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k-Hypergeometric Series Solutions to One Type of Non-Homogeneous k-Hypergeometric Equations

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Abstract: In this paper, we expound on the hypergeometric series solutions for the second-order non-homogeneous k-hypergeometric differential equation with the polynomial term. The general solutions of this equation are obtained in the form of k-hypergeometric series based on the Frobenius method. Lastly, we employ the result of the theorem to find the solutions of several non-homogeneous k-hypergeometric differential equations.

Keywords: k-hypergeometric differential equations; non-homogeneous; k-hypergeometric series; special function; general solution; Frobenius method

1. Introduction

It is well known that many phenomena in physical and technical applications are governed by a variety of differential equations. We should notice that these differential equations have appeared in many different research fields, for instance in the theory of automorphic function, in conformal mapping theory, in the theory of representations of Lie algebras, and in the theory of difference equations. Analytical and numerical methods to solve ordinary differential equations are an ancient and interesting research direction in differentiable dynamical systems and their applications. Let us consider a so-called non-homogeneous k-hypergeometric differential equation of the form:

$$kz(1-kz)\frac{d^2y}{dz^2}+[c-(k+a+b)kz]\frac{dy}{dz}-aby=f(z) \tag{1}$$

with the independent variable z, where a,b,c,k are several constants with $a,b,c\in\mathbb{R}, k\in\mathbb{R}^+$, and the function $f(z)$ is holomorphic in an interval $\mathcal{D}\subseteq\mathbb{C}$. In the case of $k=1$, if the function $f(z)$ vanishes identically, then Equation (1) degrades into a linear homogeneous hypergeometric ordinary differential equation presented by Euler [1] in 1769, which has the following normalized form:

$$z(1-z)\frac{d^2y}{dz^2}+[c-(1+a+b)z]\frac{dy}{dz}-aby=0; \tag{2}$$

such an equation has been extensively studied.

The solutions of a differential equation relate to many absorbing special functions in mathematics, physics, and engineering. For instance, the solution could be presented by power series [2,3], continued fraction [4–6], zeta function [7–10], and hypergeometric series [11–16]. Among these special functions, the hypergeometric series, denoted by:

$${}_2F_1[a,b;c;z]=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n,$$

can be applied to the solution of the differential Equation (2). For Equation (2), a hypergeometric series solution ${}_2F_1$ can be derived by the Frobenius method. The so-called hypergeometric series was researched firstly by Wallis [11] in 1655. Since then, Euler, too, had researched the topic on the hypergeometric series, but the first full systematic study was introduced by Gauss [12]. Some works and complete references concerning both the hypergeometric series and the certain equation (2) can be found in Kummer [13], Riemann [14], Bailey [15,16], Chaundy [17], Srivastava [18], Whittaker [19], Beukers [20], Gasper [21], Olde Daalhuis [22,23], Dwork [24], Chu [25], Yilmazer et al. [26], Morita et al. [27], Abramov et al. [28], Alfedeel et al. [29], and the literature therein. However, in contrast to the extensive studies on Equation (2), other hypergeometric differential equations with $k \in \mathbb{R}^+$ are very limited.

If k is not necessarily equal to one and $f(z)$ is still a zero function in Equation (1), then the associated differential equation is written as follows:

$$kz(1-kz)\frac{d^2y}{dz^2} + [c-(k+a+b)kz]\frac{dy}{dz} - aby = 0. \tag{3}$$

This differential Equation (3), called the homogeneous k-hypergeometric differential equation, has been defined only in recent years. For $k \in \mathbb{R}^+$ and $f(z) = 0$, Equation (3) has a solution in the form of k-hypergeometric series ${}_2F_{1,k}$, which will be introduced in Section 2. It is clear that the k-hypergeometric series ${}_2F_{1,k}$ has evolved from the hypergeometric series ${}_2F_1$. Hence, we mention the works of Díaz et al. [30,31], Krasniqi [32,33], Kokologiannaki [34], Mubeen et al. (see [35–40]), Rehman et al. [41,42], and the references therein for results on k-hypergeometric series and the homogeneous k-hypergeometric differential equation. In 2005, the Pochhanner k-symbol was developed by Díaz et al. [30]. Since then, for example, k-gamma and k-beta functions have been researched, and their relevant properties have been shown [30,31]. By following the works of Díaz et al., in 2010, some fascinating results with respect to k-gamma, -beta, and -zeta functions were proven in [32–34]. In 2012, a so-called k-fractional integral and its application were presented by Mubeen and Habibullah [36]. Furthermore, based on the properties of Pochhammer k-symbols, k-gamma, and k-beta functions, Mubeen et al. [35,37] suggested an integral representation of k-hypergeometric functions and some generalized confluent k-hypergeometric functions. Mubeen [37] did not introduce the second-order linear k-hypergeometric differential equation defined by Equation (3) until 2013. Furthermore, in 2014, Mubeen et al. [38,39] solved the k-hypergeometric differential equation by using the Frobenius method and gave its solution in the form of the so-called k-hypergeometric series ${}_2F_{1,k}$ introduced by Díaz et al. [30]. In the case of $k \in \mathbb{R}^+$ and $f(z) \neq 0$, the research for this question is very limited.

Motivated by the above results, in this paper, we consider the k-hypergeometric series solutions of Equation (1) when $f(z)$ is a non-vanishing function and $k \in \mathbb{R}^+$. For simplicity, we choose $f(z)$ as a polynomial $\sum_{i=0}^{m} d_i z^i$. That is, we will discuss the general solution of the so-called non-homogeneous k-hypergeometric equation:

$$kz(1-kz)\frac{d^2y}{dz^2} + [c-(k+a+b)kz]\frac{dy}{dz} - aby = \sum_{i=0}^{m} d_i z^i, \tag{4}$$

where $k \in \mathbb{R}^+$ and $d_i, i = 0, 1, 2, \ldots, m$, are some real or complex constants. The corresponding homogeneous k-hypergeometric equation of Equation (4) is denoted by Equation (3).

The aim of this paper to find general solutions of the non-homogeneous k-hypergeometric Equation (4) by means of the k-hypergeometric series. This paper is organized as follows: in Section 2, the basic definitions and facts of the k-hypergeometric series and ordinary differential equation are presented. Our results are then introduced in Section 3. Some examples are given to illustrate the applications of our results in Section 4. Some conclusions and future perspectives are given in the last section. Throughout this paper, we let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{N}^+$ stand for the set of complex numbers, the set of real numbers, the set of positive real numbers, and the set of positive integers, respectively.

2. Preliminaries

In this section, we briefly review some basic definitions and facts concerning the k-hypergeometric series and the ordinary differential equation. Some surveys and literature for k-hypergeometric series and the k-hypergeometric differential equation can be found in Díaz et al. [30,31], Krasniqi [32,33], and Mubeen et al. [38,39].

Definition 1. *Assume that $x \in \mathbb{C}$, $k \in \mathbb{R}^+$ and $n \in \mathbb{N}^+$, then the Pochhammer k-symbol $(x)_{n,k}$ is defined by:*

$$(x)_{n,k} = x(x+k)(x+2k)\ldots[x+(n-1)k]\,. \tag{5}$$

In particular, we denote $(x)_{0,k} \equiv 1$. Therefore, we have the following facts:

(i) $(x)_{n+1,k} = (x+nk)(x)_{n,k}$.

(ii) $(1)_{n,1} = n!$; $(\frac{1}{2})_{n,1} = \dfrac{(2n-1)!!}{2^n}$; $(\frac{3}{2})_{n,1} = \dfrac{(2n+1)!!}{2^n}$.

(iii) $(x)_{n,1} = \dfrac{\Gamma(x+n)}{\Gamma(x)}$, where $\Gamma(x)$ is the Gamma function defined by $\int_0^\infty e^{-t}t^{x-1}dt$.

(iv) $(1)_{n,2} = (2n-1)!!$; $(2)_{n,2} = (2n)!!$; $(3)_{n,2} = (2n+1)!!$; $(4)_{n,2} = \dfrac{(2n+2)!!}{2}$.

Definition 2. *Assume that $a, b, c \in \mathbb{C}$, $k \in \mathbb{R}^+$ and $n \in \mathbb{N}^+$, then the k-hypergeometric series with three parameters a, b, and c is defined as:*

$${}_2F_{1,k}[(a,k),(b,k);(c,k);z] = \sum_{n=0}^{\infty} \frac{(a)_{n,k}(b)_{n,k}}{(c)_{n,k}n!}z^n, \tag{6}$$

where $c \neq 0, -1, -2, -3, \ldots$ and $z \in \mathbb{C}$.

Definition 3. *Assume that $Y_0(z)$, $Y_1(z)$, and $Y_2(z)$ are three functions of z. Let a second-order ordinary differential equation be written in the following form:*

$$Y_2(z)\frac{d^2y}{dz^2} + Y_1(z)\frac{dy}{dz} + Y_0(z) = 0. \tag{7}$$

Then, the method about finding an infinite series solution of Equation (7) is called the Frobenius method.

Definition 4. *For Equation (7), let its coefficient $Y_2(z)$ satisfy $Y_2(z_0) = 0$ about the point $z_0 \in \mathcal{D} \subseteq \mathbb{C}$. Further, if this coefficient $Y_2(z)$ is holomorphic in a deleted neighborhood $\{z|\ 0 < |z-z_0| < \varepsilon\}$ for some $\varepsilon > 0$ and is meromorphic (not all holomorphic) in a neighborhood $\{z|\ |z-z_0| < \varepsilon\}$, then the point z_0 is called a singular point of Equation (7).*

Definition 5. *For the Equation (7), if the coefficient:*

$$Y_2(z) = (z-z_0)^i h(z)$$

is holomorphic at the point z_0, then the singular point z_0 of Equation (7) is said to be regular.

Dividing both sides of this Equation (7) by $Y_2(z)$ gives a differential equation of the following form:

$$\frac{d^2y}{dz^2} + \frac{Y_1(z)}{Y_2(z)}\frac{dy}{dz} + \frac{Y_0(z)}{Y_2(z)} = 0. \tag{8}$$

As we know, if either $\frac{Y_0(z)}{Y_2(z)}$ or $\frac{Y_1(z)}{Y_2(z)}$ is not analytic at any regular singular point z_0, then Equation (8) cannot be solvable with the regular power series method. However, the method of Frobenius enables us to gain a power series solution of the differential equation defined by Equation (8), provided that both $\frac{Y_0(z)}{Y_2(z)}$ and $\frac{Y_1(z)}{Y_2(z)}$ are themselves analytic at z_0 or they are analytic elsewhere and their limits exist at z_0.

3. The Solutions of Non-Homogeneous k-Hypergeometric Equations

In this section, by means of the Frobenius method, we expound upon the series solution of the second-order non-homogeneous k-hypergeometric ordinary differential equation defined by Equation (4). Before presenting the main results, in order to judge whether a series is convergent or not, we usually need to apply the following criterion.

Lemma 1 (The d'Alembert test)**.** *If the series* $\sum\limits_{n=0}^{\infty} u_n$ *satisfies the following condition:*

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \quad (\textit{resp.} > 1),$$

then this series $\sum\limits_{n=0}^{\infty} u_n$ *converges (resp. diverges).*

Proof. Let us recall the following fact: The geometric series:

$$\sum_{n=0}^{\infty} q^n \quad (q > 0),$$

converges (resp. diverges) if $q < 1$ (resp. $q > 1$). Then, the proof of Lemma 1 is a simple series exercise. □

Theorem 1. *Suppose that $k \in \mathbb{R}^+$ and all a, b, c belong to $\mathbb{R}$. Let, in addition, c and $2k - c$ be neither zero, nor negative integers. Then, the homogeneous k-hypergeometric ordinary differential Equation (3) can have a general solution in the following form:*

$$y(z) = A\,{}_2F_{1,k}[(a,k),(b,k);(c,k);z] + B\,z^{1-\frac{c}{k}}\,{}_2F_{1,k}[(a+k-c,k),(b+k-c,k);(2k-c,k);z] \tag{9}$$

for $|z| < 1/k$, where A and B are two constants in $\mathbb{C}$.

Proof. Assume that:

$$y(z) = z^g \sum_{i=0}^{\infty} u_i z^i \tag{10}$$

is any solution of the homogeneous ordinary differential Equation (3) with $u_0 \neq 0$. Then, differentiating Equation (10) directly, one has:

$$y'(z) = z^g \sum_{i=0}^{\infty} u_i (i+g) z^{i-1} \tag{11}$$

and:

$$y''(z) = z^g \sum_{i=0}^{\infty} u_i (i+g)(i+g-1) z^{i-2}. \tag{12}$$

Substituting Equations (11) and (12) into the k-hypergeometric differential Equation (3), we have:

$$\begin{aligned} &kz^g\sum_{i=0}^{\infty}u_i(i+g)(i+g-1)z^{i-1}-k^2z^g\sum_{i=0}^{\infty}u_i(i+g)(i+g-1)z^i \\ &+cz^g\sum_{i=0}^{\infty}u_i(i+g)z^{i-1}-k(a+b+k)z^g\sum_{i=0}^{\infty}u_i(i+g)z^i-abz^g\sum_{i=0}^{\infty}u_iz^i=0, \end{aligned} \tag{13}$$

or, equivalently,

$$\begin{aligned} &u_0g(k(g-1)+c)z^{g-1}+kz^g\sum_{i=1}^{\infty}u_i(i+g)(i+g-1)z^{i-1} \\ &-k^2z^g\sum_{i=1}^{\infty}u_{i-1}(i+g-1)(i+g-2)z^{i-1}+cz^g\sum_{i=1}^{\infty}u_i(i+g)z^{i-1} \\ &-k(a+b+k)z^g\sum_{i=1}^{\infty}u_{i-1}(i+g-1)z^{i-1}-abz^g\sum_{i=1}^{\infty}u_{i-1}z^{i-1}=0. \end{aligned} \tag{14}$$

By comparing the coefficients on both sides of Equation (14), one can obtain the indicial equation:

$$g[k(g-1)+c]=0 \tag{15}$$

and the difference equation:

$$(g+i+1)[k(g+i)+c]u_{i+1}=[k(g+i)+a][k(g+i)+b]u_i, \tag{16}$$

for $i=0,1,2,\ldots$.

Solving the above indicial Equation (15) for g gives:

$$g=0 \text{ and } g=1-\frac{c}{k}. \tag{17}$$

Next, we discuss the solution of Equation (3) in two cases.

- Case 1: $g=0$.

From Equation (16), we have the solution of Equation (3):

$$y_1(z)=u_0\,{}_2F_{1,k}[(a,k),(b,k);(c,k);z], \tag{18}$$

provided that c is not zero or a negative integer.

- Case 2: $g=1-\frac{c}{k}$.

In a similar manner, from Equation (16), we get the difference equation as follows:

$$(1+i)(2k-c+ki)u_{i+1}=(a+k-c+ki)(b+k-c+ki)u_i. \tag{19}$$

Therefore, it follows that the other solution of Equation (3) is written as below:

$$y_2(z)=u_0z^{1-\frac{c}{k}}\,{}_2F_{1,k}[(a+k-c,k),(b+k-c,k);(2k-c,k);z], \tag{20}$$

provided that $2k-c$ is not a negative integer or zero.

Furthermore, from Equations (18) and (20), let us consider the radius of convergence of the series:

$$\sum_{i=0}^{\infty} v_i = {}_2F_{1,k}[(a,k),(b,k);(c,k);z]$$

and:

$$\sum_{i=0}^{\infty} w_i = {}_2F_{1,k}[(a+k-c,k),(b+k-c,k);(2k-c,k);z].$$

Referring to Lemma 1, we verify that:

$$\lim_{i\to\infty}\left|\frac{v_{i+1}}{v_i}\right| = \lim_{i\to\infty}\left|\frac{(a+ki)(b+ki)}{(c+ki)(1+i)}z\right| = |kz| < 1$$

and:

$$\lim_{i\to\infty}\left|\frac{w_{i+1}}{w_i}\right| = \lim_{i\to\infty}\left|\frac{(a+k-c+ki)(b+k-c+ki)}{(2k-c+ki)(1+i)}z\right| = |kz| < 1,$$

which imply that the series $\sum_{i=0}^{\infty} v_i$ and $\sum_{i=0}^{\infty} w_i$ have the same radius of convergence $\frac{1}{k}$.

Therefore, the general solution of the k-hypergeometric differential Equation (3) can be written as:

$$\begin{aligned} y(z) =& \alpha y_1(z) + \beta y_2(z) = A\ {}_2F_{1,k}[(a,k),(b,k);(c,k);z] \\ &+ B\, z^{1-\frac{c}{k}}\ {}_2F_{1,k}[(a+k-c,k),(b+k-c,k);(2k-c,k);z] \end{aligned} \tag{21}$$

for $|z| < 1/k$, where α, β, A, and B are four constants in $\mathbb{C}$.

Therefore, we have completed the proof of Theorem 1. □

Next, when the function $f(z)$ is a polynomial, that is:

$$f(z) = \sum_{i=0}^{m} d_i z^i,\ (m = 0,1,2,\ldots), \tag{22}$$

where $d_i, i = 0,1,2,\ldots,m$, are real or complex constants, we consider the solution of the non-homogeneous k-hypergeometric ordinary differential equation. The following theorem gives the particular solution and general solution of Equation (4).

Theorem 2. *Suppose that $k \in \mathbb{R}^+$ and all a, b, c belong to $\mathbb{R}$. Let, in addition, c and $2k-c$ be neither zero, nor negative integers. Then, the non-homogeneous k-hypergeometric ordinary differential Equation (4) can have a particular solution in the following form:*

$$\overline{y}(z) = -\sum_{j=0}^{m}\left[\frac{(a)_{j,k}(b)_{j,k}}{j!(c)_{j,k}}\sum_{l=j}^{m}\frac{l!(c)_{l,k}}{(a)_{l+1,k}(b)_{l+1,k}}d_l\right]z^j. \tag{23}$$

Therefore, a general solution of Equation (4) can be written as:

$$\begin{aligned} y(z) =& A\ {}_2F_{1,k}[(a,k),(b,k);(c,k);z] \\ &+ B\, z^{1-\frac{c}{k}}\ {}_2F_{1,k}[(a+k-c,k),(b+k-c,k);(2k-c,k);z] \\ &- \sum_{j=0}^{m}\left[\frac{(a)_{j,k}(b)_{j,k}}{j!(c)_{j,k}}\sum_{l=j}^{m}\frac{l!(c)_{l,k}}{(a)_{l+1,k}(b)_{l+1,k}}d_l\right]z^j \end{aligned} \tag{24}$$

for $|z| < 1/k$, where A and B are two constants in $\mathbb{C}$.

Proof. Let us assume that:

$$\bar{y}(z) = \sum_{j=0}^{m} s_j z^j \tag{25}$$

is a particular solution of the non-homogeneous k-hypergeometric ordinary differential Equation (4), where $s_j, j = 0, 1, 2, \ldots, m$, are undetermined coefficients. Differentiating Equation (25), then we have:

$$\bar{y}'(z) = \sum_{j=0}^{m-1} (j+1) s_{j+1} z^j \tag{26}$$

and:

$$\bar{y}''(z) = \sum_{j=0}^{m-2} (j+2)(j+1) s_{j+2} z^j. \tag{27}$$

Plugging Equations (26) and (27) into Equation (4) yields:

$$kz(1-kz) \sum_{j=0}^{m-2} (j+2)(j+1) s_{j+2} z^j + [c - (k+a+b)kz] \sum_{j=0}^{m-1} (j+1) s_{j+1} z^j - ab \sum_{j=0}^{m} s_j z^j = \sum_{i=0}^{m} d_i z^i, \tag{28}$$

and it follows that:

$$\begin{aligned} & cs_1 - abs_0 + [2 \cdot 1 \cdot ks_2 + 2cs_2 - (k+a+b)ks_1 - abs_1]z \\ & \quad + [-2 \cdot 1 \cdot k^2 s_2 + 3 \cdot 2 \cdot ks_3 - 2k(k+a+b)s_2 + 3cs_3 - abs_2]z^2 \\ & \quad + \ldots \\ & \quad + [m(m-1)ks_m - (m-1)(m-2)k^2 s_{m-1} + mcs_m \\ & \qquad\qquad - (m-1)(k+a+b)ks_{m-1} - abs_{m-1}]z^{m-1} \\ & \quad + [-m(m-1)k^2 s_m - mk(k+a+b)s_m - abs_m]z^m \\ & \quad = d_0 + d_1 z + d_2 z^2 + \ldots d_m z^m. \end{aligned} \tag{29}$$

Matching the coefficients on both sides of Equation (29) gives:

$$\begin{cases} cs_1 - abs_0 = d_0, \\ 2 \cdot 1 \cdot ks_2 + 2cs_2 - (k+a+b)ks_1 - abs_1 = d_1, \\ \ldots \\ m(m-1)ks_m - (m-1)(m-2)k^2 s_{m-1} + mcs_m \\ \qquad\qquad - (m-1)(k+a+b)ks_{m-1} - abs_{m-1} = d_{m-1}, \\ -m(m-1)k^2 s_m - mk(k+a+b)s_m - abs_m = d_m. \end{cases} \tag{30}$$

Thus, Equation (30) implies that:

$$s_m = -\frac{d_m}{m(m-1)k^2 + mk(k+a+b) + ab} = -\frac{d_m}{(a+km)(b+km)}$$

and:

$$\begin{aligned} s_{m-1} &= \frac{m(m-1)ks_m + mcs_m - d_{m-1}}{k^2(m-1)(m-2) + k(m-1)(k+a+b) + ab} \\ &= \frac{m[c+(m-1)k]}{[a+(m-1)k][b+(m-1)k]} s_m - \frac{d_{m-1}}{[a+(m-1)k][b+(m-1)k]} \\ &= \frac{m[c+(m-1)k]}{[a+(m-1)k][b+(m-1)k]} \frac{-d_m}{(a+km)(b+km)} - \frac{d_{m-1}}{[a+(m-1)k][b+(m-1)k]} \\ &= -\frac{(a)_{m-1,k}(b)_{m-1,k}}{(m-1)!(c)_{m-1,k}} \frac{m!(c)_{m,k}}{(a)_{m+1,k}(b)_{m+1,k}} d_m - \frac{d_{m-1}}{[a+(m-1)k][b+(m-1)k]} \\ &= -\frac{(a)_{m-1,k}(b)_{m-1,k}}{(m-1)!(c)_{m-1,k}} \left[\frac{m!(c)_{m,k}}{(a)_{m+1,k}(b)_{m+1,k}} d_m + \frac{(m-1)!(c)_{m-1,k}}{(a)_{m,k}(b)_{m,k}} d_{m-1} \right] \\ &= -\frac{(a)_{m-1,k}(b)_{m-1,k}}{(m-1)!(c)_{m-1,k}} \sum_{l=m-1}^{m} \frac{l!(c)_{l,k}}{(a)_{l+1,k}(b)_{l+1,k}} d_l. \end{aligned}$$

Consequently, these coefficients of the particular solution (25) are:

$$s_j = -\frac{(a)_{j,k}(b)_{j,k}}{j!(c)_{j,k}} \sum_{l=j}^{m} \frac{l!(c)_{l,k}}{(a)_{l+1,k}(b)_{l+1,k}} d_l \tag{31}$$

for $j = 0, 1, 2, \ldots, m$.

Replacing Equation (25) with Equation (31), we obtain a particular solution of the equation in the following form:

$$\overline{y}(z) = -\sum_{j=0}^{m} \left[\frac{(a)_{j,k}(b)_{j,k}}{j!(c)_{j,k}} \sum_{l=j}^{m} \frac{l!(c)_{l,k}}{(a)_{l+1,k}(b)_{l+1,k}} d_l \right] z^j. \tag{32}$$

From the above Equation (32), we get a general solution of Equation (4), as shown in Equation (24). We have shown Theorem 2. □

4. Examples

Example 1. *Find the solution to the following differential equation:*

$$z(1-z)\frac{d^2y}{dz^2} + (\frac{1}{2} - 3z)\frac{dy}{dz} - y = 1 + 2z^2 \tag{33}$$

for $|z| < 1$.

From Equation (33), it is easy to see that $a = b = 1$, $c = \frac{1}{2}$, $k = 1$, $m = 2$, $d_0 = 1$, $d_1 = 0$, and $d_2 = 2$. By Equation (24) in Theorem 2, then we have:

$$\begin{aligned} {}_2F_{1,k}[(a,k),(b,k);(c,k);z] &= {}_2F_{1,1}[(1,1),(1,1);(\frac{1}{2},1);z] = \sum_{i=0}^{\infty} \frac{i! \cdot i!}{(\frac{1}{2})_{i,1} i!} z^i \\ &= \sum_{i=0}^{\infty} \frac{i!}{(\frac{1}{2})_{i,1}} z^i = 1 + \sum_{i=1}^{\infty} \frac{2^i i!}{(2i-1)!!} z^i, \end{aligned} \tag{34}$$

$$\begin{aligned} {}_2F_{1,k}[(a+k-c,k),(b+k-c,k);(2k-c,k);z] &= {}_2F_{1,1}[(\frac{3}{2},1),(\frac{3}{2},1);(\frac{3}{2},1);z] \\ &= \sum_{i=0}^{\infty} \frac{(\frac{3}{2})_{i,1}}{i!} z^i = \sum_{i=0}^{\infty} \frac{(2i+1)!!}{2^i i!} z^i \end{aligned} \tag{35}$$

and:

$$\overline{y}(z) = -\sum_{i=0}^{2}\left[\frac{(1)_{i,1}(1)_{i,1}}{i!(\frac{1}{2})_{i,1}}\sum_{l=i}^{2}\frac{l!(\frac{1}{2})_{l,1}d_l}{(1)_{l+1,1}(1)_{l+1,1}}\right]z^i = -\left(\frac{13}{12}+\frac{1}{6}z+\frac{2}{9}z^2\right). \tag{36}$$

Combining Equations (34)–(36), we obtain the general solution of Equation (33) as below:

$$\begin{aligned} y &= A\,{}_2F_{1,1}[(1,1),(1,1);(\frac{1}{2},1);z] + B\,\sqrt{z}\,{}_2F_{1,1}[(\frac{3}{2},1),(\frac{3}{2},1);(\frac{3}{2},1);z] + \overline{y}(z) \\ &= A\left(1+\sum_{i=1}^{\infty}\frac{2^i i!}{(2i-1)!!}z^i\right) + B\sqrt{z}\left(\sum_{i=0}^{\infty}\frac{(2i+1)!!}{2^i i!}z^i\right) - (\frac{13}{12}+\frac{1}{6}z+\frac{2}{9}z^2), \end{aligned} \tag{37}$$

where A and B are two constants.

Example 2. *Find the solution to the following differential equation:*

$$2z(1-2z)\frac{d^2y}{dz^2} + (1-14z)\frac{dy}{dz} - 6y = 1+2z+3z^2 \tag{38}$$

for $|z| < \frac{1}{2}$.

From Equation (38), it is clear that $k=2, c=1, m=2$, and $d_0=1, d_1=2, d_2=3$. Then, let us take $a=3, b=2$. According to Equation (24) in Theorem 2, we obtain:

$$\begin{aligned} {}_2F_{1,k}[(a,k),(b,k);(c,k);z] &= {}_2F_{1,2}[(3,2),(2,2);(1,2);z] \\ &= \sum_{i=0}^{\infty}\frac{(2i+1)!!\,(2i)!!}{(2i-1)!!\,i!}z^i = \sum_{i=0}^{\infty}2^i(2i+1)z^i, \end{aligned} \tag{39}$$

$$\begin{aligned} {}_2F_{1,k}[(a+k-c,k),(b+k-c,k);(2k-c,k);z] &= {}_2F_{1,2}[(4,2),(3,2);(3,2);z] \\ &= \sum_{i=0}^{\infty}\frac{(4)_{i,2}}{i!}z^i = \sum_{i=0}^{\infty}2^i(i+1)z^i \end{aligned} \tag{40}$$

and:

$$\overline{y}(z) = -\sum_{i=0}^{2}\left[\frac{(3)_{i,2}(2)_{i,2}}{i!(1)_{i,2}}\sum_{l=i}^{2}\frac{l!(1)_{l,2}d_l}{(3)_{l+1,2}(2)_{l+1,2}}\right]z^i = -\left(\frac{157}{840}+\frac{17}{140}z+\frac{1}{14}z^2\right). \tag{41}$$

Substituting Equations (39)–(41) into Equation (24) gives the general solution of Equation (38) as follows:

$$\begin{aligned} y &= A\,{}_2F_{1,2}[(3,2),(2,2);(1,2);z] + B\,\sqrt{z}\,{}_2F_{1,2}[(4,2),(3,2);(3,2);z] + \overline{y}(z) \\ &= A\sum_{i=0}^{\infty}2^i(2i+1)z^i + B\sqrt{z}\left(\sum_{i=0}^{\infty}2^i(i+1)z^i\right) - \frac{157}{840} - \frac{17}{140}z - \frac{1}{14}z^2, \end{aligned} \tag{42}$$

where A and B are two constants.

5. Conclusions

When $|z| < 1/k$ and $f(z) = \sum_{i=0}^{m} d_i z^i, d_i \in \mathbb{C}, i = 0, 1, 2, \ldots, m$, in this paper, we present a formula of the general solution of the non-homogeneous k-hypergeometric ordinary differential Equation (4), provided that $a, b, c \in \mathbb{R}$ with both c and $2k - c$ neither zero, nor negative integers. The solutions of this type of equations are denoted by in the form of k-hypergeometric series, and it is convenient that we can make out the corresponding computer program and put it into calculation. When $f(z)$ is not a polynomial, it is a fascinating question to derive the particular or general series solutions for the non-homogeneous k-hypergeometric ordinary differential Equation (1). It would be interesting to have more research about this case.

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