + \beta\zeta^{-1} + \frac{1}{45}\beta(1-\beta^2)\zeta^{-5} + \cdots, \quad \zeta \in \Delta.$$

Thus, $a_3 - \gamma a_1^2 = -\beta^2\gamma$ and the function $\tilde{f}_3$ which makes the equality in (52) when $6\beta|\gamma| \geq 1$. Next, let us consider a function $\tilde{f}_4 \in \mathcal{SK}_{\tilde{\Sigma}}(\beta)$ such that $\zeta\tilde{f}_4'(\zeta) = g_4(1/\zeta)$, $\zeta \in \Delta$, where g_4 is the function defined by (43) with $m = 4$. Then we have

$$\tilde{f}_4(\zeta) = \zeta + \frac{\beta}{6}\zeta^{-3} + \frac{\beta^2}{56}\zeta^{-7} + \cdots, \quad \zeta \in \Delta,$$

or $a_3 - \gamma a_1^2 = \beta/6$. Thus, it follows that the inequality (52) is sharp with the extremal function $\tilde{f}_4$ for the case $6\beta|\gamma| \leq 1$. Thus, the proof of Proposition 2 is completed. □

Now we obtain the sharp bounds for $F_{n,n-i}(f)$, $i \in \{2,3,4\}$, of f in $\mathcal{K}_{\tilde{\Sigma}}(\alpha)$ and $\mathcal{SK}_{\tilde{\Sigma}}(\beta)$.

Theorem 3. *Let $f \in \mathcal{K}_{\tilde{\Sigma}}(\alpha)$. Then the following sharp inequalities hold for $n \in \mathbb{N}$.*

(i) $|F_{n,n-2}(f)| \leq (1-\alpha)n$ *for* $n \geq 2$;
(ii) $|F_{n,n-3}(f)| \leq (1-\alpha)n/3$ *for* $n \geq 3$;
(iii) $|F_{n,n-4}(f)| \leq ((1-\alpha)n/6) \cdot \max\{1, |\alpha - 3(n-3)(1-\alpha)|\}$ *for* $n \geq 4$.

Proof. Since $F_{n,n-2}(f) = -na_1$ and $F_{n,n-3}(f) = -na_2$ for $f \in \tilde{\Sigma}$, the inequalities in (i) and (ii) follows from (47) and Lemma 1. Next we note that $|F_{n,n-4}| = n \cdot |a_3 - ((n-3)/2)a_1^2|$. Therefore, by Proposition 1 with $\gamma = (n-3)/2$, we obtain the inequality in (iii). □

Theorem 4. *Let $f \in \mathcal{SK}_{\tilde{\Sigma}}(\beta)$ be of the form (1). Then the following sharp inequalities hold for $n \in \mathbb{N}$.*

(i) $|F_{n,n-2}(f)| \leq \beta n$ *for* $n \geq 2$;
(ii) $|F_{n,n-3}(f)| \leq \beta n/3$ *for* $n \geq 3$;
(iii) $|F_{n,n-4}(f)| \leq (\beta n/6) \cdot \max\{1, 3\beta(n-3)\}$ *for* $n \geq 4$.

We will finish our paper by giving the sharp bounds of $F_{n,n-i}(f)$, $i \in \{2,3,4\}$, for a starlike function $f \in \tilde{\Sigma}$ of order α ($\alpha \in [0,1)$), or a strongly starlike function $f \in \tilde{\Sigma}$ of order β ($\beta \in (0,1]$).

Theorem 5. *Let $f \in \mathcal{S}^*_\Sigma(\alpha) \cap \tilde{\Sigma}$. Then the following sharp inequalities hold for $n \in \mathbb{N}$.*

(i) $|F_{n,n-2}(f)| \le (1-\alpha)n$ *for* $n \ge 2$;
(ii) $|F_{n,n-3}(f)| \le 2(1-\alpha)n/3$ *for* $n \ge 3$;
(iii) $|F_{n,n-4}(f)| \le ((1-\alpha)n/2) \cdot \max\{1, |\alpha(4-n)+n-3|\}$ *for* $n \ge 4$.

Proof. Let

$$g(\zeta) = \int_{\zeta_0}^{\zeta} \frac{f(t)}{t} \mathrm{d}t, \quad \zeta \in \Delta,$$

where ζ_0 is determined so that $g(\zeta) = \zeta + \sum_{n=1}^{\infty} b_n \zeta^{-n}$. From $f \in \mathcal{S}^*_\Sigma(\alpha) \cap \tilde{\Sigma}$, we have $g \in \mathcal{K}_{\tilde{\Sigma}}(\alpha)$. Furthermore we have $a_n = -nb_n$ for $n \in \mathbb{N}$. Therefore, the relations $F_{n,n-2}(f) = -F_{n,n-2}(g)$ and $F_{n,n-3}(f) = -2F_{n,n-3}(g)$ hold. Hence, by Theorem 3, we obtain the inequalities in (i) and (ii). Next, we note that

$$|F_{n,n-4}(f)| = \left| \frac{1}{2} n(n-3) a_1^2 - n a_3 \right| = 3n \left| b_3 + \frac{1}{6}(n-3) b_1^2 \right|.$$

Then it follows from Proposition 1 with $\gamma = -(n-3)/6$ that the inequality in (iii) holds. □

Theorem 6. *Let $f \in \mathcal{SS}^*_\Sigma(\beta) \cap \tilde{\Sigma}$ be of the form (1). Then the following sharp inequalities hold for $n \in \mathbb{N}$.*

(i) $|F_{n,n-2}(f)| \le \beta n$ *for* $n \ge 2$;
(ii) $|F_{n,n-3}(f)| \le 2\beta n/3$ *for* $n \ge 3$;
(iii) $|F_{n,n-4}(f)| \le (n\beta/2) \cdot \max\{1, \beta(n-3)\}$ *for* $n \ge 4$.

Proof. The assertions given above can be proved by similar processes with the proof of Theorem 5. □

5. Conclusions

In the present paper, we obtained the sharp inequalities for $F_{n,n-i}(f)$, $n \in \mathbb{N}_0$, $i \in \{1,2,3,4\}$, where $F_{n,i}(f)$ is the ith coefficient of the Faber polynomial of a meromorphic function $f \in \Sigma$, which are starlike (or convex) functions of order α ($\alpha \in [0,1)$) and strongly starlike (or convex) functions of order β ($\beta \in (0,1]$). In particular, we observed that the sharp inequality $|F_{n,n-i}(f)| \le |F_{n,n-i}(f_1)|$, where f_1 is the function defined by (21), holds for $i \in \{1,2,3,4\}$ and $f \in \mathcal{S}^*_\Sigma(\alpha)$. Hence, it can be naturally expected that this sharp inequalty would hold for all $i \le n-1$.

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References

1. Pommerenke, C. On meromorphic starlike functions. *Pac. J. Math.* **1963**, *13*, 221–235. [CrossRef]
2. Brannan, D.A.; Clunie, J.; Kirwan, W.E. Coefficient estimates for a class of star-like functions. *Can. J. Math.* **1970**, *22*, 476–485. [CrossRef]
3. Aouf, M.K.; El-Ashwah, R.M.; Zayed, H.M. Fekete–Szegö inequalities for certain class of meromorphic functions. *J. Egypt. Math. Soc.* **2013**, *21*, 197–200. [CrossRef]
4. Silverman, H.; Suchithra, K.; Stephen, B.; Gangadharan, A. Coefficient bounds for certain classes of meromorphic functions. *J. Inequal. Pure Appl. Math.* **2008**, *2018*, 1–9. [CrossRef]
5. Ali, R.M.; Ravichandran, V. Classes of meromorphic α-convex functions. *Taiwan J. Math.* **2010**, *14*, 1479–1490. [CrossRef]
6. Pommerenke, C. *Univalent Functions*; Vandenhoeck and Ruprecht: Göttingen, Germany, 1975.

7. Libera, R.J.; Zlotkiewicz, E.J. Early coefficients of the inverse of a regular convex function. *Proc. Am. Math. Soc.* **1982**, *85*, 225–230. [CrossRef]
8. Libera, R.J.; Zlotkiewicz, E.J. Coefficient bounds for the inverse of a function with derivatives in $\mathcal{P}$. *Proc. Am. Math. Soc.* **1983**, *87*, 251–257. [CrossRef]
9. Kwon, O.S.; Lecko, A.; Sim, Y.J. On the fourth coefficient of functions in the Carathéodory class. *Comp. Meth. Funct. Theory* **2018**, *18*, 307–314. [CrossRef]
10. Goodman, A.W. *Univalent Functions*; Mariner Publishing Company: Orlando, FL, USA, 1983.
11. Prokhorov, D.V.; Szynal, J. Inverse Coefficients for (α,β)-Convex Functions. Available online: https://www.researchgate.net/publication/265427362_Inverse_coefficients_for_a_b_-convex_functions (accessed on 25 July 2019).