symmetry

MDPI

Article

Quantum Cosmologies Under Geometrical Unification of Gravity and Dark Energy

Carlos A. Rubio [1,†], Felipe A. Asenjo [2,*,†] and Sergio A. Hojman [3,4,5,†]

1 Facultad de Física, Pontificia Universidad Católica de Chile, Santiago 7820436, Chile
2 Facultad de Ingeniería y Ciencias, Universidad Adolfo Ibáñez, Santiago 7941169, Chile
3 Departamento de Ciencias, Facultad de Artes Liberales, Universidad Adolfo Ibáñez, Santiago 7941169, Chile
4 Departamento de Física, Facultad de Ciencias, Universidad de Chile, Santiago 7800003, Chile
5 Centro de Recursos Educativos Avanzados, CREA, Santiago 7500018, Chile
* Correspondence: felipe.asenjo@uai.cl
† These authors contributed equally to this work.

Received: 7 June 2019; Accepted: 26 June 2019; Published: 2 July 2019

Abstract: A Friedmann–Robertson–Walker Universe was studied with a dark energy component represented by a quintessence field. The Lagrangian for this system, hereafter called the Friedmann–Robertson–Walker–quintessence (FRWq) system, was presented. It was shown that the classical Lagrangian reproduces the usual two (second order) dynamical equations for the radius of the Universe and for the quintessence scalar field, as well as a (first order) constraint equation. Our approach naturally unified gravity and dark energy, as it was obtained that the Lagrangian and the equations of motion are those of a relativistic particle moving on a two-dimensional, conformally flat spacetime. The conformal metric factor was related to the dark energy scalar field potential. We proceeded to quantize the system in three different schemes. First, we assumed the Universe was a spinless particle (as it is common in literature), obtaining a quantum theory for a Universe described by the Klein–Gordon equation. Second, we pushed the quantization scheme further, assuming the Universe as a Dirac particle, and therefore constructing its corresponding Dirac and Majorana theories. With the different theories, we calculated the expected values for the scale factor of the Universe. They depend on the type of quantization scheme used. The differences between the Dirac and Majorana schemes are highlighted here. The implications of the different quantization procedures are discussed. Finally, the possible consequences for a multiverse theory of the Dirac and Majorana quantized Universe are briefly considered.

Keywords: quantum cosmology; quintessence; Dirac and Majorana quantization

1. Introduction

Quintessence is the name of one model put forward in order to explain the increment of the rate of expansion of the Universe. The model modifies the equations of General Relativity by adding a Lagrangian density for a massless scalar (quintessence) field rolling down a potential minimally-coupled to the usual Einstein–Hilbert Lagrangian density [1,2]. The equations of motion for the Friedmann–Robertson–Walker–quintessence (FRWq) system are obtained from Einstein's equations modified by the addition of the quintessence field. They consist of a set of two ordinary dynamical second order differential equations which govern the evolution of the dynamical variables (the radius of the universe and the scalar quintessence field) and one ordinary first order differential equation which constraints the initial values and velocities of the dynamical variables. This system has been studied extensively both in classical [1–3], as well as in quantum [4,5], cosmologies. Its importance is justified as this model is used to give an explanation on dark energy.

Several of the articles which deal with the subject, write down a Lagrangian formulation for the FRWq system. Nevertheless, to the best of our knowledge, all of the work published up to now is based on a classical Lagrangian, which gives rise to the two dynamical equations but does not yield the contraint equation [6]. In this work, we proposed a model for a FRWq system that could be studied as a relativistic pointlike particle. This would allow us to quantize the system, creating different quantum cosmological models. The first step in this model consisted of constructing a new Lagrangian without the previously mentioned problem. The Lagrangian presented in this article gives rise to two dynamical equations and one constraint equation [6]. We show that the FRWq system may be completely described in terms of a Lagrangian similar to that of a relativistic pointlike particle moving on a two-dimensional, conformally flat gravitational field. This is presented in Sections 2 and 3. The two-dimensional conformal factor is a function of the radius of the universe and of the scalar quintessence field, which naturally plays the role of coordinates in this two-dimensional mini-superspace. Hence, the relativistic particle description of the FRWq system is only possible with the merging of spacetime and quintessence.

The previous concepts and identifications between the classical cosmological spacetime and the quintessence field will lead us to a straightforward construction of a quantum theory in Section 4. Notice that this is only possible because of the unification achieved through the relativistic particle description. Quantizing the model as a spinless particle will give rise to a Klein–Gordon theory, which is a generalization of the Wheeler–DeWitt equation. Moreover, following Breit's prescription [7], we can quantize the model in a way analogous to that of a spin particle, producing Dirac and Majorana theories for the cosmological model. Both models are in agreement with the principle of manifest covariance, as the found quantum cosmology theories can be written in terms of four-tensor forms. Finally, in Section 5, different aspects of our models are discussed. The validity of the Dirac quantization scheme is highlighted in that section, thereby showing how the same theory can be obtained using different approaches. Also, a connection of this theory with Multiverses is discussed.

2. Lagrangian for FRWq System

In general, it is possible to write a Lagrangian density $\mathcal{L}$ for the evolution of the spacetime metric $g_{\mu\nu}(x^\alpha)$ in interaction with a massless scalar field $\phi(x^\beta)$, which may be identified with the quintessence field. The general Lagrangian is:

$$\mathcal{L} = \sqrt{-g}\left(\frac{R-2\Lambda}{2\mathcal{G}} + \mathcal{L}_\phi\right) , \tag{1}$$

where g stands for the determinant of the metric $g_{\mu\nu}$, $\mathcal{G} = 8\pi G/c^4$ (with G as the gravitational constant and c the speed of light), Λ is the cosmological constant, and $\mathcal{L}_\phi$ is the Lagrangian density for the massless scalar field:

$$\mathcal{L}_\phi = \epsilon\left(\frac{1}{2}g^{\mu\nu}\phi_{,\mu}\,\phi_{,\nu} - \mathcal{V}(\phi)\right) . \tag{2}$$

Here $\mathcal{V}(\phi)$ is an unspecified potential dependent on the scalar field ϕ. Also, ϵ is a parameter that determines the nature of the scalar field: $\epsilon = 1$ produces the Lagrangian density for usual scalar fields, while $\epsilon = -1$ defines the gravitational Lagrangian modified with the quintessence field [4].

As it is well known, the Lagrangian density $\mathcal{L}$ defined by Equation (1) is singular. The action S:

$$S = \int \mathcal{L}\, d^4x\,, \tag{3}$$

gives rise (upon variation with respect to the metric tensor $g_{\mu\nu}$) to gauge invariant (generally covariant) and constrained Einstein field equations coupled to matter, i.e.,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \mathcal{G}\, T_{\mu\nu}\,, \tag{4}$$

where $G^{\mu\nu}$ is the Einstein tensor, and $T^{\mu\nu}$ is the energy–momentum tensor of matter:

$$T_{\mu\nu} = g_{\mu\nu}\mathcal{L}_\phi - 2\frac{\delta\mathcal{L}_\phi}{\delta g^{\mu\nu}}\,. \tag{5}$$

Variation of the action S with respect to ϕ yields the Klein–Gordon equation for the massless scalar field:

$$\Box\phi + \frac{d\mathcal{V}(\phi)}{d\phi} = 0\,, \tag{6}$$

where $\Box$ is the d'Alembert operator in curved spacetimes.

In order to study a cosmological model with quintessence, let us take the line element for an isotropic and homogeneous FRW spacetime, with the metric defined by [8]:

$$ds^2 = -dt^2 + a(t)^2\left[\frac{d\rho^2}{1-k\rho^2} + \rho^2(d\theta^2 + \sin^2\theta d\phi^2)\right]\,, \tag{7}$$

where $a(t)$ is the scale factor of the studied cosmology, and ρ, θ and ϕ are spherical coordinates. The number k is the curvature constant, and it can take the values $k = -1, 0, 1$ measuring the negative, zero, or positive curvature of the Universe.

When Einstein equations coupled to matter are written in terms of the line element (7), and under the assumption that the scalar field ϕ depends on time only, we get two second order dynamical and one first order constraint. Setting $\mathcal{G} = 1$, the dynamical equations read:

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} - \epsilon\left(\frac{1}{2}\,\dot{\phi}^2 - V(\phi)\right) = 0, \tag{8}$$

and:

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \frac{dV(\phi)}{d\phi} = 0\,, \tag{9}$$

where we have introduced the new potential $V(\phi) = \mathcal{V}(\phi) - \epsilon\Lambda$. The constraint equation is:

$$3\left(\frac{\dot{a}}{a}\right)^2 + 3\frac{k}{a^2} + \epsilon\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) = 0\,. \tag{10}$$

Another useful equation can be obtained manipulating Equations (8) and (10) to get:

$$\frac{\ddot{a}}{a} = \frac{\epsilon}{3}\left(\dot{\phi}^2 - V\right)\,. \tag{11}$$

For $\epsilon = -1$, the set (8)–(11) becomes the FRWq system. This set of equations has been already studied and solved for quintessence by Capozziello and Roshan [3] for different scenarios and configurations of matter.

In this article, we explore the analogy of this system with a relativistic particle [6], which implies, as we will show, a geometric unification of gravity and quintessence fields. With this purpose in mind, we examine a Lagrangian L that gives rise to the dynamical Equations (8) and (9). This Lagrangian is:

$$L = 3a\dot{a}^2 - 3ka + \epsilon a^3\left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right)\,. \tag{12}$$

We emphasize that the Lagrangian (12) does not give rise to the constraint Equation (10). This constraint is equivalent to imposing the vanishing of the Hamiltonian H associated to L, i.e.,

$$H \equiv \frac{\partial L}{\partial\dot{a}}\dot{a} + \frac{\partial L}{\partial\dot{\phi}}\dot{\phi} - L = 0\,. \tag{13}$$

It is a remarkable fact that the change of variables:

$$r = \frac{2\sqrt{6}}{3}\, a^{3/2}\,, \qquad \theta = \frac{3}{2\sqrt{6}}\, \phi\,, \tag{14}$$

re-writte the Lagrangian L in a "kinetic energy minus potential energy $(T - V)$" form for the quintessence case (without considering that both T and V have the "wrong signs" in the θ associated terms)

$$L \to \bar{L} = \frac{1}{2}(\dot{r}^2 + \epsilon r^2 \dot{\theta}^2) - \bar{V}(r,\theta)\,, \tag{15}$$

where $\bar{V}(r,\theta)$ is a general potential:

$$\bar{V}(r,\theta) = 3\left(\frac{3}{8}\right)^{\frac{1}{3}} k\, r^{2/3} + \epsilon \frac{3}{8} r^2 V(\theta)\,. \tag{16}$$

Note that the Lagrangian $\bar{L}$ defined in Equation (15), describing the evolution of a FRWq Universe in the presence of geometry—represented either by r or by a—and dark energy—represented by $\epsilon = -1$, and either θ or ϕ—shows that the Universe evolves as a relativistic particle moving on a two-dimensional surface under the influence of the potential (16). Nevertheless, the Lagrangian (15) does not produce the constraint Equation (10). To construct a Lagrangian which gives rise to all three equations—Equations (8)–(10), it is enough to use the Jacobi–Maupertuis and Fermat principles. Those produce identical equations of motion in classical mechanics and geometrical (ray) optics, but the Fermat principle also produces a constraint equation [6,9]. We stress that exactly the same results are obtained for the case of a relativistic particle moving on a two-dimensional conformally flat spacetime.

3. Quintessence and Fermat-Like Lagrangian

From now on, we just focus in the case of $\epsilon = -1$, which describes the dark energy (quintessence) scenario [4]. The description of the FRWq system in terms of a Fermat-type Lagrangian is established by defining the relation between the potential $\bar{V}(r,\theta)$ and the ("refraction index") conformal factor $n(r,\theta)$:

$$\bar{V}(r,\theta) \equiv -\frac{1}{2}\,[n(r,\theta)]^2\,, \tag{17}$$

and the constraint:

$$\bar{H} \equiv \frac{\partial \bar{L}}{\partial \dot{r}}\dot{r} + \frac{\partial \bar{L}}{\partial \dot{\theta}}\dot{\theta} - \bar{L} = 0\,. \tag{18}$$

The Fermat–like Lagrangian L_F which gives rise to all three equations—Equations (8)–(10)—is:

$$L_F = n(r,\theta)\sqrt{\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\theta}{d\lambda}\right)^2}\,, \tag{19}$$

where λ is, in principle, an arbitrary parameter, and $n(r,\theta) = \sqrt{-2\bar{V}(r,\theta)}$, with $\bar{V}(r,\theta)$ defined in Equation (16) as:

$$\bar{V}(r,\theta) = 3\left(\frac{3}{8}\right)^{\frac{1}{3}} k\, r^{2/3} - \frac{3}{8} r^2 V(\theta)\,, \tag{20}$$

with $V(\theta) = \mathcal{V}(\theta) + \Lambda$. Thus, the Lagrangian (19) may be appropriately rewritten as:

$$L_F = \sqrt{-2\bar{V}(r,\theta)\left[\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\theta}{d\lambda}\right)^2\right]}\,. \tag{21}$$

To reproduce the relativistic equations of motion, λ is defined by Lüneburg's parameter choice [9,10]:

$$\sqrt{\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\theta}{d\lambda}\right)^2} = n(r,\theta)\,. \tag{22}$$

It is straightforward to prove that, varying the Lagrangian (21) with respect to r and θ, one gets Equations (8) and (9)—when rewritten in terms of r and θ—, while the constraint (10) arises because the Hamiltonian associated to a Lagrangian, which is a homogeneous function of degree one in the velocities, vanishes identically, much in the same way it happens in the Lagrangian description of the dynamics of a relativistic particle. This statement may equivalently be related to the eikonal equation in geometrical optics.

4. Quantization

As the FRWq dynamics may be described completely in terms of a Lagrangian similar to that of a relativistic particle, we can proceed further in our scheme to quantize this model described by FRWq Lagrangian. The quantization of cosmological Universes has been a prosperous field for decades. This field, called quantum cosmology, attempts to construct a quantum theory for the entire Universe. However, there is not a unique form to achieve this. Several possibilities have been carried out (for a comprehensive review, we refer the reader to Reference [5]). The most famous procedure corresponds to do a canonical quantization of the classical dynamical equations for the FRW Universe. This is done by replacing the momentum by a derivative operator on the scale factor variable. The final equation is known as the Wheeler–DeWitt equation [11,12]. This equation will depend on the main features we want to study of the Universe. Thus, quantum cosmologies have been studied for Universes with a dynamical vacuum in de Sitter cosmologies [13], in anti-de Sitter spacetimes [14], in radiation-dominated Universe [15], in Universes with cosmological constant [16,17], in $f(R)$ gravity [18,19], in conformal theory [20], and with a massive scalar field [21], among plenty of other works.

Below we will proceed quantizing in three different ways. First, we will use canonical quantization of the classical Lagrangian (21) modeling the Universe as a relativistic particle—producing a Klein–Gordon equation. This is the standard quantization for a relativistic point particle. Secondly, we quantize the FRWq Universe as a relativistic Dirac particle—Dirac or Majorana theories—given a proper physical justification for this procedure.

First, let us rewrite the Lagrangian (21) as:

$$L_F = \sqrt{\bar{V}(\xi,\theta)\, e^{2\xi}\left(\dot{\theta}^2 - \dot{\xi}^2\right)}\,, \tag{23}$$

where we have introduced the new variable $\xi = \ln r$, and $\bar{V} \equiv \bar{V}(\xi,\theta) = \bar{V}(r,\theta)$ with $\dot{\theta} = d\theta/d\lambda$, $\dot{\xi} = d\xi/d\lambda$. The previous Lagrangian coincides with the one for a relativistic particle on a two-dimensional, conformally flat spacetime. The corresponding conformal flat metric is:

$$g_{00} = \Omega^2\,,\ g^{00} = \frac{1}{\Omega^2}\,,\ g_{11} = -\Omega^2\,,\ g^{11} = -\frac{1}{\Omega^2}\,, \tag{24}$$

where:

$$\Omega \equiv \sqrt{\bar{V}} e^{\xi}\,, \tag{25}$$

such that the Lagrangian (23) is written as:

$$L_F = \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,, \tag{26}$$

with $x^0 = \theta$ and $x^1 = \xi$.

In order to avoid problems with the procedure of canonical quantization of the FRWq system, we restrict ourselves to the cases where $\bar{V} > 0$, and consider static manifolds only [22], where there exists a family of spacelike surfaces which are always orthogonal to a timelike Killing vector. This implies that $\partial_\theta g_{\mu\nu} = 0$, or:

$$\frac{\partial \bar{V}}{\partial \theta} = 0\,, \tag{27}$$

which means that the original potential $V(\theta)$ is a constant. Thereby, for the current quantization process, $V(\theta)$ is essentially equal to a cosmological constant. We have just restricted ourselves to the cases in which the FRWq Lagrangian is θ independent. The associated Noether conservation law forces that $\dot{\theta}$ cannot change sign (see Equation (9) for a ϕ independent potential). Thereby, θ may be used as the evolution (time) variable, in the form that the variational principle and quantization procedure suggest. Notice that the quintessence field has now the functionality of a super-time in this new description, where the particle evolves in an effective two-dimensional conformally flat spacetime.

Classically, it can be rigorously shown [23] that the Hamiltonian for the system described by Lagrangian (23) is:

$$H = \sqrt{g_{00}}\sqrt{1 - g^{11}\pi^2} = \sqrt{g_{00}}\sqrt{1 + \frac{\pi^2}{\Omega^2}}\,, \tag{28}$$

where π is the canonical momentum. We used this Hamiltonian to construct the quantum theory for the FRWq system. In order to avoid factor ordering issues, the quantum Hamiltonian operator $\mathcal{H}$ may be constructed from its classical analogue (28) as:

$$\mathcal{H} = \Omega^{1/2}\sqrt{1 + \hat{p}^2}\,\Omega^{1/2}\,, \tag{29}$$

where $\hat{p}$ is the momentum operator defined as

$$\hat{p} = \sqrt{-g^{11}}\,\hat{\pi} = \frac{\hat{\pi}}{\Omega} = -\frac{i}{\Omega}\frac{\partial}{\partial \xi}\,, \tag{30}$$

because of $\hat{\pi} = -i\partial_\xi$. In this way, the quantum equation that describe the quantization of the FRWq system is:

$$i\hbar\frac{\partial \Psi}{\partial \theta} = \mathcal{H}\Psi\,, \tag{31}$$

where Ψ is the wavefunction of the FRWq system.

In principle, one may ask whether there are ways to construct other Hamiltonian operators that differ from Equation (29), giving rise to quantum theories which are not equivalent to the one described by Equation (31) (see, for instance [24]). This point is subtle, and the answer is affirmative; however, the operator (29) has the advantage in that it reproduces results from quantum field theory in curved spacetimes, as we will see below. In what follows, we proceed to quantize the FRWq theory in two majorly different ways. In the first case, we quantize the system using a procedure devised for spinless particles. This approach produces a Klein–Gordon equation for the wavefunction of the FRWq Universe. The second case corresponds to the quantization of the FRWq system as a Dirac particle.

4.1. Quantization of the FRWq System as a Klein–Gordon Particle

One way to canonically quantize the relativistic spinless particle can be obtained following the method developed by Gavrilov and Gitman [25]. This procedure is a consistent way to construct the quantum theory along Dirac's theory for gauge and constrained systems [26,27]. We will not reproduce the calculations for this quantization scheme here, but we limit ourselves to exhibit the results of applying this method. The quantization for the FRWq system produces the quantum equation [25]:

$$i\partial_\theta \Psi = \hat{h}\Psi\,, \tag{32}$$

(taking $\hbar = 1$ for convenience) where Ψ is the spinor:

$$\Psi = \begin{pmatrix} \chi \\ \psi \end{pmatrix}, \tag{33}$$

and $\hat{h}$ is a matrix Hamiltonian:

$$\hat{h} = \begin{pmatrix} 0 & -\partial_\xi^2 + \Omega^2 \\ 1 & 0 \end{pmatrix}. \tag{34}$$

Let us notice that these are not two dynamical equations for a spinor, but Equation (32) will produce the constraint $i\partial_\theta \psi = \chi$. Therefore, Equation (32) gives rise to a Klein–Gordon equation:

$$0 = \frac{\partial^2 \psi}{\partial \theta^2} - \frac{\partial^2 \psi}{\partial \xi^2} + \Omega^2 \psi, \tag{35}$$

that it also can be written as:

$$0 = g^{\mu\nu}\partial_\mu \partial_\nu \psi + \psi, \tag{36}$$

with the notation from previous section, Equations (24) and (25), $x^0 = \theta$ and $x^1 = \xi$. The wavefunction ψ represents the probability amplitude obtained by using the quantization of the FRWq Universe system as a Klein–Gordon particle. It obeys the same equation of the Klein–Gordon field in Minkowski space but now with an effective mass term whose origin is the conformal metric. Equation (35)—or (36)—is consistent with the principle of manifest covariance, as this quantum theory emerges from general relativity. A similar result is obtained when a scalar field is quantized in an expanding curved spacetime background [28], obtaining a mass-corrected term due to a conformal time. However, we must emphasize that our treatment is different because, in our approach, it is the spacetime itself that is quantized.

The conserved Klein–Gordon probability density is not positive definitive. For the case at hand, one can calculate, from Equation (35), the probability density ρ_ψ for the Klein–Gordon field as:

$$\rho_\psi = \int_0^{a_f} \frac{da}{ia} \left(\psi^* \frac{\partial \psi}{\partial \theta} - \frac{\partial \psi^*}{\partial \theta} \psi \right), \tag{37}$$

where ψ^* is the complex conjugated of the wavefunction (41), and a_f is some value of the scale factor that can be equal or larger than its present value. We notice that the probability density depends on θ. In the same way, for our case, the expected value of the scale factor is [29]:

$$\langle a(\theta) \rangle = \frac{1}{|\psi|^2} \int_0^{a_f} \frac{da}{i} \left(\psi^* \frac{\partial \psi}{\partial \theta} - \frac{\partial \psi^*}{\partial \theta} \psi \right), \tag{38}$$

which depends on θ, as well.

We can solve Equation (35) exactly for $k = 0$ and any constant V. Assuming the dependence $\psi(\xi) = \phi(\xi)\, e^{iE\theta}$, for a constant parameter E, the previous equation becomes:

$$\frac{\partial^2 \phi}{\partial \xi^2} = \left(\Omega^2 - E^2 \right) \phi. \tag{39}$$

For a spatially flat Universe $k = 0$, where $\Omega^2 = -3Ve^{4\xi}/8$—being $V = V(\theta)$ a constant related to the cosmological constant—, the general solution is found to be:

$$\begin{aligned} \phi(\xi) &= C_1 J_{-iE/2}\left(\frac{e^{2\xi}}{4}\sqrt{\frac{3V}{2}} \right) \Gamma\left(1 - \frac{iE}{2}\right) \\ &+ C_2 J_{iE/2}\left(\frac{e^{2\xi}}{4}\sqrt{\frac{3V}{2}} \right) \Gamma\left(1 + \frac{iE}{2}\right), \end{aligned} \tag{40}$$

where Γ is the Euler gamma function, and J_n is the modified Bessel function of the first kind of order n. By appropiate choice of constants C_1 and C_2, the wavefunction ϕ can be real.

To find solutions for $k \neq 0$ for the wavefunction ψ, we proceed assuming that the wavefunction can be decomposed as:

$$\psi(\xi,\theta) = \psi(a,\theta) = \sum_{n=0}^{\infty} \psi_n(a)\, e^{iE_n\theta}\,, \tag{41}$$

where the reason to go back to the a-representation in the variables will be clear in the following. Here, E_n is a parameter that, as we will see below, can be associated with the energy of the n-state of the Klein–Gordon field ψ_n. Using this decomposition, we can find a suitable equation to be solved. First, let us define the field $u_n(a) = \sqrt{a}\,\psi_n(a)$. This new field follows the differential equation:

$$-\frac{d^2 u_n}{da^2} + V_{\text{eff}}(a)\, u_n = \frac{9E_n^2}{4a^2} u_n\,, \tag{42}$$

where we have introduced the effective potential:

$$V_{\text{eff}}(a) = \frac{9\Omega^2 - 1}{4a^2}\,, \tag{43}$$

that depends on the value of k. Equation (42) can be converted in the following Ricatti equation:

$$-i\frac{d\mu_n}{da} + \mu_n^2 + \left(V_{\text{eff}} - \frac{9E_n^2}{4a^2}\right) = 0\,, \tag{44}$$

where:

$$\mu_n(a) = -i\frac{d}{da}\ln u_n\,. \tag{45}$$

The above Ricatti equation can be generally solved if a particular solution can be found for any k.

This is not a trivial task. An approximated solution of Equation (42) for the field u_n can be obtained using the Spectral Method (SM) [30,31] in the expansion (41). This method is usually utilized in similar quantum theories for the Universe [17]. As the wavefunction of the Universe must vanish at the origin, as well as in infinity, the SM uses the approximation that the wavefunction vanishes at some length L (as large as we require). Thus, the SM allows us to expand the wavefunction u_n in a Fourier series:

$$u_n(a) \approx \sum_{m=1}^{N} A_m^{(n)} \sin\left(\frac{m\pi}{L}a\right)\,, \tag{46}$$

where $A_m^{(n)}$ are constant coefficients that depends on the n-state. Notice that this expansion implies, from Equation (41), that the wavefunction $\psi(a \to 0, \theta) \to 0$, which is the desired behavior. Here, N is a number that can be chosen arbitrarily. The approximation improves as N increases. According to the SM, we can expand also the following functions:

$$\begin{aligned} V_{\text{eff}}(a)\, u_n(a) &\approx \sum_{m=1}^{N} B_m^{(n)} \sin\left(\frac{m\pi}{L}a\right)\,, \\ \frac{9}{4a^2} u_n(a) &\approx \sum_{m=1}^{N} C_m^{(n)} \sin\left(\frac{m\pi}{L}a\right)\,, \end{aligned} \tag{47}$$

where again $B_m^{(n)}$ and $C_m^{(n)}$ are coefficients depending on the n-state. It is straightforward to find the relation between $B_m^{(n)}$ and $C_m^{(n)}$ with $A_m^{(n)}$. Using Equation (46) in Equation (47), we get that $B_m^{(n)} = \sum_{l=1}^{N} D_{ml} A_l^{(n)}$, and $C_m^{(n)} = \sum_{l=1}^{N} F_{ml} A_l^{(n)}$, where:

$$
\begin{aligned}
D_{ml} &= \frac{2}{L}\int_0^L \sin\left(\frac{m\pi}{L}a\right) V_{\text{eff}}(a) \sin\left(\frac{l\pi}{L}a\right) da\,, \\
F_{ml} &= \frac{9}{2L}\int_0^L \sin\left(\frac{m\pi}{L}a\right) \frac{1}{a^2} \sin\left(\frac{l\pi}{L}a\right) da\,.
\end{aligned} \tag{48}
$$

Now, with the expansion in Equations (46) and (47) in Equation (42), we can finally obtain the eigenvector equation:

$$
\mathbb{F}^{-1}\cdot\mathbb{K}\cdot\mathbb{A}^{(n)} = E_n^2\mathbb{A}^{(n)}\,, \tag{49}
$$

for the $\mathbb{A}^{(n)}$ vector (with N components $A_l^{(n)}$) and where E_n corresponds to the eigenvalues. Here, $\mathbb{F}^{-1}$ is the $N\times N$ inverse matrix of $\mathbb{F}$ (with components F_{ml}), and the $N\times N$ matrix $\mathbb{K}$ has the components $D_{ml}+(m\pi/L)^2\delta_{ml}$. The dimensions of the matrices $\mathbb{F}$ and $\mathbb{K}$ will be fixed once a cut on the series Equation (46) in Equation (47) will be done, while the better the approximation, the larger will be the matrices. Thus, the system is completed solved. The values of E_n correspond to the energies of the different possible states of the Universe. This also allows us to identify the variable θ as a super-time, as it was previously discussed.

If one is being less restrictive with the assumption that the potential $V(\theta)$ is constant, Equation (35) can be written in the form of the Wheeler–DeWitt Super-Hamiltonian formalism [12,21]. For example, for a closed universe $k=1$—and the case quinteessence $\epsilon=-1$—, it is possible to rewrite Equation (35) as:

$$
\frac{\partial^2\psi}{\partial\mathcal{A}^2}-\frac{\partial^2\psi}{\partial\varphi^2}+\left(\hat{m}^2\varphi^2e^{6\alpha}-e^{4\alpha}\right)\psi\equiv\mathcal{H}\psi=0, \tag{50}
$$

where $\mathcal{A}=\ln a$, and $\varphi=2\theta/3$. To obtain this equation, we chose $\hat{m}^2=1/18$ and $V(\theta)=3\hat{m}^2\varphi^2$, where $\hat{m}$ is the mass of the field. Here, $\mathcal{H}$ is usually called the Wheeler–DeWitt Super-Hamiltonian [12,21]. Thus, the quantization of the FRWq system as a Klein–Gordon particle proposed here can reproduce known results of quantization using the Super-Hamiltonian formalism.

On the other hand, another possible solution of Equation (42) could be achieved using the Frobenius (polynomial) method, which is different from the SM. This solution corresponds to a polynomial expansion in a, where all the coefficients can be found from recurrence relations. A polynomial solution for u_n has the form:

$$
u_n(a)\approx a^y\sum_{m=0}^{\infty}b_m^{(n)}a^m\,, \tag{51}
$$

where $y>0$ and $b_m^{(n)}$ are constants. For the sake of simplicity, we choose $b_0^{(n)}=1$. Using the previous expansion in Equation (42) we find:

$$
\begin{aligned}
&\sum_{m=0}^{\infty}\left[(m+y)(m+y-1)+\frac{9E_n^2+1}{4}\right]b_m^{(n)}a^{m+y-2} \\
&-\sum_{m=4}^{\infty}18k\,b_{m-4}^{(n)}a^{m+y-2}+\sum_{m=6}^{\infty}6V\,b_{m-6}^{(n)}a^{m+y-2}=0\,,
\end{aligned} \tag{52}
$$

where V is defined after Equation (9). Equating every term to zero, we can readily find the solutions:

$$
\begin{aligned}
y &= \frac{1}{2} \pm \frac{3i}{2} E_n\,, \\
b_4^{(n)} &= \frac{72k}{(3iE_n + 9)\,(3iE_n + 7)}\,, \\
b_6^{(n)} &= \frac{-24V}{(3iE_n + 13)\,(3iE_n + 11)}\,, \\
b_8^{(n)} &= \frac{72k}{(3iE_n + 17)\,(3iE_n + 15)}\,,
\end{aligned} \tag{53}
$$

with also $b_2^{(n)} = 0$, $b_{2m+1}^{(n)} = 0$, and the recurrence relation for $m \geq 5$ is:

$$
b_{2m}^{(n)} = \frac{72kb_{2m-4}^{(n)} - 24Vb_{2m-6}^{(n)}}{(3iE_n + 4m + 1)\,(3iE_n + 4m - 1)}\,, \tag{54}
$$

where now the problem is completely solved. The weakness of this method is that it does not give any physical meaning to the constant E_n in the ansatz (41).

4.2. Quantization of the FRWq System à la Dirac-Pauli

Basically, the quantization process proposed here consists in finding the square root of the Hamiltonian operator (29). In principle, one can use matrices to find the square root, but its use implies the notion that the cosmological model behaves as a Dirac particle. At a first glance, it may appear strange to quantize a model for a relativistic pointlike particle with a quantum spin theory. However, in 1928, Breit [7] showed that there exists a correspondence between the Dirac and the relativistic pointlike particle Hamiltonians. In that work, it is shown that one can obtain the Dirac equation via a prescription of replacement of the particle velocity and the Dirac matrices, as well as the prescription in Schrödinger or Klein–Gordon theories where the energy and momentum and can be replaced by the time and space derivatives. Thus, Breit's prescription implies a classical and geometrical interpretation of the spin. We leave the calculations and the deep discussion of this idea to Section 5. For now, in this section, we restrict ourselves to follow Breit's interepretation and perform the quantization of the FRWq Universe using Dirac matrices.

We propose that the Hamiltonian (29) can be written using Dirac matrices (α and β). This will give us the Hamiltonian operator:

$$
\mathcal{H} = \Omega^{1/2}\,(\alpha \cdot \hat{p} + \beta)\,\Omega^{1/2}\,. \tag{55}
$$

In Section 5, we justify this choice. This Hamiltonian allows us to quantize a FRWq Universe as if it were a relativistic spin particle. Using the operator (55), the quantum mechanical Equation (31) now reads:

$$
i\frac{\partial \Psi}{\partial \theta} = -i\alpha^{\xi}\left[\frac{\partial}{\partial \xi} + \frac{1}{2}\frac{\partial \ln \Omega}{\partial \xi}\right]\Psi + \beta\Omega\Psi\,, \tag{56}
$$

where Ψ now is a bi-spinor. Here, α^{ξ} stands for any of the Dirac matrices α. Choosing the Dirac representation $\gamma^0 = \beta$ and $\gamma^{\xi} = \gamma^0\alpha^{\xi}$, we can operate Equation (56) by γ^0 by the left to find that:

$$
i\gamma^0\frac{\partial \Psi}{\partial \theta} + i\gamma^{\xi}\left(\frac{\partial}{\partial \xi} + \frac{1}{2}\frac{\partial \ln \Omega}{\partial \xi}\right)\Psi = \Omega\Psi\,. \tag{57}
$$

This equation describes the quantum theory for the FRWq Universe modeled as a Dirac particle. Also, it can be obtained directly from the theory of the Dirac equation in curved spacetimes, thus giving validity to our quantization scheme (shown in Appendix A).

Dirac matrices are 4×4, and as we are describing a two-dimensional conformal system, we may anticipate that the above equation is reducible. This means that Dirac Equation (57) couples the wavefunction in pairs, implying that the two pairs of wavefunctions satisfy the same equation. This gives us a hint that a completely equivalent quantization formalism to the previous one can be achieved using Pauli matrices. Solving the Hamiltonian (29) using Pauli matrices—notice that there is no restriction to this ansatz—, the Hamiltonian operator (29) can be written as:

$$\mathcal{H} = \Omega^{1/2} \left(\sigma_x + \sigma_y \hat{p} \right) \Omega^{1/2} \,, \tag{58}$$

where σ_x and σ_y can be any two different Pauli matrices. The quantum mechanical equation (31) describing the FRWq system becomes (setting $\hbar = 1$):

$$i\mathbf{1}\frac{\partial \Psi}{\partial \theta} = \Omega \, \sigma_x \Psi - i\sigma_y \left[\frac{\partial}{\partial \xi} + \frac{1}{2}\frac{\partial \ln \Omega}{\partial \xi} \right] \Psi \,, \tag{59}$$

where $\mathbf{1}$ is the unit matrix, and Ψ represents a spinor field. We can use the Pauli matrices properties to put the previous equation in the following form:

$$i\sigma_x \frac{\partial \Psi}{\partial \theta} - \sigma_z \left[\frac{\partial}{\partial \xi} + \frac{1}{2}\frac{\partial \ln \Omega}{\partial \xi} \right] \Psi = \Omega \, \Psi \,, \tag{60}$$

where $i\sigma_z = \sigma_x \sigma_y$. It can be proven that choosing $\sigma_x = \sigma_3$ and $\sigma_z = \sigma_2$ gives the same dynamical equation as choosing $\gamma^\xi = \gamma^1$ in the Dirac equation.

On the other hand, from Equation (57), we notice a that defining the bi-spinor $\Psi' = \sqrt{\Omega}\Psi$, we can obtain the equation:

$$i \left(\gamma^0 \frac{\partial}{\partial \theta} + \gamma^\xi \frac{\partial}{\partial \xi} \right) \Psi' = \Omega \Psi' \,, \tag{61}$$

which is a flat 1+1 spacetime massless Dirac equation with a scalar potential. The principle of manifest covariance of this quantum cosmology model can be explicitly seen here, as we could have re-written the previous equation as:

$$i\gamma^\mu \partial_\mu \Psi' = \Omega \Psi' \,. \tag{62}$$

These kinds of equations have been extensively studied and approximated solutions have been found [32–35].

Finally, using the wavefunction Ψ (given by solving either the Dirac or Pauli equations), we can calculate the probability density of the Dirac field as:

$$|\Psi(\theta)|^2 = \int_0^{a_f} \frac{da}{a} \Omega \, \Psi^\dagger \Psi \,, \tag{63}$$

where $\Psi^\dagger$ is the transpose conjugated of the wavefunction Ψ. In the previous expression, Ψ and $\Psi^\dagger$ are written in terms of a. In a similar fashion, the expected value of the scale factor for the Universe under the Dirac quantization is obtained:

$$\langle a(\theta) \rangle = \frac{1}{|\Psi|^2} \int_0^{a_f} da \, \Omega \, \Psi^\dagger \Psi \,, \tag{64}$$

depending again on the values of super-time θ.

To further study the system, let us do the bi-spinor descomposition:

$$\Psi(\theta,\xi) = \sum_{n=0}^{\infty} e^{iE_n\theta} \begin{pmatrix} \psi_n(\xi) \\ \zeta_n(\xi) \\ \varphi_n(\xi) \\ \chi_n(\xi) \end{pmatrix}, \tag{65}$$

where E_n are constants. Using Equation (65) in Equation (57), gives:

$$\begin{aligned} 0 &= \frac{d\chi_n}{d\xi} + \frac{1}{2\Omega}\frac{d\Omega}{d\xi}\chi_n + i(E_n+\Omega)\psi_n\,, \\ 0 &= \frac{d\psi_n}{d\xi} + \frac{1}{2\Omega}\frac{d\Omega}{d\xi}\psi_n + i(E_n-\Omega)\chi_n\,, \end{aligned} \tag{66}$$

where we have made the particular choice of the Dirac matrix:

$$\gamma^{\xi} = \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{67}$$

The same mathematical equations can be obtained for the fields ζ_n and φ_n, reflecting that the system can also be studied using Pauli matrices. Now the fields ψ_n and χ_n appear coupled. From those equations, it is impossible to recover the Klein–Gordon Equation (39). The reason is that wavefunctions are coupled to the spacetime metric through the potential due to the quintessence field.

Let us notice that a simple exact solution of Equation (66) can be found when the fields do not depend on quintessence, i.e., when $E_n = 0$. In this case, the solutions are:

$$\psi_n(\xi) = \frac{i}{\sqrt{\Omega}}\exp\left(\Omega_I\right) = i\chi_n(\xi)\,, \tag{68}$$

where we define:

$$\Omega_I(\xi) \equiv \left(\frac{1}{2} - 2\frac{3^{1/3}k}{V}e^{-4\xi/3}\right)\sqrt{\frac{3^{4/3}k}{2}e^{8\xi/3} - \frac{3Ve^{4\xi}}{8}}\,. \tag{69}$$

However, these solutions are not well-behaved at $\xi \to -\infty$ ($a \to 0$) as it diverges. Therefore, we will seek solutions with $E_n \neq 0$.

A more general solution can be obtained in the following way. Let us define $\phi_n^+(\xi) = \sqrt{\Omega}\,\psi_n(\xi)$, and $\phi_n^-(\xi) = \sqrt{\Omega}\,\chi_n(\xi)$. Thus, the Equation (66) can be re-written as:

$$\frac{d\phi_n^{\pm}}{d\xi} = \Omega_{\mp}\phi_n^{\mp}\,, \tag{70}$$

where $\Omega_{\pm} = \pm i\left(E_n \pm \Omega\right)$, such that $\Omega_+\Omega_- = E_n^2 - \Omega^2$. The previous equations are coupled, but we can find the following second-order equation, which holds for each of the fields:

$$\left(-\frac{d^2}{d\xi^2} + \frac{1}{\Omega_{\mp}}\frac{d\Omega^{\mp}}{d\xi}\frac{d}{d\xi} + \Omega_+\Omega_-\right)\phi_n^{\pm} = 0\,. \tag{71}$$

The above equation can be reduced to familiar expressions doing the change $\phi_n^{\pm} = \sqrt{\Omega_{\mp}}\exp\left(i\int \mu_{\pm}(\xi')d\xi'\right)$. The equation for μ is reduced to a Ricatti equation:

$$\frac{d\mu_{\pm}}{d\xi} + i\mu_{\pm}^2 - iV_{\pm} = 0\,, \tag{72}$$

with:

$$V_{\pm} = -\frac{3}{4\Omega_{\mp}^2}\left(\frac{d\Omega_{\mp}}{d\xi}\right)^2 + \frac{1}{2\Omega_{\mp}}\frac{d^2\Omega_{\mp}}{d\xi^2} - \Omega_{-}\Omega_{+}\,. \tag{73}$$

A general solution of the Ricatti Equation (72) can be found if we are able to find a particular solution.

On the other hand, when the solutions depend on the quintessence field, we can use the SM to completely solve the Dirac Equation (66), as well as in the previous section. As it was shown before, this method allows us to reduce the complicated Equation (66) to an eigenvalue problem for any k. First, we define the wavefunctions $u_n(a) = \sqrt{\Omega}\,\psi_n(a)/a$ and $v_n(a) = \sqrt{\Omega}\,\chi_n(a)/a$, where now ψ_n and χ_n should be written in terms of a; similar changes can be done for the fields ζ_n and φ_n. These two new functions satisfy the equations:

$$\begin{aligned} 0 &= a\frac{dv_n}{da} + v_n + i\,(E_n + \Omega)\,u_n\,, \\ 0 &= a\frac{du_n}{da} + u_n + i\,(E_n - \Omega)\,v_n\,. \end{aligned} \tag{74}$$

We can now use the SM for the functions u_n and v_n. Notice that every term in Equation (74)'s approach to zero as a goes to zero. With the SM, we can assume the following dependence for the different functions:

$$\begin{aligned} u_n(a) &\approx \sum_{m=1}^{N} A_m^{(n)} \sin\left(\frac{m\pi}{L}a\right)\,, \\ v_n(a) &\approx \sum_{m=1}^{N} B_m^{(n)} \sin\left(\frac{m\pi}{L}a\right)\,, \end{aligned} \tag{75}$$

where again $A_m^{(n)}$ and $B_m^{(n)}$ are constant coefficients that depend on the n-state. Anew, the wavefunction (65) behaves as $\Psi(\theta, a \to 0) \to 0$. Similarly, we get:

$$\begin{aligned} \Omega(a)\,u_n(a) &\approx \sum_{m=1}^{N} C_m^{(n)} \sin\left(\frac{m\pi}{L}a\right)\,, \\ \Omega(a)\,v_n(a) &\approx \sum_{m=1}^{N} D_m^{(n)} \sin\left(\frac{m\pi}{L}a\right)\,, \\ a\frac{du_n(a)}{da} &\approx \sum_{m=1}^{N} E_m^{(n)} \sin\left(\frac{m\pi}{L}a\right)\,, \\ a\frac{dv_n(a)}{da} &\approx \sum_{m=1}^{N} F_m^{(n)} \sin\left(\frac{m\pi}{L}a\right)\,, \end{aligned} \tag{76}$$

where the relations between the coefficients are $C_l^{(n)} = \sum_{l=1}^{N} G_{lm} A_m^{(n)}$, $D_l^{(n)} = \sum_{l=1}^{N} G_{lm} B_m^{(n)}$, $E_l^{(n)} = \sum_{l=1}^{N} H_{lm} A_m^{(n)}$ and $F_l^{(n)} = \sum_{l=1}^{N} H_{lm} B_m^{(n)}$, with the matrix elements:

$$\begin{aligned} G_{lm} &= \frac{2}{L}\int_0^L \sin\left(\frac{l\pi}{L}a\right)\Omega(a)\sin\left(\frac{m\pi}{L}a\right)da\,, \\ H_{lm} &= \frac{2m\pi}{L^2}\int_0^L a\sin\left(\frac{l\pi}{L}a\right)\cos\left(\frac{m\pi}{L}a\right)da\,. \end{aligned} \tag{77}$$

Using Equation (75) in Equation (76), and the previous relations on Equation (74), we find, after some algebra, the eigenvector equation:

$$\mathbb{M}\cdot\mathbb{V}^{(n)} = E_n\mathbb{V}^{(n)}\,, \tag{78}$$

where E_n corresponds to the eigenvalues (associated to the energy), and the eigenvector $\mathbb{V}^{(n)}$ is formed by the $2N$ components $A_m^{(n)}$ and $B_m^{(n)}$ as:

$$\mathbb{V}^{(n)} = \begin{pmatrix} A^{(n)} \\ B^{(n)} \end{pmatrix}, \tag{79}$$

and the $2N \times 2N$ matrix $\mathbb{M}$ is such that:

$$\mathbb{M} = \begin{pmatrix} -\mathbb{G} & i\,(\mathbb{I} + \mathbb{H}) \\ i\,(\mathbb{I} + \mathbb{H}) & \mathbb{G} \end{pmatrix}, \tag{80}$$

where the $N \times N$ matrices $\mathbb{G}$ and $\mathbb{H}$ are constructed by the components given in Equation (77); $\mathbb{I}$ is the identity matrix. The evolution of the system is completely determined by solving the eigenvector and eigenvalue Equation (78). The approximated solution improves by increasing N.

4.3. Quantization of the FRWq System à la Majorana

Strictly speaking, the Dirac description of the FRQW system implies that the Universe can interact with self-electromagnetic fields, as the particle modeled can have charge. One way to avoid this issue is to use Majorana matrices instead of Dirac matrices. In this way, the quantization scheme models a Universe with quintessence as a neutral relativistic quantum particle. The quantum mechanical equation is:

$$i\gamma_M^0 \frac{\partial \Psi_M}{\partial \theta} + i\gamma_M^\xi \left(\frac{\partial}{\partial \xi} + \frac{1}{2}\frac{\partial \ln \Omega}{\partial \xi} \right) \Psi_M = \Omega \Psi_M\,, \tag{81}$$

where now γ_M are the Majorana matrices, and Ψ_M represents the wavefunction of the FRWq system in the Majorana scheme of quantization.

Similar to the previous case, the expected value of the scale factor for the Universe under the Majorana quantization is:

$$\langle a(\theta) \rangle = \frac{1}{|\Psi_M|^2} \int_0^{a_f} da\, \Omega\, \Psi_M^\dagger \Psi_M\,, \tag{82}$$

where $\Psi_M^\dagger$ is the transpose conjugated of the wavefunction Ψ_M, and the probability density is $|\Psi_M|^2 = \int_0^{a_f} da\, \Omega\, \Psi_M^\dagger \Psi_M / a$. As well as all the previous cases, the expected value of the scale factor depends on θ.

Analytical representation of the solutions can be obtained by performing the descomposition:

$$\Psi_M(\theta, \xi) = \sum_{n=0}^{\infty} e^{iE_n\theta} \begin{pmatrix} \psi_n(\xi) \\ \zeta_n(\xi) \\ \varphi_n(\xi) \\ \chi_n(\xi) \end{pmatrix}, \tag{83}$$

and choosing:

$$\gamma_M^\xi = \gamma_M^1 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}; \tag{84}$$

then Equation (81) may be rewritten as:

$$\begin{aligned} 0 &= \frac{d\psi_n}{d\xi} + \left(\frac{1}{2\Omega}\frac{d\Omega}{d\xi} + \Omega \right) \psi_n - iE_n \chi_n\,, \\ 0 &= \frac{d\chi_n}{d\xi} + \left(\frac{1}{2\Omega}\frac{d\Omega}{d\xi} - \Omega \right) \chi_n - iE_n \psi_n\,. \end{aligned} \tag{85}$$

Again the wavefunction ψ_n is coupled to χ_n. Similarly, ζ_n and φ_n are coupled by the same mathematical equations.

Notice that unlike the Dirac case, when $E_n = 0$, the equations in the Majorana scheme decouple. In this case, the simple solutions can be obtained as:

$$\psi_n(\xi) = \frac{1}{\sqrt{\Omega}} \exp\left(-\Omega_I\right) , \quad \chi_n(\xi) = \frac{1}{\sqrt{\Omega}} \exp\left(\Omega_I\right) , \tag{86}$$

where we use the definition (69) for Ω_I. As in Dirac scheme, these solutions again diverge for $\xi \to -\infty$ ($a \to 0$). For $E_n \neq 0$, we can find a modified Klein–Gordon equation for ψ_n:

$$\begin{aligned} 0 \;\; = \;\; & \frac{d^2\psi_n}{d\xi^2} + \frac{1}{\Omega}\frac{d\Omega}{d\xi}\frac{d\psi_n}{d\xi} \\ & + \left(\frac{1}{2\Omega}\frac{d^2\Omega}{d\xi^2} - \frac{1}{4\Omega^2}\left(\frac{d\Omega}{d\xi}\right)^2 + \frac{d\Omega}{d\xi} - \Omega^2 + E_n^2 \right)\psi_n , \end{aligned} \tag{87}$$

which contains an effective mass term depending on curvature variations, which can be compared with Equation (39). For the special case of $k = 0$, it can be shown that this equation reduces to the equation for a diatomic molecule decribed by the Morse potential [6]. Performing the change of variables $\psi = \exp(-\xi)\, \phi'$, Equation (87) can be simplified to:

$$\frac{\partial^2\phi'}{\partial\xi^2} = \left(-\frac{3}{8}Ve^{4\xi} - E^2 - \sqrt{-\frac{3}{2}Ve^{2\xi}} \right)\phi' . \tag{88}$$

As discussed in the begining of Section 4, the general potential $\bar{V}$ must be positive. Therefore, we can choose a representation of $V = V(\theta) = -(8/3)\exp(2x_e)$, where x_e is a constant. Making another change of variables $x = -2\xi$ and defining $\mathcal{E} = E^2/4$, Equation (88) can be put in the form:

$$\mathcal{E}\phi' = \left[-\frac{\partial^2}{\partial x^2} + \frac{1}{4}\left(e^{-2(x-x_e)} - 2e^{-(x-x_e)} \right) \right]\phi' , \tag{89}$$

which is the quantum equation for a diatomic molecule described by the Morse potential. The wavefunctions and energy spectrum of this problem are already known.

Similar to previous sections, analytical approximated solutions for any k can be found using the SM approach. As before, we can perform this task by defining the variables $u_n = \sqrt{\Omega}\,\psi_n/a$ and $v_n = \sqrt{\Omega}\,\chi_n/a$ that satisfy the following equations:

$$\begin{aligned} 0 &= a\frac{du_n}{da} + (1+\Omega)\, u_n - iE_n v_n , \\ 0 &= a\frac{dv_n}{da} + (1-\Omega)\, v_n - iE_n u_n . \end{aligned} \tag{90}$$

Again, notice that the SM allows us to have a well-defined behavior of the wavefunction (83) as $\Psi_M(\theta, a \to 0) \to 0$. Applying the SM means we have to use similar decompositions, as in Equations (75)–(77), to the Majorana case. For simplicity, we use the same notation as before. Equation (90) can be finally written as:

$$\mathbb{K} \cdot \mathbb{V}^{(n)} = E_n \mathbb{V}^{(n)} , \tag{91}$$

where again E_n are the eigenvalues and $\mathbb{V}^{(n)}$ is the vector (79). The $2N \times 2N$ matrix $\mathbb{K}$ is now:

$$\mathbb{K} = -i \begin{pmatrix} \mathbb{O} & \mathbb{I} + \mathbb{H} - \mathbb{G} \\ \mathbb{I} + \mathbb{H} + \mathbb{G} & \mathbb{O} \end{pmatrix} , \tag{92}$$

where again $\mathbb{G}$ and $\mathbb{H}$ are $N \times N$ matrices constructed by Equation (77), and $\mathbb{O}$ is the $N \times N$ zero matrix.

Another important feature of Equation (85) deserves to be highlighted. Defining the new wavefunctions $\phi_n^+ = \sqrt{\Omega}\psi_n$ and $\phi_n^- = \sqrt{\Omega}\chi_n$, then Equation (85) can be re-expressed as:

$$Z_\pm \phi_n^\pm = \pm i E_n \phi_n^\mp \,, \tag{93}$$

where the operators are defined as [6]

$$Z_\pm = \pm \frac{d}{d\xi} + \Omega \,. \tag{94}$$

Notice that Equation (93) represents the equations for supersymmetric quantum mechanics [35–37]. Each wavefunction satisfies:

$$H_1 \phi_n^+ = \frac{E_n^2}{2} \phi_n^+ \,, \qquad H_2 \phi_n^- = \frac{E_n^2}{2} \phi_n^- \,, \tag{95}$$

where the Hamiltonians:

$$\begin{aligned} H_1 &= -\frac{1}{2}\frac{d^2}{d\xi^2} + \frac{1}{2}\left(-\frac{d\Omega}{d\xi} + \Omega^2\right) \,, \\ H_2 &= -\frac{1}{2}\frac{d^2}{d\xi^2} + \frac{1}{2}\left(\frac{d\Omega}{d\xi} + \Omega^2\right) \,, \end{aligned} \tag{96}$$

can be used to define the Super-Hamiltonian:

$$H_M = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \,, \tag{97}$$

used to write the above Equation (95) as:

$$H_M \begin{pmatrix} \phi_n^+ \\ \phi_n^- \end{pmatrix} = \frac{E_n^2}{2} \begin{pmatrix} \phi_n^+ \\ \phi_n^- \end{pmatrix} \,. \tag{98}$$

The operators (94) can be used to define the supercharges:

$$Q = \begin{pmatrix} 0 & 0 \\ Z_+ & 0 \end{pmatrix} \,, \qquad Q^\dagger = \begin{pmatrix} 0 & Z_- \\ 0 & 0 \end{pmatrix} \,, \tag{99}$$

which are operators that can change bosonic (fermionic) states into fermionic (bosonic) ones. The above supersymmetric system exhibits the same features of any other supersymmetric quantum theory [37].

5. Discussion

The quantization schemes presented here (Klein–Gordon, Dirac, or Majorana) are only possible due the unification between the FRW geometry and the quintessence scalar field with a Fermat-like Lagrangian for a relativistic particle moving in a two-dimensional, conformally flat spacetime. The quantum equations obtained for every case can be considered as generalizations of the Wheeler–DeWitt Super-Hamiltonian formalism, and they are consistent with the principle of manifest covariant. Our proposal establishes that the quintessence field could be necessary as a first step to construct a geometrically unified theory for the quantization of an expanding universe (with a quintessence type of dark energy).

The quantum theory for the FRW Universe using a relativistic quantum theory for Dirac particles follows the quantization scheme for a relativistic pointlike particle model developed by Breit [7]. In

Breit's interpretation, the spin can emerge as a geometrical and dynamical interpretation of the classical velocity of the particle. Dirac matrices follow a replacement prescription similar to those largely used in non-relativistic quantum mechanics and spinless relativistic quantum mechanics. Originally, this prescription is shown for a particle on flat spacetime; however, we can generalize it for a relativistic particle in a conformally two-dimensional flat space. From the classical Hamiltonian (28), we can obtain the relation $H^2 = g_{00} + \pi^2$, where we have made use of $g_{00} = \Omega^2$. From here we can obtain that:

$$\frac{\sqrt{g_{00}}}{H} = \sqrt{1 - q^2}\,, \tag{100}$$

where we have defined the velocity variable:

$$q = \frac{\pi}{H}\,. \tag{101}$$

Using this variable, the Hamiltonian (28) can be re-written as $H = g_{00}/H + \pi^2/H$, or:

$$H = \sqrt{g_{00}}\left(\frac{q\pi}{\Omega} + \sqrt{1 - q^2}\right)\,. \tag{102}$$

Breit's interpretation corresponds to the identification of the Dirac matrices as [7]:

$$q \to \alpha\,, \qquad \sqrt{1 - q^2} \to \beta\,. \tag{103}$$

This allows us to construct the quantum Hamitonian (55)—with the definition (30) for momentum operator—from its classical analogue (28). Breit showed that the identification (103) is consistent with the postulates of Dirac theory. The implications of this prescription have been investigated with the purpose of understanding the underlying nature of the spin or antiparticles [38,39]. Therefore, the previous interpretation gives validity to the quantum theories developed in Sections 4.2 and 4.3 and also, as it is shown in Appendix A, both treatments are completely equivalent to the well-established Dirac theory in curved spacetimes.

It is important to discuss a common concept appearing in the three different quantum theories of Section 4, and it is related with the emergent gravity phenomenon, where classical observables can be consequence of these quantum theories [40,41]. In all models, we can identify a new time variable θ for the quantized cosmological model, suggesting that the Universe evolves along this super-time, and not with the usual time coordinate. Thus, the super-time θ is the quintessence field $\phi = \sqrt{8/3}\,\theta$, given by Equation (14). This implies that the evolution of this quantum cosmological model is not possible in the absence of quintessence. This emegent gravity phenomenon can explain the spacetime properties of time and dark energy, as they follow from the evolution of the quantum cosmology theories along this super-time. An argument can be given in the following way: As the quantum Universe evolves through to the super-time, we classically detect the quantum evolution as a quintessence field which produces a cosmological negative pressure as dark energy.

On the other hand, we have limited ourselves to present solutions of the quantum cosmological theories developed in previous section. In principle, with the quantum Equations (35), (57) and (81), we can study their Bohm dynamics and their trajectory-based dynamics. This has been already done for the Wheeler–DeWitt equation [42–46], thus using similar tools, the trajectory-based dynamics analysis can be done for Equations (35), (57) and (81). In the same fashion, we can use the Klein–Gordon, Dirac, and Majorana models to study their statistical features of those quantum space-time dynamics in an analogue fashion to Wheeler–DeWitt equation [47–49]. These works are left for the future.

Another interesting feature of the quantization theories of Sections 4.2 and 4.3 is the physical meaning of the wavefunctions components of the bi-spinor. In comparison with the Dirac theory for physical particles, we can recognize that every component of the wavefunctions introduced along this work has the dynamics of an entity in interaction with the others. Hence, we can argue that each component represents a Universe, evidencing that the Dirac bi-spinor wavefunction (65), or the Majorana bi-spinor wavefunction (83), represents a description of a Multiverse. The different Universes ψ and χ (or ζ and φ) appear coupled, implying an interaction between the Universes. In this way, the Multiverse behaves as a di-atomic molecule under some potential. The interaction between the wavefunction of the Universes produces the dynamical evolution of the expected value of the scale factor, as it can be seen from Equations (64) and (82). Therefore, the expansion rate of the scale factor, produced by quintessence (dark energy), could be a direct consequence of the existence of a Multiverse. Also, according to Equation (93), we can infer that the Multiverse has the structure of a supersymmetric system, being that the two Universes are the super-partners of each other. In principle, we can use all the well-known tools of supersymmetry [37] to study the main characteristics of this Multiverse theory. This idea will be explored in forthcoming works, comparing it with other versions of the Multiverse idea [50–61].

Finally, we would like to remark that the three quantization schemes are only possible due to the geometrical unification between spacetime and the quintessence field. In the case of the Klein–Gordon theory, the unification leads to results that coincide with similar ones in literature. We expect that this unification can bring new insights in the field of quantum cosmology.

Author Contributions: All authors contribute equally to this research.

Funding: C.A.R. was supported by Conicyt PhD fellowship No 21150314. F.A.A. was supported by Fondecyt-Chile Grant No. 1180139.

Conflicts of Interest: The authors declare no conflict of interest. The founding sponsors had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, and in the decision to publish the results.

Appendix A Dirac Equation in Curved Spacetimes.

In general, the curved spacetime Dirac equation is:

$$ie^{\mu}{}_{d}\gamma^{d}\left(\partial_{\mu}+\frac{1}{8}\omega_{ab\mu}[\gamma^{a},\gamma^{b}]\right)\Psi=\Psi\,, \tag{A1}$$

where we defined the vierbein as:

$$g_{\mu\nu}=e_{\mu}{}^{a}e_{\nu}{}^{b}\eta_{ab}\,, \tag{A2}$$

with the flat-spacetime metric η_{ab}. Here, $e^{\mu}{}_{d}$ is the inverse vierbein in the sense that $e_{\mu}{}^{a}e^{\mu}{}_{b}=\delta^{a}_{b}$. We also define the spin connection $\omega_{ab\mu}=\eta_{ac}\omega^{c}{}_{b\mu}$, with:

$$\omega^{c}{}_{b\mu}=e^{c}{}_{\nu}e^{\nu}{}_{b,\mu}+e^{c}{}_{\nu}e^{\sigma}{}_{b}\Gamma^{\nu}{}_{\sigma\mu}\,, \tag{A3}$$

where $\Gamma^{\nu}{}_{\sigma\mu}$ are the Christoffel symbols. Because the antisymmetry of the spin connection in its first two indices, we have $\omega_{ab\mu}[\gamma^{a},\gamma^{b}]=2\omega_{ab\mu}\gamma^{a}\gamma^{b}$.

For a given metric, the Dirac equation for that curved spacetime can be straightforwardly constructed. In our two-dimensional, conformally flat case of the FRWq system, the vierbeins are:

$$e_{0}{}^{0}=\Omega\,,\;\; e_{1}{}^{1}=\Omega\,,\;\; e^{0}{}_{0}=\frac{1}{\Omega}\,,\;\; e^{1}{}_{1}=\frac{1}{\Omega}\,, \tag{A4}$$

Thus, the Dirac equation becomes:

$$i\gamma^{0}\left(\partial_{0}+\frac{1}{4}\omega_{ab0}\gamma^{a}\gamma^{b}\right)\Psi+i\gamma^{1}\left(\partial_{\xi}+\frac{1}{4}\omega_{ab1}\gamma^{a}\gamma^{b}\right)\Psi=\Omega\Psi\,. \tag{A5}$$

We also have that $\omega_{ab0}\gamma^{a}\gamma^{b}=2\omega_{010}\gamma^{0}\gamma^{1}$, $\omega_{ab1}\gamma^{a}\gamma^{b}=2\omega_{011}\gamma^{0}\gamma^{1}$, and:

$$\omega_{010}=\Gamma^{0}{}_{10}=\partial_{\xi}\ln\Omega\,,\qquad \omega_{011}=\Gamma^{0}{}_{11}=0\,, \tag{A6}$$

for the two-dimensional, conformally flat metric. Then, the Dirac equation is written as:

$$i\gamma^0\left(\partial_0+\frac{1}{2}\frac{\partial\ln\Omega}{\partial\xi}\gamma^0\gamma^1\right)\Psi+i\gamma^1\partial_\xi\Psi=\Omega\Psi\,, \tag{A7}$$

or:

$$i\gamma^0\partial_0\Psi+i\gamma^1\left(\partial_\xi+\frac{1}{2}\frac{\partial\ln\Omega}{\partial\xi}\right)\Psi=\Omega\Psi\,, \tag{A8}$$

which is exactly the same equation than (57) for appropiated choices of matrices, and identifying the time with θ.

References

1. Tsujikawa, S. Quintessence: A review. *Class. Quantum Grav.* **2013**, *30*, 214003. [CrossRef]
2. Copeland, E.J.; Sami, M.; Tsujikawa, S. Dynamics of dark energy. *Int. J. Mod. Phys. D* **2006**, *15*, 1753. [CrossRef]
3. Capozziello, S.; Roshan, M. Exact cosmological solutions from Hojman conservation quantities. *Phys. Lett. B* **2013**, *726*, 471. [CrossRef]
4. Steinhardt, P.J. A quintessential introduction to dark energy. *Phil. Trans. R. Soc. Lond. A* **2003**, *361*, 2497. [CrossRef] [PubMed]
5. Bojowald, M. Quantum cosmology: a review. *Rep. Prog. Phys.* **2015**, *78*, 023901. [CrossRef] [PubMed]
6. Hojman, S.A.; Asenjo, F.A. Supersymmetric Majorana quantum cosmologies. *Phys. Rev. D* **2015**, *92*, 083518. [CrossRef]
7. Breit, G. An Interpretation of Dirac's Theory of the Electron. *Proc. Natl. Acad. Sci. USA* **1928**, *14*, 553. [CrossRef]
8. Ryden, B. *Introduction to Cosmology*; Addison Wesley: Boston, MA, USA, 2003.
9. Hojman, S.A.; Chayet, S.; Núñez, D.; Roque, M.A. An algorithm to relate general solutions of different bidimensional problems. *J. Math. Phys.* **1991**, *32*, 1491. [CrossRef]
10. Luneburg, R.K. *Mathematical Theory of Optics*; University of California at Berkeley: Berkeley, CA, USA, 1948.
11. DeWitt, B.S. Quantum Theory of Gravity. I. The Canonical Theory. *Phys. Rev.* **1967**, *160*, 1113. [CrossRef]
12. Halliwell, J.J.; Hawking, S.W. Origin of structure in the Universe. *Phys. Rev. D* **1985**, *31*, 1777. [CrossRef]
13. Alvarenga, F.G.; Lemos, N.A. Dynamical vacuum in quantum cosmology. *Gen. Rel. Grav.* **1998**, *30*, 681. [CrossRef]
14. Oliveira-Neto, G. No-boundary wave function of the anti–de Sitter space-time and the quantization of Λ. *Phys. Rev. D* **1998**, *58*, 107501. [CrossRef]
15. Lemos, N.A. Radiation-dominated quantum Friedmann models. *J. Math. Phys.* **1996**, *37*, 1449. [CrossRef]
16. Monerat, G.A.; Corrêa Silva, E.V.; Oliveira-Neto, G.; Ferreira Filho, L.G.; Lemos, N.A. Quantization of Friedmann-Robertson-Walker spacetimes in the presence of a negative cosmological constant and radiation. *Phys. Rev. D* **2006**, *73*, 044022. [CrossRef]
17. Oliveira-Neto, G.; Monerat, G.A.; Corrêa Silva, E.V.; Neves, C.; Ferreira Filho, L.G. Quantization of Friedmann-Robertson-Walker Spacetimes in the presence of a cosmological constant and stiff matter. *Int. J. Theor. Phys.* **2013**, *52*, 2991. [CrossRef]
18. Vakili, B. Noether symmetric minisuperspace model of $f(R)$ cosmology *Ann. Phys. (Berlin)* **2010**, *19*, 359. [CrossRef]
19. Huang, R.-N. The Wheeler-DeWitt equation of $f(R, L_m)$ gravity in minisuperspace. *arXiv* **2013**, arXiv:1304.5309v2.
20. Barbour, J.B.; Murchadha, N.O. Classical and quantum gravity on conformal superspace. *arXiv* **1999**, arXiv:gr-qc/9911071v1.
21. Hawking, S.W.; Wu, Z.C. Numerical calculations of minisuperspace cosmological models. *Phys. Lett.* **1985**, *151B*, 15. [CrossRef]
22. Saa, A. Canonical quantization of the relativistic particle in static spacetimes . *Class. Quantum Grav.* **1996**, *13*, 553. [CrossRef]
23. Hanson, A.; Regge, T.; Teitelboim, C. *Constrained Hamiltonian Systems*; Accademia Nazionale dei Lincei: Roma, Italy, 1976.

24. Hojman, S.; Montemayor, R. s-Equivalent Lagrangians for free particles and canonical quantization. *Hadron. J.* **1980**, *3*, 1644.
25. Gavrilov, S.P.; Gitman, D.M. Quantization of the relativistic particle. *Class. Quantum Grav.* **2000**, *17*, L133. [CrossRef]
26. Dirac, P.A.M. *Lectures on Quantum Mechanics*; Dover Publications, Inc.: Mineola, NY, USA, 2001.
27. Henneaux, M.; Teitelboim, C. *Quantization of Gauge Systems*; Princeton University: Princeton, NJ, USA, 1992.
28. Mukhanov, V.F.; Winitzki, S. *Introduction to Quantum Effects in Gravity*; Cambridge University Press: Cambridge, UK, 2007.
29. Strange, P. *Relativistic Quantums Mechanic, with Applications in Condensed Matter and Atomic Physics*; Cambridge University Press: Cambridge, UK, 1998.
30. Pedram, P.; Mirzaei, M.; Gousheh, S.S. Using spectral method as an approximation for solving hyperbolic PDEs. *Comput. Phys. Commun.* **2007**, *176*, 581. [CrossRef]
31. Pedram, P.; Mirzaei, M.; Jalalzadeh, S.; Gousheh, S.S. Perfect fluid quantum Universe in the presence of negative cosmological constant. *Gen. Relativ. Gravit.* **2008**, *40*, 1663. [CrossRef]
32. Sánchez-Monroy, J.A.; Quimbay, C.J. Dirac equation in low dimensions: The factorization method. *Ann. Phys.* **2014**, *350*, 69. [CrossRef]
33. Yeşiltaş, Ö. $\mathcal{PT}$ symmetric Hamiltonian model and Dirac equation in 1+1 dimensions. *J. Phys. A Math. Theor.* **2013**, *46*, 015302. [CrossRef]
34. Eleuch, H.; Alhaidari, A.D.; Bahlouli, H. Analytical solutions to the Dirac equation in 1+1 Space-Time Dimension. *Appl. Math. Inf. Sci.* **2012**, *6*, 149.
35. Cooper, F.; Khare, A.; Musto, R.; Wipf, A. Supersymmetry and the Dirac equation. *Ann. Phys.* **1988**, *187*, 1. [CrossRef]
36. De Crombrugghe, M.; Rittenberg, V. Supersymmetric quantum mechanics. *Ann. Phys.* **1983**, *151*, 99. [CrossRef]
37. Cooper, F.; Khare, A.; Sukhatme, U. *Supersymmetry in Quantum Mechanics*; World Scientific: Singapore, 2001.
38. Savasta, S.; Di Stefano, O. Classical origin of the spin of relativistic pointlike particles and geometric interpretation of Dirac solutions. *arXiv* **2008**, arXiv:0803.4013v1.
39. Savasta, S.; Di Stefano, O.; Maragò, O.M. Quantum-classical correspondence of the Dirac matrices: The Dirac Lagrangian as a Total Derivative. *arXiv* **2009**, arXiv:0905.4741v1.
40. Barceló, C.; Visser, M.; Liberati, S. Einstein gravity as an emergent phenomenon? *Int. J. Mod. Phys. D* **2001**, *10*, 799. [CrossRef]
41. Padmanabhan, T. Emergent perspective of Gravity and Dark Energy. *Res. Astron. Astrophys.* **2012**, *12*, 891. [CrossRef]
42. Horiguchi, T. Quantum potential interpretation of the Wheeler-DeWitt equation. *Mod. Phys. Lett. A* **1994**, *9*, 1429. [CrossRef]
43. Vink, J.C. Quantum potential interpretation of the wave function of the universe. *Nucl. Phys. B* **1992**, *369*, 707. [CrossRef]
44. Halliwell, J.J. Derivation of the Wheeler-DeWitt equation from a path integral for minisuperspace models. *Phys. Rev. D* **1988**, *38*, 2468. [CrossRef]
45. Barvinsky, A.O.; Ponomariov, V.N. Quantum geometrodynamics: The path integral and the initial value problem for the wave function of the universe. *Phys. Lett. B* **1986**, *167*, 289. [CrossRef]
46. Arisue, H.; Fujiwara, T.; Kato, M.; Ogawa, K. Path-integral and operator formalism in quantum gravity. *Phys. Rev. D* **1987**, *35*, 2309. [CrossRef]
47. Faizal, M. Deformation of the Wheeler–DeWitt equation. *Int. J. Mod. Phys. A* **2014**, *29*, 1450106. [CrossRef]
48. Garattini, R.; Faizal, M. Cosmological constant from a deformation of the Wheeler–DeWitt equation. *Nuc. Phys. B* **2016**, *905*, 313. [CrossRef]
49. Pedram, P. Generalized uncertainty principle and the conformally coupled scalar field quantum cosmology. *Phys. Rev. D* **2015**, *91*, 063517. [CrossRef]
50. Everett, H. Theory of the Universal Wavefunction. Ph.D. Thesis, Princeton University, Princeton, NJ, USA, 1956.
51. Everett, H. "Relative State" formulation of quantum mechanics. *Rev. Mod. Phys.* **1957**, *29*, 454. [CrossRef]
52. Weinberg, S. *Living in the Multiverse, Universe or Multiverse?*; Carr, B., Ed.; Cambridge University Press: Cambridge, UK, 2009.

53. Tegmark, M. The interpretation of quantum mechanics: Many worlds or many words? *Fortschr. Phys.* **1998**, *46*, 855–862. [CrossRef]
54. Tegmark, M.; Aguirre, A.; Rees, M.J.; Wilczek, F. Dimensionless constants, cosmology, and other dark matters. *Phys. Rev. D* **2006**, *73*, 023505. [CrossRef]
55. Tegmark, M. *Parallel Universes*, Science and Ultimate Reality; Barrow, J.D., Davies, P.C.W., Harper, C.L., Eds.; Cambridge University Press: Cambridge, UK, 2004.
56. Feeney, S.M.; Johnson, M.C.; Mortlock, D.J.; Peiris, H.V. First observational tests of eternal inflation. *Phys. Rev. Lett.* **2011**, *107*, 071301. [CrossRef]
57. Hall, M.J.W.; Deckert, D.-A.; Wiseman, H.M. Quantum phenomena modeled by interactions between many classical worlds. *Phys. Rev. X* **2014**, *4*, 041013. [CrossRef]
58. Antonov, A.A. Hidden Multiverse: Explanation of Dark Matter and Dark Energy phenomena. *Int. J. Phys.* **2015**, *3*, 84.
59. Vilekin, A. A quantum measure of the multiverse. *arXiv* **2013**, arXiv:1312.0682.
60. Caroll, S.M. Is our Universe natural? *Nature* **2006**, *440*, 1132. [CrossRef]
61. Carroll, S.M.; Sebens, C.T. Many Worlds, the Born Rule, and Self-Locating uncertainty. In *Quantum Theory: A Two-Time Success Story*; Struppa, D.C., Tollaksen, J.M., Eds.; Springer: Milan, Italy, 2014.