

## Article

# Some Inequalities Using Generalized Convex Functions in Quantum Analysis

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**Abstract:** In the present work, the Hermite–Hadamard inequality is established in the setting of quantum calculus for a generalized class of convex functions depending on three parameters: a number in  $(0, 1]$  and two arbitrary real functions defined on  $[0, 1]$ . From the proven results, various inequalities of the same type are deduced for other types of generalized convex functions and the methodology used reveals, in a sense, a symmetric mathematical phenomenon. In addition, the definition of dominated convex functions with respect to the generalized class of convex functions aforementioned is introduced, and some integral inequalities are established.

**Keywords:** integral inequalities;  $(m, h_1, h_2)$ –convex functions; dominated convexity; quantum calculus

**MSC:** primary 26A51; 26D15; 32F17; secondary 81Q99

## 1. Introduction

The quantum calculus was initiated by Euler in the 18th century (1707–1783), it is known as calculus with no limits. In 1910, a systematic study of  $q$ -calculus was presented by F. H. Jackson [1], in which the definition of  $q$ -integral was introduced. The studies made by T. Ernst in [2,3], H. Gauchman in [4], V. Kac in [5], and recently by M.E.H. Ismail [6,7] have enriched some branches of Mathematics and Physics.

The Hermite–Hadamard inequality provides a lower and an upper estimation for the integral average of any convex function defined on a compact interval, involving the midpoint and the endpoints of the domain. More precisely:

Let  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be a convex and integrable function on an interval  $I$ . Then the following inequality holds

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2} \quad (1)$$

for all  $a, b \in I$ .

Due to modern analysis involving the applications of convexity, this concept has been extended and generalized in several directions. Various types of generalized convexity have appeared in different research works, some of them modify the domain or range of the function, always maintaining the basic structure of a convex function. Among them are:  $m$ –convexity [8,9],  $s$ –convexity in the first

and second sense [10,11],  $P$ -convexity [12],  $(s, m)$ -convexity in the first and second sense [8,13–15],  $MT$ -convexity [16],  $tgs$ -convexity [17], and others. In particular, the types of generalized convexity called  $(m, h_1, h_2)$ -convexity [18] and convexity dominated by a  $(m, h_1, h_2)$ -convex function  $g$  are of interest to this work.

In recent years several works have been published that relate the concepts of  $q$ -calculus with those of generalized convexity, and oriented towards the area of inequalities. In particular, the quantum Hermite–Hadamard inequality and its variant forms are useful for quantum physics where lower and upper bounds of natural phenomena modelled and described by integrals are frequently required [19,20].

Following the steps of the excellent works presented by the aforementioned authors, the Hermite–Hadamard inequality for  $(m, h_1, h_2)$ -convex functions is established, the concept of dominated convexity by a  $(m, h_1, h_2)$ -convex function  $g$  is introduced, and some integral inequalities involving other types of generalized convexity are deduced. The methodology used reveals, in a sense, a symmetric mathematical phenomenon.

## 2. Preliminaries

This section contains the basic information required for the development of this work, and is divided into two subsections so that its contents are organized according to the topics addressed in this study.

### 2.1. About $q$ -Calculus

The following basics about  $q$ -calculus can be found in [5].

**Definition 1.** Let  $f$  be an arbitrary function. Its  $q$ -differential is defined as

$$d_q f(x) = f(qx) - f(x).$$

In the particular case  $f(x) = x$  we have  $d_q x = (q - 1)x$ .

**Definition 2.** Let  $f : I \rightarrow \mathbb{R}$  be a continuous function on an interval  $I$ , and  $x \in I$ . Then the  $q$ -derivative of  $f$  in  $x$  is given by

$$D_q f(x) = \frac{d_q f(x)}{d_q x} = \frac{f(qx) - f(x)}{(q - 1)x}.$$

It is said that  $f$  is  $q$ -differentiable on  $I$  if  $D_q f(x)$  exists for all  $x \in I$ . Note that

$$\lim_{q \rightarrow 1} D_q f(x) = \frac{df}{dx}(x)$$

if  $f$  is differentiable on  $I$ .

The action of  $q$ -differentiation is a linear operator. Indeed, if  $f, g$  are functions and  $\alpha, \beta$  are arbitrary constants it follows that

$$D_q (\alpha f(x) + \beta g(x)) = \alpha D_q f(x) + \beta D_q g(x).$$

**Example 1.** The  $q$ -derivative of  $f(x) = x^n$ , where  $n$  is a positive integer is given by

$$D_q f(x) = \frac{q^n x^n - x^n}{(q - 1)x} = \frac{q^n - 1}{q - 1} x^{n-1}$$

using the notation

$$[n] = \frac{q^n - 1}{q - 1}$$

it can be written

$$D_q f(x) = [n] x^{n-1}.$$

Taking limit when  $q \rightarrow 1$  and using L'Hopital rule it is obtained that

$$\lim_{q \rightarrow 1} \frac{q^n - 1}{q - 1} = \lim_{q \rightarrow 1} nq^{n-1} = n.$$

**Proposition 1.** Let  $f, g : I \rightarrow \mathbb{R}$  be  $q$ -differentiable functions then

$$D_q(fg)(x) = f(qx)D_q g(x) + g(x)D_q f(x) \quad (2)$$

or

$$D_q(fg)(x) = g(qx)D_q f(x) + f(x)D_q g(x) \quad (3)$$

being equivalent expressions, also,

$$D_q \left( \frac{f(x)}{g(x)} \right) = \frac{g(x)D_q f(x) - f(x)D_q g(x)}{g(x)g(qx)} \quad (4)$$

or

$$D_q \left( \frac{f(x)}{g(x)} \right) = \frac{g(qx)D_q f(x) - f(qx)D_q g(x)}{g(x)g(qx)} \quad (5)$$

being equivalent expressions.

The chain rule does not have a general form but the following is known.

**Proposition 2.** Let  $f : I \rightarrow \mathbb{R}$  be a  $q$ -differentiable function and  $g(x) = ax + b$  then

$$D_q(f(g(x))) = D_q f(g(x))D_q g(x)$$

and if  $g(x) = \alpha x^\beta$  it is hold

$$D_q(f(g(x))) = D_{q^\beta} f(g(x))D_q g(x).$$

A function  $F$  is called a  $q$ -antiderivative of  $f$  if  $D_q F(x) = f(x)$ . It is denoted by

$$\int f(x) d_q x.$$

The following is known as the Jackson integral of  $f(x)$

$$\int f(x) d_q x = (1-q)x \sum_{n=0}^{\infty} q^n f(q^n x).$$

A rule for a change of variable  $u(x) = \alpha x^\beta$  for a  $q$ -antiderivative is known ([5] p. 66). Suppose that  $F$  is a  $q$ -antiderivative of  $f$ , then for any  $q'$

$$\begin{aligned} F(u(x)) &= \int f(u(x)) d_q x = \int D_{q'}(F(u(x))) d_{q'} x \\ &= \int (D_{q'^\beta} F)(u(x)) D_{q'} u(x) d_{q'} x \\ &= \int (D_{q'^\beta} F)(u(x)) d_{q'} u(x) \end{aligned}$$

choosing  $q = (q')^{1/\beta}$  we have,

$$\int f(u) d_q u = \int f(u(x)) d_{q^{1/\beta}} u(x).$$

similarly if  $u(x) = ax + b$ , we obtain

$$F(u(x)) = \int (D_{q^a} F)(u(x)) d_q u(x).$$

**Definition 3.** Let  $0 < a < b$ . The definite  $q$ -integral is defined by

$$\int_0^b f(x) d_q(x) = (1-q)b \sum_{n=0}^{\infty} q^n f(q^n b)$$

and

$$\int_a^b f(x) d_q(x) = \int_0^b f(x) d_q(x) - \int_0^a f(x) d_q(x).$$

It is useful recall, at this stage, a particular (case  $u(x) = \alpha x^\beta$ ) rule of change of variable for the definite Jackson integral ([5]p. 71):

$$\int_{u(a)}^{u(b)} f(u) d_q u = \int_a^b f(u(x)) d_{q^{1/\beta}} x$$

**Remark 1.** The previous paragraph is valid for functions such as  $u(x) = \alpha x^\beta + \gamma$ .

**Theorem 1.** (The fundamental theorem of  $q$ -calculus) If  $F(x)$  is a  $q$ -antiderivative of  $f(x)$  and  $F(x)$  is continuous in  $x = 0$  then

$$\int_a^b f(x) d_q(x) = F(b) - F(a)$$

where  $0 \leq a \leq b \leq \infty$ .

The  $q$ -integration by parts is given by the formula

$$\int_a^b f(x) d_q g(x) = f(b)g(b) - f(a)g(a) - \int_a^b g(qx) d_q f(x).$$

## 2.2. About Generalized Convexity

In this part some definitions about generalized convexity are presented. We start with a definition introduced by Z. Pavić and Z. Ardic in [9].

**Definition 4.** A function  $f : [0, \infty) \rightarrow \mathbb{R}$  is said to be a  $m$ -convex function for  $m \in (0, 1]$  if the following inequality holds for all  $a, b \in [0, \infty)$  with  $a < b$ ,

$$f(ta + m(1-t)b) \leq tf(a) + m(1-t)f(b). \quad (6)$$

Obviously when  $m = 1$ , the inequality (6) coincides with the classical convexity. When  $m = 0$  it becomes

$$f(tx) \leq tf(x)$$

and this kind of function is called “starshaped” function.

In addition, in the works of W.W. Breckner [21], M. Alomari [10], and E. Set [22] the concept of  $s$ -convex functions in the first and second sense were introduced.

**Definition 5.** A function  $f : [0, \infty) \rightarrow \mathbb{R}$  is said to be  $s$ -convex in the first sense if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t^s) f(y)$$

and it is said to be  $s$ -convex in the second sense if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y),$$

both inequalities hold for all  $x, y \in [0, \infty)$  and  $t \in [0, 1]$  and some fixed  $s \in (0, 1]$ .

The definition of  $s$ -convex functions in the first sense was introduced by W. Orlicz and used in the theory of Orlicz spaces [23,24].

Hudzik and Maligranda [25] exposed the following example. Let  $0 < s \leq 1$ ,  $a, b, c \in \mathbb{R}$  and

$$f(x) = \begin{cases} a & \text{if } x = 0 \\ bx^s + c & \text{if } x > 0 \end{cases}$$

then

1. If  $b \geq 0$  and  $c \leq a$  the function  $f$  is  $s$ -convex in the first sense.
2. If  $b \geq 0$  and  $0 \leq c \leq a$  the function  $f$  is  $s$ -convex in the second sense.

S.S. Dragomir et al., in [26], introduced the concept of  $P$ -convex functions and A. O. Akdemir in [12] used it for functions of several variables.

**Definition 6.** Let  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be a non negative function, where  $I$  is an interval, is said to be  $P$ -convex if for all  $x, y \in I$  and  $t \in [0, 1]$  the function satisfies the following inequality

$$f(tx + (1-t)y) \leq f(x) + f(y).$$

As an example, the author exposed that  $f(x) = x^p$ ,  $0 < p \leq 1$  and  $x \in [0, \infty)$  is a  $P$ -convex function.

When Definitions 4 and 5 are combined, as shown in the works of Bakula [8] and N. Eftekhari [15], the following definition is obtained.

**Definition 7.** The function  $f : [0, b] \rightarrow \mathbb{R}$ , with  $b > 0$ , is said to be  $(s, m)$ -convex in the first sense if  $f$  satisfies

$$f(tx + m(1-t)y) \leq t^s f(x) + m(1-t^s) f(y)$$

and it is  $(s, m)$ -convex in the second sense if  $f$  satisfies

$$f(tx + m(1-t)y) \leq t^s f(x) + m(1-t)^s f(y),$$

both inequalities hold for all  $x, y \in [0, b]$  and  $t \in [0, 1]$  for some fixed  $s, m \in (0, 1]$ .

M. Tunc et al. [17] introduced the following definition of generalized convex function.

**Definition 8.** Let  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be a non-negative function. We say that  $f$  is  $tgs$ -convex function on  $I$ , if

$$f((1-t)u + tv) \leq t(1-t)(f(u) + f(v))$$

for all  $u, v \in I$  and  $t \in (0, 1)$ .

In addition, S.S. Dragomir and N.M. Ionescu, in [27] introduced the so-called dominated convexity.

**Definition 9.** Let  $g : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be a given convex function. The function  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  is called  $g$ -convex dominated on  $I$  if the inequality

$$|tf(x) + (1-t)f(y) - f(tx + (1-t)y)| \leq tg(x) + (1-t)g(y) - g(tx + (1-t)y)$$

holds for all  $x, y \in I$  and  $t \in [0, 1]$ .

As S.S. Dragomir established in [27], the class of  $g$ -convex dominated on an interval  $I$  is non-empty. Indeed there are concave functions  $g$ -convex dominated, for example  $-g$ .

D-P Shi, B-Y Xi, and F Qi introduced in [18] the class of generalized convex functions called  $(m, h_1, h_2)$ -convex functions, and they were used by Cristescu G. et al. in [28].

**Definition 10.** Let  $f : [0, \infty) \rightarrow \mathbb{R}$  be a function,  $h_1, h_2 : [0, 1] \rightarrow \mathbb{R}$  non negative functions and  $m \in (0, 1]$ . Then  $f$  is called a  $(m, h_1, h_2)$ -convex function if for all  $x, y \in [0, \infty)$  and  $t \in [0, 1]$  the following inequality holds

$$f(tx + m(1-t)y) \leq h_1(t)f(x) + mh_2(t)f(y).$$

This work introduces a new class of generalized convex functions, these will be called  $g - (m, h_1, h_2)$ -convex dominated functions.

**Definition 11.** Let  $g : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be a given  $(m, h_1, h_2)$ -convex function, where  $h_1, h_2 : [0, 1] \rightarrow \mathbb{R}$  are non-negative functions and  $m \in (0, 1]$ . A function  $f : [0, \infty) \rightarrow \mathbb{R}$  is called  $g - (m, h_1, h_2)$ -convex dominated if for all  $x, y \in [0, \infty)$  and  $t \in [0, 1]$  the following inequality holds

$$|h_1(t)f(x) + mh_2(t)f(y) - f(tx + m(1-t)y)| \leq h_1(t)g(x) + mh_2(t)g(y) - g(tx + m(1-t)y).$$

### 3. Main Results

This section is divided into two subsections—in the first of them some results about the quantum Hermite–Hadamard inequality for  $(m, h_1, h_2)$ -convex functions are presented and in the second some properties and results about the same inequality for  $g - (m, h_1, h_2)$ -convex dominated functions are presented.

#### 3.1. Quantum Hermite–Hadamard Inequality for Generalized Convex Functions

**Theorem 2.** Let  $h_1, h_2 : [0, 1] \rightarrow \mathbb{R}$  be two positive functions and  $m \in (0, 1]$ . Let  $f : [0, \infty) \rightarrow \mathbb{R}$  be a  $(m, h_1, h_2)$ -convex function and  $a, b \in [0, \infty)$  with  $a < b$ . If  $f$  is  $q$ -integrable on  $[0, \infty)$  then

$$f\left(\frac{a+b}{2}\right) \leq h_1\left(\frac{1}{2}\right) \frac{1}{b-a} \int_a^b f(u) d_q u + h_2\left(\frac{1}{2}\right) \frac{m^2}{b-a} \int_{a/m}^{b/m} f(u) d_q u, \quad (7)$$

and

$$\frac{1}{b-a} \int_a^b f(u) d_q u \leq f(a) \int_0^1 h_1(t) d_q t + mf\left(\frac{b}{m}\right) \int_0^1 h_2(t) d_q t. \quad (8)$$

**Proof.** From Definition 10 it follows that

$$f\left(\frac{x+y}{2}\right) \leq h_1\left(\frac{1}{2}\right) f(x) + mh_2\left(\frac{1}{2}\right) f\left(\frac{y}{m}\right).$$

So, choosing

$$x = ta + (1-t)b \text{ and } y = (1-t)a + tb$$

we have that

$$f\left(\frac{a+b}{2}\right) \leq h_1\left(\frac{1}{2}\right) f(ta + (1-t)b) + mh_2\left(\frac{1}{2}\right) f\left(\frac{(1-t)a + tb}{m}\right)$$

for all  $t \in [0, 1]$ .

Taking the Jackson integral over  $t \in [0, 1]$  it follows that

$$f\left(\frac{a+b}{2}\right) \int_0^1 d_q t \leq h_1\left(\frac{1}{2}\right) \int_0^1 f(ta + (1+t)b) d_q t \\ + mh_2\left(\frac{1}{2}\right) \int_0^1 f\left(\frac{(1-t)a + tb}{m}\right) d_q t.$$

With the change  $u(t) = ta + (1+t)b$  and  $v(t) = ((1-t)a + tb)/m$ , and using Remark 1, then

$$f\left(\frac{a+b}{2}\right) \leq h_1\left(\frac{1}{2}\right) \frac{1}{b-a} \int_a^b f(u) d_q u + h_2\left(\frac{1}{2}\right) \frac{m^2}{b-a} \int_{a/m}^{b/m} f(u) d_q u.$$

In order to proof the inequality (8), from Definition 10 we have

$$f(ta + (1+t)b) \leq h_1(t) f(a) + mh_2(t) f\left(\frac{b}{m}\right).$$

Taking the Jackson integral over  $t \in [0, 1]$

$$\int_0^1 f(ta + (1-t)b) d_q t \leq f(a) \int_0^1 h_1(t) d_q t + mf\left(\frac{b}{m}\right) \int_0^1 h_2(t) d_q t,$$

and with the change of variable  $u(t) = ta + (1-t)b$  we obtain

$$\frac{1}{b-a} \int_a^b f(u) d_q u \leq f(a) \int_0^1 h_1(t) d_q t + mf\left(\frac{b}{m}\right) \int_0^1 h_2(t) d_q t.$$

The proof is complete.  $\square$

The following corollaries are obtained using Theorem 2 with a particular choice of the parameters  $m, h_1$  and  $h_2$ .

**Corollary 1.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a convex function. If  $f$  is  $q$ -integrable then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(u) d_q u \leq \frac{f(a)}{q+1} + \frac{qf(b)}{q+1}. \quad (9)$$

**Proof.** Letting  $m = 1$  and  $h_1 = t, h_2 = 1 - t$  for  $t \in [0, 1]$  in Theorem 2 it is obtained that

$$\int_0^1 h_1(t) d_q t = \int_0^1 t d_q t = \frac{1}{q+1}$$

and replacing these values in the inequalities (7) and (8) it is obtained

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(u) d_q u \leq \frac{f(a)}{q+1} + \frac{qf(b)}{q+1}.$$

The proof is complete.  $\square$

**Remark 2.** If in corollary we take limit when  $q \rightarrow 1$  it is attained

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(u) du \leq \frac{f(a) + f(b)}{2}$$

making coincidence with the classical Hermite–Hadamard inequality (1).

**Corollary 2.** Let  $s \in (0, 1]$  and  $f : [a, b] \rightarrow \mathbb{R}$  be a  $s$ -convex function in the second sense. If  $f$  is  $q$ -integrable then

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(u) d_q u \leq \frac{q-1}{q^{s+1}-1} (f(a) + f(b)). \quad (10)$$

**Proof.** Letting  $m = 1$  and  $h_1 = t^s, h_2 = (1-t)^s$  for  $t \in [0, 1]$  in Theorem 2 it is had that

$$\int_0^1 h_1(t) d_q t = \int_0^1 t^s d_q t = \frac{1}{[s+1]} = \frac{q-1}{q^{s+1}-1}$$

and

$$\int_0^1 h_2(t) d_q t = \int_0^1 (1-t)^s d_q t = \frac{q-1}{q^{s+1}-1}.$$

Replacing these values in the inequalities (7) and (8) the inequality (10) is obtained.  $\square$

**Remark 3.** If in corollary it is taken limit when  $q \rightarrow 1$

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(u) du \leq \frac{1}{s+1} (f(a) + f(b)).$$

This coincides with Theorem 2.1 in [29].

**Corollary 3.** Let  $m \in (0, 1]$  and  $f : [a, b] \rightarrow \mathbb{R}$  be a  $m$ -convex function. If  $f$  is  $q$ -integrable then

$$f\left(\frac{a+b}{2}\right) \leq \left(\frac{1}{2}\right) \frac{1}{b-a} \int_a^b f(u) d_q u + \frac{m^2}{2(b-a)} \int_{a/m}^{b/m} f(u) d_q u, \quad (11)$$

and

$$\frac{1}{b-a} \int_a^b f(u) d_q u \leq \frac{1}{q+1} \left( f(a) + m f\left(\frac{b}{m}\right) \right) \quad (12)$$

**Proof.** If in Theorem 2 it is chosen  $h_1 = t, h_2 = 1-t$  for  $t \in [0, 1]$  it is had that

$$\int_0^1 h_1(t) d_q t = \int_0^1 t d_q t = \frac{t^2}{[2]} \Big|_0^1 = \frac{1}{[2]} = \frac{1}{q+1}$$

and

$$\int_0^1 h_2(t) d_q t = \int_0^1 (1-t) d_q t = \frac{1}{q+1}$$

Replacing in the inequalities (7) and (8) the inequalities (11) and (12) are attained.  $\square$

**Remark 4.** If in Corollary 3 is chosen  $m = 1$  then it is obtained the quantum Hermite Hadamard inequality for convex functions (9). If it is taken limit when  $q \rightarrow 1$  the classical Hermite–Hadamard for  $m$ -convex functions is obtained

$$f\left(\frac{a+b}{2}\right) \leq \left(\frac{1}{2}\right) \frac{1}{b-a} \int_a^b f(u) du + \frac{m^2}{2(b-a)} \int_{a/m}^{b/m} f(u) du,$$

and

$$\frac{1}{b-a} \int_a^b f(u) du \leq \frac{1}{2} \left( f(a) + m f\left(\frac{b}{m}\right) \right)$$

**Corollary 4.** Let  $f : [0, \infty) \rightarrow \mathbb{R}$  be a  $(s, m)$ -convex function in the second sense,  $m, s \in (0, 1]$  and  $a, b \in [0, \infty)$  with  $a < b$ . If  $f$  is  $q$ -integrable on  $[0, b/m]$  then



$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2^s} \frac{1}{b-a} \int_a^b f(u) d_q u + \frac{m^2}{2(b-a)} \int_{a/m}^{b/m} f(u) d_q u, \quad (13)$$

and

$$\frac{1}{b-a} \int_a^b f(u) d_q u \leq \frac{1}{[s+1]} \left[ f(a) + m f\left(\frac{b}{m}\right) \right]. \quad (14)$$

**Proof.** Letting  $h_1 = t^s, h_2 = (1-t)^s$  for  $t \in [0, 1]$  in Theorem 2, and using the scheme of proof in the above corollaries the inequalities (13) and (14) are obtained.  $\square$

**Remark 5.** If in Corollary 4  $m = 1$  then we have the quantum Hermite Hadamard inequality for  $s$ -convex function in the second sense (10), and if it is chosen  $s = 1$  then it is obtained the quantum Hermite Hadamard inequalities (11) and (12) for  $m$ -convex functions. In addition, if  $s = 1$  and  $m = 1$  then is obtained the quantum Hermite Hadamard inequality for convex functions.

**Corollary 5.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a  $P$ -convex function. If  $f$  is  $q$ -integrable then

$$f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a} \int_a^b f(u) d_q u \leq 2(f(a) + f(b)).$$

**Proof.** Letting  $m = 1, h_1(t) = h_2(t) = 1$  for  $t \in [0, 1]$  in Theorem 2 then it is obtained

$$\int_0^1 h_2(t) d_q t = \int_0^1 h_1(t) d_q t = 1$$

so, replacing in inequalities (7) and (8) the inequality (5) is obtained.  $\square$

**Remark 6.** If it is taken limit when  $q \rightarrow 1$  then it is had that

$$f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a} \int_a^b f(u) \leq 2(f(a) + f(b))$$

making coincidence with Theorem 3.1 in [26].

**Corollary 6.** Let  $f : I = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$  be a  $tg$ s-convex function. If  $f$  is  $q$ -integrable on  $[a, b]$  then

$$2f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(u) d_q u \leq \left( \frac{1}{q+1} - \frac{1}{1+q+q^2} \right) (f(a) + f(b)).$$

**Proof.** Letting  $m = 1$ , and  $h_1(t) = h_2(t) = t(1-t)$  for all  $t \in [0, 1]$  in Theorem 2, then

$$h_1(1/2) = h_2(1/2) = \frac{1}{4}$$

and

$$\int_0^1 h_1(t) d_q t = \int_0^1 h_2(t) d_q t = \frac{1}{q+1} - \frac{1}{1+q+q^2},$$

therefore, by replacement of these values we have the desired result.  $\square$

This last result coincides with Theorem 2.1 in [30].

### 3.2. Quantum Hermite–Hadamard Inequality for Generalized $g$ -convex Dominated Functions

First it is given a characterization of  $g - (m, h_1, h_2)$  convex dominated function.

**Proposition 3.** Let  $g$  a  $(m, h_1, h_2)$  convex function on an interval  $I$  and  $f : I \rightarrow \mathbb{R}$ . The following statements are equivalent

- (i)  $f$  is a  $g - (m, h_1, h_2)$  convex dominated function
- (ii) The mappings  $g + f$  and  $g - f$  are  $(m, h_1, h_2)$  convex function
- (iii) There exists two  $(m, h_1, h_2)$  convex function  $h, k$  defined on  $I$  such that  $f = (h - k) / 2$  and  $g = (h + k) / 2$ .

**Proof.** Let  $g$  a  $(m, h_1, h_2)$  convex function on an interval  $I$ . The statement  $f$  is a  $g - (m, h_1, h_2)$ -convex dominated function is equivalent to

$$\begin{aligned} g(tx + m(1-t)y) - h_1(t)g(x) - mh_2(t)g(y) \\ \leq h_1(t)f(x) + mh_2(t)f(y) - f(tx + m(1-t)y) \\ \leq h_1(t)g(x) + mh_2(t)g(y) - g(tx + m(1-t)y) \end{aligned}$$

for all  $x, y \in I$  and  $t \in [0, 1]$ . These two inequalities may be rearranged as

$$\begin{aligned} h_1(t)(f(x) + g(x)) + mh_2(t)(f(y) + g(y)) \\ \geq f(tx + m(1-t)y) + g(tx + m(1-t)y) \end{aligned}$$

and

$$\begin{aligned} h_1(t)(g(x) - f(x)) + mh_2(t)(g(y) - f(y)) \\ \geq g(tx + m(1-t)y) - f(tx + m(1-t)y) \end{aligned}$$

for all  $x, y \in I$  and  $t \in [0, 1]$ , which are equivalent to the  $(m, h_1, h_2)$  convexity of  $g + f$  and  $g - f$ . This prove the equivalence between (i) and (ii).

The equivalence between (ii) and (iii) is proved taking  $h = f + g$  and  $k = g - f$ .

The proof is complete.  $\square$

**Theorem 3.** Let  $g : I \rightarrow \mathbb{R}$  be a  $(m, h_1, h_2)$  convex function on an interval  $I$  and  $f : [a, b] \rightarrow \mathbb{R}$  be a  $g - (m, h_1, h_2)$  convex dominated function. Then

$$\begin{aligned} \left| h_1 \left( \frac{1}{2} \right) \frac{1}{b-a} \int_a^b f(u) d_q u + m^2 h_2 \left( \frac{1}{2} \right) \frac{1}{b-a} \int_{a/m}^{b/m} f(u) d_q u - f \left( \frac{a+b}{2} \right) \right| \\ \leq h_1 \left( \frac{1}{2} \right) \frac{1}{b-a} \int_a^b g(u) d_q u + m^2 h_2 \left( \frac{1}{2} \right) \frac{1}{b-a} \int_{a/m}^{b/m} g(u) d_q u - g \left( \frac{a+b}{2} \right) \end{aligned}$$

and

$$\begin{aligned} \left| f(a) \int_0^1 h_1(t) d_q t + m f \left( \frac{b}{m} \right) \int_0^1 h_2(t) d_q t - \frac{1}{b-a} \int_a^b f(u) d_q u \right| \\ \leq g(a) \int_0^1 h_1(t) d_q t + m g \left( \frac{b}{m} \right) \int_0^1 h_2(t) d_q t - \frac{1}{b-a} \int_a^b g(u) d_q u. \end{aligned}$$

**Proof.** From the Proposition 3 and using the Theorem 2 we have

$$(f + g) \left( \frac{a+b}{2} \right) \leq h_1 \left( \frac{1}{2} \right) \frac{1}{b-a} \int_a^b (f + g)(u) d_q u + m^2 h_2 \left( \frac{1}{2} \right) \frac{1}{b-a} \int_{a/m}^{b/m} (f + g)(u) d_q u$$

and

$$\frac{1}{b-a} \int_a^b (f + g)(u) d_q u \leq (f + g)(a) \int_0^1 h_1(t) d_q t + m (f + g) \left( \frac{b}{m} \right) \int_0^1 h_2(t) d_q t;$$

also,

$$(g-f)\left(\frac{a+b}{2}\right) \leq h_1\left(\frac{1}{2}\right) \frac{1}{b-a} \int_a^b (g-f)(u) d_q u + m^2 h_2\left(\frac{1}{2}\right) \frac{1}{b-a} \int_{a/m}^{b/m} (g-f)(u) d_q u$$

and

$$\frac{1}{b-a} \int_a^b (g-f)(u) d_q u \leq (g-f)(a) \int_0^1 h_1(t) d_q t + m (g-f)\left(\frac{b}{m}\right) \int_0^1 h_2(t) d_q t.$$

This inequalities are equivalent to those in the enunciate.  $\square$

**Corollary 7.** Let  $g : I \rightarrow \mathbb{R}$  be a convex function on an interval  $I$  and  $f : [a, b] \rightarrow \mathbb{R}$  be a  $g$  convex dominated function. Then

$$\left| \frac{1}{b-a} \int_a^b f(u) d_q u - f\left(\frac{a+b}{2}\right) \right| \leq \frac{1}{b-a} \int_a^b g(u) d_q u - g\left(\frac{a+b}{2}\right)$$

and

$$\left| \frac{f(a) + f(b)}{q+1} - \frac{1}{b-a} \int_a^b f(u) d_q u \right| \leq \frac{g(a) + g(b)}{q+1} - \frac{1}{b-a} \int_a^b g(u) d_q u.$$

**Proof.** Letting  $m = 1$ ,  $h_1(t) = t$  and  $h_2(t) = 1 - t$  for  $t \in [0, 1]$  in Theorem 3 then the desired result is attained.

The proof is complete.  $\square$

**Remark 7.** Taking limit when  $q \rightarrow 1$  it is had that

$$\left| \frac{1}{b-a} \int_a^b f(u) du - f\left(\frac{a+b}{2}\right) \right| \leq \frac{1}{b-a} \int_a^b g(u) du - g\left(\frac{a+b}{2}\right)$$

and

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(u) du \right| \leq \frac{g(a) + g(b)}{2} - \frac{1}{b-a} \int_a^b g(u) du$$

which coincides with Theorem 1 in [31].

**Corollary 8.** Let  $g : I \rightarrow \mathbb{R}$  be a  $s$ -convex function in the second sense on an interval  $I$  and  $f : [a, b] \rightarrow \mathbb{R}$  be a  $g$ - $s$ -convex dominated function then

$$\left| \frac{1}{b-a} \int_a^b f(u) d_q u - 2^{s-1} f\left(\frac{a+b}{2}\right) \right| \leq \frac{1}{b-a} \int_a^b g(u) d_q u - 2^{s-1} g\left(\frac{a+b}{2}\right)$$

and

$$\left| \frac{1}{[s+1]} (f(a) + f(b)) - \frac{1}{b-a} \int_a^b f(u) d_q u \right| \leq \frac{1}{[s+1]} (g(a) + g(b)) - \frac{1}{b-a} \int_a^b g(u) d_q u.$$

**Proof.** Letting  $m = 1$ ,  $h_1(t) = t^s$  and  $h_2(t) = (1-t)^s$  for  $t \in [0, 1]$  and some fixed  $s \in (0, 1]$  in Theorem 3, the desired result is attained.

The proof is complete.  $\square$

**Remark 8.** Taking limit when  $q \rightarrow 1$  we have

$$\left| \frac{1}{b-a} \int_a^b f(u) du - 2^{s-1} f\left(\frac{a+b}{2}\right) \right| \leq \frac{1}{b-a} \int_a^b g(u) du - 2^{s-1} g\left(\frac{a+b}{2}\right)$$

and

$$\left| \frac{1}{s+1} (f(a) + f(b)) - \frac{1}{b-a} \int_a^b f(u) du \right| \leq \frac{1}{s+1} (g(a) + g(b)) - \frac{1}{b-a} \int_a^b g(u) du.$$

**Corollary 9.** Let  $g : I \rightarrow \mathbb{R}$  be a  $m$ -convex function on an interval  $I$  and  $f : [a, b] \rightarrow \mathbb{R}$  be a  $g - m$ -convex dominated function then

$$\begin{aligned} \left| \frac{1}{2(b-a)} \int_a^b f(u) d_q u + \frac{m^2}{2(b-a)} \int_{a/m}^{b/m} f(u) d_q u - f\left(\frac{a+b}{2}\right) \right| \\ \leq \left(\frac{1}{2}\right) \frac{1}{b-a} \int_a^b g(u) d_q u + m^2 \frac{1}{2} \frac{1}{b-a} \int_{a/m}^{b/m} g(u) d_q u - g\left(\frac{a+b}{2}\right) \end{aligned}$$

and

$$\begin{aligned} \left| \frac{1}{q+1} \left( f(a) + m f\left(\frac{b}{m}\right) \right) - \frac{1}{b-a} \int_a^b f(u) d_q u \right| \\ \leq \frac{1}{q+1} \left( g(a) + m f\left(\frac{b}{m}\right) \right) - \frac{1}{b-a} \int_a^b g(u) d_q u. \end{aligned}$$

**Proof.** Letting  $h_1(t) = t$  and  $h_2(t) = 1 - t$  for  $t \in [0, 1]$  in Theorem 3 the desired result is attained. The proof is complete.  $\square$

**Remark 9.** Taking limit when  $q \rightarrow 1$  we have

$$\begin{aligned} \left| \frac{1}{2(b-a)} \int_a^b f(u) du + \frac{m^2}{2(b-a)} \int_{a/m}^{b/m} f(u) du - f\left(\frac{a+b}{2}\right) \right| \\ \leq \frac{1}{2(b-a)} \int_a^b g(u) du + \frac{m^2}{2(b-a)} \int_{a/m}^{b/m} g(u) du - g\left(\frac{a+b}{2}\right) \end{aligned}$$

and

$$\left| \frac{1}{2} \left( f(a) + m f\left(\frac{b}{m}\right) \right) - \frac{1}{b-a} \int_a^b f(u) du \right| \leq \frac{1}{2} \left( g(a) + m f\left(\frac{b}{m}\right) \right) - \frac{1}{b-a} \int_a^b g(u) du.$$

**Corollary 10.** Let  $g : I \rightarrow \mathbb{R}$  be a  $(s, m)$ -convex function in the second sense on an interval  $I$  and  $f : [a, b] \rightarrow \mathbb{R}$  be a  $g - (s, m)$ -convex dominated function then

$$\left| \frac{1}{2^s(b-a)} \int_a^b f(u) d_q u - f\left(\frac{a+b}{2}\right) \right| \leq \frac{1}{2^s(b-a)} \int_a^b g(u) d_q u - g\left(\frac{a+b}{2}\right)$$

and

$$\begin{aligned} \left| \frac{1}{[s+1]} \left[ f(a) + m f\left(\frac{b}{m}\right) \right] - \frac{1}{b-a} \int_a^b f(u) d_q u \right| \\ \leq \frac{1}{[s+1]} \left[ g(a) + m g\left(\frac{b}{m}\right) \right] - \frac{1}{b-a} \int_a^b g(u) d_q u. \end{aligned}$$

**Remark 10.** Taking limit when  $q \rightarrow 1$  it is had the inequality for ordinary integral

$$\left| \frac{1}{2^s(b-a)} \int_a^b f(u) du - f\left(\frac{a+b}{2}\right) \right| \leq \frac{1}{2^s(b-a)} \int_a^b g(u) du - g\left(\frac{a+b}{2}\right)$$

and

$$\left| \frac{1}{s+1} \left[ f(a) + mf\left(\frac{b}{m}\right) \right] - \frac{1}{b-a} \int_a^b f(u) du \right| \leq \frac{1}{s+1} \left[ g(a) + mg\left(\frac{b}{m}\right) \right] - \frac{1}{b-a} \int_a^b g(u) du.$$

**Corollary 11.** Let  $g : I \rightarrow \mathbb{R}$  be a  $P$ -convex function in the second sense on an interval  $I$  and  $f : [a, b] \rightarrow \mathbb{R}$  be a  $g$ - $P$ -convex dominated function then

$$\left| \frac{2}{b-a} \int_a^b f(u) d_q u - f\left(\frac{a+b}{2}\right) \right| \leq \frac{2}{b-a} \int_a^b g(u) d_q u - g\left(\frac{a+b}{2}\right)$$

and

$$\left| (f(a) + f(b)) - \frac{1}{b-a} \int_a^b f(u) d_q u \right| \leq (g(a) + g(b)) - \frac{1}{b-a} \int_a^b g(u) d_q u.$$

**Remark 11.** Taking limit when  $q \rightarrow 1$  then it is had that

$$\left| \frac{2}{b-a} \int_a^b f(u) du - f\left(\frac{a+b}{2}\right) \right| \leq \frac{2}{b-a} \int_a^b g(u) du - g\left(\frac{a+b}{2}\right)$$

and

$$\left| (f(a) + f(b)) - \frac{1}{b-a} \int_a^b f(u) du \right| \leq (g(a) + g(b)) - \frac{1}{b-a} \int_a^b g(u) du.$$

#### 4. Conclusions

In the development of this work the quantum Hermite–Hadamard inequality for  $(m, h_1 h_2)$ -convex function was established and from this result some inequalities of the same type for  $s$ -convex functions in the second sense,  $m$ -convex functions,  $(s, m)$ -convex functions were deduced, making coincidence with particularized results found in [26,29]. In addition, the definition of dominated  $(m, h_1 h_2)$ -convexity function by a function  $g$  which is  $(m, h_1 h_2)$ -convex function was introduced, and with this definition some inequalities of Hadamard type were deduced, and some others for generalized convex functions were established.

The authors hope that this work contributes to the development of this line of research.

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