

Review

Modified Gravity in Higher Dimensions, Flux Compactification, and Cosmological Inflation

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Abstract: We review a possible origin of cosmological inflation from higher (D) spacetime dimensions in the context of modified gravity theory. It is demonstrated that it requires a spontaneous warped compactification of higher (D) spacetime dimensions together with the stabilization of extra ($D - 4$) dimensions by Freund–Rubin mechanism. The relevant tools include an extra gauge ($D/2 - 1$)-form field with a non-vanishing flux in compact dimensions and a positive cosmological constant in D dimensions. Those features are illustrated on the specific example in eight spacetime dimensions compactified on a four-sphere with a warped factor and a flux, which leads to a viable Starobinsky-like inflationary model in four (non-compact) spacetime dimensions.

Keywords: inflation; extra dimensions; modified gravity

1. Introduction

Einstein’s theory of general relativity is the best example of power of the general covariance. It is, therefore, quite natural to exploit the same symmetry or just gravitational interactions, in order to theoretically describe cosmological phenomena such as inflation. Since Einstein’s theory of gravity is non-renormalizable as a quantum field theory, its ultra-violet completion may require extra spacetime dimensions, as is the case in string theory.

In the seminal paper [1] Freund and Rubin pointed out the need of dynamical description of extra dimensions and their spontaneous compactification, and offered the tools for stabilization of moduli in compact extra dimensions by using the p -form fluxes. It was the important extension of the old Kaluza–Klein (KK) idea motivated by the unification of fundamental forces because extra dimensions appear in field theory and gravity, supersymmetry and supergravity, as well as string theory and braneworld.

Extra dimensions offer a lot of possibilities that should be constrained both theoretically and experimentally. Amongst the natural constraints to be imposed are (i) spontaneous compactification, i.e., the requirement of a compactified theory to obey the equations of motion of the higher-dimensional theory, (ii) stabilization of extra dimensions, in order to ensure the visible four-dimensional universe with a long lifetime, (iii) a viable hierarchy of fundamental scales, and (iv) consistency with observations. This list can be extended by demanding a realization of viable inflation in the early universe from higher dimensions.

In regards the theoretical *tools* needed for the realization of the concept of extra dimensions, the key developments in the past were (a) *flux* compactifications suggested by Freund and Rubin [1], which resolved several “no-go” problems in supergravity and string theory, and (b) *warped* metrics

and compactifications proposed by Randall and Sundrum as a possible solution to the hierarchy problem [2,3].

The Planck observations [4] of the Cosmic Microwave Background (CMB) radiation favor a *single-field* inflation driven by a real scalar field called *inflaton*, while extra dimensions always give rise to more scalar fields (called *moduli*) that must be stabilized because they are mixed with inflaton in four dimensions. In addition, the mass hierarchy $M_{\text{inf.}} \ll M_{\text{KK}} \ll M_{\text{Pl}}$ should be satisfied. The Freund–Rubin compactification Ansatz [1] solves those uneasy problems for certain values of the parameters [5–7] after assuming that compactification took place before inflation. In this paper we review the inflationary models based on modified gravity in various dimensions, and emphasize the role of warped metric and flux compactification for theoretical consistency and phenomenological applications.

The paper is organized as follows. The Starobinsky $(R + R^2)$ model of inflation is reviewed in Section 2. The $(R + R^n)$ gravity in D dimensions is considered in Section 3. The existence of a plateau in the potential (needed for slow-roll inflation and favored by observations [4]) leads to the condition $n = D/2$ with D as a multiple of 4. Section 4 is devoted to a warped compactification of the modified gravity in 8 dimensions on a 4-sphere and a derivation of the mixed potential depending on inflaton and the volume modulus of the sphere. Section 5 describes the Freund–Rubin compactification in $D = 8$ after adding a single $(p - 1)$ -form gauge field with a non-vanishing flux in compact dimensions under another necessary condition $p = n$ [6,7]. In Section 6 we investigate the mixed (two-field) scalar potential and demonstrate that the modulus can be stabilized in the certain range of the parameters. In Section 7 we apply our model to a description of inflation in the early universe. Section 8 is our conclusion.

2. Starobinsky Model of Inflation

The Starobinsky inflationary model of modified $(R + R^2)$ gravity in four dimensions is defined by the action [8]:

$$S_{\text{Starobinsky}} = \frac{1}{2} \int d^4x \sqrt{-g} \left[R + \frac{1}{6M^2} R^2 \right]. \quad (1)$$

We use the natural units $\hbar = c = 1$ with the reduced Planck mass $M_{\text{Pl}} = 1$ and spacetime signature $(-, +, +, +)$ in $D = 4$. The Starobinsky model is an excellent model of inflation, in very good agreement with the Planck data [4]. The Starobinsky model has one real parameter M that can be identified with the inflaton mass, whose value is fixed by the CMB data as $M \approx 10^{-5} (\frac{50}{N_e})$, where N_e is the e-foldings number. The corresponding scalar potential of the (canonically normalized) inflaton field ϕ in the scalar-tensor reformulation of $f(R)$ gravity (see the next Section 3) is given by [9]:

$$V(\phi) = \frac{3}{4} M^2 \left(1 - e^{-\sqrt{\frac{2}{3}} \phi} \right)^2. \quad (2)$$

The scalar potential in Equation (2) has a plateau with a positive height and describes slow roll inflation. During the slow roll the scalar potential in Equation (2) can be simplified to:

$$V(\phi) \approx V_0 (1 - 2e^{-\alpha_s \phi}), \quad (3)$$

where we have kept only the leading (exponentially small) correction to the emergent cosmological constant $V_0 = \frac{3}{4} M^2$ and have introduced the notation $\alpha_s = \sqrt{\frac{2}{3}}$. The scalar potential in Equation (2) is the particular case of a class of inflationary scalar potentials having a plateau and taking the form:

$$V(\phi) = V_0 - V_1 e^{-\alpha \phi} \quad (4)$$

with generic positive real parameters V_0 , V_1 , and α . The V_1 can be changed by a shift of the field ϕ and, therefore, is irrelevant. The V_0 determines the scale of inflation. The value of α determines the key observable r related to primordial gravity waves and known as the tensor-to-scalar ratio,

$$r = \frac{8}{\alpha^2 N_e^2}. \quad (5)$$

The Planck data [4] gives the upper bound $r < 0.08$ that yields:

$$\alpha > \frac{10}{N_e} = 0.2 \left(\frac{50}{N_e} \right). \quad (6)$$

The scalar spectral index n_s and its running $dn_s/d\ln k$ derived from the potential in Equation (2) are [10]:

$$n_s \approx 1 - \frac{2}{N_e} \quad \text{and} \quad \frac{dn_s}{d\ln k} \approx -\frac{(1-n_s)^2}{2} \approx -\frac{2}{N_e^2}. \quad (7)$$

3. Modified Gravity and Inflaton Scalar Potential in D Dimensions

Let us denote spacetime vector indices in D dimensions by capital latin letters $A, B, \dots = 0, 1, \dots, D-1$, and spacetime vector indices in four spacetime dimensions by lower case greek letters $\alpha, \beta, \dots = 0, 1, 2, 3$.

We propose the following (modified) gravity action in D spacetime dimensions:

$$S_{\text{grav.}} = \frac{1}{2\kappa^2} \int d^D x \sqrt{-g_D} (R + \gamma R^n - 2\Lambda), \quad (8)$$

where κ is the gravitational coupling constant of (mass) dimension $\frac{1}{2}(-D+2)$, $\gamma > 0$ is the new coupling constant of (mass) dimension $(-2n+2)$, and Λ is the cosmological constant of (mass) dimension 2 in D dimensions. A substitution:

$$R + \gamma R^n \longrightarrow (1+B)R - \left(\frac{1}{\gamma n}\right)^{\frac{1}{n-1}} \left(\frac{n-1}{n}\right) B^{\frac{n}{n-1}}, \quad (9)$$

where we have introduced the auxiliary scalar field B , allows us to rewrite the action in Equation (8) as:

$$S = \frac{1}{2\kappa^2} \int d^D x \sqrt{-g_D} \left[(1+B)R - \left(\frac{1}{\gamma n}\right)^{\frac{1}{n-1}} \left(\frac{n-1}{n}\right) B^{\frac{n}{n-1}} - 2\Lambda \right]. \quad (10)$$

Varying this action with respect to the auxiliary field B yields $B = \gamma n R^{n-1}$, and a substitution of that into the action of Equation (10) gives back the original action in Equation (8).

Next, in order to get the canonical (Einstein–Hilbert) gravity action, a Weyl transformation of metric with the space-time-dependent parameter $\Omega(x)$,

$$g_{AB} = \Omega^{-2} \tilde{g}_{AB}, \quad \sqrt{-g} = \Omega^{-D} \sqrt{-\tilde{g}}, \quad (11)$$

can be used. It implies:

$$R = \Omega^2 [\tilde{R} + 2(D-1)\tilde{\square}f - (D-1)(D-2)\tilde{g}^{AB}f_{-A}f_{-B}], \quad (12)$$

where we have introduced the notation:

$$f = \ln \Omega, \quad f_A = \frac{\partial_A \Omega}{\Omega}, \quad (13)$$

and the covariant wave operator $\tilde{\square} = \tilde{D}^A \tilde{D}_A$ in D spacetime dimensions. The transformed action reads:

$$S = \frac{1}{2\kappa^2} \int d^D x \sqrt{-\tilde{g}_D} \Omega^{-D} [(1+B)\Omega^2(\tilde{R} + 2(D-1)\tilde{\square}f - (D-1)(D-2)\tilde{g}^{AB}f_A f_B) - \left(\frac{1}{\gamma^n}\right)^{\frac{1}{n-1}} \frac{n-1}{n} B^{\frac{n}{n-1}} - 2\Lambda], \quad (14)$$

so that we should choose the local parameter Ω as:

$$\Omega^{D-2} = e^{(D-2)f} = 1 + B. \quad (15)$$

It yields:

$$f = \frac{1}{D-2} \ln(1+B) \quad (16)$$

and,

$$S = \frac{1}{2\kappa^2} \int d^D x \sqrt{-\tilde{g}_D} \left[\tilde{R} - (D-1)(D-2)\tilde{g}^{AB} \partial_A f \partial_B f - e^{-Df} \left(\frac{1}{\gamma^n} \right)^{\frac{1}{n-1}} \frac{n-1}{n} B^{\frac{n}{n-1}} - 2e^{-Df} \Lambda \right]. \quad (17)$$

The scalar field f does not have a canonical kinetic term, so that it should be rescaled as:

$$\phi = \sqrt{\frac{(D-1)(D-2)}{\kappa^2}} f. \quad (18)$$

In terms of the canonical scalar ϕ , we have:

$$B = e^{(D-2)\kappa\phi/\sqrt{(D-1)(D-2)}} - 1, \quad (19)$$

and the scalar potential reads:

$$2\kappa^2 V(\phi) = \left(\frac{1}{\gamma^n} \right)^{\frac{1}{n-1}} \left(\frac{n-1}{n} \right) \left[e^{(D-2)\kappa\phi/\sqrt{(D-1)(D-2)}} - 1 \right]^{\frac{n}{n-1}} \times \\ \times e^{-D\kappa\phi/\sqrt{(D-1)(D-2)}} + 2\Lambda e^{-D\kappa\phi/\sqrt{(D-1)(D-2)}}. \quad (20)$$

As a result, we end up with the standard scalar-tensor gravity action in the Einstein frame in D dimensions,

$$S = \frac{1}{2\kappa^2} \int d^D x \sqrt{-\tilde{g}_D} \tilde{R} + \int d^D x \sqrt{-\tilde{g}_D} \left[-\frac{1}{2} \tilde{g}^{AB} \partial_A \phi \partial_B \phi - V(\phi) \right]. \quad (21)$$

A dimensional reduction from D spacetime dimensions to 4 spacetime dimensions (with $(D-4)$ compact dimensions) yields:

$$\int d^D x = V_{D-4} \int d^4 x, \quad \phi = \phi_4 / \sqrt{V_{D-4}}, \quad \kappa = \kappa_4 \sqrt{V_{D-4}}, \quad V = V_4 / V_{D-4}. \quad (22)$$

It implies $\kappa\phi = \kappa_4\phi_4$ and $\kappa^2 V = \kappa_4^2 V_4$, where we have introduced the volume V_{D-4} of the compact space (we use the notation $\kappa_4 = 1/M_{\text{Pl}} = 1$). Then the four-dimensional action reads:

$$S_{\text{inf}}[\tilde{g}_4, \phi_4] = \frac{1}{2} \int d^4 x \sqrt{-\tilde{g}_4} \tilde{R}_4 + \int d^4 x \sqrt{-\tilde{g}_4} \left[-\frac{1}{2} \tilde{g}_4^{\mu\nu} \partial_\mu \phi_4 \partial_\nu \phi_4 - V_4(\phi_4) \right] \quad (23)$$

and defines our inflationary model in four spacetime dimensions, where the initial dimension D and the power n are the parameters of the four-dimensional scalar potential $V_4(\phi_4)$.

The scalar potential in 4 spacetime dimensions with the notation:

$$\lambda = \left(\frac{n}{n-1} \right) \left(\frac{1}{\gamma^m} \right)^{-\frac{1}{n-1}} 2\Lambda, \quad (24)$$

$$\tilde{\phi} = \frac{\phi_4}{\sqrt{(D-1)(D-2)}}, \quad (25)$$

$$\tilde{V}(\tilde{\phi}) = \frac{2V_4(\phi_4)}{\left(\frac{1}{\gamma^m} \right)^{\frac{1}{n-1}} \left(\frac{n-1}{n} \right)}, \quad (26)$$

reads as follows:

$$\tilde{V}(\tilde{\phi}) = \left[e^{(D-2)\tilde{\phi}} - 1 \right]^{\frac{n}{n-1}} e^{-D\tilde{\phi}} + \lambda e^{-D\tilde{\phi}}, \quad (27)$$

where we have used Equations (20) and (22). To get slow roll inflation, we demand this scalar potential to have a plateau of a positive height at $\tilde{\phi} \rightarrow \infty$, which implies:

$$\left[e^{(D-2)\tilde{\phi}} \right]^{\frac{n}{n-1}} e^{-D\tilde{\phi}} = 1, \quad (28)$$

and, hence, (cf. Ref. [11]):

$$n = \frac{D}{2}. \quad (29)$$

A substitution of this condition into Equation (27) gives rise to the potential:

$$\tilde{V}(\tilde{\phi}) = \left[1 - e^{-(D-2)\tilde{\phi}} \right]^{\frac{D}{D-2}} + \lambda e^{-D\tilde{\phi}}. \quad (30)$$

It is convenient to represent the power $D/(D-2) = p/q$ in terms of mutually prime positive integers p and q . Should q be even, it implies the global obstruction $\tilde{\phi} \geq 0$ on the real scalar field $\tilde{\phi}$ because its scalar potential becomes imaginary for $\tilde{\phi} < 0$ (it happens, for example, when $D = 6$ and $D = 10$). The n is the power of R in Equation (8), while the scalar curvature R is arbitrary and can take negative values. The value of n thus should be an even integer. Therefore, as a result, the allowed spacetime dimensions D should be multiples of four.

We demand a minimum of the scalar potential to correspond either a Minkowski or a de Sitter vacuum with a positive second derivative (or a positive inflaton mass squared). This implies:

$$\lambda \geq 0, \quad (31)$$

i.e., the cosmological constant in D spacetime dimensions should be positive. Then we arrive at the scalar potential in four dimensions as:

$$V_4(\phi_4) = \left(\frac{2}{\gamma^D} \right)^{\frac{2}{D-2}} \left(\frac{D-2}{2D} \right) \left(1 - e^{-\sqrt{\frac{D-2}{D-1}}\phi_4} \right)^{\frac{D}{D-2}} + \Lambda e^{-\sqrt{\frac{D^2}{(D-1)(D-2)}}\phi_4}. \quad (32)$$

The simplest case beyond four dimensions is in $D = 8$ dimensions, and it has the action:

$$S_{8,\text{grav.}} = \frac{1}{2\kappa_8^2} \int d^8 X \sqrt{-g_8} (R_8 + \gamma_8 R_8^4 - 2\Lambda_8). \quad (33)$$

The combined (Legendre–Weyl) transformation (see this Section below) in $D = 8$ yields the action:

$$S_{8,\text{grav.}}[\tilde{g}_{AB}, f] = \frac{1}{2\kappa_8^2} \int d^8 X \sqrt{-\tilde{g}_8} \left[\tilde{R}_8 - 42\tilde{g}^{AB} \partial_A f \partial_B f - \frac{3}{4} \left(\frac{1}{4\gamma_8} \right)^{\frac{1}{3}} \left(1 - e^{-6f} \right)^{4/3} - 2e^{-8f} \Lambda_8 \right]. \quad (34)$$

We find convenient to redefine the coupling constants as:

$$\kappa_8 \equiv M_8^{-3}, \quad \gamma_8 \equiv M_8^{-6} \tilde{\gamma}_8, \quad \Lambda_8 \equiv M_8^2 \tilde{\Lambda}_8, \quad \frac{3}{4} \left(\frac{1}{4\tilde{\gamma}_8} \right)^{\frac{1}{3}} \equiv a^{-2}, \quad (35)$$

in terms of the new (mass) parameter $M_8 > 0$ of dimension $(+1)$, and the dimensionless parameters $\tilde{\Lambda}_8$ and $a > 0$. Then the action in Equation (34) reads as:

$$S_{8,\text{grav.}}[\tilde{g}_{AB}, f] = \frac{M_8^6}{2} \int d^8 X \sqrt{-\tilde{g}_8} \left[\tilde{R}_8 - 42\tilde{g}^{AB} \partial_A f \partial_B f - M_8^2 \tilde{V}(f) \right] \quad (36)$$

with the (dimensionless) scalar potential:

$$\tilde{V}(f) = a^{-2} (1 - e^{-6f})^{\frac{4}{3}} + 2e^{-8f} \tilde{\Lambda}_8. \quad (37)$$

For simplicity, we only consider the case of $D = 8$ dimensions in the next Sections.

4. Spontaneous Compactification of 8 Dimensions to 4 Dimensions with a Warp Factor

Our considerations in the previous Section ignored dynamics of moduli associated with the compactified dimensions. The moduli may easily spoil inflation unless their masses exceed the inflationary scale. This problem is known as moduli stabilization. Therefore, we have to take the moduli into account in our case also, and stabilize them (see [12] for the earlier attempts to derive inflation from multi-dimensional cosmology). Here is the place where Freund–Rubin-type compactification [1] truly helps, but it has to be supplemented by a warp factor.

In this Section we describe a compactification of the $(R + R^4)$ modified gravity in $D = 8$ dimensions on a four-sphere S^4 with the warp factor χ and derive its effective action in four spacetime dimensions [6,7]. We separate eight-dimensional coordinates (X^A) into the four-dimensional coordinates (x^α) with $\alpha = 0, 1, 2, 3$, and the coordinates (y^a) of four compact dimensions with $a, b, \dots = 1, 2, 3, 4$. The standard compactification ansatz with a warp factor reads [2]:

$$ds_8^2 = \tilde{g}_{AB} dX^A dX^B = g_{\alpha\beta} dx^\alpha dx^\beta + e^{2\chi} g_{ab} dy^a dy^b, \quad (38)$$

where $g_{\alpha\beta} = g_{\alpha\beta}(x)$, $g_{ab} = g_{ab}(y)$ and $\chi = \chi(x)$. We use the normalization:

$$\int d^4 y \sqrt{g_y} = M_8^{-4}. \quad (39)$$

The Euler number of S^4 is equal to 2, so that we also have:

$$\int d^4 y \sqrt{g_y} R_y = 2M_8^{-2}, \quad (40)$$

where R_y is the scalar curvature of the sphere S^4 . The decomposition in Equation (38) yields:

$$\sqrt{-\tilde{g}_8} = e^{4\chi} \sqrt{-g_4} \sqrt{g_y} \quad (41)$$

and

$$\tilde{R}_8 = R + e^{-2\chi} R_y - 8e^{-\chi} \tilde{\square} e^\chi - 12e^{-2\chi} g^{\alpha\beta} \partial_\alpha e^\chi \partial_\beta e^\chi, \quad (42)$$

where we have introduced the Ricci scalar R and the covariant wave operator $\tilde{\square} = g^{\alpha\beta} \nabla_\alpha \nabla_\beta$ in four spacetime dimensions.

The volume \mathcal{V} of four compact dimensions is given by:

$$\mathcal{V} = \int d^4y \sqrt{\det(e^{2\chi} g_{ab})} = e^{4\chi} M_8^4. \quad (43)$$

Therefore, the warp factor χ is simply related to the volume modulus of S^4 .

A substitution of Equations (38), (41), and (42) into the action of Equation (36), and an integration over the compact dimensions by using Equations (39) and (40), give rise to the action:

$$S_4[g_{\alpha\beta}, f, \chi] = \frac{M_8^2 e^{4\chi_0}}{2} \int d^4x \sqrt{-g} \left(\frac{e^\chi}{e^{\chi_0}} \right)^4 \left[R + 2M_8^2 e^{-2\chi} \right. \\ \left. + 12g^{\alpha\beta} \partial_\alpha \chi \partial_\beta \chi - 42g^{\alpha\beta} \partial_\alpha f \partial_\beta f - M_8^2 \tilde{V}(f) \right], \quad (44)$$

where we have introduced the vacuum expectation value $\langle \chi \rangle_0 = \chi_0 = \text{const.}$

The action in Equation (44) is clearly in Jordan frame, so that the wrong sign of the kinetic term of the field χ is not yet a problem. The Weyl transformation of metric with the parameter $h(x)$ to Einstein frame is given by:

$$g_{\alpha\beta} = e^{-2h} \hat{g}_{\alpha\beta}, \quad h = 2(\chi - \chi_0), \quad (45)$$

which implies:

$$g^{\alpha\beta} = e^{2h} \hat{g}^{\alpha\beta}, \quad \sqrt{-g} = e^{-4h} \sqrt{-\hat{g}}, \quad (46)$$

and

$$R = e^{2h} \left[\hat{R} + 6\hat{g}^{\alpha\beta} \nabla_\alpha \nabla_\beta h - 6\hat{g}^{\alpha\beta} \partial_\alpha h \partial_\beta h \right]. \quad (47)$$

The action of Equation (44) now takes the form:

$$S_4[\hat{g}_{\alpha\beta}, f, \chi] = \frac{M_8^2 e^{4\chi_0}}{2} \int d^4x \sqrt{-\hat{g}} \left\{ \hat{R} - 12\hat{g}^{\alpha\beta} \partial_\alpha \chi \partial_\beta \chi \right. \\ \left. - 42\hat{g}^{\alpha\beta} \partial_\alpha f \partial_\beta f - \left(\frac{e^\chi}{e^{\chi_0}} \right)^{-4} M_8^2 \left[\tilde{V}(f) - 2e^{-2\chi} \right] \right\}, \quad (48)$$

with the physical signs in front of all the kinetic terms. The four-dimensional (reduced) Planck mass is also fixed as:

$$M_{\text{Pl}}^2 \equiv \kappa^{-2} = M_8^2 e^{4\chi_0}. \quad (49)$$

As a result, we arrive at the final effective action in four spacetime dimensions in the form:

$$S_4[\hat{g}_{\alpha\beta}, f, \chi] = \frac{M_{\text{Pl}}^2}{2} \int d^4x \sqrt{-\hat{g}} \left[\hat{R} - 12\hat{g}^{\alpha\beta} \partial_\alpha \chi \partial_\beta \chi \right. \\ \left. - 42\hat{g}^{\alpha\beta} \partial_\alpha f \partial_\beta f - e^{-4\chi} M_{\text{Pl}}^2 \left(\tilde{V}(f) - 2e^{-2\chi} \right) \right]. \quad (50)$$

This equation clearly shows that the volume modulus (or the warp factor) χ has its own dynamics. In addition, there is mixing of the volume modulus with the inflaton f in the scalar potential. Therefore, stabilization of the modulus during (single-field) inflation is needed.

5. Freund–Rubin Compactification

Stabilization of moduli can be achieved by flux compactification [1], while it is the standard approach in string theory [13,14]. In our case it requires an introduction of at least one p -form gauge field obeying the condition $p = n = D/2$. Therefore, the minimal inflationary model in the $D = 8$ modified gravity should be defined by the action:

$$S = \frac{M_8^6}{2} \int d^8 X \sqrt{-g_8} \left[R_8 + \gamma_8 R_8^4 - 2\Lambda_8 - g^{A_1 B_1} g^{A_2 B_2} g^{A_3 B_3} g^{A_4 B_4} F_{A_1 A_2 A_3 A_4} F_{B_1 B_2 B_3 B_4} \right] \quad (51)$$

that depends upon two fields, a metric g_{AB} , and a 3-form gauge potential A_{ABC} , whose field strength 4-form is $F = dA$. The gauge field strength F has (mass) dimension (+1), whereas the gauge field A is dimensionless.

It is important to observe that under the Weyl transformation in Equation (11), the Ω -factors are cancelled in the p -form action, so that the latter is unchanged,

$$S_8[\tilde{g}_{AB}, F_4] = -\frac{M_8^6}{2} \int d^8 X \sqrt{-\tilde{g}_8} \tilde{g}^{A_1 B_1} \dots \tilde{g}^{A_4 B_4} F_{A_1 \dots A_4} F_{B_1 \dots B_4}. \quad (52)$$

The compactification ansatz of Equation (38) for the action in Equation (52) gives rise to:

$$S_{8,F}[\tilde{g}_{AB}, F] = -\frac{M_8^6}{2} \int d^4 x \sqrt{-g} \int d^4 y \sqrt{g_y} e^{-4\chi} g^{a_1 b_1} \dots g^{a_4 b_4} F_{a_1 \dots a_4} F_{b_1 \dots b_4}. \quad (53)$$

We define the dimensionless *flux* parameter F^2 as follows:

$$\int d^4 y \sqrt{g_y} g^{a_1 b_1} \dots g^{a_4 b_4} F_{a_1 \dots a_4} F_{b_1 \dots b_4} = M_8^{-2} F^2 = \text{const.}, \quad (54)$$

and use the Weyl transformation in Equation (45) in order to reduce the action in Equation (53) to:

$$\begin{aligned} S_{4,F}[\hat{g}_{AB}, \chi] &= -\frac{M_8^2 e^{4\chi_0}}{2} \int d^4 x \sqrt{-g} \left(\frac{e^\chi}{e^{\chi_0}} \right)^4 e^{-8\chi} M_8^2 F^2 \\ &= -\frac{M_8^2 e^{4\chi_0}}{2} \int d^4 x \sqrt{-\hat{g}} e^{-4h} \left(\frac{e^\chi}{e^{\chi_0}} \right)^4 e^{-8\chi} M^2 F^2 \\ &= -\frac{M_{\text{Pl}}^4}{2} \int d^4 x \sqrt{-\hat{g}} e^{-12\chi} F^2. \end{aligned} \quad (55)$$

Then the effective action in four spacetime dimensions is given by a sum of Equations (50) and (55), which reads:

$$\begin{aligned} S_4[\hat{g}_{AB}, \chi, f] &= \frac{M_{\text{Pl}}^2}{2} \int d^4 x \sqrt{-g} \left[\hat{R} - 12 \hat{g}^{\alpha\beta} \partial_\alpha \chi \partial_\beta \chi \right. \\ &\quad \left. - 42 \hat{g}^{\alpha\beta} \partial_\alpha f \partial_\beta f - M_{\text{Pl}}^2 \left(e^{-4\chi} \tilde{V}(f) - 2e^{-6\chi} - e^{-12\chi} F^2 \right) \right]. \end{aligned} \quad (56)$$

The canonical scalar fields $\hat{\chi}$ and \hat{f} are obtained after a simple renormalization,

$$\hat{\chi} = 2\sqrt{3} M_{\text{Pl}} \chi \quad \text{and} \quad \hat{f} = \sqrt{42} M_{\text{Pl}} f, \quad (57)$$

so that their scalar potential in four spacetime dimensions is given by:

$$M_{\text{Pl}}^{-4} V(\chi, f) = \left[a^{-2} (1 - e^{-6f})^{\frac{4}{3}} + 2\tilde{\Lambda}_8 e^{-8f} \right] e^{-4\chi} - 2e^{-6\chi} + F^2 e^{-12\chi}. \quad (58)$$

This two-field scalar potential for the purpose of single-field inflation is studied in the next Section.

6. Study of the Scalar Potential

The derived scalar potential in Equation (58) depends on two fields, the inflaton f and the volume modulus χ , and has the three parameters $(a^{-2}, F^2, \tilde{\Lambda}_8)$ originating from the higher (eight) dimensions. There is a Minkowski vacuum at (f_0, χ_0) defined by the equations:

$$\left. \frac{\partial V}{\partial f} \right|_{f=f_0} = \left. \frac{\partial V}{\partial \chi} \right|_{\chi=\chi_0} = V|_{f=f_0, \chi=\chi_0} = 0. \quad (59)$$

It is not difficult to verify that a solution to vacuum equations is given by:

$$e^{6f_0} = 1 + (2\tilde{\Lambda}_8 a^2)^3 \quad \text{and} \quad e^{6\chi_0} = 2F^2, \quad (60)$$

under the following condition on the parameters:

$$\frac{2}{3}\tilde{\Lambda}_8 = \left(\frac{1}{16F^2 - 256\tilde{\gamma}_8} \right)^{1/3}, \quad (61)$$

where we have used the third relation in Equation (35) between $\tilde{\gamma}_8$ and a .

The masses of the canonically normalized scalars in Equation (57) are determined by the second derivatives of the scalar potential Equation (58) at the critical point of Equation (60),

$$m_{f_0}^2 = \left. \frac{\partial^2 V}{\partial f^2} \right|_{f=f_0} \frac{1}{42M_{\text{Pl}}^2} = \frac{M_{\text{Pl}}^2}{56F^2} \left(\frac{F^2}{\tilde{\gamma}_8} - 16 \right) \quad (62)$$

and

$$m_{\chi_0}^2 = \left. \frac{\partial^2 V}{\partial \chi^2} \right|_{\chi=\chi_0} \frac{1}{12M_{\text{Pl}}^2} = \frac{M_{\text{Pl}}^2}{F^2}, \quad (63)$$

where we have also used Equation (61). Equations (61) and (62) also imply that:

$$\frac{F^2}{\tilde{\gamma}_8} > 16 \quad (64)$$

is needed for the existence of a Minkowski vacuum and its stability.

At the onset of inflation ($f = +\infty$), the scalar potential of the modulus χ ,

$$M_{\text{Pl}}^{-4}V(\chi) = a^{-2}e^{-4\chi} - 2e^{-6\chi} + F^2e^{-12\chi}, \quad (65)$$

depends on only two parameters (a^{-2}, F^2) .

The critical points of the scalar potential in Equation (65) are determined by the condition:

$$a^{-2} - 3e^{2\chi_c} + 3F^2e^{-8\chi_c} = 0 \quad (66)$$

that is the depressed quartic equation:

$$z^4 + qz + r = 0 \quad (67)$$

in the notation:

$$z = e^{-2\chi_c}, \quad q = \frac{-1}{F^2} < 0, \quad r = \frac{1}{3a^2F^2} > 0. \quad (68)$$

The associated quartic discriminant is given by:

$$\frac{\Delta_4}{27 \cdot 256} = (r/3)^3 - (q/4)^4, \quad (69)$$

so that a solution to Equation (67) crucially depends upon the sign of the Δ_4 .

The auxiliary (Ferrari) resolvent cubic equation:

$$m^3 - rm - q^2/8 = 0 \quad (70)$$

can be used to factorize the left-hand-side of the quartic Equation (67) as follows:

$$\left(z^2 + m + \sqrt{2m}z - \frac{q}{2\sqrt{2m}}\right) \left(z^2 + m - \sqrt{2m}z + \frac{q}{2\sqrt{2m}}\right) = 0. \quad (71)$$

Each term in the first factor is positive in our case, so that we arrive at a quadratic equation from the vanishing second factor, whose two roots are:

$$z_{1,2} = \sqrt{\frac{m}{2}} \left[1 \pm \sqrt{-\frac{q}{m} - \sqrt{2m}} \right]. \quad (72)$$

These two roots precisely correspond to a local (meta-stable) minimum and a local maximum of the potential of Equation (65), with $-\infty < \chi_{\min.} < \chi_{\max.} < +\infty$.

The cubic discriminant $\Delta_3 = 4r^3 - 27(q^2/8)^2$ of the depressed cubic Equation (70) is related to Δ_4 as follows:

$$\frac{\Delta_3}{4 \cdot 27} = (r/3)^3 - (q/4)^4 = \frac{\Delta_4}{27 \cdot 256}. \quad (73)$$

Given $\Delta_{3,4} \geq 0$, the three real solutions to the cubic Equation (70) are given by the Vieté formula:

$$m_k = 2\sqrt{r/3} \cos \theta_k, \quad k = 0, 1, 2, \quad (74)$$

whose angles are:

$$\theta_k = \frac{1}{3} \arccos \left(\frac{3q^2}{16r} \sqrt{3/r} \right) - \frac{2\pi k}{3}. \quad (75)$$

We have to choose the highest (positive) root in our case. The condition $\Delta_{3,4} \geq 0$ implies:

$$\frac{F^2}{\tilde{\gamma}_8} \geq 27. \quad (76)$$

Given $\Delta_{3,4} \leq 0$ or, equivalently, $F^2/\tilde{\gamma}_8 \leq 27$, the angle of Equation (75) does not exist. Instead, we should use Vieté's substitution in Ferrari's equation with:

$$m = w + \frac{r}{3w}, \quad r > 0. \quad (77)$$

It yields a quadratic equation for w^3 in the form:

$$(w^3)^2 - \frac{q^2}{8} w^3 + \frac{r^3}{27} = 0, \quad (78)$$

whose roots are given by:

$$w_{1,2}^3 = (q/4)^2 \left[1 \pm \sqrt{1 - \frac{(r/3)^3}{(q/4)^4}} \right]. \quad (79)$$

Inserting the critical condition in Equation (66) in the form:

$$F^2 = e^{6\chi_c} \left[1 - \frac{1}{3} a^{-2} e^{2\chi_c} \right] \quad (80)$$

into the potential of Equation (65) yields the height of the inflationary potential V_{plateau} at the onset of inflation as:

$$M_{\text{Pl}}^{-4} V_{\text{plateau}} = e^{-6\chi_c} \left[\frac{2}{3} a^{-2} e^{2\chi_c} - 1 \right]. \quad (81)$$

Requiring positivity of V_{plateau} gives us the restriction Equation (64) again.

The second derivative of the potential in Equation (65) at the critical point of Equation (66) reads:

$$\left. \frac{\partial^2 V}{\partial \chi^2} \right|_{\chi=\chi_c} = 8e^{-6\chi_c} (9 - 4a^{-2} e^{2\chi_c}). \quad (82)$$

Its positivity (needed for stability) implies the condition:

$$\frac{F^2}{\tilde{\gamma}_8} < 54. \quad (83)$$

Taken together with Equations (64) and (76), it leads to the following restrictions on the values of the ratio $F^2/\tilde{\gamma}_8$:

$$\begin{aligned} 16 < \frac{F^2}{\tilde{\gamma}_8} &\leq 27, & \Delta_{3,4} &\leq 0, \\ 27 &\leq \frac{F^2}{\tilde{\gamma}_8} < 54, & \Delta_{3,4} &\geq 0. \end{aligned} \quad (84)$$

Since $1 < F^2/(16\tilde{\gamma}_8) \equiv 1 + \delta < (\frac{3}{2})^3$, it is illuminating to investigate the case of $0 < \delta \ll 1$ describing strong stabilization of the modulus χ . In this case, Equations (60) and (80) give rise to:

$$0 < \chi_c - \chi_0 \approx \frac{1}{12} \delta \ll 1, \quad (85)$$

leading to a single-field inflation driven by the inflaton (scalaron) f .

The physical scale hierarchy is given by:

$$m_{\hat{f}_0} < m_{\hat{\chi}_0} \ll M_{\text{KK}} \ll M_{\text{Pl}}. \quad (86)$$

The KK scale is given by $M_{\text{KK}} \approx e^{-\chi_0} M_{\text{Pl}}$, where the presence of the warp factor is dictated by the compactification ansatz of Equation (38).

The condition $M_{\text{KK}} \ll M_{\text{Pl}}$ also implies:

$$2F^2 \gg 1 \quad (87)$$

due to Equation (60). The condition $m_{\hat{\chi}_0} \ll M_{\text{KK}}$ leads to:

$$F^2 \gg \sqrt{2} \quad (88)$$

that is slightly stronger than Equation (87). Both conditions can be satisfied by taking $F^2 \gg 1$.

The remaining condition $m_{\hat{f}_0} < m_{\hat{\chi}_0}$ implies $F^2/\tilde{\gamma}_8 < 72$ that is already satisfied under the conditions of Equation (84). It is impossible to achieve $m_{\hat{f}_0} \ll m_{\hat{\chi}_0}$.

A profile of the scalar potential in four spacetime dimensions is given in Figure 1. The cosmological constant in $D = 8$ dimensions is given by Equation (61), which leads to:

$$\tilde{\Lambda}_8 = \frac{\delta^{-1/3}}{2a^2}. \quad (89)$$

In particular, it means that δ cannot vanish.

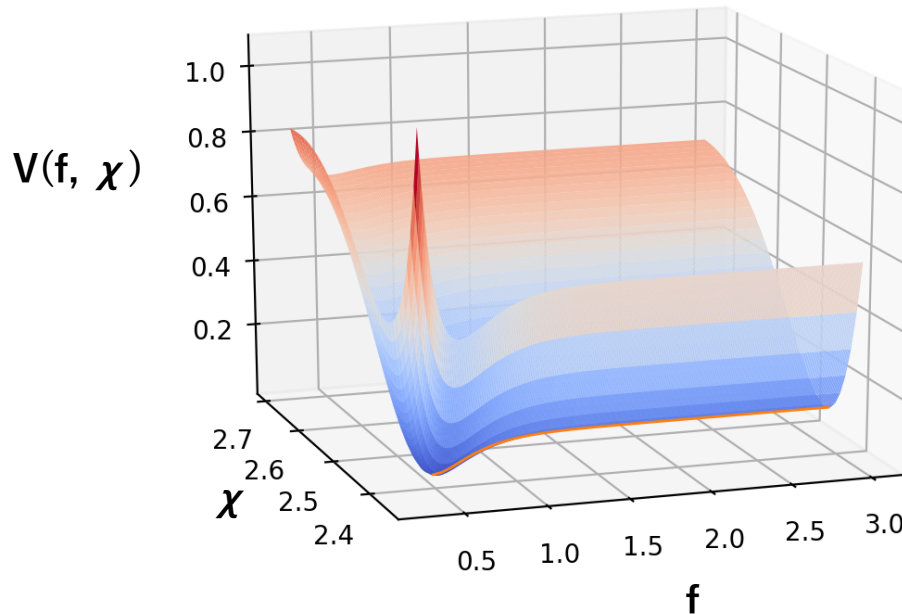


Figure 1. The profile of the scalar potential of Equation (58) with the input values $F^2 = 10^6$, $\tilde{\gamma}_8 = 6 \cdot 10^4$ and $\tilde{\Lambda}_8 \approx 0.0174$. The bottom (red) line shows the inflationary trajectory.

7. The Inflationary Observables

After the modulus χ is strongly stabilized, the inflaton scalar potential in Equation (58) takes the form ($M_{\text{Pl}} = 1$)

$$e^{4\chi_0} a^2 V(f) = \left(1 - e^{-6f}\right)^{\frac{4}{3}} + \lambda e^{-8f} - \lambda(1 + \lambda^3)^{-\frac{1}{3}}, \quad (90)$$

where we have introduced the parameter $\lambda = 2a^2 \tilde{\Lambda}_8 = \delta^{-1/3}$. The potential has the absolute minimum at:

$$f_0 = \frac{1}{6} \ln(1 + \lambda^3), \quad (91)$$

where it vanishes (Minkowski vacuum). A profile of the scalar potential in Equation (90) is given in Figure 2.

During inflation along the plateau the scalar potential in Equation (90) can be approximated by Equation (4) with:

$$\alpha = \sqrt{\frac{6}{7}}. \quad (92)$$

This value of α determines the observable tensor-to-scalar ratio r as:

$$r = \frac{8}{\alpha^2 N_e^2} = \frac{28}{3N_e^2} \quad (93)$$

that is very close to the Starobinsky value in Equation (5). This also applies to the scalar spectral index n_s and its running $dn_s/d\ln k$ in Equation (7).

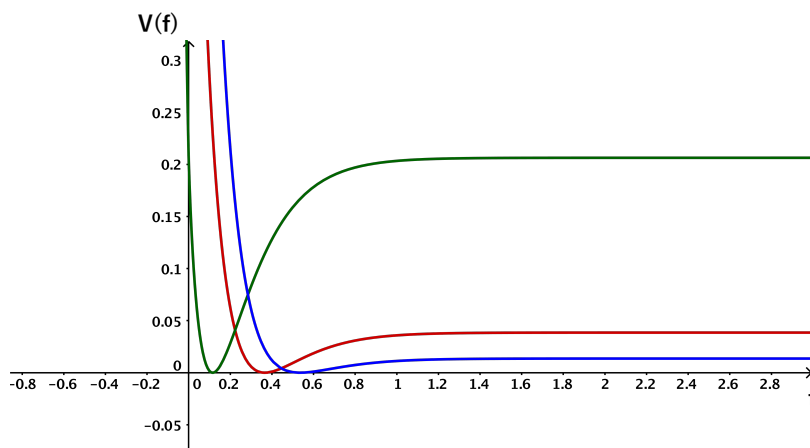


Figure 2. The profile of the scalar potential in Equation (90) for the values $\lambda = 1$ (green), $\lambda = 2$ (red), and $\lambda = 2.88$ (blue).

The microscopic parameters of the modified gravity in higher dimensions can be easily tuned to get the required inflaton mass M , so that the effective inflationary model is almost indistinguishable from the Starobinsky model having $\alpha_s = \sqrt{2/3}$.

When a conventional matter action is added, Weyl rescaling of the metric gives rise the universal couplings (via the covariant derivatives) of the inflaton f to all matter fields with the powers of $\exp(-\alpha\kappa_4 f)$. The value in Equation (92) of α derived from $D = 8$ dimensions is slightly different from the Starobinsky value $\alpha_s = \sqrt{2/3}$, while all matter couplings to the inflaton are suppressed by the Planck mass. Therefore, the impact of higher dimensions on reheating is expected to be negligible.

8. Conclusions

We used the Starobinsky inflationary model (1) in four spacetime dimensions as the prototype for deriving new inflationary models of modified gravity descending from higher dimensions. It is worth mentioning that any $(R + \gamma R^n)$ modified gravity model in four spacetime dimensions with an integer power n different from two is not viable for inflation [15] (having n to be a non-integer close to 2 is possible [16], though it can be reduced to the R^2 inflation with the logarithmic corrections).

The advantages of our approach to inflation from higher dimensions are as follows: (i) Its geometrical nature because only gravitational interactions are used, (ii) its consistency with the current astronomical observations of CMB, and (iii) its clear physical nature of the inflaton originating from metric. We focused on the case of $D = 8$ spacetime dimensions as the simplest non-trivial example. In our cosmological scenario, the universe was born multi-dimensional, and then four spacetime dimensions became infinite, while the others curled up by unknown mechanism before inflation. Inflation happened after compactification and moduli stabilization.

A positive cosmological constant and a gauge (form) field in higher dimensions are necessary, while there are the strong conditions on the number of extra dimensions, the power n of the scalar curvature in the modified gravity term, and the rank of the gauge form. The moduli stabilization and the physical scale hierarchy are possible to achieve, though both are non-trivial.

It may be possible to embed our $D = 8$ modified gravity model into the modified $D = 8$ supergravity and then into the modified $D = 11$ supergravity, see [7] for details.

As regards the observational predictions of our approach, it results in the certain value of Equation (93) of the CMB tensor-to-scalar ratio that is slightly different from that of the Starobinsky model.

Our results can be used for studying inflation and moduli stabilization in the more general frameworks, such as unification of fields and forces, KK theories of gravity, supergravity and superstrings, and braneworld. It was demonstrated in [17] that the modified $(R + R^2)$ gravity in the Randall–Sundrum (RSII) braneworld [2,3] does not destabilize the Randall–Sundrum solution to the hierarchy problem in high-energy particle physics.

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