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Degenerate Stirling Polynomials of the Second Kind and Some Applications

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Abstract: Recently, the degenerate λ-Stirling polynomials of the second kind were introduced and investigated for their properties and relations. In this paper, we continue to study the degenerate λ-Stirling polynomials as well as the r-truncated degenerate λ-Stirling polynomials of the second kind which are derived from generating functions and Newton's formula. We derive recurrence relations and various expressions for them. Regarding applications, we show that both the degenerate λ-Stirling polynomials of the second and the r-truncated degenerate λ-Stirling polynomials of the second kind appear in the expressions of the probability distributions of appropriate random variables.

Keywords: degenerate λ-Stirling polynomials of the second kind; r-truncated degenerate λ-Stirling polynomials of the second kind; probability distribution

1. Introduction

For $n \geq 0$, the Stirling numbers of the second kind are defined as (see [1–26])

$$x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k, \tag{1}$$

where $(x)_0 = 1, (x)_n = x(x-1)\cdots(x-n+1), \ (n \geq 1)$.

From (1), we note that the generating function for $S_2(n,k)$ is given by (see [8,9,23,26])

$$\frac{1}{k!}(e^t - 1)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}, \ \ (k \geq 0). \tag{2}$$

For $\lambda \in \mathbb{R}$, the degenerate exponential function is defined by (see [10,11,27,28])

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}}, \ \ e_\lambda(t) = e_\lambda^1(t) = (1+\lambda t)^{\frac{1}{\lambda}}. \tag{3}$$

In view of (2), the *degenerate λ-Stirling polynomials of the second kind* are defined by the generating function

$$\frac{1}{k!}(e_\lambda(t) - 1)^k e_\lambda^x(t) = \sum_{n=k}^{\infty} S_{2,\lambda}^{(x)}(n,k)\frac{t^n}{n!}, \tag{4}$$

where $x \in \mathbb{R}$ and k is a nonnegative integer, (see [10,11]).

When $x = 0$, $S_{2,\lambda}^{(0)}(n,k) = S_{2,\lambda}(n,k)$ are called the degenerate λ-Stirling numbers of the second kind. Note that $\lim_{\lambda \to 0} S_{2,\lambda}(n,k) = S_2(n,k), \ \ (n,k \geq 0)$.

By letting $\lambda \to 0$ in (4), we have the generating function for the Stirling polynomials of the second kind $S_2^{(x)}(n,k)$ (see [11,13,14,22]):

$$\frac{1}{k!}(e^t-1)^k e^{xt} = \sum_{n=k}^{\infty} S_2^{(x)}(n,k)\frac{t^n}{n!}. \tag{5}$$

Let X be a discrete random variable with probability mass function $P[X=x]=p(x)$. Then, the probability generating function of X is given by (see [4,17,18,20,22])

$$G(t) = E[t^X] = \sum_{x=0}^{\infty} t^x p(x). \tag{6}$$

Suppose that $X = (X_1, X_2, \cdots, X_k)$ is a discrete random variable taking values in the k-dimensional nonnegative integer lattice. Then, the probability generating function of X is defined as (see [4])

$$\begin{aligned} G(t) &= G(t_1,t_2,\cdots,t_k) = E[t_1^{X_1}t_2^{X_2}\cdots t_k^{X_k}] \\ &= \sum_{x_1,x_2,\cdots,x_k=0}^{\infty} t_1^{x_1}t_2^{x_2}\cdots t_k^{x_k} p(x_1,x_2,\cdots x_k), \end{aligned} \tag{7}$$

where $p(x_1,x_2,\cdots x_k)$ is the probability mass function of $X=(X_1,X_2,\cdots,X_k)$.

X is the random variable with the zero-truncated Poisson distribution with parameter λ if the probability mass function of X is (see [4,17,18,20,22])

$$P[X=x] = p(x) = \frac{1}{1-e^{-\lambda}}e^{-\lambda}\frac{\lambda^x}{x!}, \tag{8}$$

where x is a positive integer.

For $r \in \mathbb{N}$, X is the random variable with r-truncated Poisson distribution with parameter λ if the probability mass function of X is

$$P[X=x] = p(x) = C(\lambda,r)e^{-\lambda}\frac{\lambda^x}{x!}, \tag{9}$$

where $C(\lambda,r) = (1-e^{-\lambda}\sum_{x=0}^{r-1}\frac{\lambda^x}{x!})^{-1}$.

In this paper, we study the degenerate λ-Stirling polynomials, as a continuation of the previous work in [10], and also the r-truncated degenerate λ-Stirling polynomials of the second kind which are derived from generating functions and Newton's formula. We derive recurrence relations and various expressions for them. As applications, we show that both the degenerate λ-Stirling polynomials of the second kind and the r-truncated degenerate λ-Stirling polynomials of the second kind appear in the expressions of the probability distributions of appropriate random variables.

2. The Degenerate λ-Stirling Polynomials of the Second Kind

Let x,t be real numbers and let n be a nonnegative integer. The difference operator $\triangle$ is defined by $\triangle f(x) = f(x+1) - f(x)$. It is easy to show that

$$(I+\triangle)^n f(0) = f(n) = \sum_{k=0}^{n}\binom{n}{k}\triangle^k f(0). \tag{10}$$

From (10), we can derive Newton's formula which is given by

$$f(t) = \sum_{k=0}^{\infty}\binom{t}{k}\triangle^k f(0). \tag{11}$$

Let us take $f(t) = (t+x)_{n,\lambda}, \ (n \geq 0)$. Then, by (11), we get

$$\begin{aligned}(t+x)_{n,\lambda} &= \sum_{k=0}^{\infty} \binom{t}{k} \left[\triangle^k (t+x)_{n,\lambda}\right]_{t=0} \\ &= \sum_{k=0}^{n} \binom{t}{k} \left[\triangle^k (t+x)_{n,\lambda}\right]_{t=0} \\ &= \sum_{k=0}^{n} \frac{1}{k!} \left[\triangle^k (t+x)_{n,\lambda}\right]_{t=0} (t)_k.\end{aligned} \tag{12}$$

Here, the generalized factorial sequence $(x)_{n,\lambda}$ is given by

$$(x)_{0,\lambda} = 1, (x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda), \ (n \geq 1). \tag{13}$$

Let

$$S_{2,\lambda}^{(x)}(n,k) = \frac{1}{k!}\left[\triangle^k (t+x)_{n,\lambda}\right]_{t=0} = \frac{1}{k!}\triangle^k (x)_{n,\lambda}, \ (n,k \geq 0). \tag{14}$$

Then, by (12) and (14), we get

$$(t+x)_{n,\lambda} = \sum_{k=0}^{n} S_{2,\lambda}^{(x)}(n,k)(t)_k, \ (n \geq 0). \tag{15}$$

In (18) below, we show that the $S_{2,\lambda}^{(x)}(n,k)$ in (14) or (15) is really the degenerate λ-Stirling polynomials of the second kind defined in (4).

Using (13), (15), and interchanging the order of summations, we note that

$$\begin{aligned}e_\lambda^{x+y}(t) &= \sum_{n=0}^{\infty} (x+y)_{n,\lambda} \frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} S_{2,\lambda}^{(x)}(n,k)(y)_k \frac{t^n}{n!} \\ &= \sum_{k=0}^{\infty}\left(\sum_{n=k}^{\infty} S_{2,\lambda}^{(x)}(n,k)\frac{t^n}{n!}\right)(y)_k.\end{aligned} \tag{16}$$

On the other hand,

$$e_\lambda^{x+y}(t) = e_\lambda^x(t)(e_\lambda(t)-1+1)^y = \sum_{k=0}^{\infty} \frac{1}{k!}(e_\lambda(t)-1)^k e_\lambda^x(t)(y)_k. \tag{17}$$

By (16) and (17), we get

$$\frac{1}{k!}(e_\lambda(t)-1)^k e_\lambda^x(t) = \sum_{n=k}^{\infty} S_{2,\lambda}^{(x)}(n,k)\frac{t^n}{n!}, \ (k \geq 0). \tag{18}$$

Thus, we have shown that the definition of the degenerate λ-Stirling polynomials of the second kind can be given as (14) or equivalently (15) or equivalently (18).

Note that

$$S_{2,\lambda}^{(x)}(0,0) = 1, \ S_{2,\lambda}^{(x)}(n,0) = (x)_{n,\lambda}, \ S_{2,\lambda}^{(x)}(0,k) = 0, \ (k \neq 0). \tag{19}$$

We would like to derive a recurrence relation for the degenerate λ-Stirling polynomials of the second kind. Now, we observe that

$$\begin{aligned}
&\sum_{k=0}^{n+1} S_{2,\lambda}^{(x)}(n+1,k)(t)_k = (t+x)_{n+1,\lambda} = (t+x)_{n,\lambda}(t+x-n\lambda) \\
=&t\sum_{k=0}^{n} S_{2,\lambda}^{(x)}(n,k)(t)_k + (x-n\lambda)\sum_{k=0}^{n} S_{2,\lambda}^{(x)}(n,k)(t)_k \\
=&t\sum_{k=1}^{n+1} S_{2,\lambda}^{(x)}(n,k-1)(t)_{k-1} + (x-n\lambda)\sum_{k=0}^{n} S_{2,\lambda}^{(x)}(n,k)(t)_k \\
=&\sum_{k=1}^{n+1}(t-k+1+k-1) S_{2,\lambda}^{(x)}(n,k-1)(t)_{k-1} + (x-n\lambda)\sum_{k=0}^{n} S_{2,\lambda}^{(x)}(n,k)(t)_k \\
=&\sum_{k=1}^{n+1} S_{2,\lambda}^{(x)}(n,k)(t)_k + \sum_{k=0}^{n} k S_{2,\lambda}^{(x)}(n,k)(t)_k + (x-n\lambda)\sum_{k=0}^{n} S_{2,\lambda}^{(x)}(n,k)(t)_k \\
=&\sum_{k=0}^{n+1}\left\{ S_{2,\lambda}^{(x)}(n,k-1) + (k+x-n\lambda) S_{2,\lambda}^{(x)}(n,k)\right\}(t)_k.
\end{aligned} \tag{20}$$

Therefore, by comparing the coefficients on both sides of (20), we obtain the following theorem.

Theorem 1. *Let n,k be nonnegative integers. Then, we have*

$$S_{2,\lambda}^{(x)}(n+1,k) = S_{2,\lambda}^{(x)}(n,k-1) + (k+x-n\lambda) S_{2,\lambda}^{(x)}(n,k), \ \ (n \geq k-1).$$

Note that

$$S_{2,\lambda}^{(x)}(n,k) = 0, \ \text{ if } k > n, \ \ S_{2,\lambda}^{(x)}(n,n) = 1.$$

From (14), we have

$$S_{2,\lambda}^{(0)}(n,k) = \frac{1}{k!}\triangle^k (0)_{n,\lambda} = S_{2,\lambda}(n,k), \ \ (n,k \geq 0). \tag{21}$$

Here, we want to derive an explicit expression for the degenerate λ-Stirling polynomials of the second kind. By (18), we get

$$\begin{aligned}
&\sum_{n=k}^{\infty} S_{2,\lambda}^{(x)}(n,k)\frac{t^n}{n!} = \frac{1}{k!} e_\lambda^x(t)(e_\lambda(t)-1)^k \\
=&\frac{1}{k!}\sum_{m=0}^{\infty}\frac{(x)_{m,\lambda}}{m!}t^m \sum_{l=k}^{\infty}\left(\sum_{l_1+\cdots+l_k=l}\frac{(1)_{l_1,\lambda}(1)_{l_2,\lambda}\cdots(1)_{l_k,\lambda}}{l_1!l_2!\cdots l_k!}\right)t^l \\
=&\sum_{n=k}^{\infty}\left(\frac{n!}{k!}\sum_{l=k}^{n}\frac{(x)_{n-l,\lambda}}{(n-l)!}\sum_{l_1+\cdots+l_k=l}\frac{(1)_{l_1,\lambda}(1)_{l_2,\lambda}\cdots(1)_{l_k,\lambda}}{l_1!l_2!\cdots l_k!}\right)\frac{t^n}{n!},
\end{aligned} \tag{22}$$

where all l_j's are positive integers.

Therefore, by comparing the coefficients on both sides of (22), we obtain the following theorem.

Theorem 2. *For $n,k \in \mathbb{N}\cup\{0\}$, with $n \geq k$, we have*

$$S_{2,\lambda}^{(x)}(n,k) = \frac{n!}{k!}\sum_{l=k}^{n}\frac{(x)_{n-l,\lambda}}{(n-l)!}\sum_{l_1+\cdots+l_k=l}\frac{(1)_{l_1,\lambda}(1)_{l_2,\lambda}\cdots(1)_{l_k,\lambda}}{l_1!l_2!\cdots l_k!},$$

where in the inner sum, all l_j's are positive integers.

Next, we want to derive more explicit expressions for the degenerate λ-Stirling polynomials of the second kind.

From (18), we note that

$$\begin{aligned}&\sum_{n=k}^{\infty} S_{2,\lambda}^{(x)}(n,k)\frac{t^n}{n!} = e_\lambda^x(t)\frac{1}{k!}(e_\lambda(t)-1)^k\\ =&\sum_{m=0}^{\infty}\frac{(x)_{m,\lambda}}{m!}t^m\sum_{l=k}^{\infty}S_{2,\lambda}(l,k)\frac{t^l}{l!}\\ =&\sum_{n=k}^{\infty}\left(\sum_{l=k}^{n}\binom{n}{l}S_{2,\lambda}(l,k)(x)_{n-l,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{23}$$

On the other hand, we have

$$\begin{aligned}\frac{1}{k!}e_\lambda^x(t)(e_\lambda(t)-1)^k &= \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}e_\lambda^{l+x}(t)\\ &=\sum_{n=0}^{\infty}\left(\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}(l+x)_{n,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{24}$$

Therefore, by (23) and (24), we obtain the following theorem.

Theorem 3. *Let n,k be nonnegative integers with $n \geq k$. Then, we have*

$$S_{2,\lambda}^{(x)}(n,k) = \sum_{l=k}^{n}\binom{n}{l}S_{2,\lambda}(n,k)(x)_{n-l,\lambda} = \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}(l+x)_{n,\lambda}.$$

For $r \in \mathbb{N}$, we define the *r-truncated degenerate λ-Stirling polynomials of the second kind*, $S_{2,\lambda}^{(x)}(n,k|r)$, by the generating function

$$e_\lambda^x(t)\frac{1}{k!}\left(e_\lambda(t)-\sum_{j=0}^{r-1}\frac{(1)_{j,\lambda}}{j!}t^j\right)^k = \sum_{n=rk}^{\infty}S_{2,\lambda}^{(x)}(n,k|r)\frac{t^n}{n!}. \tag{25}$$

Now, we derive an explicit expression for the r-truncated degenerate λ-Stirling polynomials of the second kind. From (25), we note that

$$\begin{aligned}&\sum_{n=rk}^{\infty}S_{2,\lambda}^{(x)}(n,k|r)\frac{t^n}{n!} = e_\lambda^x(t)\frac{1}{k!}\left(e_\lambda(t)-\sum_{j=0}^{r-1}\frac{(1)_{j,\lambda}}{j!}t^j\right)^k\\ =&\frac{1}{k!}\sum_{m=0}^{\infty}(x)_{m,\lambda}\frac{t^m}{m!}\sum_{n=rk}^{\infty}\left(\sum_{l_1+\cdots+l_k=l}\frac{(1)_{l_1,\lambda}(1)_{l_2,\lambda}\cdots(1)_{l_k,\lambda}}{l_1!l_2!\cdots l_k!}\right)t^l\\ =&\frac{n!}{k!}\sum_{l=rk}^{\infty}\left(\sum_{l=rk}^{n}\frac{(x)_{n-l,\lambda}}{(n-l)!}\sum_{l_1+\cdots+l_k=l}\frac{(1)_{l_1,\lambda}(1)_{l_2,\lambda}\cdots(1)_{l_k,\lambda}}{l_1!l_2!\cdots l_k!}\right)\frac{t^n}{n!},\end{aligned} \tag{26}$$

where all l_j's are integers with $l_j \geq r$.

Therefore, by comparing the coefficients on both sides of (26), we obtain the following theorem.

Theorem 4. *Let n,k be nonnegative integers. Then, we have*

$$S_{2,\lambda}^{(x)}(n,k|r) = \frac{n!}{k!}\sum_{l=rk}^{n}\frac{(x)_{n-l,\lambda}}{(n-l)!}\sum_{l_1+\cdots+l_k=l}\frac{(1)_{l_1,\lambda}(1)_{l_2,\lambda}\cdots(1)_{l_k,\lambda}}{l_1!l_2!\cdots l_k!}, \quad (n \geq rk),$$

where the inner sum runs over all integers $l_1,\cdots,l_k \geq r$, with $l_1+\cdots+l_k = l$.

Remark 1. *When $x = 0$, $S_{2,\lambda}^{(0)}(n,k|r) = S_{2,\lambda}(n,k|r)$ are called the r-truncated degenerate λ-Stirling numbers of the second kind.*

From (25), we note that

$$\begin{aligned}\sum_{n=rk}^{\infty} S_{2,\lambda}^{(x)}(n,k|r)\frac{t^n}{n!} &= e_\lambda^x(t)\frac{1}{k!}\left(e_\lambda(t) - \sum_{j=0}^{r-1}\frac{(1)_{j,\lambda}}{j!}t^j\right)^k \\ &= \sum_{m=0}^{\infty}\frac{(x)_{m,\lambda}}{m!}t^m \sum_{l=rk}^{\infty} S_{2,\lambda}(l,k|r)\frac{t^l}{l!} \\ &= \sum_{n=rk}^{\infty}\left(\sum_{l=rk}^{n}\binom{n}{l}S_{2,\lambda}(l,k|r)(x)_{n-l,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{27}$$

Thus, by (27), we get

$$S_{2,\lambda}^{(x)}(n,k|r) = 0, \text{ if } n < kr, \tag{28}$$

and

$$S_{2,\lambda}^{(x)}(n,k|r) = \sum_{l=rk}^{n}\binom{n}{l}S_{2,\lambda}(l,k|r)(x)_{n-l,\lambda}, \text{ if } n \geq kr. \tag{29}$$

Next, we deduce a recurrence relation for the r-truncated degenerate λ-Stirling polynomials of the second kind. Now, we observe that

$$\begin{aligned}\sum_{n=rk}^{\infty} S_{2,\lambda}^{(x)}(n,k|r+1)\frac{t^n}{n!} &= \sum_{n=k(r+1)}^{\infty} S_{2,\lambda}^{(x)}(n,k|r+1)\frac{t^n}{n!} \\ &= \frac{e_\lambda^x(t)}{k!}\left(e_\lambda(t) - \sum_{j=0}^{r}\frac{(1)_{j,\lambda}}{j!}t^j\right)^k \\ &= \frac{1}{k!}e_\lambda^x(t)\left(e_\lambda(t) - \sum_{j=0}^{r-1}\frac{(1)_{j,\lambda}}{j!}t^j - \frac{(1)_{r,\lambda}}{r!}t^r\right)^k \\ &= \frac{1}{k!}e_\lambda^x(t)\sum_{l=0}^{k}\binom{k}{l}\left(e_\lambda(t) - \sum_{j=0}^{r-1}\frac{(1)_{j,\lambda}}{j!}t^j\right)^{k-l}(-1)^l\frac{(1)_{j,\lambda}^l}{(r!)^l}t^{rl} \\ &= \sum_{l=0}^{k}\frac{(-1)^l(1)_{j,\lambda}^l}{l!(r!)^l}t^{rl}\frac{1}{(k-l)!}e_\lambda^x(t)\left(e_\lambda(t) - \sum_{j=0}^{r-1}\frac{(1)_{j,\lambda}}{j!}t^j\right)^{k-l} \\ &= \sum_{l=0}^{k}\frac{(-1)^l(1)_{j,\lambda}^l}{l!(r!)^l}t^{rl}\sum_{n=(k-l)r}^{\infty} S_{2,\lambda}^{(x)}(n,k-l|r)\frac{t^n}{n!} \\ &= \sum_{l=0}^{k}\frac{(-1)^l(1)_{j,\lambda}^l}{l!(r!)^l}t^{rl}\sum_{n=kr}^{\infty} S_{2,\lambda}^{(x)}(n-lr,k-l|r)\frac{t^{n-lr}}{(n-lr)!} \\ &= \sum_{n=kr}^{\infty}\sum_{l=0}^{k}\frac{(-1)^l(1)_{j,\lambda}^l}{l!}\frac{(n)_{lr}}{(r!)^l}S_{2,\lambda}^{(x)}(n-lr,k-l|r)\frac{t^n}{n!}.\end{aligned} \tag{30}$$

Comparing the coefficients on both sides of (30), we obtain the following theorem.

Theorem 5. *Let n, k be nonnegative integers. For $n \geq kr$, we have*

$$S_{2,\lambda}^{(x)}(n,k|r+1) = \sum_{l=0}^{k}\frac{(-1)^l(1)_{j,\lambda}^l}{l!}\frac{(n)_{lr}}{(r!)^l}S_{2,\lambda}^{(x)}(n-lr,k-l|r).$$

Definition 1. *We call a random variable Y the degenerate Poisson random variable with parameter α if the probability mass function of Y is given by*

$$P[Y=y|Y\geq 0]=p(y)=e_{\lambda}^{-1}(\alpha)\frac{\alpha^{y}(1)_{y,\alpha}}{y!}. \tag{31}$$

X is called the zero-truncated degenerate Poisson random variable with parameter α if the probability mass function of X is given by

$$P[X=x|X>0]=p(x)=\frac{1}{1-e_{\lambda}^{-1}(\alpha)}e_{\lambda}^{-1}(\alpha)\frac{(1)_{x,\alpha}\alpha^{x}}{x!}. \tag{32}$$

Note that

$$\sum_{y=0}^{\infty}p(y)=e_{\lambda}^{-1}(\alpha)\sum_{y=0}^{\infty}\frac{\alpha^{y}(1)_{y,\alpha}}{y!}=1,$$

and

$$\sum_{x=1}^{\infty}p(x)=\frac{1}{e_{\lambda}(\alpha)-1}\sum_{x=1}^{\infty}\frac{(1)_{x,\alpha}\alpha^{x}}{x!}=1.$$

As an application, we show that the degenerate λ-Stirling polynomials of the second kind appear in the expression of the probability distributions of appropriate random variables. Suppose that $X_1, X_2, \cdots, X_k$ are independent random variables with degenerate zero-truncated Poisson distribution with parameter α and that Y is another random variable with degenerate Poisson distribution with parameter α. If Y is independent of $X = X_1 + X_2 + \cdots + X_k$, then we have

$$E[t^{X+Y}]=E[t^{X}]E[t^{Y}]=\left(\prod_{j=1}^{k}E[t^{X_j}]\right)E[t^{Y}].$$

From (32), we note that

$$\begin{aligned}E[t^{X_j}]&=\sum_{x=1}^{\infty}P[X_j=x]t^{x}=\frac{1}{e_{\lambda}(\alpha)-1}\sum_{x=1}^{\infty}\frac{(1)_{x,\alpha}\alpha^{x}}{x!}t^{x}\\&=\frac{1}{e_{\lambda}(\alpha)-1}(e_{\lambda}(\alpha t)-1).\end{aligned} \tag{33}$$

By (33), we get

$$\prod_{j=1}^{k}E[t^{X_j}]=\left(\frac{1}{e_{\lambda}(\alpha)-1}\right)^{k}\left(e_{\lambda}(\alpha t)-1\right)^{k}=\frac{k!}{(e_{\lambda}(\alpha)-1)^{k}}\frac{1}{k!}\left(e_{\lambda}(\alpha t)-1\right)^{k}. \tag{34}$$

By (31), we get

$$\begin{aligned}E[t^{Y}]&=\sum_{y=0}^{\infty}P[Y=y]t^{y}=e_{\lambda}^{-1}(\alpha)\sum_{y=0}^{\infty}\frac{\alpha^{y}(1)_{y,\alpha}}{y!}t^{y}\\&=e_{\lambda}^{-1}(\alpha)e_{\lambda}(\alpha t).\end{aligned} \tag{35}$$

From (34) and (35), we have

$$\begin{aligned} E[t^{X+Y}] &= \left(\prod_{j=1}^{k} E[t^{X_j}]\right) E[t^Y] \\ &= \frac{k!}{e_\lambda(\alpha)(e_\lambda(\alpha)-1)^k} \frac{1}{k!} \Big(e_\lambda(\alpha t) - 1\Big)^k e_\lambda(\alpha t) \\ &= \sum_{n=k}^{\infty} \frac{k!}{e_\lambda(\alpha)(e_\lambda(\alpha)-1)^k} S_{2,\lambda}^{(1)}(n,k)\alpha^n \frac{t^n}{n!}. \end{aligned} \tag{36}$$

On the other hand,

$$E[t^{X+Y}] = \sum_{n=k}^{\infty} P[X+Y=n]t^n. \tag{37}$$

Therefore, by (36) and (37), we obtain the following theorem.

Theorem 6. *Suppose that $X_1, X_2, \cdots, X_k$ are independent random variables with degenerate zero-truncated Poisson distribution with parameter α, and Y is another random variable with degenerate Poisson distribution with parameter α. If Y is independent of $X = X_1 + X_2 + \cdots + X_k$, then the probability distribution of $X+Y$ is given by*

$$P[X+Y=n] = \frac{k!}{e_\lambda(\alpha)(e_\lambda(\alpha)-1)^k} \frac{\alpha^n}{n!} S_{2,\lambda}^{(1)}(n,k),$$

where $n \geq k$.

For $r \in \mathbb{N}$, we define X as the random variable with the r-truncated degenerate Poisson distribution with parameter α if the probability mass function of X is given by

$$P[X=x|X\geq r] = p(x) = \frac{e_\lambda^{-1}(\alpha)}{1 - e_\lambda^{-1}(\alpha)\sum_{x=0}^{r-1}\frac{\alpha^x(1)_{x,\alpha}}{x!}} \frac{\alpha^x(1)_{x,\alpha}}{x!}. \tag{38}$$

As an application, we show that the r-truncated degenerate λ-Stirling polynomials of the second kind appear in the expression of the probability distribution of appropriate random variables. Suppose that $X_1, X_2, \cdots, X_k$ are independent random variables with r-truncated degenerate Poisson distribution with parameter α, and Y is a random variable with degenerate Poisson distribution with parameter α. If Y is independent of $X = X_1 + X_2 + \cdots + X_k$, then we have

$$E[t^{X+Y}] = E[t^X]E[t^Y] = \left(\prod_{j=1}^{k} E[t^{X_j}]\right) E[t^Y]. \tag{39}$$

From (38), we have

$$\begin{aligned} E[t^{X_j}] &= \sum_{n=r}^{\infty} P[X_j = n]t^n \\ &= \sum_{n=r}^{\infty} \left(\frac{1}{e_\lambda(\alpha) - \sum_{j=0}^{r-1}\frac{(1)_{j,\alpha}\alpha^j}{j!}}\right) \frac{\alpha^n(1)_{n,\alpha}}{n!} t^n \\ &= \left(\frac{1}{e_\lambda(\alpha) - \sum_{j=0}^{r-1}\frac{(1)_{j,\alpha}\alpha^j}{j!}}\right) \left(e_\lambda(\alpha t) - \sum_{j=0}^{r-1}\frac{(1)_{j,\alpha}\alpha^j}{j!}t^j\right) \\ &= C_\lambda(\lambda, r)\left(e_\lambda(\alpha t) - \sum_{j=0}^{r-1}\frac{(1)_{j,\alpha}\alpha^j}{j!}t^j\right), \end{aligned} \tag{40}$$

where $C_\lambda(\lambda, r) = \frac{1}{e_\lambda(\alpha) - \sum_{j=0}^{r-1} \frac{(1)_{j,\alpha}\alpha^j}{j!}}$.

By (40), we get

$$\prod_{j=1}^{k} E[t^{X_j}] = C_\lambda^k(\lambda, r)\left(e_\lambda(\alpha t) - \sum_{j=0}^{r-1} \frac{(1)_{j,\alpha}\alpha^j}{j!} t^j\right)^k. \tag{41}$$

From (35) and (41), we note that

$$\begin{aligned} E[t^{X+Y}] &= k! C_\lambda^k(\lambda, r)\frac{1}{k!}\left(e_\lambda(\alpha t) - \sum_{j=0}^{r-1} \frac{(1)_{j,\alpha}\alpha^j}{j!} t^j\right)^k e_\lambda^{-1}(\alpha) e_\lambda(\alpha t) \\ &= k! C_\lambda^k(\lambda, r) e_\lambda^{-1}(\alpha)\frac{1}{k!}\left(e_\lambda(\alpha t) - \sum_{j=0}^{r-1} \frac{(1)_{j,\alpha}\alpha^j}{j!} t^j\right)^k e_\lambda(\alpha t) \\ &= \sum_{n=kr}^{\infty} \frac{k! C_\lambda^k(\lambda, r)}{e_\lambda(\alpha)} S_{2,\lambda}^{(1)}(n, k|r) \frac{\alpha^n}{n!} t^n. \end{aligned}$$

On the other hand,

$$E[t^{X+Y}] = \sum_{n=k}^{\infty} P[X+Y=n] t^n. \tag{42}$$

Therefore, by (41) and (42), we obtain the following theorem.

Theorem 7. *For $r \in \mathbb{N}$, suppose that $X_1, X_2, \cdots, X_k$ are independent random variables with r-truncated degenerate Poisson distribution with parameter α and that Y is a random variable with degenerate Poisson distribution with parameter α. If Y is independent of $X = X_1 + X_2 + \cdots + X_k$, then the probability distribution of $X+Y$ is given by*

$$P[X+Y=n] = \frac{k! C_\lambda^k(\lambda, r)}{e_\lambda(\alpha)} S_{2,\lambda}^{(1)}(n, k|r) \frac{\alpha^n}{n!},$$

where $n \geq kr$.

Remark 2. *Suppose that $X_1, X_2, \cdots, X_k$ are independent random variables with r-truncated degenerate Poisson distribution with parameter α, and $Y_1, Y_2, \cdots, Y_m$ are independent random variables with degenerate Poisson distribution with parameter α. If $Y = Y_1 + Y_2 + \cdots + Y_m$ is independent of $X = X_1 + X_2 + \cdots + X_k$, then we have*

$$E[t^{X+Y}] = E[t^X]E[t^Y] = \left(\prod_{j=1}^{k} E[t^{X_j}]\right)\left(\prod_{i=1}^{m} E[t^{Y_i}]\right). \tag{43}$$

From (35) and (40), we have

$$\begin{aligned} E[t^{X+Y}] &= k! C_\lambda^k(\lambda, r)\frac{1}{k!}\left(e_\lambda(\alpha t) - \sum_{j=0}^{r-1} \frac{(1)_{j,\alpha}\alpha^j}{j!} t^j\right)^k e_\lambda^{-m}(\alpha) e_\lambda^m(\alpha t) \\ &= \frac{k! C_\lambda^k(\lambda, r)}{e_\lambda^m(\alpha)} \frac{1}{k!}\left(e_\lambda(\alpha t) - \sum_{j=0}^{r-1} \frac{(1)_{j,\alpha}\alpha^j}{j!} t^j\right)^k e_\lambda^m(\alpha t) \\ &= \sum_{n=kr}^{\infty} \frac{k! C_\lambda^k(\lambda, r)}{e_\lambda^m(\alpha)} S_{2,\lambda}^{(m)}(n, k|r) \frac{\alpha^n}{n!} t^n. \end{aligned} \tag{44}$$

On the other hand,

$$E[t^{X+Y}] = \sum_{n=k}^{\infty} P[X+Y=n] t^n. \tag{45}$$

From (44) and (45), we obtain the probability distribution of $X+Y$ to be

$$P[X+Y=n] = \frac{k! C_{\lambda}^{k}(\lambda, r)}{e_{\lambda}^{m}(\alpha)} S_{2,\lambda}^{(m)}(n,k|r) \frac{\alpha^n}{n!},$$

where $n \geq kr$.

3. Conclusions

The degenerate λ-Stirling polynomials of the second kind were introduced and investigated for their properties and relations in [10]. In this paper, we continued to study the degenerate λ-Stirling polynomials as well as the r-truncated degenerate λ-Stirling polynomials of the second kind which are derived from generating functions and Newton's formula. We derived recurrence relations and various expressions for them. Regarding applications, we showed that both the degenerate λ-Stirling polynomials of the second kind and the r-truncated degenerate λ-Stirling polynomials of the second kind appear in the expressions of the probability distributions of appropriate random variables. Indeed, the degenerate λ-Stirling polynomials of the second kind (more precisely, the value at 1 of them) appear in the probability distribution of the random variable given as the sum of a finite number of random variables with degenerate zero-truncated Poisson distributions and a random variable with degenerate Poisson distribution, all having the same parameter. Similarly, the r-truncated degenerate λ-Stirling polynomials of the second kind (more precisely, the value at 1 of them) appear in the probability distribution of the random variable given as the sum of a finite number of random variables with r-truncated Poisson distributions and a random variable with degenerate Poisson distribution, all having the same parameter. As one of our future projects, we will continue to pursue this line of research, namely study certain special polynomials and numbers and their applications as regards probability theory.

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References

1. Araci, S.; Acikgoz, M. A note on the Frobenius-Euler numbers and polynomials associated with Bernstein polynomials. *Adv. Stud. Contemp. Math. (Kyungshang)* **2012**, *22*, 399–406.
2. Carlitz, L. Degenerate Stirling, Bernoulli and Eulerian numbers. *Util. Math.* **1979**, *15*, 51–88.
3. Dolgy, D.V.; Kim, T. Some explicit formulas of degenerate Stirling numbers associated with the degenerate special numbers and polynomials. *Proc. Jangjeon Math. Soc.* **2018**, *21*, 309–317.
4. Feller, W. *An Introduction to Probability Theory and Its Application;* John Wiley: Hoboken, NJ, USA, 1970.
5. He, Y.; Pan, J. Some recursion formulae for the number of derangements and Bell numbers. *J. Math. Res. Appl.* **2016**, *36*, 15–22.
6. Howard, F.T. Bell polynomials and degenerate Stirling numbers. *Rend. Sem. Mat. Univ. Padova.* **1979**, *61*, 203–219.
7. Jeong, J.; Rim, S.-H.; Kim, B.M. On finite-times degenerate Cauchy numbers and polynomials. *Adv. Differ. Equ.* **2015**, *2015*, 321. [CrossRef]
8. Kim, D.S.; Kim, T.; Jang, G.-W. A note on degenerate Stirling numbers of the first kind. *Proc. Jangjeon Math. Soc.* **2018**, *21*, 393–404.

9. Kim, D.S.; Dolgy, D.V.; Kim, D.; Kim, T. Some identities on r-central factorial numbers and r-central Bell polynomials. *Adv. Differ. Equ.* **2019**, *2019*, 245. [CrossRef]
10. Kim, T. A note on degenerate Stirling polynomials of the second kind. *Proc. Jangjeon Math. Soc.* **2017**, *20*, 319–331.
11. Kim, T.; Kim, D.S.; Jang, G.-W. Extended Stirling polynomials of the second kind and extended Bell polynomials. *Proc. Jangjeon Math. Soc.* **2017**, *20*, 365–376.
12. Kim, T.; Jang, G.-W. A note on degenerate gamma function and degenerate Stirling number of the second kind. *Adv. Stud. Contemp. Math. (Kyungshang)* **2018**, *28*, 207–214.
13. Kim, T.; Yao, Y.; Kim, D.S.; Kwon, H.-I. Some identities involving special numbers and moments of random variables. *Rocky Mt. J. Math.* **2019**, *49*, 521–538. [CrossRef]
14. Kim, T.; Kim, D.S. Extended Stirling numbers of the first kind associated with Daehee numbers and polynomials. *Adv. Stud. Contemp. Math. (Kyungshang)* **2018**, *28*, 127–138. [CrossRef]
15. Kim, T.; Kim, D.S.; Kwon, H.-I. A note on degenerate Stirling numbers and their applications. *Proc. Jangjeon Math. Soc.* **2018**, *21*, 195–203.
16. Kim,T.; Kim, D.S. On λ-Bell polynomials associated with umbral calculus. *Russ. J. Math. Phys.* **2017**, *24*, 69–78. [CrossRef]
17. Koutras, M. Non-central stirling numbers and some applications. *Discret. Math.* **1982**, *42*, 73–89. [CrossRef]
18. Ross, S.M. *Introduction to Probability Models*; Academic Press: Cambridge, MA, USA, 2007.
19. Brualdi, R.A. *Introductory Combinatorics*; China Machine Press: Beijing, China, 2009.
20. Kotz, S.; Kozubowski, T.J.; Podgorski, K. *The Laplace Distribution and Generalizations*; Birkhäuser Boston, Inc.: Boston, MA, USA, 2001.
21. Kim, D.S.; Kim, H.Y.; Kim, D.; Kim, T. Identities of symmetry for type 2 Bernoulli and Euler polynomials. *Symmetry* **2019**, *11*, 613. [CrossRef]
22. Dolgy, D.V.; Kim, D.S.; Kwon, J.; Kim, T. Some identities of ordinary and degenerate Bernoulli numbers and polynomials. *Symmetry* **2019**, *11*, 847. [CrossRef]
23. Kim, D.S.; Dolgy, D.V.; Kwon, J.; Kim, T. Note on type 2 degenerate q-Bernoulli polynomials. *Symmetry* **2019**, *11*, 914. [CrossRef]
24. Kim, T.; Kim, D.S. Degenerate Bernstein polynomials. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **2019**, *113*, 2913–2920. [CrossRef]
25. Kim, T.; Kim, D.S. A note on type 2 Changhee and Daehee polynomials. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **2019**, *113*, 2783–2791. [CrossRef]
26. Kim, T.; Kim, D.S. Degenerate central Bell numbers and polynomials. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **2019**, *113*, 2507–2513. [CrossRef]
27. Kim, T.; Kim, D.S. Identities for degenerate Bernoulli polynomials and Korobov polynomials of the first kind. *Sci. China Math.* **2019**, *62*, 999–1028. [CrossRef]
28. Kim, T.; Kim, D.S. Degenerate Laplace transform and degenerate gamma function. *Russ. J. Math. Phys.* **2017**, *24*, 241–248. [CrossRef]