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Numerical Analysis of Boundary Layer Flow Adjacent to a Thin Needle in Nanofluid with the Presence of Heat Source and Chemical Reaction

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Abstract: The steady boundary layer flow of a nanofluid past a thin needle under the influences of heat generation and chemical reaction is analyzed in the present work. The mathematical model has been formulated by using Buongiorno's nanofluid model which incorporates the effect of the Brownian motion and thermophoretic diffusion. The governing coupled partial differential equations are transformed into a set of nonlinear ordinary differential equations by using appropriate similarity transformations. These equations are then computed numerically through MATLAB software using the implemented package called bvp4c. The influences of various parameters such as Brownian motion, thermophoresis, velocity ratio, needle thickness, heat generation and chemical reaction parameters on the flow, heat and mass characteristics are investigated. The physical characteristics which include the skin friction, heat and mass transfers, velocity, temperature and concentration are further elaborated with the variation of governing parameters and presented through graphs. It is observed that the multiple (dual) solutions are likely to exist when the needle moves against the direction of the fluid flow. It is also noticed that the reduction in needle thickness contributes to the enlargement of the region of the dual solutions. The determination of the stable solution has been done using a stability analysis. The results indicate that the upper branch solutions are linearly stable, while the lower branch solutions are linearly unstable. The study also revealed that the rate of heat transfer is a decreasing function of heat generation parameter, while the rate of mass transfer is an increasing function of heat generation and chemical reaction parameters.

Keywords: numerical analysis; heat generation; chemical reaction; thin needle; nanofluid

1. Introduction

In recent decades, the performance of heat transfer of conventional fluids like ethylene glycol, lubricants, oil, kerosene and water, etc., has become less favorable in certain applications. Hence, new kinds of fluid are needed to reach the thermal efficiency for heat exchangers in the future. Choi [1] came out with a tactful idea to resolve the problem by adding dilute suspension of nanoparticles into conventional fluids and this mixture is known as 'nanofluid'. Normally, nanoparticles consist of metals, carbides, oxides, nitrides or non-metals and have dimensions from 1 to 100 nm. Due to the tiny size of nanoelements, nanofluids possess strong suspension stability and able to move without clogging the flow system. Since nanoparticles have higher thermal conductivity than the base fluid, nanofluids are regarded as better coolants particularly in nuclear reactors, domestic refrigerators, transportations, cancer therapy, microelectronic devices, lubricants and also thin film solar energy

collectors. A comprehensive literature on the nanofluid applications can be found in the works by Wong and Leon [2], Saidur et al. [3], Huminic and Huminic [4] and Colangelo et al. [5].

In view of above relevant applications, many researchers started to employ nanofluid as an alternative way to enhance the heat transfer efficacy. For instance, Buongiorno [6] established the non-homogeneous equilibrium model which is comprised of seven slip mechanisms; thermophoresis, Brownian diffusion, diffusiophoresis, inertia, gravity Magnus effect and fluid drainage. In this model, the Brownian movement and thermophoretic diffusion of nanoparticles are two notable effects that enhances the thermal conductivity of ordinary liquids. A year after the published work of Buongiorno [6], Tiwari and Das [7] proposed a homogeneous model which taking into account the effect of nanopartilces volume fractions. It is reported that the boundary layer flow of nanofluids over a stretching surface has been studied by Khan and Pop [8]. Makinde and Aziz [9] investigated the boundary layer flow of a nanofluid towards a stretching sheet with convective boundary conditions. Some relevant works on the homogeneous and non-homogeneous models can be seen in the references [10–17].

The study of chemical reactions has amazingly increased due to their wide range useful industrial and technological applications in polymer processing and electrochemistry. Such applications include chemical processing equipment, glass manufacturing, creation and distribution of fog, food processing, energy transfer in a wet cooling tower and evaporation at the surface of the water body [18–20]. The consideration of mixed convection flow past a vertical surface implanted in a porous medium carries species that are relatively soluble in the fluid. In fact, chemical reactions occur due to the presence of a foreign mass in a fluid. In many chemical reactions, the reaction rate relies on the concentration of the species itself. A chemical reactions between the conventional liquid and nanoparticles can be classified as a homogeneous reaction or heterogeneous reaction. Homogeneous reaction is a chemical reaction that occurs consistency in a single phase (gaseous, liquid, or solid). In addition, a heterogeneous reaction is a reaction that involves two or more phases (solid and gas, solid and liquid, two immiscible liquids) and takes place within the boundary of a phase. It is worth mentioning that a chemical reaction is said to be a first order reaction if the reaction rate is directly proportional to the concentration [21,22]. Some applications for the diffusion of species in the boundary layer flow include fibrous insulation, pollution studies and oxidation and synthesis materials. Furthermore, Mabood et al. [23] investigated the influence of chemical reaction on magnetohydrodynamics (MHD) stagnation point flow of nanofluid in porous medium by considering the additional effects of viscous dissipation and thermal radiation. Eid [24] analyzed the chemical reaction effect on MHD nanofluid flow past a stretching sheet with heat generation. It is noticed from his study that the presence of heat source and chemical reaction decrease the heat transfer rate and increase the mass transfer rate. The influence of chemical reaction and heat generation on mixed convection flow of a Casson nanofluid towards a permeable stretching sheet has been studied by Ibrahim et al. [25]. Inspired by the previous works, many authors have considered the chemical reaction effects on different flow concepts as can be seen in the references [26–29]. Very recently, Hayat et al. [30] discussed the mixed convection flow of Williamson nanofluid subject to chemical reaction.

Moreover, the boundary layer flow over a thin needle is of considerable importance in the biomedical and engineering purposes. For instance, it is commonly used in hot wire anemometer or protected thermocouple for calculating the wind velocity, transportations, circulatory problems and wire coating. The topic of thin needle seems very famous due to the movement of the needle that distracts the free-stream flow. This criterion is a primary point of the flow and heat transfer process to calculate the velocity and temperature distributions in experimental studies. Thin (or slender) needle is categorized as a rebellious body whose thicknesses are comparable to that of boundary layer or smaller. The boundary layer development adjacent to a thin needle in viscous fluid is first considered by Lee [31]. Narain and Uberoi [32] analyzed free and mixed convection flow along a thin needle. In extension to which, many works regarding slender needle in a viscous fluid are found in the existing literature [33–36]. Furthermore, the literature shows that researchers have also devoted

their attention to the study of boundary layer flow near a slender needle in nanofluid. These situations are caused by the usage of nanofluid that enhance the heat transfer rate. In 2011, the study of the forced convection flow with variable surface temperature over a slender needle has been done by Grosan and Pop [37]. A collection of the boundary layer flow over a thin needle with various physical effects in nanofluid can be found in the work by Trimbilas et al. [38], Hayat et al. [39], Soid et al. [40], Krishna et al. [41] and Ahmad et al. [42]. Very recently, Salleh et al. [43] performed the numerical analysis of magnetohydrodynamics flow over a moving vertical slender needle in nanofluid using Buongiorno's model with the revised boundary conditions.

Therefore, the novelty of the present work is to analyze the problem of the steady laminar nanofluid flow adjacent to a slender needle by considering the additional effects of chemical reaction and also heat source. Buongiorno's model is chosen in the simulation of the nanofluid. The system of nonlinear ordinary differential equations is computed numerically using bvp4c package in MATLAB software. The graphical results are presented and discussed for the varying effect of emerging parameters.

2. Governing Formula and Modeling

A steady nanofluid flow past a horizontal thin needle is examined. The geometry of the problem is illustrated in Figure 1 with u and v denoting x and r components of velocity, respectively, and $r = R(x) = (\nu cx/U)^{1/2}$ represents the needle radius. The needle is considered to move with uniform velocity U_w in the same or reverse direction of the external flow of constant velocity U_∞ with the composite velocity $U = U_w + U_\infty$. It is assumed that T_w and C_w are the constant wall temperature and nanoparticle concentration and as $r \rightarrow \infty$, the ambient temperature and nanoparticle fraction are T_∞ and C_∞ such that $T_w > T_\infty$ and $C_w > C_\infty$. In view of small needle size, the pressure gradient is ignored, however, the transverse curvature effect is required.

By using Buongiorno's nanofluid model, the relevant governing boundary layer systems for the flow are [6,42]

$$\frac{\partial}{\partial x}(ru) + \frac{\partial}{\partial r}(rv) = 0, \quad (1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} = \frac{\nu}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right), \quad (2)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial r} = \frac{\alpha}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \kappa \left[D_B \frac{\partial T}{\partial r} \frac{\partial C}{\partial r} + \frac{D_T}{T_\infty} \left(\frac{\partial T}{\partial r} \right)^2 \right] + \frac{Q^*}{\rho C_p} (T - T_\infty), \quad (3)$$

$$u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial r} = \frac{D_B}{r} \frac{\partial}{\partial r} \left(r \frac{\partial C}{\partial r} \right) + \frac{D_T}{T_\infty} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) - K^* (C - C_\infty). \quad (4)$$

The physical boundary restrictions are

$$\begin{aligned} u = U_w, \quad v = 0, \quad T = T_w, \quad C = C_w \quad \text{at} \quad r = R(x), \\ u \rightarrow U_\infty, \quad T \rightarrow T_\infty, \quad C \rightarrow C_\infty \quad \text{as} \quad r \rightarrow \infty, \end{aligned} \quad (5)$$

in which ν is kinematic viscosity, T is the temperature of nanofluid, C is the concentration of nanoparticles, α is the thermal diffusivity, ρ is the density, C_p is the heat capacity at uniform pressure, $\kappa = (\rho C_p)_s / (\rho C_p)_f$ is the proportion of effectual heat capacity of nanofluid in which subscripts 's' and 'f' refer to solid nanoparticle and base fluid, $Q^* = Q_0/x$ is the dimensionless heat generation, $K^* = K_0/x$ is the dimensionless reaction rate, Q_0 is the heat generation coefficient and K_0 is the chemical reaction coefficient. It is worth mentioning that the dimensionless parameters Q^* and K^* are the function of x and its value varies locally throughout the flow motion. Besides, D_B and D_T are Brownian and thermophoresis diffusion coefficients, respectively.

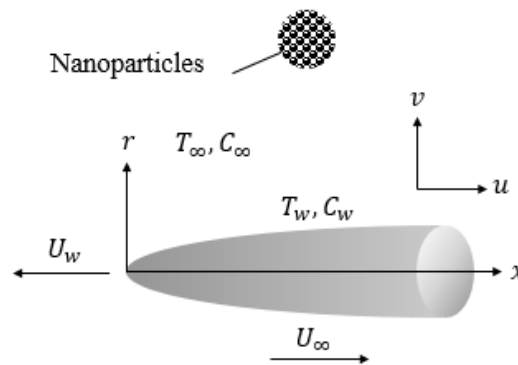


Figure 1. Schematic view of the present study.

The similarity transformation technique has been used for obtaining the ordinary differential equations. Hence, the following non-dimensional parameters are introduced

$$\psi = \nu x f(\eta), \quad \eta = \frac{U r^2}{\nu x}, \quad \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \quad \phi(\eta) = \frac{C - C_\infty}{C_w - C_\infty}, \quad (6)$$

where the stream functions are given as

$$u = r^{-1} \frac{\partial \psi}{\partial r}, \quad v = -r^{-1} \frac{\partial \psi}{\partial x}. \quad (7)$$

The stream functions (7) satisfies the continuity Equation (1). Using Equations (6) and (7), we obtain the following equations

$$2\eta f''' + 2f'' + f f'' = 0, \quad (8)$$

$$\frac{2}{Pr} (\eta \theta')' + f \theta' + 2\eta (Nt \theta'^2 + Nb \theta' \phi') + \frac{1}{2} Q \theta = 0, \quad (9)$$

$$2(\eta \phi')' + 2 \frac{Nt}{Nb} (\eta \theta')' + Le f \phi' - \frac{1}{2} Le K \phi = 0. \quad (10)$$

Also, the boundary condition could be rewritten as

$$f(c) = \frac{\varepsilon}{2} c, \quad f'(c) = \frac{\varepsilon}{2}, \quad \theta(c) = 1, \quad \phi(c) = 1,$$

$$f'(\eta) \rightarrow \frac{1}{2}(1 - \varepsilon), \quad \theta(\eta) \rightarrow 0, \quad \phi(\eta) \rightarrow 0 \quad \text{as } \eta \rightarrow \infty, \quad (11)$$

where prime denotes the differentiation with regard to similarity variable η . Besides, assume $\eta = c$ to represent size or thickness of the needle.

Here, Pr , Nt , Nb , Q , Le and ε represent the Prandtl number, thermophoresis parameter, Brownian motion parameter, heat generation parameter, Lewis number and velocity ratio parameter. K is the chemical reaction parameter with $K > 0$ represents a destructive reaction, and $K < 0$ represents generative reaction. These non-dimension parameters are defined as follows:

$$Pr = \frac{\nu}{\alpha}, \quad Nt = \frac{\kappa D_T (T_w - T_\infty)}{\nu T_\infty}, \quad Nb = \frac{\kappa D_B C_\infty}{\nu}, \quad Q = \frac{Q_0}{\rho C_p U}, \quad Le = \frac{\nu}{D_B},$$

$$K = \frac{K_0}{U}, \quad \varepsilon = \frac{U_w}{U}, \quad (12)$$

The skin friction coefficient C_f , local Nusselt number Nu_x and local Sherwood number Sh_x that relate to the shear stress, heat transfer rate and mass transfer rate are defined as

$$C_f = \frac{\mu}{\rho U^2} \left(\frac{\partial u}{\partial r} \right)_{r=c} = 4Re_x^{-1/2} c^{1/2} f''(c), \quad (13)$$

$$Nu_x = \frac{x}{(T_w - T_\infty)} \left(\frac{\partial T}{\partial r} \right)_{r=c} = -2Re_x^{1/2} c^{1/2} \theta'(c), \quad (14)$$

$$Sh_x = \frac{-x}{C_\infty} \left(\frac{\partial C}{\partial r} \right)_{r=c} = -2Re_x^{1/2} c^{1/2} \phi'(c). \quad (15)$$

where $Re_x = Ux/\nu$ is the local Reynold number.

3. Stability Analysis

The idea of the stability analysis came from Weidman et al. [44]. In their study, they noticed that there exists more than one solution called dual solutions. It is important to note that this analysis is introduced to determine which solution provides a good physical meaning to the flow (stable solution). Since we obtained the dual solutions, thus we are encouraged to determine which solutions are stable. To carry out this analysis, Equations (2)–(4) must be in unsteady case. Hence, the new dimensionless time variable is taken as $\tau = 2Ut/x$. Thus, we have

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} = \frac{\nu}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right), \quad (16)$$

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial r} = \frac{\alpha}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \kappa \left[D_B \frac{\partial T}{\partial r} \frac{\partial C}{\partial r} + \frac{D_T}{T_\infty} \left(\frac{\partial T}{\partial r} \right)^2 \right] + \frac{Q^*}{\rho C_p} (T - T_\infty), \quad (17)$$

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial r} = \frac{D_B}{r} \frac{\partial}{\partial r} \left(r \frac{\partial C}{\partial r} \right) + \frac{D_T}{T_\infty} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) - K^* (C - C_\infty), \quad (18)$$

and the new similarity transformations take the following form

$$\psi = \nu x f(\eta, \tau), \quad \eta = \frac{Ur^2}{\nu x}, \quad \theta(\eta, \tau) = \frac{T - T_\infty}{T_w - T_\infty}, \quad \phi(\eta, \tau) = \frac{C - C_\infty}{C_w - C_\infty}, \quad \tau = \frac{2Ut}{x}. \quad (19)$$

Please note that the use of τ is related to an initial value problem that is consistent with the solution that will be attained in practice (physically realizable). Afterwards, incorporating Equation (19) into Equations (16)–(18), we obtain

$$2 \frac{\partial}{\partial \eta} \left(\eta \frac{\partial^2 f}{\partial \eta^2} \right) + f \frac{\partial^2 f}{\partial \eta^2} - \frac{\partial^2 f}{\partial \eta \partial \tau} + \tau \frac{\partial f}{\partial \eta} \frac{\partial^2 f}{\partial \eta \partial \tau} - \tau \frac{\partial^2 f}{\partial \eta^2} \frac{\partial f}{\partial \tau} = 0, \quad (20)$$

$$\frac{2}{Pr} \frac{\partial}{\partial \eta} \left(\eta \frac{\partial \theta}{\partial \eta} \right) + f \frac{\partial \theta}{\partial \eta} + 2\eta \left[Nt \left(\frac{\partial \theta}{\partial \eta} \right)^2 + Nb \frac{\partial \theta}{\partial \eta} \frac{\partial \phi}{\partial \eta} \right] + \frac{1}{2} Q\theta - \frac{\partial \theta}{\partial \tau} + \tau \frac{\partial f}{\partial \eta} \frac{\partial \theta}{\partial \tau} - \tau \frac{\partial \theta}{\partial \eta} \frac{\partial f}{\partial \tau} = 0, \quad (21)$$

$$2 \frac{\partial}{\partial \eta} \left(\eta \frac{\partial \phi}{\partial \eta} \right) + 2 \frac{Nt}{Nb} \frac{\partial}{\partial \eta} \left(\eta \frac{\partial \theta}{\partial \eta} \right) + Le \left(f \frac{\partial \phi}{\partial \eta} - \frac{\partial \phi}{\partial \tau} \right) - \frac{1}{2} LeK\phi + Le\tau \frac{\partial f}{\partial \eta} \frac{\partial \phi}{\partial \tau} - Le\tau \frac{\partial \phi}{\partial \eta} \frac{\partial f}{\partial \tau} = 0, \quad (22)$$

together with the boundary conditions

$$f(c, \tau) = \frac{\varepsilon}{2} c + \tau \frac{\partial f}{\partial \tau}(c, \tau), \quad \frac{\partial f}{\partial \eta}(c, \tau) = \frac{\varepsilon}{2}, \quad \theta(c, \tau) = 1, \quad \phi(c, \tau) = 1, \\ \frac{\partial f}{\partial \eta}(\eta, \tau) \rightarrow \frac{1}{2}(1 - \varepsilon), \quad \theta(\eta, \tau) \rightarrow 0, \quad \phi(\eta, \tau) \rightarrow 0 \quad \text{as } \eta \rightarrow \infty. \quad (23)$$

Subsequently, we assume [44,45]

$$\begin{aligned} f(\eta, \tau) &= f_0(\eta) + e^{-\gamma\tau}F(\eta, \tau), \\ \theta(\eta, \tau) &= \theta_0(\eta) + e^{-\gamma\tau}G(\eta, \tau), \\ \phi(\eta, \tau) &= \phi_0(\eta) + e^{-\gamma\tau}H(\eta, \tau) \end{aligned} \quad (24)$$

in order to specify the stability of solutions $f = f_0(\eta)$, $\theta = \theta_0(\eta)$ and $\phi = \phi_0(\eta)$ which meets the boundary value problem (20)–(23). Also, functions $F(\eta, \tau)$, $G(\eta, \tau)$ and $H(\eta, \tau)$ represent small relative to $f_0(\eta)$, $\theta_0(\eta)$ and $\phi_0(\eta)$, respectively, and γ denotes an unknown eigenvalue parameter.

Then, introducing Equation (24) into Equations (20)–(23) yields the linear eigenvalue equations below:

$$2(\eta F_0'')' + f_0 F_0'' + f_0'' F_0 + \gamma F_0' = 0, \quad (25)$$

$$\frac{2}{Pr}(\eta G_0')' + f_0 G_0' + F_0 \theta_0' + 2\eta Nb(\theta_0' H_0' + \phi_0' G_0') + 4\eta Nt\theta_0' G_0' + \frac{1}{2}QG_0 + \gamma G_0 = 0, \quad (26)$$

$$2(\eta H_0')' + 2\frac{Nt}{Nb}(\eta G_0')' + Le(f_0 H_0' + F_0 \phi_0') - \frac{1}{2}LeKH_0 + Le\gamma H_0 = 0. \quad (27)$$

The corresponds boundary conditions for these equations are given by

$$F_0(c) = 0, \quad F_0'(c) = 0, \quad G_0(c) = 0, \quad H_0(c) = 0,$$

$$F_0'(\eta) \rightarrow 0, \quad G_0(\eta) \rightarrow 0, \quad H_0(\eta) \rightarrow 0 \quad \text{as} \quad \eta \rightarrow \infty. \quad (28)$$

Next, to identify an early growth or decomposition of the solution (24), we have to set $\tau = 0$. Hence, functions F , G and H can be expressed as $F_0(\eta)$, $G_0(\eta)$ and $H_0(\eta)$, respectively (see Weidman et al. [44] for the detail).

In the present work, we computed the numerical results for Equations (25)–(27) associated with conditions (28) by using the new boundary condition which is $F_0''(c) = 1$. This condition is obtained by relaxing the condition $F_0(\eta) \rightarrow 0$ as $\eta \rightarrow \infty$ (see [46] for details). It is to be noted that the flow will be stable if γ is positive, while the flow will be unstable if γ is negative.

4. Graphical Results and Discussion

In this section, the graphical outputs of our problem are interpreted for various effects of the involved parameters. All the computations have been carried out for a wide range of values of the governing parameters; $c(0.1 \leq c \leq 0.2)$, $\varepsilon(-4.3 \leq \varepsilon \leq 0.8)$, $K(-0.1 \leq K \leq 0.2)$, $Q(0 \leq Q \leq 0.4)$, $Nb(0.1 \leq Nb \leq 0.5)$, $Nt(0.1 \leq Nt \leq 0.5)$ and for a fixed values of $Pr = 2$ and $Le = 1$. Equations (8)–(10) along with the conditions (11) are computed numerically via bvp4c function that implemented in MATLAB software. Besides the shooting method, there is a new effective method for solving the boundary value problem for ordinary differential equations that is bvp4c package. Mathematically, this package uses the finite difference methods, in which the output is attained using an initial guess provided at the starting mesh point and resize the step to obtain the particular certainty. Nevertheless, to use this package, the boundary value problem must reduce to first order system of ordinary differential equations. To validate the accuracy of the present results, we have initially compared our results to those of Ahmad et al. [42] and Salleh et al. [43]. In this respect, Table 1 shows a comparison value of shear stress $f''(c)$ for $\varepsilon = Le = Q = K = 0$ for some of the thickness of the needle c when $Pr = 1$. An excellent agreement is observed in these studies.

The effect of needle thickness c on the velocity, temperature and concentration profiles are graphically presented in Figure 2a–c. It is noticed from the plots that the velocity, temperature and concentration profiles for upper branch solution increase with the increasing value of needle thickness. Similar observation is found for momentum, thermal and concentration boundary layer thicknesses for the upper branch as the c increase. Mathematically, the shape of graphs obtained in these profiles

has asymptote behaviors and it fulfills the requirement of boundary condition (11). One can see that an increment in the needle thickness decreases the numerical values of surface shear stress $f''(c)$, heat flux $-\theta'(c)$ and also mass flux $-\phi'(c)$. These phenomena are clearly shown in Figure 3a–c. This situation occurs due to an increase in the momentum, thermal and concentration boundary layer thicknesses on the surface, and consequently decline the shear stress and slow down the heat and mass transfers from the surface to the flow. Physically, the slender surface of the needle makes heat and mass to diffuse through it quickly compared to thick surface. In addition, the critical values of ε , by which the upper and lower branch solutions connected, are noticed to decrease as the needle thickness reduces. In other words, we can say that the needle thickness has a significant effect on the existence of the dual solutions.

Table 1. Comparison values of shear stress $f''(c)$ when $\varepsilon = Le = Q = K = 0$ for some of the thickness of the needle c when $Pr = 1$.

c	Ahmad et al. [42]	Salleh et al. [43]	Current Study
0.01	8.4924360	8.4924452	8.4924453
0.1	1.2888171	1.2888299	1.2888300
0.15	-	-	0.9383388
0.2	-	-	0.7515725

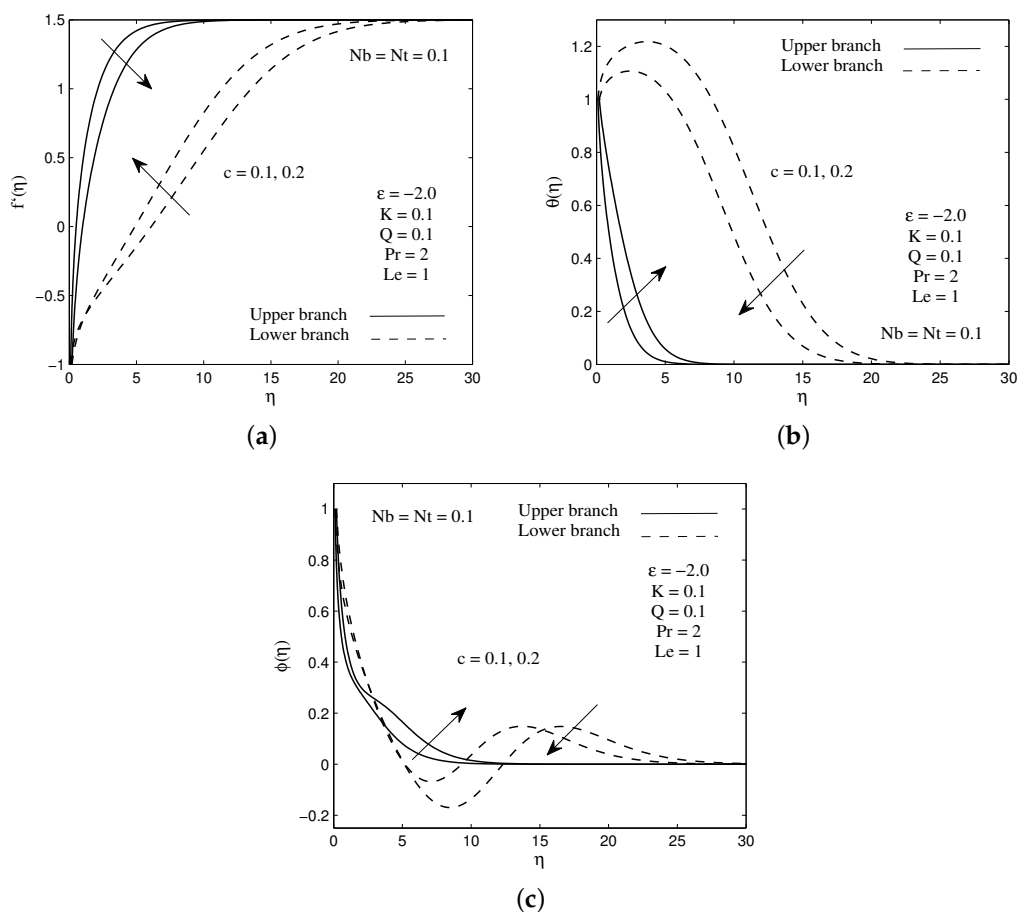


Figure 2. Sample of (a) velocity, (b) temperature and (c) concentration profiles for several values of needle thickness c .

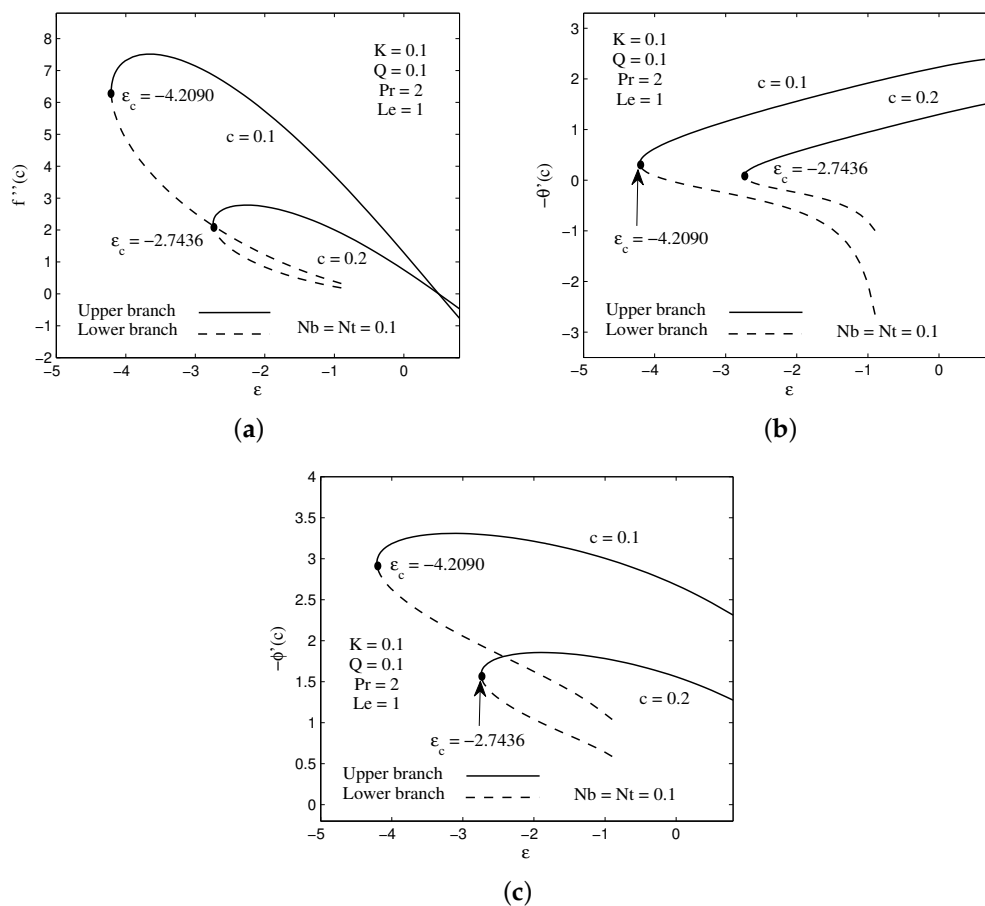


Figure 3. Variation of surface (a) shear stress, (b) local heat flux and (c) local mass flux with velocity ratio parameter ε for several values of needle thickness c .

The distributions of the temperature and concentration for several values of heat generation parameter Q are illustrated in Figure 4a–b when $K = 0.1$. It is worth noticing from Figure 4a that the fluid temperature enhances significantly within the thermal boundary layer for higher values of heat generation parameter. However, the opposite effect is observed for the fluid concentration as the heat generation parameter increases. This characteristic can be seen in Figure 4b. Figure 5a–b visualize the effect of the heat generation parameter Q on the local heat and mass fluxes with velocity ratio parameter ε . It is found from the figures that increasing values of Q reduce the heat flux at the surface, while an opposite criterion is observed for the mass flux. Generally, the presence of the heat generator produces a hot fluid layer near the needle surface due to mechanism of heat generation. As a consequence, the rate of heat transfer decreases from the needle surface to the fluid flow. In addition, the decrement of heat transfer is also due to an increment in the thermal boundary layer thickness as Q increases (see Figure 4a). Moreover, the existence of the hot fluid in the system will accelerate the motion of nanoparticles, and as the result increases the rate of mass transfer on the needle surface. It is worth mentioning that the existence of the dual solutions is noted when the needle moves against the free stream direction, $\varepsilon < 0$. In these variations, the presence of heat generation does not affect the flow. This statement can be proved by looking at Equation (8), where it does not contain the parameter Q inside.

The concentration distributions for some values of chemical reaction parameter K are plotted in Figure 6 when $Q = 0.1$. It is discerned from this figure that the fluid concentration is decreasing function of the chemical reaction parameter. It is important to know that the nanoparticle concentration as well as boundary layer thickness diminishes with the destructive chemical reaction, $K > 0$. Noteworthy, with the existence of destructive reaction, the change of the species as a cause of chemical

reaction will reduce the nanoparticle concentration in the boundary layer thickness. Other than that, Figure 7 elucidates the influence of chemical reaction parameter on the local mass flux. This figure explains that increasing the chemical reaction parameter K results in an increase in the mass transfer rate on the surface. The reason behind this is that the reduction of the concentration boundary layer thickness causes the mass transfer takes place quickly between the needle surface and the fluid flow. Since the chemical reaction parameter exists in Equation (10), thus we present only the result for the mass flux here.

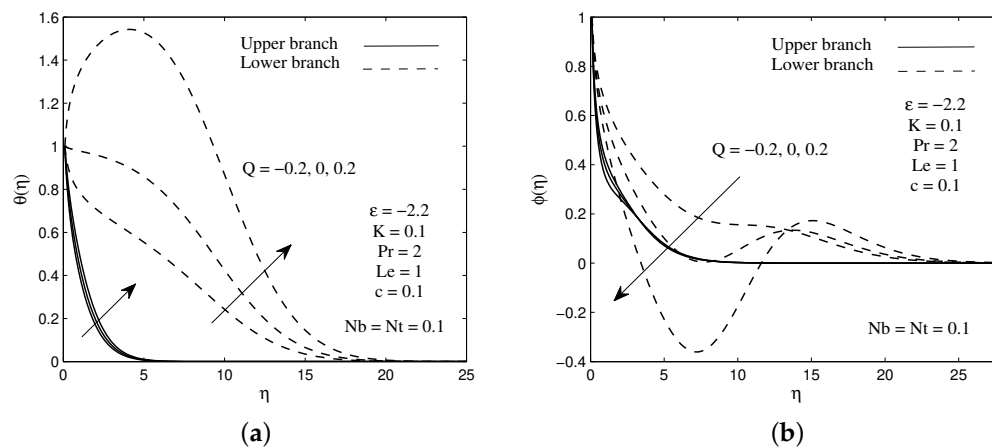


Figure 4. Sample of (a) temperature and (b) concentration profiles for several values of heat generation parameter Q .

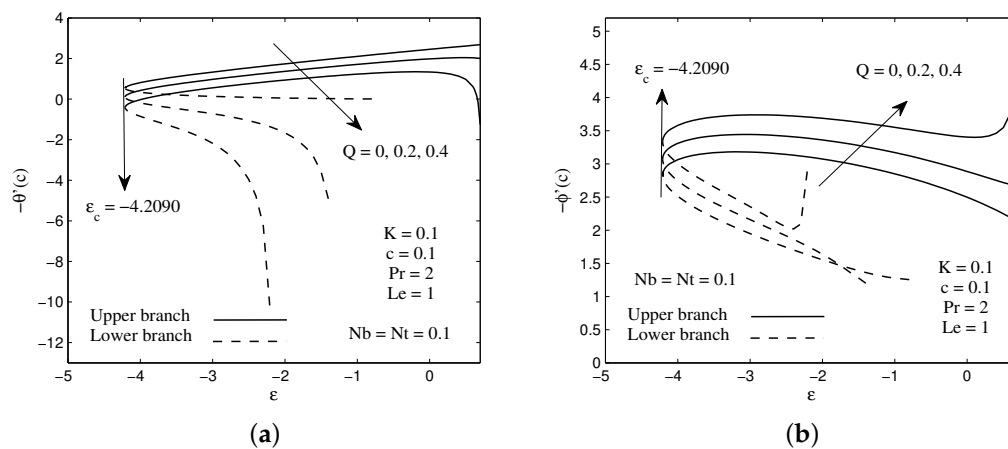


Figure 5. Variation of surface (a) local heat flux and (b) local mass flux with velocity ratio parameter ϵ for several values of heat generation parameter Q .

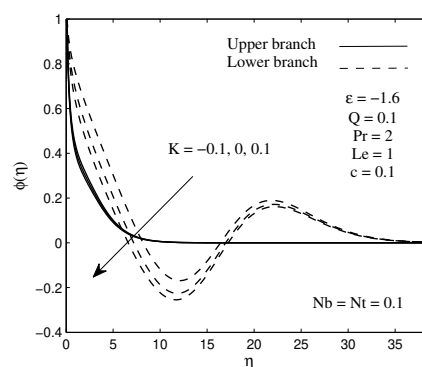


Figure 6. Sample of concentration profiles for several values of chemical reaction parameter K .

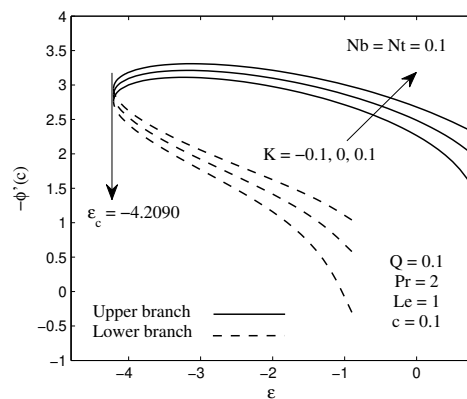


Figure 7. Variation of surface local mass flux for several values of chemical reaction parameter K .

The variations of the local Nusselt number $(Re_x)^{-1/2}Nu_x$ and local Sherwood number $(Re_x)^{-1/2}Sh_x$ with thermophoresis parameter Nt and Brownian motion parameter Nb are presented in Tables 2 and 3. Table 2 indicates that the higher values of Nt and Nb decrease the local Nusselt number or rate of heat transfer occurs on the needle surface. The same features can be seen as the heat generation parameter increases. It is quite clear from Figures 8a and 9a that the higher rate of Brownian motion and thermophoresis enhance the temperature of the fluid as well as the thermal boundary layer thickness. This increment in the boundary layer thickness minimizes the rate of heat transfer from the needle to the flow. Furthermore, Table 3 clarifies that the rate of mass transfer (or local Sherwood number) increases with an increase in thermophoresis and chemical reaction parameters. Noticeably, the higher value of the Brownian motion parameter tends to slow down the rate of mass transfer in the system. This happens due to the continuous collision of base fluid particles and nanoparticles which cause the random movement of those particles in the fluid. In addition, it can be observed in Figure 8b that the presence of higher value of Nb enhance the concentration profiles as well as the concentration boundary layer thickness. This criterion leads to the decrement in the mass transfer rate. In addition, as thermophoretic effect intensifies, nanoparticles with high thermal conductivity penetrate deeper in the fluid, hence, decreases the concentration boundary layer thickness as well as concentration profiles as can be seen in Figure 9b. This behavior leads to an increase in the local Sherwood Number.

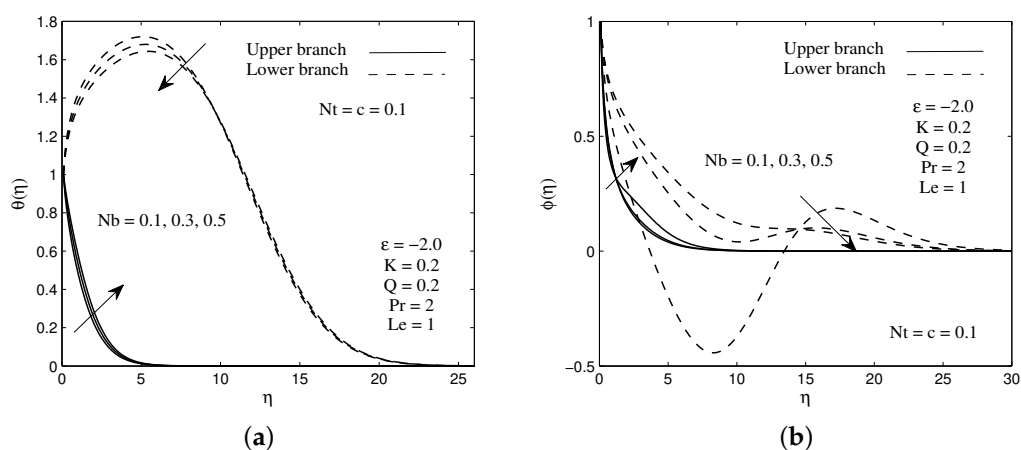


Figure 8. Sample of (a) temperature and (b) concentration profiles for several values of Brownian motion parameter Nb .

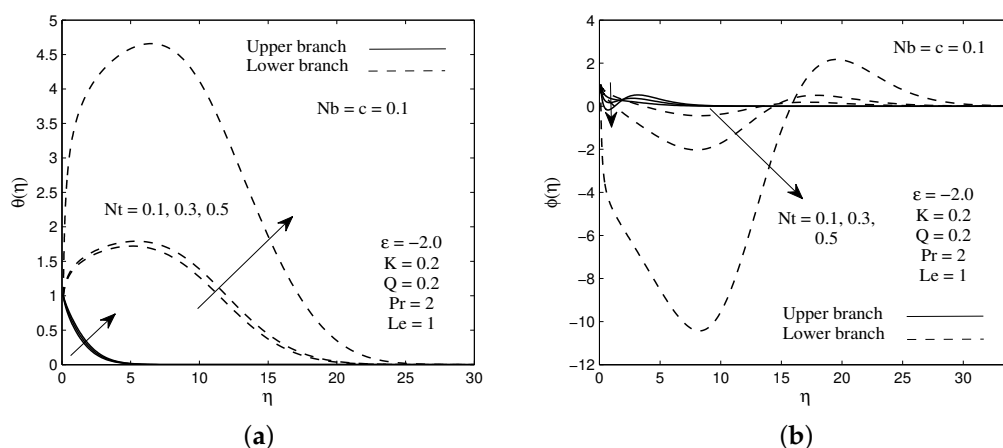


Figure 9. Sample of (a) temperature and (b) concentration profiles for several values of thermophoresis parameter Nt .

Table 2. Effects of thermophoresis parameter Nt and Brownian motion parameter Nb on the numerical values of local Nusselt number, $(Re_x)^{-1/2}Nu_x$ for $Q = 0.1$ and $Q = 0.2$ when $\varepsilon = -1.0$, $K = 0.2$, $c = 0.1$, $Pr = 2$ and $Le = 1$.

Heat Generation Parameter	Thermophoresis Parameter	$(Re_x)^{-1/2}Nu_x = -2c^{1/2}\theta'(c)$		
		$Nb = 0.1$	$Nb = 0.3$	$Nb = 0.5$
0.1	0.1	1.208880	0.959223	0.749362
	0.3	0.989170	0.777343	0.601318
	0.5	0.805833	0.627349	0.480535
0.2	0.1	1.078738	0.832762	0.628219
	0.3	0.863788	0.656596	0.486510
	0.5	0.685682	0.512462	0.371923

Table 3. Effects of thermophoresis parameter Nt and Brownian motion parameter Nb on the numerical values of local Sherwood number, $(Re_x)^{-1/2}Sh_x$ for $K = 0.1$ and $K = 0.2$ when $\varepsilon = -1.0$, $Q = 0.2$, $c = 0.1$, $Pr = 2$ and $Le = 1$.

Chemical Reaction Parameter	Thermophoresis Parameter	$(Re_x)^{-1/2}Sh_x = -2c^{1/2}\phi'(c)$		
		$Nb = 0.1$	$Nb = 0.3$	$Nb = 0.5$
0.1	0.1	2.005009	1.825444	1.781448
	0.3	3.271205	2.343083	2.135890
	0.5	5.055702	2.990494	2.546082
0.2	0.1	2.085015	1.898138	1.852426
	0.3	3.362939	2.418572	2.207592
	0.5	5.144680	3.063527	2.615317

Since this study has more than one solution, we need to verify which of the solutions obtained are physically relevant (stable solution) by solving Equations (25)–(28). The determination of the stable solution count on the sign of the smallest eigenvalue γ gained through this analysis. Table 4 presents the smallest eigenvalue γ for several values of chemical reaction, heat generation and velocity ratio parameters when $c = 0.1$ and $c = 0.2$. Table 4 indicates that the positive sign of γ for upper branch solution represents an initial decomposition of disturbance, while the negative sign of γ for lower branch solution represents an initial growth of disturbance in the system. Please note that the flow is said to be stable and physically relevant, if there is an initial decay of disturbance in the boundary layer separation. Otherwise, the flow is said to be unstable and not physically relevant.

Table 4. Smallest eigenvalues γ for several values of chemical reaction parameter K , heat generation parameter Q and velocity ratio parameter ε for $c = 0.1$ and $c = 0.2$ when $Nb = Nt = 0.1$, $Pr = 2$ and $Le = 1$.

$K = Q$	c	ε	Upper Branch	Lower Branch
0.1	0.1	−4.1994	0.0471	−0.0449
		−4.199	0.0481	−0.0458
		−4.19	0.0668	−0.0625
	0.2	−2.7424	0.0150	−0.0147
		−2.742	0.0175	−0.0170
		−2.74	0.0265	−0.0255
0.2	0.1	−4.1246	0.1444	−0.1254
		−4.124	0.1449	−0.1258
		−4.12	0.1484	−0.1284
	0.2	−2.7136	0.0793	−0.0706
		−2.713	0.0801	−0.0713
		−2.71	0.0841	−0.0744

5. Final Remarks

In this work, the numerical model is developed to study the boundary layer flow of two-phase nanofluid on a moving slender needle. The influences of chemical reaction and heat generation on the flow have been taken into consideration. The governing flow equations are solved and validated numerically by applying bvp4c package through MATLAB software. The key findings of this analysis can be summarized as follows:

- The heat generation parameter reduces the local heat flux as well as the rate of heat transfer.
- The presence of a chemical reaction increases the rate of mass transfer on the needle surface.
- The Brownian motion parameter diminishes the rate of heat and mass transfers from the needle surface to the flow.
- An increase in the thermophoresis parameter results in an increase in the mass transfer rate, while the reverse effect is noted for the heat transfer rate.
- An increment in the needle thickness leads to decrease the magnitudes of the surface shear stress, local heat flux and local mass flux.
- The dual solutions are likely to exist when the needle surface moves against the free-stream direction, $\varepsilon < 0$.
- The upper branch solution exhibits stable flow (or solution) and lower branch solution exhibits unstable flow.

Author Contributions: S.N.A.S. and N.B. designed the research; S.N.A.S. formulated the mathematical model and computed the numerical results; S.N.A.S. and N.B. analyzed the results; S.N.A.S. wrote the manuscript; S.N.A.S., N.B., N.M.A. and F.M.A. have read and approved this manuscript.

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Abbreviations

c	Needle size
C	Fluid concentration (kg m^{-3})
C_f	Skin friction coefficient
C_∞	Ambient nanoparticle volume fraction
C_w	Surface volume fraction
C_p	Specific heat at constant pressure
D_B	Brownian diffusion coefficient ($\text{m}^2 \text{s}^{-1}$)
D_T	Thermophoretic diffusion coefficient ($\text{m}^2 \text{s}^{-1}$)
f	Similarity function for velocity
K	Chemical reaction parameter
K_0	Chemical reaction coefficient
K^*	Dimensionless reaction rate
Le	Lewis number
Nb	Brownian motion parameter
Nt	Thermophoresis parameter
Nu_x	Local Nusselt number
Pr	Prandtl number
Q	Heat generation parameter
Q_0	Heat generation coefficient
Q^*	Dimensionless heat generation
r	Cartesian coordinate
Re_x	Local Reynolds number
Sh_x	Local Sherwood number
T	Fluid temperature (K)
T_w	Wall temperature (K)
T_∞	Ambient temperature (K)
U	Composite velocity (ms^{-1})
U_w	Wall velocity (ms^{-1})
U_∞	Ambient velocity (ms^{-1})
u	Velocity in x direction (ms^{-1})
v	Velocity in r direction (ms^{-1})
x	Cartesian coordinate
α	Thermal diffusivity ($\text{m}^2 \text{s}^{-1}$)
η	Similarity independent variable
θ	Dimensionless temperature
ε	Velocity ratio parameter
κ	Ratio of effective heat capacity of nanofluid
ρC_p	Volumetric heat capacity (J K^{-1})
ν	Kinematic viscosity ($\text{m}^2 \text{s}^{-1}$)
μ	Dynamic viscosity ($\text{kg m}^{-1} \text{s}^{-1}$)
ρ	Fluid density (kg m^{-3})
ϕ	Dimensionless solid volume fraction
w	Condition at the wall
∞	Ambient condition
$'$	Differentiative with respect to η

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