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Anti-Periodic Boundary Value Problems for Nonlinear Langevin Fractional Differential Equations

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Abstract: In this paper, we focus on the existence of solutions of the nonlinear Langevin fractional differential equations involving anti-periodic boundary value conditions. By using some techniques, formulas of solutions for the above problem and some properties of the Mittag-Leffler functions $E_{\alpha,\beta}(z)$, $\alpha, \beta \in (1,2), z \in \mathbb{R}$ are presented. Moreover, we utilize the fixed point theorem under the weak assumptions for nonlinear terms to obtain the existence result of solutions and give an example to illustrate the result.

Keywords: nonlinear Langevin fractional differential equations; anti-periodic boundary value problem; Mittag-Leffler functions

1. Introduction

The application of fractional calculus is very broad, and the differential equations involving Riemann-Liouville and Caputo operators of fractional orders arise in many scientific disciplines, such as the mathematical modeling of earthquake analysis, mechanics and electricity, the memory of many kinds of material, electrolysis chemical, electronic circuits, etc. [1–6]. In recent years, the subject of fractional differential equations is gaining much importance and attention. For details, see [1–4,6–11] and the references therein.

In 1908, Langevin [12] applied Newton's second law to a Brownian particle to give an elaborate description of Brownian motion which is now called the "Langevin equation" [13].

The classical Langevin equation for the apparently random movement of a Brownian particle in a fluid due to collisions with the molecules of the fluid is described by

$$m\frac{d^2x}{dt^2} = f = -\lambda\frac{dx}{dt} + \eta(t),$$

where x denotes the position of the particle, m denotes the particle's mass, and f denotes the force acting on the particle from molecules of the fluid surrounding the Brownian particle. The force f may be written as a sum of two parts. The first one is the viscous force proportional to the particle's velocity with coefficient λ. The second one denoted by $\eta(t)$ is the random force arising from rapid thermal fluctuation [14].

The fractional Langevin equation was introduced by Mainardi et al. in the early 1990s [14,15]. Much work since then has been devoted to the study of the fractional Langevin equations in the field physics (e.g., [16–22]). Moreover, the fractional Langevin equations have been applied to describe various anomalous diffusive process, such as single file diffusion and crossover dynamics between different diffusive regimes (see, e.g., [23–26]).

Recently, there has been a significant development in solving fractional Langevin equation (see [7–10] and the references therein). To the best of our knowledge, there are few papers dealing with anti-periodic BVP involving fractional Langevin equation with two fractional orders ${}^cD^{\alpha}_{0+}{}^cD^{\beta}_{0+}u(\alpha \in (0,1),\ \beta, \alpha+\beta \in (1,2))$ [10].

In this paper, we study the following anti-periodic boundary value problem of nonlinear fractional Langevin equations:

$$ {}^cD^{\alpha}_{0+}({}^cD^{\beta}_{0+} + \lambda)u(t) = f(t, u(t)), \quad t \in J := (0,1], \tag{1} $$

$$ u(0) + u(1) = 0,\ u'(0) + u'(1) = 0,\ \lim_{t\to 0^+} t^{\alpha}({}^HD^{\xi,\alpha}_{0+}u)(t) = 0, \tag{2} $$

where $\alpha, \xi \in (0,1),\ \beta, \alpha+\beta \in (1,2), \lambda > 0, 0 < \alpha + \xi - \alpha\xi < 1$. ${}^cD^{*}_{0+}$ is the standard Caputo fractional derivative, ${}^HD^{\xi,\alpha}_{0+}$ is the Hilfer fractional derivative, $f : J \times \mathbb{R} \to \mathbb{R}$ is an appropriate function to be specified later.

As mentioned in [7] and the references therein, the existence results of fractional differential equations involving Caputo differential operator of order $\alpha, \beta \in (0,1)$ are obtained by Mittag-Leffler functions $E_{\alpha}(z)$ and $E_{\alpha,\beta}(z)$, since $E_{\alpha}(z)$ and $E_{\alpha,\beta}(z)$ have "good" properties, such as explicit boundedness, monotonicity and nonnegativity. However, for $\alpha, \beta \in (1,2)$, the above properties do not hold anymore, which leads to difficulties for the theoretical analysis. In this paper, using some techniques, we study the properties of $E_{\beta}(z)$ and $E_{\beta,\theta}(z)(\beta, \theta \in (1,2))$ and obtain the existence result of solutions to (1) and (2) under the weak assumptions on $f(t, u(t))$.

The plan of this paper is as follows. In Section 2, we present some basic concepts, notations about fractional calculus. In Section 3, we prove some properties of Mittag-Leffler functions. In Section 4, we present the definition of solution to (1) and (2). In Section 5, we employ Krasnoselskii's fixed point theorem to obtain the existence of solutions to problem (1) and (2). An example is given in Section 6 to demonstrate the application of our result.

2. Preliminaries

In this paper, we denote by $C(J, \mathbb{R})$ the Banach space of all continuous functions from J to $\mathbb{R}$, $L^p(J, \mathbb{R})$ the Banach space of all Lebesgue measurable functions $l : J \to \mathbb{R}$ with the norm $\|l\|_{L^p} = \left(\int_J |l(t)|^p dt\right)^{\frac{1}{p}} < \infty$ and by $AC([a,b], \mathbb{R})$ the space of all absolutely continuous functions defined on $[a,b]$. Moreover, for $n = 1, 2$,

$$ AC^n([a,b], \mathbb{R}) = \{f : f \in C^{n-1}([a,b], \mathbb{R}) \text{ and } f^{(n-1)} \in AC([a,b], \mathbb{R})\}. $$

In particular, $AC^1([a,b], \mathbb{R}) = AC([a,b], \mathbb{R})$.

Definition 1. *[3,4] The fractional integral of order γ with the lower limit a for a function $x(t) \in L^1([a,+\infty), \mathbb{R})$ is defined as*

$$ (I^{\gamma}_{a+}x)(t) = \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} x(s) ds, \quad t > a, \quad \gamma > 0, $$

where $\Gamma(\cdot)$ is the gamma function.

Definition 2. *[3,4] If $x(t) \in AC^n([a,b], \mathbb{R})$, then the Riemann-Liouville fractional derivative ${}^LD^{\gamma}_{a+}x(t)$ of order γ exists almost everywhere on $[a,b]$ and can be written as*

$$ ({}^LD^{\gamma}_{a+}x)(t) = \frac{1}{\Gamma(n-\gamma)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\gamma-1} x(s) ds = \left(\frac{d}{dt}\right)^n (I^{n-\gamma}_{a+}x)(t), \quad t > a, \quad n-1 < \gamma < n. $$

Definition 3. *[3,4] If $x(t) \in AC^n([a,b],\mathbb{R})$, then the Caputo derivative ${}^cD^{\gamma}_{a+}x(t)$ of order γ exists almost everywhere on $[a,b]$ and can be written as*

$$({}^cD^{\gamma}_{a+}x)(t) = \left({}^LD^{\gamma}_{a+}\left[x(s) - \sum_{k=0}^{n-1}\frac{x^{(k)}(a)}{k!}(s-a)^k\right]\right)(t), \quad t > a, \quad n-1 < \gamma < n.$$

Moreover, if $x(a) = x'(a) = \cdots = x^{(n-1)}(a) = 0$, then $({}^cD^{\gamma}_{a+}x)(t) = ({}^LD^{\gamma}_{a+}x)(t)$.

Definition 4. *[5] The Hilfer fractional derivative of order $0 \le \gamma \le 1$ and $0 < \alpha < 1$ with lower limit a is defined as*

$$({}^HD^{\gamma,\alpha}_{a+}x)(t) = I^{\gamma(1-\alpha)}_{a+}({}^LD^{\gamma+\alpha-\gamma\alpha}_{a+}x)(t).$$

We introduce the following fixed point theorem.

Theorem 1. *(Krasnoselskii's fixed point theorem) Let B be a closed, convex, and nonempty subset of a Banach space U, and let A_1, A_2 be operators such that:*

(i) $A_1u + A_2v \in B$ *whenever* $u, v \in B$,
(ii) A_1 *is compact and continuous,*
(iii) A_2 *is a contraction mapping.*

Then there exists $z \in B$ such that $z = A_1z + A_2z$.

3. Properties of Mittag-Leffler Functions

In this section, we prove some properties of the Mittag-Leffler functions.

Definition 5. *[3,4] For $\mu,\nu > 0$, $z \in \mathbb{R}$, the classical Mittag-Leffler functions $E_\mu(z)$ and the generalized Mittag-Leffler functions $E_{\mu,\nu}(z)$ are defined by*

$$E_\mu(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu k+1)}, \quad E_{\mu,\nu}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu k+\nu)}.$$

Clearly, $E_{\mu,1}(z) = E_\mu(z)$.

Lemma 1. *[3] If $1 < \beta < 2$, γ is an arbitrary real number, $\frac{\pi\beta}{2} < \mu < \pi\beta$, then*

$$|E_{\beta,\gamma}(z)| \le C, \qquad \mu \le |argz| \le \pi, \quad |z| \ge 0,$$

where C is a positive constant.

Lemma 2. *[4,11] For $\gamma,\mu,\nu,\lambda > 0$, $t > 0$, $g \in L(0,t)$, the usual derivatives of $E_{\mu,\nu}$ and the Riemann-Liouville integration of $E_{\mu,\nu}$ are expressed by*

(i) $\left(\frac{d}{dt}\right)^n[t^{\nu-1}E_{\mu,\nu}(-\lambda t^\mu)] = t^{\nu-n-1}E_{\mu,\nu-n}(-\lambda t^\mu), \quad n \ge 1;$
(ii) $\frac{d}{dt}E_\mu(-\lambda t^\mu) = -\lambda t^{\mu-1}E_{\mu,\mu}(-\lambda t^\mu);$
(iii) $I^{\gamma}_{0+}(s^{\nu-1}E_{\mu,\nu}(-\lambda s^\mu))(t) = \frac{1}{\Gamma(\gamma)}\int_0^t(t-s)^{\gamma-1}s^{\nu-1}E_{\mu,\nu}(-\lambda s^\mu)ds = t^{\gamma+\nu-1}E_{\mu,\gamma+\nu}(-\lambda t^\mu);$
(iv)

$$\begin{aligned}&\frac{1}{\Gamma(\gamma)}\int_0^t\int_0^s(t-s)^{\gamma-1}(s-\tau)^{\mu-1}E_{\nu,\mu}(-\lambda(s-\tau)^\nu)g(\tau)d\tau ds\\ =\ &\int_0^t(t-\tau)^{\gamma+\mu-1}E_{\nu,\gamma+\mu}(-\lambda(t-\tau)^\nu)g(\tau)d\tau.\end{aligned}$$

Lemma 3. *For $\lambda > 0$, $1 < \beta \leq 2$, $\theta > \beta$, the generalized Mittag-Leffler functions have the following properties:*

(i) $E_{\beta,\beta}(-\lambda t^\beta) = \frac{1}{\Gamma(\beta-1)}\int_0^1 E_\beta(-\lambda t^\beta s^\beta)(1-s)^{\beta-2}ds;$

(ii) $E_{\beta,\theta}(-\lambda t^\beta) = \frac{1}{\Gamma(\theta-\beta)}\int_0^1 E_{\beta,\beta}(-\lambda t^\beta s^\beta)s^{\beta-1}(1-s)^{\theta-\beta-1}ds;$

(iii) $tE_{\beta,2}(-\lambda t^\beta) = \int_0^t E_\beta(-\lambda s^\beta)ds.$

Proof. We denote the beta function by $\mathbb{B}(\cdot,\cdot)$, then

$$\begin{aligned} E_{\beta,\beta}(-\lambda t^\beta) &= \sum_{k=0}^{\infty}\frac{(-\lambda t^\beta)^k}{\Gamma(\beta k+\beta)} = \frac{1}{\Gamma(\beta-1)}\sum_{k=0}^{\infty}\frac{(-\lambda t^\beta)^k\mathbb{B}(\beta k+1,\beta-1)}{\Gamma(\beta k+1)} \\ &= \frac{1}{\Gamma(\beta-1)}\int_0^1\sum_{k=0}^{\infty}\frac{(-\lambda t^\beta s^\beta)^k}{\Gamma(\beta k+1)}(1-s)^{\beta-2}ds \\ &= \frac{1}{\Gamma(\beta-1)}\int_0^1 E_\beta(-\lambda t^\beta s^\beta)(1-s)^{\beta-2}ds, \end{aligned} \tag{3}$$

and

$$\begin{aligned} E_{\beta,\theta}(-\lambda t^\beta) &= \sum_{k=0}^{\infty}\frac{(-\lambda t^\beta)^k}{\Gamma(\beta k+\theta)} = \frac{1}{\Gamma(\theta-\beta)}\sum_{k=0}^{\infty}\frac{(-\lambda t^\beta)^k\mathbb{B}(\beta k+\beta,\theta-\beta)}{\Gamma(\beta k+\beta)} \\ &= \frac{1}{\Gamma(\theta-\beta)}\int_0^1\sum_{k=0}^{\infty}\frac{(-\lambda t^\beta s^\beta)^k}{\Gamma(\beta k+\beta)}s^{\beta-1}(1-s)^{\theta-\beta-1}ds \\ &= \frac{1}{\Gamma(\theta-\beta)}\int_0^1 E_{\beta,\beta}(-\lambda t^\beta s^\beta)s^{\beta-1}(1-s)^{\theta-\beta-1}ds. \end{aligned} \tag{4}$$

Using Lemma 2 (i) for $\nu = 2$, $n = 1$, we obtain (iii). □

Similar to the arguments in [3,4], we can obtain the following results.

Lemma 4. *For $\lambda > 0$, $\alpha, \xi, \alpha+\xi-\alpha\xi \in (0,1)$, $\theta > \alpha$, $\beta, \alpha+\beta \in (1,2)$, then*

(i) $\left[{}^cD_{0^+}^\alpha E_\beta(-\lambda s^\beta)\right](t) = -\lambda t^{\beta-\alpha}E_{\beta,\beta-\alpha+1}(-\lambda t^\beta).$

(ii) $\left[{}^cD_{0^+}^\alpha s^\theta E_{\beta,\theta+1}(-\lambda s^\beta)\right](t) = t^{\theta-\alpha}E_{\beta,\theta-\alpha+1}(-\lambda t^\beta),$

(iii) $\left[{}^cD_{0^+}^\beta s^\beta E_{\beta,\beta+1}(-\lambda s^\beta)\right](t) = E_\beta(-\lambda t^\beta),\quad \left[{}^cD_{0^+}^\beta sE_{\beta,2}(-\lambda s^\beta)\right](t) = -\lambda tE_{\beta,2}(-\lambda t^\beta).$

(iv) $\left[{}^HD_{0^+}^{\xi,\alpha} s^\theta E_{\beta,\theta+1}(-\lambda s^\beta)\right](t) = t^{\theta-\alpha}E_{\beta,\theta-\alpha+1}(-\lambda t^\beta).$

Proof. From Lemma 2.7 of [11], (i) holds. From Definition 2–4 and Lemma 2, it follows that

$$\begin{aligned} \left[{}^cD_{0^+}^\alpha s^\theta E_{\beta,\theta+1}(-\lambda s^\beta)\right](t) &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t(t-s)^{-\alpha}s^\theta E_{\beta,\theta+1}(-\lambda s^\beta)ds \\ &= \frac{d}{dt}\left[t^{\theta-\alpha+1}E_{\beta,\theta-\alpha+2}(-\lambda t^\beta)\right] \\ &= t^{\theta-\alpha}E_{\beta,\theta-\alpha+1}(-\lambda t^\beta), \end{aligned}$$

and

$$\begin{aligned} \left[{}^cD_{0^+}^\beta s^\beta E_{\beta,\beta+1}(-\lambda s^\beta)\right](t) &= \frac{1}{\Gamma(2-\beta)}\left(\frac{d}{dt}\right)^2\int_0^t(t-s)^{1-\beta}s^\beta E_{\beta,\beta+1}(-\lambda s^\beta)ds \\ &= \frac{d^2}{dt^2}[t^2E_{\beta,3}(-\lambda t^\beta)] = E_\beta(-\lambda t^\beta), \end{aligned}$$

and

$$
\begin{aligned}
\left[{}^{c}D_{0^+}^{\beta} sE_{\beta,2}(-\lambda s^{\beta})\right](t) &= \frac{1}{\Gamma(2-\beta)}\left(\frac{d}{dt}\right)^2 \int_0^t (t-s)^{1-\beta}(sE_{\beta,2}(-\lambda s^{\beta})-s)ds \\
&= \left(\frac{d}{dt}\right)^2 \left[t^{3-\beta}\left(E_{\beta,4-\beta}(-\lambda t^{\beta}) - \frac{1}{\Gamma(4-\beta)}\right)\right] \\
&= \left(\frac{d}{dt}\right)^2 \left[-\lambda t^3 \sum_{k=0}^{\infty} \frac{(-\lambda t^{\beta})^k}{\Gamma(\beta k+4)}\right] \\
&= -\lambda t E_{\beta,2}(-\lambda t^{\beta}),
\end{aligned}
$$

and

$$
\begin{aligned}
&\left[{}^{H}D_{0^+}^{\xi,\alpha}\left(s^{\theta}E_{\beta,\theta+1}(-\lambda s^{\beta})\right)\right](t) \\
= &\ \frac{1}{\Gamma(1-\xi-\alpha+\alpha\xi)}\left[I_{0^+}^{\xi(1-\alpha)}\frac{d}{ds}\left(\int_0^s (s-\tau)^{\alpha\xi-\alpha-\xi}\tau^{\theta}E_{\beta,\theta+1}(-\lambda\tau^{\beta})d\tau\right)\right](t) \\
= &\ \left[I_{0^+}^{\xi(1-\alpha)}(s^{\theta-\alpha-\xi+\alpha\xi}E_{\beta,1+\theta-\alpha-\xi+\alpha\xi}(-\lambda s^{\beta}))\right](t) \\
= &\ t^{\theta-\alpha}E_{\beta,\theta-\alpha+1}(-\lambda t^{\beta}).
\end{aligned}
$$

□

For $\lambda > 0$, $\beta \in (1,2)$ and $\gamma > 0$, from Lemma 3.2 in [10], we find that

$$
E_{\beta,\gamma}(-\lambda t^{\beta}) = \int_0^{+\infty} Q(r,t)dr + V(t), \tag{5}
$$

where

$$
\begin{aligned}
Q(r,t) &= \frac{r^{\frac{1-\gamma}{\beta}}\exp(-r^{\frac{1}{\beta}})}{\pi\beta}\cdot\frac{r\sin(\pi(1-\gamma))+\lambda t^{\beta}\sin(\pi(1-\gamma+\beta))}{U(t)}, \\
U(t) &= r^2 + 2\lambda r t^{\beta}\cos(\pi\beta) + \lambda^2 t^{2\beta}, \\
V(t) &= \frac{2t^{1-\gamma}}{\beta\lambda^{1-\frac{1}{\beta}}}\exp(t\lambda^{\frac{1}{\beta}}\cos\frac{\pi}{\beta})\cos\left[t\lambda^{\frac{1}{\beta}}\sin\frac{\pi}{\beta} - \frac{\pi}{\beta}(\gamma-1)\right].
\end{aligned}
$$

Lemma 5. *Let $\beta \in (1,2)$, $\zeta \in (0,1]$, $\theta > \beta$ be arbitrary. Then the functions E_{β}, $E_{\beta,\beta}$, $E_{\beta,\zeta}$ and $E_{\beta,\theta}$ have the following properties:*

(i) For any $t \in J$, $E_{\beta}(-\lambda t^{\beta}) \le 1$, $E_{\beta,\beta}(-\lambda t^{\beta}) \le \frac{1}{\Gamma(\beta)}$, $E_{\beta,\theta}(-\lambda t^{\beta}) \le \frac{1}{\Gamma(\theta)}$.

(ii) For any t_1, $t_2 \in J$,

$$
\begin{aligned}
|E_{\beta}(-\lambda t_2^{\beta}) - E_{\beta}(-\lambda t_1^{\beta})| &= O(|t_2-t_1|), \quad as \quad t_2 \to t_1, \\
|E_{\beta,\beta}(-\lambda t_2^{\beta}) - E_{\beta,\beta}(-\lambda t_1^{\beta})| &= O(|t_2-t_1|), \quad as \quad t_2 \to t_1, \\
|E_{\beta,\zeta}(-\lambda t_2^{\beta}) - E_{\beta,\zeta}(-\lambda t_1^{\beta})| &= O(|t_2-t_1|), \quad as \quad t_2 \to t_1, \\
|E_{\beta,\theta}(-\lambda t_2^{\beta}) - E_{\beta,\theta}(-\lambda t_1^{\beta})| &= O(|t_2-t_1|), \quad as \quad t_2 \to t_1, \\
|t_2E_{\beta,2}(-\lambda t_2^{\beta}) - t_1E_{\beta,2}(-\lambda t_1^{\beta})| &= O(|t_2-t_1|), \quad as \quad t_2 \to t_1.
\end{aligned}
$$

Proof. (i) From Lemma 3.4 of [10], we have $E_{\beta}(-\lambda t^{\beta}) \le 1$. Using Lemma 3, the second and the third inequalities hold.

(ii) From (5), for $\beta \in (1, 2)$, $\zeta \in (0, 1]$, we can see

$$E_{\beta,\zeta}(-\lambda t^{\beta}) = \int_0^{+\infty} \widetilde{Q}(r, t)dr + \widetilde{V}(t),$$

where

$$\begin{aligned}
\widetilde{Q}(r, t) &= \frac{r^{\frac{1-\zeta}{\beta}} \exp(-r^{\frac{1}{\beta}})}{\pi\beta} \cdot \frac{r \sin \pi\zeta + \lambda t^{\beta} \sin(\pi(\zeta - \beta))}{U(t)}, \\
\widetilde{V}(t) &= \frac{2t^{1-\zeta}}{\beta\lambda^{1-\frac{1}{\beta}}} \exp(t\lambda^{\frac{1}{\beta}} \cos \frac{\pi}{\beta}) \cos\left[t\lambda^{\frac{1}{\beta}} \sin \frac{\pi}{\beta} - \frac{\pi}{\beta}(\zeta - 1)\right].
\end{aligned}$$

Clearly,

$$U(t) = (r + \lambda t^{\beta} \cos(\pi\beta))^2 + \lambda^2 t^{2\beta} \sin^2(\pi\beta) \geq \lambda^2 t^{2\beta} \sin^2(\pi\beta) > 0, \tag{6}$$

and

$$U(t) = (r - \lambda t^{\beta})^2 + 2\lambda r t^{\beta}(\cos(\pi\beta) + 1) \geq 4\lambda r t^{\beta} \cos^2(\frac{\pi\beta}{2}) > 0. \tag{7}$$

Assume that $0 < t_1 < t_2 \leq 1$, by (6), (7) and Lagrange's mean value theorem, we obtain

$$\left|\frac{1}{U(t_2)} - \frac{1}{U(t_1)}\right| = \frac{|U(t_1) - U(t_2)|}{U(t_1)U(t_2)} \leq \frac{\beta(\lambda + r)(t_2 - t_1)}{8\lambda r^2 t_1^{2\beta} \cos^4(\frac{\pi\beta}{2})} := O(|t_2 - t_1|), \tag{8}$$

$$\begin{aligned}
\frac{|t_1^{\beta}U(t_2) - t_2^{\beta}U(t_1)|}{U(t_1)U(t_2)} &\leq \frac{r^2(t_2^{\beta} - t_1^{\beta}) + \lambda^2 t_1^{\beta} t_2^{\beta}(t_2^{\beta} - t_1^{\beta})}{U(t_1)U(t_2)} \\
&\leq \left(\frac{1}{16\lambda^2 \cos^4(\frac{\pi\beta}{2})} + \frac{1}{\lambda^2 \sin^4(\pi\beta)}\right) \frac{t_2^{\beta} - t_1^{\beta}}{t_1^{\beta} t_2^{\beta}} \\
&\leq \left[\frac{1}{16 \cos^4(\frac{\pi\beta}{2})} + \frac{1}{\sin^4(\pi\beta)}\right] \frac{\beta(t_2 - t_1)}{\lambda^2 t_1^{2\beta}} \\
&:= O(|t_2 - t_1|),
\end{aligned} \tag{9}$$

and

$$\begin{aligned}
& |\widetilde{V}(t_2) - \widetilde{V}(t_1)| \\
\leq\ & \frac{2|t_2^{1-\zeta} - t_1^{1-\zeta}|}{\beta\lambda^{1-\frac{1}{\beta}}} + \frac{2}{\beta\lambda^{1-\frac{1}{\beta}}} \left\{ \left| \exp(t_2\lambda^{\frac{1}{\beta}} \cos \frac{\pi}{\beta}) - \exp(t_1\lambda^{\frac{1}{\beta}} \cos \frac{\pi}{\beta}) \right| \right. \\
& \left. + \exp(t_1\lambda^{\frac{1}{\beta}} \cos \frac{\pi}{\beta}) \left| \cos\left[\lambda^{\frac{1}{\beta}} t_2 \sin \frac{\pi}{\beta} - \frac{\pi}{\beta}(\zeta - 1)\right] - \cos\left[\lambda^{\frac{1}{\beta}} t_1 \sin \frac{\pi}{\beta} - \frac{\pi}{\beta}(\zeta - 1)\right] \right| \right\} \\
\leq\ & \left(\frac{2}{\beta\lambda^{1-\frac{1}{\beta}} t_1^{\zeta}} + \frac{4\lambda^{\frac{2}{\beta}-1}}{\beta}\right)(t_2 - t_1) := O(|t_2 - t_1|).
\end{aligned} \tag{10}$$

According to (8)–(10), as $t_2 \to t_1$, we arrive that

$$\begin{aligned}
|E_{\beta,\zeta}(-\lambda t_2^{\beta}) - E_{\beta,\zeta}(-\lambda t_1^{\beta})| &\leq \int_0^{\infty} |\widetilde{Q}(r,t_2) - \widetilde{Q}(r,t_1)|dr + |\widetilde{V}(t_2) - \widetilde{V}(t_1)| \\
&\leq \frac{1}{\pi\beta}\int_0^{\infty} r^{\frac{1-\zeta+\beta}{\beta}} \exp(-r^{\frac{1}{\beta}})\Big(\Big|\frac{1}{U(t_2)} - \frac{1}{U(t_1)}\Big|\Big)dr \\
&\quad + \frac{\lambda}{\pi\beta}\int_0^{\infty} r^{\frac{1-\zeta}{\beta}} \exp(-r^{\frac{1}{\beta}})\frac{|t_1^{\beta}U(t_2) - t_2^{\beta}U(t_1)|}{U(t_1)U(t_2)}dr + |\widetilde{V}(t_2) - \widetilde{V}(t_1)| \\
&\leq \frac{[\Gamma(2\beta-\zeta+1) + \lambda\Gamma(\beta-\zeta+1) + \pi]}{\pi} \cdot O(|t_2 - t_1|) =: O(|t_2 - t_1|).
\end{aligned}$$

In particular, when $\zeta = 1$, we get $|E_{\beta}(-\lambda t_2^{\beta}) - E_{\beta}(-\lambda t_1^{\beta})| = O(|t_2 - t_1|)$. From Lemma 3, the remain estimates can be proved. □

Remark 1. *Obviously, for $\beta \in (1,2]$, $z \in \mathbb{R}$, $E_{\beta}(z)$ and $E_{\beta,\beta}(z)$ are not always nonnegative.*

Lemma 6. *If $\lambda > 0$, $\alpha \in (0,1)$, β, $\alpha+\beta \in (1,2)$, $t \in J$, then*

$$|t^{\beta}E_{\beta,\beta+1}(-\lambda t^{\beta})| \leq \frac{\Gamma(\beta)}{\lambda^2\pi|\sin(\pi\beta)|}\cdot\frac{1}{t^{\beta}} + \frac{2}{\beta\lambda^{1-\frac{1}{\beta}}}\exp(t\lambda^{\frac{1}{\beta}}\cos\frac{\pi}{\beta}); \tag{11}$$

$$|tE_{\beta,2}(-\lambda t^{\beta})| \leq \frac{\Gamma(\beta-1)}{\lambda\pi|\sin(\pi\beta)|}\cdot\frac{1}{t^{\beta}} + \frac{2}{\beta\lambda^{1-\frac{1}{\beta}}}\exp(t\lambda^{\frac{1}{\beta}}\cos\frac{\pi}{\beta}); \tag{12}$$

$$|t^{\alpha+\beta-1}E_{\beta,\alpha+\beta}(-\lambda t^{\beta})| \leq \frac{\Gamma(1-\alpha)}{\lambda\pi}\Big[\frac{1}{4\cos^2(\frac{\pi\beta}{2})} + \frac{1}{\sin^2(\pi\beta)}\Big]\frac{1}{t^{1-\alpha}} + \frac{2}{\beta\lambda^{1-\frac{1}{\beta}}}\exp(t\lambda^{\frac{1}{\beta}}\cos\frac{\pi}{\beta}). \tag{13}$$

Proof. From (5), we have

$$\begin{aligned}
E_{\beta,\beta+1}(-\lambda t^{\beta}) &= \int_0^{+\infty} Q_1(r,t)dr + \frac{2t^{-\beta}}{\beta\lambda^{1-\frac{1}{\beta}}}\exp(t\lambda^{\frac{1}{\beta}}\cos\frac{\pi}{\beta})\cos\Big[t\lambda^{\frac{1}{\beta}}\sin\frac{\pi}{\beta} - \pi\Big], \\
E_{\beta,2}(-\lambda t^{\beta}) &= \int_0^{+\infty} Q_2(r,t)dr + \frac{2t^{-1}}{\beta\lambda^{1-\frac{1}{\beta}}}\exp(t\lambda^{\frac{1}{\beta}}\cos\frac{\pi}{\beta})\cos\Big[t\lambda^{\frac{1}{\beta}}\sin\frac{\pi}{\beta} - \frac{\pi}{\beta}\Big], \\
E_{\beta,\alpha+\beta}(-\lambda t^{\beta}) &= \int_0^{+\infty} Q_3(r,t)dr + \frac{2t^{1-\alpha-\beta}}{\beta\lambda^{1-\frac{1}{\beta}}}\exp(t\lambda^{\frac{1}{\beta}}\cos\frac{\pi}{\beta})\cos\Big[t\lambda^{\frac{1}{\beta}}\sin\frac{\pi}{\beta} - \frac{\pi}{\beta}(\alpha+\beta-1)\Big],
\end{aligned}$$

where

$$\begin{aligned}
Q_1(r,t) &= \frac{r^{-1}\exp(-r^{\frac{1}{\beta}})}{\pi\beta}\cdot\frac{r\sin(-\pi\beta)}{U(t)}, \\
Q_2(r,t) &= \frac{r^{-\frac{1}{\beta}}\exp(-r^{\frac{1}{\beta}})}{\pi\beta}\cdot\frac{\lambda t^{\beta}\sin(\pi(\beta-1))}{U(t)}, \\
Q_3(r,t) &= \frac{r^{\frac{1-\alpha-\beta}{\beta}}\exp(-r^{\frac{1}{\beta}})}{\pi\beta}\cdot\frac{r\sin(\pi(1-\alpha-\beta)) + \lambda t^{\beta}\sin(\pi(1-\alpha))}{U(t)}.
\end{aligned}$$

Using (6) and (7), we derive that

$$\begin{aligned}
|Q_1(r,t)| &\leq \frac{\exp(-r^{\frac{1}{\beta}})}{\lambda^2\pi\beta t^{2\beta}|\sin(\pi\beta)|}, \quad |Q_2(r,t)| \leq \frac{r^{-\frac{1}{\beta}}\exp(-r^{\frac{1}{\beta}})}{\lambda\pi\beta t^{\beta}|\sin(\pi\beta)|}, \\
|Q_3(r,t)| &\leq \frac{1}{\lambda\pi\beta t^{\beta}}\Big[\frac{1}{4\cos^2(\frac{\pi\beta}{2})} + \frac{1}{\sin^2(\pi\beta)}\Big]r^{\frac{1-\alpha-\beta}{\beta}}\exp(-r^{\frac{1}{\beta}}),
\end{aligned}$$

then

$$\begin{aligned}
|t^{\beta}E_{\beta,\beta+1}(-\lambda t^{\beta})| &\leq \frac{\Gamma(\beta)}{\lambda^2\pi|\sin(\pi\beta)|}\cdot\frac{1}{t^{\beta}}+\frac{2}{\beta\lambda^{1-\frac{1}{\beta}}}\exp(t\lambda^{\frac{1}{\beta}}\cos\frac{\pi}{\beta}),\\
|tE_{\beta,2}(-\lambda t^{\beta})| &\leq \frac{\Gamma(\beta-1)}{\lambda\pi|\sin(\pi\beta)|}\cdot\frac{1}{t^{\beta}}+\frac{2}{\beta\lambda^{1-\frac{1}{\beta}}}\exp(t\lambda^{\frac{1}{\beta}}\cos\frac{\pi}{\beta}),\\
|t^{\alpha+\beta-1}E_{\beta,\alpha+\beta}(-\lambda t^{\beta})| &\leq \frac{\Gamma(1-\alpha)}{\lambda\pi}\left[\frac{1}{4\cos^2(\frac{\pi\beta}{2})}+\frac{1}{\sin^2(\pi\beta)}\right]\frac{1}{t^{1-\alpha}}+\frac{2}{\beta\lambda^{1-\frac{1}{\beta}}}\exp(t\lambda^{\frac{1}{\beta}}\cos\frac{\pi}{\beta}).
\end{aligned}$$

□

4. Solutions of BVP

In this section, we present the formulas of solutions to problem (1) and (2).

Lemma 7. *[4] For $\theta > 0$, a general solution of the fractional differential equation ${}^cD^{\theta}_{0+}u(t) = 0$ is given by*

$$u(t) = \sum_{i=0}^{n-1} C_i t^i,$$

where $C_i \in \mathbb{R}$, $i = 0, 1, 2, \cdots, n-1 (n = [\theta]+1)$, and $[\theta]$ denotes the integer part of the real number θ.

Lemma 8. *For $\alpha \in (0,1)$, $\beta \in (1,2)$, $h \in L^1(0,1)$, if ${}^cD^{\alpha}_{0+}\left({}^cD^{\beta}_{0+}+\lambda\right)u(t) = h(t)$, $t \in J$, then*

$$u(t) = \sum_{i=1}^{2} C_i t^{i-1}E_{\beta,i}(-\lambda t^{\beta}) + C_0 t^{\beta}E_{\beta,\beta+1}(-\lambda t^{\beta}) + \int_0^t (t-s)^{\alpha+\beta-1}E_{\beta,\alpha+\beta}(-\lambda(t-s)^{\beta})h(s)ds, \quad t \in J,$$

where $C_i \in \mathbb{R}$, $i = 0, 1, 2$.

Formally, by Lemma 7, for $C_i \in \mathbb{R}(i = 0,1,2)$, we have $\left({}^cD^{\beta}_{0+}+\lambda\right)u(t) = C_0 + (I^{\alpha}_{0+}h)(t)$ and

$$u(t) = -\lambda(I^{\beta}_{0+}u)(t) + I^{\beta}_{0+}(C_0 + (I^{\alpha}_{0+}h))(t) + C_1 + C_2 t.$$

Based on the arguments of (see [4], pp. 222–223) and Lemma 2, we obtain

$$\begin{aligned}
&u(t)\\
=\ & C_1E_{\beta}(-\lambda t^{\beta}) + C_2tE_{\beta,2}(-\lambda t^{\beta}) + \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-\lambda(t-s)^{\beta})(C_0 + (I^{\alpha}_{0+}h)(s))ds\\
=\ & C_1E_{\beta}(-\lambda t^{\beta}) + C_2tE_{\beta,2}(-\lambda t^{\beta}) + C_0t^{\beta}E_{\beta,\beta+1}(-\lambda t^{\beta}) + \int_0^t\int_0^{t-s}\frac{\tau^{\beta-1}E_{\beta,\beta}(-\lambda\tau^{\beta})(t-s-\tau)^{\alpha-1}h(s)}{\Gamma(\alpha)}d\tau ds\\
=\ & C_1E_{\beta}(-\lambda t^{\beta}) + C_2tE_{\beta,2}(-\lambda t^{\beta}) + C_0t^{\beta}E_{\beta,\beta+1}(-\lambda t^{\beta}) + \int_0^t (t-s)^{\alpha+\beta-1}E_{\beta,\alpha+\beta}(-\lambda(t-s)^{\beta})h(s)ds.
\end{aligned}$$

We define $C_{\beta}([0,1],\mathbb{R}) = \{u \in C(J,\mathbb{R}) : t^{\beta}u(t) \in C([0,1],\mathbb{R})\}$ with the norm $\|u\|_{\beta} = \max\limits_{t\in[0,1]} t^{\beta}|u(t)|$ and we abbreviate $C_{\beta}([0,1],\mathbb{R})$ to C_{β}.

(H1) $f : J \times \mathbb{R} \to \mathbb{R}$ satisfies $f(\cdot,u) : J \to \mathbb{R}$ is measurable for all $u \in \mathbb{R}$ and $f(t,\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous for a.e. $t \in J$, and there exists a function $\varphi \in L^{\frac{1}{p_1}}(J,\mathbb{R}^+)(0 < p_1 < \min\{\alpha, \frac{\alpha+\beta-1}{2}\}$ such that $|f(t,u(t))| \leq \varphi(t)$.

Definition 6. *A function $u : J \to \mathbb{R}$ is said to be a solution of (1) and (2) if*

(i) $u \in AC^2(J,\mathbb{R})$;

(ii) u *satisfies the equation* ${}^cD^{\alpha}_{0^+}({}^cD^{\beta}_{0^+}+\lambda)u(t)=f(t,u(t))$ *on* J*;*
(iii) $u(0)+u(1)=0,\ u'(0)+u'(1)=0,\ \lim_{t\to 0^+} t^{\alpha}({}^HD^{\xi,\alpha}_{0^+}u)(t)=0.$

Lemma 9. *For all* $s_1,s_2\in J,\ s_1<s_2$*,*

$$\int_0^{s_1}[(s_2-\tau)^{\alpha+\beta-2}-(s_1-\tau)^{\alpha+\beta-2}]\varphi(\tau)d\tau\to 0,\quad (s_1\to s_2).$$

Proof. From the Hölder inequality, we have

$$\begin{aligned}
&\left|\int_0^{s_1}[(s_2-\tau)^{\alpha+\beta-2}-(s_1-\tau)^{\alpha+\beta-2}]\varphi(\tau)d\tau\right|\\
\le\ & \|\varphi\|_{L^{\frac{1}{p_1}}}\left[\int_0^{s_1}|(s_2-\tau)^{\alpha+\beta-2}-(s_1-\tau)^{\alpha+\beta-2}|^{\frac{1}{1-p_1}}d\tau\right]^{1-p_1}\\
=\ & (2-\alpha-\beta)\|\varphi\|_{L^{\frac{1}{p_1}}}\left[\int_0^{s_1}|\int_{s_2}^{s_1}(\xi-\tau)^{\alpha+\beta-3}d\xi|^{\frac{1}{1-p_1}}d\tau\right]^{1-p_1}\\
\le\ & \widetilde{M}\left[\int_0^{s_1}((s_1-\tau)^{\delta}-(s_2-\tau)^{\delta})d\tau\right]^{1-p_1}\\
=\ & \frac{\widetilde{M}}{(1+\delta)^{1-p_1}}\left[(s_2-s_1)^{1+\delta}-s_2^{1+\delta}+s_1^{1+\delta}\right]^{1-p_1}\to 0,\quad \text{as}\quad s_2\to s_1.
\end{aligned}$$

where $\widetilde{M}>0$ is a constant, $\delta=\frac{\alpha+\beta-2-p_1}{1-p_1}\in(-1,0)$. □

For $t\in J,\ y>p_1$, using the Hölder inequality, we have

$$\int_0^t(t-s)^{y-1}\varphi(s)ds\le\left(\int_0^t(t-s)^{\frac{y-1}{1-p_1}}ds\right)^{1-p_1}\|\varphi\|_{L^{\frac{1}{p_1}}}=\left(\frac{1-p_1}{y-p_1}\right)^{1-p_1}t^{y-p_1}\|\varphi\|_{L^{\frac{1}{p_1}}}. \tag{14}$$

For convenience, we define

$$(F^{\varsigma}u)(t)=\int_0^t(t-s)^{\varsigma-1}E_{\beta,\varsigma}(-\lambda(t-s)^{\beta})f(s,u(s))ds.$$

Lemma 10. *Assume that (H1) holds. For* $u\in C_{\beta}$*,* $t\in J$*, we have*

(i) $(F^{\alpha+\beta}u)(t)\in AC^2(J,\mathbb{R})$*;*
(ii) $\left[{}^cD^{\beta}_{0^+}(F^{\alpha+\beta}u)\right](t)=(F^{\alpha}u)(t),\quad \left[{}^cD^{\alpha}_{0^+}(F^{\alpha+\beta}u)\right](t)=(F^{\beta}u)(t)$*;*
(iii) $\left[{}^cD^{\alpha}_{0^+}(F^{\alpha}u)\right](t)=-\lambda(F^{\beta}u)(t)+f(t,u(t))$*;*
(iv) $\left[{}^HD^{\xi,\alpha}_{0^+}(F^{\alpha+\beta}u)\right](t)=(F^{\beta}u)(t)$*.*

Proof. It follows from the definition of derivative for the Lebesgue integration and (14) that

$$\frac{d}{dt}(F^{\alpha+\beta}u)(t)=(F^{\alpha+\beta-1}u)(t). \tag{15}$$

Next, we show that $(F^{\alpha+\beta-1}u)(t) \in AC(J,\mathbb{R})$. For every finite collection $\{(a_j,b_j)\}_{1\le j\le n}$ on J with $\sum_{j=1}^{n}(b_j-a_j) \to 0$, noting Lemmas 1, 5 (ii), 9 and (14), we derive

$$
\begin{aligned}
& \sum_{j=1}^{n}\left|(F^{\alpha+\beta-1}u)(b_j)-(F^{\alpha+\beta-1}u)(a_j)\right| \\
\le\ & \sum_{j=1}^{n}\int_0^{a_j}|(b_j-s)^{\alpha+\beta-2}E_{\beta,\alpha+\beta-1}(-\lambda(b_j-s)^{\beta})-(a_j-s)^{\alpha+\beta-2}E_{\beta,\alpha+\beta-1}(-\lambda(a_j-s)^{\beta})|\cdot|f(s,u(s))|ds \\
+\ & \sum_{j=1}^{n}\int_{a_j}^{b_j}|b_j-s|^{\alpha+\beta-2}|E_{\beta,\alpha+\beta-1}(-\lambda(b_j-s)^{\beta})|\cdot|f(s,u(s))|ds \\
\le\ & \sum_{j=1}^{n}\int_0^{a_j}|(b_j-s)^{\alpha+\beta-2}-(a_j-s)^{\alpha+\beta-2}|\cdot|E_{\beta,\alpha+\beta-1}(-\lambda(b_j-s)^{\beta})|\varphi(s)ds \\
+\ & \sum_{j=1}^{n}\int_0^{a_j}|a_j-s|^{\alpha+\beta-2}|E_{\beta,\alpha+\beta-1}(-\lambda(b_j-s)^{\beta})-E_{\beta,\alpha+\beta-1}(-\lambda(a_j-s)^{\beta})|\varphi(s)ds \\
+\ & \sum_{j=1}^{n}\int_{a_j}^{b_j}|b_j-s|^{\alpha+\beta-2}|E_{\beta,\alpha+\beta-1}(-\lambda(b_j-s)^{\beta})|\varphi(s)ds \\
\to\ & 0
\end{aligned}
$$

Hence, $(F^{\alpha+\beta-1}u)(t)$ is absolutely continuous on J. Furthermore, for almost all $t \in J$, $\left[{}^cD_{0+}^{\beta}(F^{\alpha+\beta}u)(s)\right](t)$ and $\left[{}^cD_{0+}^{\alpha}(F^{\alpha+\beta}u)(s)\right](t)$ exist. Similarly, $\left[{}^cD_{0+}^{\alpha}(F^{\alpha}u)(s)\right](t)$ exists.

Moreover, similar to (15), one has

$$\frac{d^2}{dt^2}(F^{\alpha+2}u)(t)=(F^{\alpha}u)(t),\quad \frac{d}{dt}(F^{\beta+1}u)(t)=(F^{\beta}u)(t). \tag{16}$$

Noting that Lemma 2 and (14) we can see

$$\left[I_{0+}^{\gamma}(F^{\xi}u)\right](t)=(F^{\gamma+\xi}u)(t),\quad \text{for}\quad \gamma,\xi>0 \quad\text{and}\quad \gamma+\xi>p_1. \tag{17}$$

From the Definition 2, (16) and (17), we get

$$\left[{}^cD_{0+}^{\beta}(F^{\alpha+\beta}u)\right](t)=\left(\frac{d}{dt}\right)^2\left[I_{0+}^{2-\beta}(F^{\alpha+\beta}u)\right](t)=\left(\frac{d}{dt}\right)^2(F^{\alpha+2}u)(t)=(F^{\alpha}u)(t).$$

and

$$\left[{}^cD_{0+}^{\alpha}(F^{\alpha+\beta}u)\right](t)=\frac{d}{dt}\left[I_{0+}^{1-\alpha}(F^{\alpha+\beta}u)\right](t)=\frac{d}{dt}(F^{\beta+1}u)(t)=(F^{\beta}u)(t), \tag{18}$$

and

$$
\begin{aligned}
\left[{}^cD_{0+}^{\alpha}(F^{\alpha}u)\right](t) &= \frac{d}{dt}\left[I_{0+}^{1-\alpha}(F^{\alpha}u)\right](t)=\frac{d}{dt}\int_0^t E_{\beta}(-\lambda(t-\tau)^{\beta})f(\tau,u(\tau))d\tau \\
&= -\lambda\int_0^t(t-\tau)^{\beta-1}E_{\beta,\beta}(-\lambda(t-\tau)^{\beta})f(\tau,u(\tau))d\tau+f(t,u(t)) \\
&= -\lambda(F^{\beta}u)(t)+f(t,u(t)),
\end{aligned}
$$

and

$$\left[{}^cD_{0+}^{\alpha+\xi-\alpha\xi}(F^{\alpha+\beta}u)\right](t)=\frac{d}{dt}\left[I_{0+}^{1-\alpha-\xi+\alpha\xi}(F^{\alpha+\beta}u)\right](t)=\frac{d}{dt}(F^{1+\beta-\xi+\alpha\xi}u)(t)=(F^{\beta-\xi+\alpha\xi}u)(t). \tag{19}$$

Noting that Definitions 3 and 4, (17) and (19) we obtain

$$\begin{aligned} & \left[{}^{H}D_{0^+}^{\xi,\alpha}(F^{\alpha+\beta}u)\right](t) = \left[I_{0^+}^{\xi(1-\alpha)}({}^{L}D_{0^+}^{\alpha+\xi-\alpha\xi}(F^{\alpha+\beta}u)\right](t) \\ = & \left[I_{0^+}^{\xi(1-\alpha)}({}^{c}D_{0^+}^{\alpha+\xi-\alpha\xi}F^{\alpha+\beta}u)\right](t) \\ = & \left[I_{0^+}^{\xi(1-\alpha)}(F^{\beta-\xi+\alpha\xi}u)\right](t) \\ = & (F^{\beta}u)(t). \end{aligned}$$

□

For convenience, we shall use the following notation:

$$M(\lambda) \quad = \quad E_{\beta,2}(-\lambda)\cdot E_{\beta,\beta}(-\lambda) - E_{\beta,\beta+1}(-\lambda)\cdot(1+E_{\beta}(-\lambda)).$$

Lemma 11. *Assume that (H1) holds. A function u is a solution of the following fractional integral equation*

$$u(t) = (Pu)(t) + (Qu)(t) + (F^{\alpha+\beta}u)(t) \tag{20}$$

if and only if u is a solution of the problem (1) *and* (2), *where*

$$\begin{aligned} (Pu)(t) &= \frac{t^{\beta}E_{\beta,\beta+1}(-\lambda t^{\beta})(1+E_{\beta}(-\lambda)) - tE_{\beta,2}(-\lambda t^{\beta})E_{\beta,\beta}(-\lambda)}{M(\lambda)}(F^{\alpha+\beta}u)(1), \\ (Qu)(t) &= \frac{tE_{\beta,2}(-\lambda t^{\beta})E_{\beta,\beta+1}(-\lambda) - t^{\beta}E_{\beta,\beta+1}(-\lambda t^{\beta})E_{\beta,2}(-\lambda)}{M(\lambda)}(F^{\alpha+\beta-1}u)(1). \end{aligned}$$

Proof. (Sufficiency) Let u be the solution of (1) and (2), Lemmas 8, 2 and 4 imply

$$\begin{aligned} u(t) &= aE_{\beta}(-\lambda t^{\beta}) + btE_{\beta,2}(-\lambda t^{\beta}) + ct^{\beta}E_{\beta,\beta+1}(-\lambda t^{\beta}) + (F^{\alpha+\beta}u)(t), \\ u'(t) &= (-\lambda a + c)t^{\beta-1}E_{\beta,\beta}(-\lambda t^{\beta}) + bE_{\beta}(-\lambda t^{\beta}) + (F^{\alpha+\beta-1}u)(t), \\ ({}^{H}D_{0^+}^{\xi,\alpha}u)(t) &= at^{-\alpha}E_{\beta,1-\alpha}(-\lambda t^{\beta}) + bt^{1-\alpha}E_{\beta,2-\alpha}(-\lambda t^{\beta}) + ct^{\beta-\alpha}E_{\beta,\beta-\alpha+1}(-\lambda t^{\beta}) + (F^{\beta}u)(t), \end{aligned}$$

where a, b, c are constants. Using the boundary value condition (2), we derive that $a = 0$ and

$$\begin{cases} bE_{\beta,2}(-\lambda) + cE_{\beta,\beta+1}(-\lambda) + (F^{\alpha+\beta}u)(1) = 0, \\ b(1+E_{\beta}(-\lambda)) + cE_{\beta,\beta}(-\lambda) + (F^{\alpha+\beta-1}u)(1) = 0, \end{cases}$$

then

$$\begin{cases} b = \dfrac{-(F^{\alpha+\beta}u)(1)\cdot E_{\beta,\beta}(-\lambda) + (F^{\alpha+\beta-1}u)(1)\cdot E_{\beta,\beta+1}(-\lambda)}{M(\lambda)}, \\ c = \dfrac{-(F^{\alpha+\beta-1}u)(1)\cdot E_{\beta,2}(-\lambda) + (F^{\alpha+\beta}u)(1)\cdot(1+E_{\beta}(-\lambda))}{M(\lambda)}. \end{cases}$$

Now we can see that (20) holds.

(Necessity) Let u satisfy (20). Noting that Lemma 10, $({}^cD^{\alpha}_{0+}{}^cD^{\beta}_{0+}u)(t)$ exists and

$$\begin{aligned}
&({}^cD^{\alpha}_{0+}{}^cD^{\beta}_{0+}u)(t)\\
= & \frac{-\lambda t^{\beta-\alpha}E_{\beta,\beta-\alpha+1}(-\lambda t^{\beta})(1+E_{\beta}(-\lambda))+\lambda t^{1-\alpha}E_{\beta,2-\alpha}(-\lambda t^{\beta})E_{\beta,\beta}(-\lambda)}{M(\lambda)}(F^{\alpha+\beta}u)(1)\\
&+\frac{-\lambda t^{1-\alpha}E_{\beta,2-\alpha}(-\lambda t^{\beta})E_{\beta,\beta+1}(-\lambda)+\lambda t^{\beta-\alpha}E_{\beta,\beta-\alpha+1}(-\lambda t^{\beta})E_{\beta,2}(-\lambda)}{M(\lambda)}(F^{\alpha+\beta-1}u)(1)\\
&-\lambda(F^{\beta}u)(t)+f(t,u(t)),\\
&\lambda({}^cD^{\alpha}_{0+}u)(t)\\
= & \lambda\Big[\frac{t^{\beta-\alpha}E_{\beta,\beta-\alpha+1}(-\lambda t^{\beta})(1+E_{\beta}(-\lambda))-t^{1-\alpha}E_{\beta,2-\alpha}(-\lambda t^{\beta})E_{\beta,\beta}(-\lambda)}{M(\lambda)}(F^{\alpha+\beta}u)(1)\\
&+\frac{t^{1-\alpha}E_{\beta,2-\alpha}(-\lambda t^{\beta})E_{\beta,\beta+1}(-\lambda)-t^{\beta-\alpha}E_{\beta,\beta-\alpha+1}(-\lambda t^{\beta})E_{\beta,2}(-\lambda)}{M(\lambda)}(F^{\alpha+\beta-1}u)(1)\\
&+(F^{\beta}u)(t)\Big],
\end{aligned}$$

then ${}^cD^{\alpha}_{0+}({}^cD^{\beta}_{0+}+\lambda)u(t)=f(t,u(t))$ for $t\in J$. Clearly, the boundary value condition (2) holds and hence the necessity is proved. □

For convenience of the following presentation, set

$$\begin{aligned}
A(\lambda) &= \frac{\Gamma(1-\alpha)}{\lambda\pi}\left[\frac{1}{4\cos^2(\frac{\pi\beta}{2})}+\frac{1}{\sin^2(\pi\beta)}\right],\\
B(\lambda) &= \frac{2}{\beta\lambda^{1-\frac{1}{\beta}}},\quad C(\lambda)=\frac{\Gamma(\beta)}{\lambda^2\pi|\sin\pi\beta|},\\
N(\lambda) &= \frac{(C(\lambda)+B(\lambda))\,|E_{\beta}(-\lambda)|+\left(\frac{\lambda C(\lambda)}{\beta-1}+B(\lambda)\right)|E_{\beta,\beta}(-\lambda)|}{|M(\lambda)|},\\
R(\lambda) &= \frac{\left(\frac{\lambda C(\lambda)}{\beta-1}+B(\lambda)\right)|E_{\beta,\beta+1}(-\lambda)|+(C(\lambda)+B(\lambda))|E_{\beta,2}(-\lambda)|}{|M(\lambda)|},\\
L(\lambda) &= \left(A(\lambda)\left(\frac{1-p_1}{\alpha-p_1}\right)^{1-p_1}+B(\lambda)\right)\|\varphi\|_{L^{\frac{1}{p_1}}},\\
\widetilde{L}(\lambda) &= \left(A(\lambda)\left(\frac{1-p_2}{\alpha-p_2}\right)^{1-p_2}+B(\lambda)\right)\|\psi\|_{L^{\frac{1}{p_2}}}.
\end{aligned}$$

5. Existence Result

In this section, we deal with the existence of solutions to the problem (1) and (2). To this end, we consider the following assumption.

(H2) There exists a function $\psi\in L^{\frac{1}{p_2}}(J,\mathbb{R}^+)(p_2\in(0,\alpha))$ such that

$$|f(t,x)-f(t,y)|\le\psi(t)\|x-y\|_{\beta}.$$

Theorem 2. *Assume that (H1) and (H2) are satisfied, then the problem* (1) *and* (2) *has at least a solution* $u\in C_{\beta}(J)$ *if* $\widetilde{L}(\lambda)<1$.

Proof. We consider an operator $\mathcal{F}: C_\beta \to C_\beta$ defined by

$$(\mathcal{F}u)(t) = (Pu)(t) + (Qu)(t) + (F^{\alpha+\beta}u)(t).$$

Clearly, $\mathcal{F}$ is well defined. Obviously, the fixed point of $\mathcal{F}$ is the solution of problem (1) and (2).

By (11)–(14) and (H1), the following inequalities hold:

$$\begin{aligned} &\frac{t^\beta|t^\beta E_{\beta,\beta+1}(-\lambda t^\beta)(1+E_\beta(-\lambda)) - tE_{\beta,2}(-\lambda t^\beta)E_{\beta,\beta}(-\lambda)|}{|M(\lambda)|} \le N(\lambda), \\ &\frac{t^\beta|tE_{\beta,2}(-\lambda t^\beta)E_{\beta,\beta+1}(-\lambda) - t^\beta E_{\beta,\beta+1}(-\lambda t^\beta)E_{\beta,2}(-\lambda)|}{|M(\lambda)|} \le R(\lambda), \\ &|(F^{\alpha+\beta}u)(t)| \le A(\lambda)\int_0^t (t-s)^{\alpha-1}\varphi(s)ds + B(\lambda)\int_0^t \varphi(s)ds \le L(\lambda). \end{aligned} \tag{21}$$

Moreover, from Lemma 1, there exists a constant $\mathcal{C}$ such that $|E_{\beta,\alpha+\beta-1}(-\lambda t^\beta)| \le \mathcal{C}$, then

$$|(F^{\alpha+\beta-1}u)(t)| \le \mathcal{C}\int_0^t (t-s)^{\alpha+\beta-2}\varphi(s)ds \le \mathcal{C}\left(\frac{1-p_1}{\alpha+\beta-1-p_1}\right)^{1-p_1}\|\varphi\|_{L^{\frac{1}{p_1}}}. \tag{22}$$

Furthermore

$$\|Pu\|_\beta \le N(\lambda)L(\lambda), \quad \|Qu\|_\beta \le \mathcal{C}R(\lambda)\left(\frac{1-p_1}{\alpha+\beta-1-p_1}\right)^{1-p_1}\|\varphi\|_{L^{\frac{1}{p_1}}}. \tag{23}$$

Let $B_r = \{u \in C_\beta : \|u\|_\beta \le r\}$, where $r \ge L(\lambda)(1+N(\lambda)) + \mathcal{C}R(\lambda)\left(\frac{1-p_1}{\alpha+\beta-1-p_1}\right)^{1-p_1}\|\varphi\|_{L^{\frac{1}{p_1}}}$.

It follows from (21) and (23) that $\|(\mathcal{F}u)\|_\beta \le r$. Now, we can see that $(\mathcal{F}u)(t) \in B_r$ for any $u \in B_r$ and $t \in J$.

Setting

$$(\mathcal{F}_1u)(t) = (Pu)(t) + (Qu)(t), \quad (\mathcal{F}_2u)(t) = (F^{\alpha+\beta}u)(t).$$

According to (H2), (13) and (14), we obtain

$$\begin{aligned} \|\mathcal{F}_2u - \mathcal{F}_2v\|_\beta &\le \left(A(\lambda)\int_0^t (t-s)^{\alpha-1}\psi(s)ds + B(\lambda)\int_0^t \psi(s)ds\right)\cdot\|u-v\|_\beta \\ &\le \widetilde{L}(\lambda)\|u-v\|_\beta, \quad \text{for} \quad u,v \in B_r. \end{aligned}$$

This implies that $\mathcal{F}_2$ is a contraction mapping.

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in C_β, then there exists $\varepsilon > 0$ such that $\|u_n - u\|_\beta < \varepsilon$ for n sufficiently large. By (H2), we have

$$|f(t,u_n(t)) - f(t,u(t))| \le \psi(t)\varepsilon.$$

Moreover, f satisfies (H1), we get $f(t,u_n(t)) \to f(t,u(t))$ as $n \to \infty$ for almost every $t \in J$. Then (13), $\int_0^t (t-s)^{\alpha-1}\psi(s)ds \le \left(\frac{1-p_2}{\alpha-p_2}\right)^{1-p_2}\|\psi\|_{L^{\frac{1}{p_2}}}$ and the Lebesgue dominated convergence theorem imply that $|(F^{\alpha+\beta}u_n)(t) - (F^{\alpha+\beta}u)(t)| \to 0$, furthermore,

$$\|Pu_n - Pu\|_\beta \to 0, \quad \text{as} \quad n \to \infty.$$

Similarly, from Lemma 1 and $\int_0^t (t-s)^{\alpha+\beta-2}\psi(s)ds \leq \left(\frac{1-p_2}{\alpha+\beta-1-p_2}\right)^{1-p_2} \|\psi\|_{L^{\frac{1}{p_2}}}$, we derive $|(F^{\alpha+\beta-1}u_n)(t) - (F^{\alpha+\beta-1}u)(t)| \to 0$, then $\|Qu_n - Qu\|_\beta \to 0$, as $n \to \infty$. Now we see that $\mathcal{F}_1$ is continuous.

Moreover, by Lemmas 1 and 5, (21) and (22), $\{(\mathcal{F}_1 u)(t) : u \in B_r\}$ is an equicontinuous and uniformly bounded set. Then, $\mathcal{F}_1$ is a completely continuous operator on B_r. The proof now can be finished by using Theorem 1. □

6. Application

In this section, we give an example to illustrate our result.

$$\begin{cases} {}^cD_{0^+}^{\frac{2}{5}}({}^cD_{0^+}^{\frac{3}{2}} + 10)u(t) = \dfrac{\sin(2 + t^{\frac{3}{2}}u(t))}{\sqrt[10]{t}}, \quad t \in J := (0,1] \\ u(0) + u(1) = 0,\ u'(0) + u'(1) = 0,\ \lim\limits_{t\to 0^+} t^{\frac{2}{5}}(D_{0^+}^{\frac{1}{3},\frac{2}{5}}u)(t) = 0. \end{cases} \tag{24}$$

Corresponding to (1) and (2), we have $\alpha = \frac{2}{5}, \xi = \frac{1}{3}, \beta = \frac{3}{2}, \lambda = 10, f(t,u(t)) = (\sin(2 + t^{\frac{3}{2}}u(t)))/\sqrt[10]{t}$. The space $C_\beta := \{u \in C(J,\mathbb{R}) : t^{\frac{3}{2}}u(t) \in C([0,1],\mathbb{R})\}$ with the norm $\|u\|_{\frac{3}{2}} = \max\limits_{t\in[0,1]} t^{\frac{3}{2}}|u(t)|$.

Obviously, $|f(t,u(t))| \leq \varphi(t)$ and $|f(t,u(t)) - f(t,v(t))| \leq \psi(t)\|u-v\|_{\frac{3}{2}}$, where $\varphi(t) = \psi(t) = \frac{1}{\sqrt[10]{t}} \in L^{\frac{1}{p_2}}[0,1](p_1 = p_2 = \frac{1}{5})$ and $\|\psi\|_{L^{\frac{1}{p_2}}} = 2^{\frac{1}{5}}$. By direct computation, we have

$$\begin{aligned} A(\lambda) &= \frac{\Gamma(1-\alpha)}{\lambda\pi}\left[\frac{1}{4\cos^2(\frac{\pi\beta}{2})} + \frac{1}{\sin^2(\pi\beta)}\right] = \frac{3\Gamma(\frac{3}{5})}{20\pi}, \\ B(\lambda) &= \frac{2}{\beta\lambda^{1-\frac{1}{\beta}}} = \frac{4}{3\times\sqrt[3]{10}}, \quad \left(\frac{1-p_2}{\alpha-p_2}\right)^{1-p_2} = 4^{\frac{4}{5}}, \\ \widetilde{L}(\lambda) &= \left(A(\lambda)\left(\frac{1-p_2}{\alpha-p_2}\right)^{1-p_2} + B(\lambda)\right)\|\psi\|_{L^{\frac{1}{p_2}}} = \left(\frac{3\times 4^{\frac{4}{5}}\times\Gamma(\frac{3}{5})}{20\pi} + \frac{4}{3\times\sqrt[3]{10}}\right)\times 2^{\frac{1}{5}} \approx 0.96 < 1. \end{aligned}$$

Thus, by Theorem 2, problem (24) has at least one solution.

7. Conclusions

In this paper, we have presented existence results to the nonlinear Langevin fractional differential equations with the anti-periodic boundary value conditions and some properties of the Mittag-Leffler functions $E_\beta(z)$ and $E_{\beta,\theta}(z)(\beta,\theta \in (1,2))$. We prove the equivalence of the problem (1) and (2) and the integral Equation (20) under the weak assumption (H1). Moreover, when $\beta,\theta \in (1,2)$, $E_\beta(z)$ and $E_{\beta,\theta}(z)$ do not possess the monotonicity and nonnegativity, using Lemma 6, we successfully obtain some estimates for the Mittag-Leffler functions. Our results are new and significantly contribute to the existing literature on fractional order differential equation with anti-periodic boundary value conditions. In fact, our approach is simple and can easily be applied to a variety of real world problems.

In this area, our future work will focus on studying the more complex model, such as the boundary value problem for the mixed type fractional differential equations with the Caputo and the Riemann-Liouville fractional derivative.

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