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Some Applications of the $(G'/G, 1/G)$-Expansion Method for Finding Exact Traveling Wave Solutions of Nonlinear Fractional Evolution Equations

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Abstract: In this paper, the $(G'/G, 1/G)$-expansion method is applied to acquire some new, exact solutions of certain interesting, nonlinear, fractional-order partial differential equations arising in mathematical physics. The considered equations comprise the time-fractional, (2+1)-dimensional extended quantum Zakharov-Kuznetsov equation, and the space-time-fractional generalized Hirota-Satsuma coupled Korteweg-de Vries (KdV) system in the sense of the conformable fractional derivative. Applying traveling wave transformations to the equations, we obtain the corresponding ordinary differential equations in which each of them provides a system of nonlinear algebraic equations when the method is used. As a result, many analytical exact solutions obtained of these equations are expressed in terms of hyperbolic function solutions, trigonometric function solutions, and rational function solutions. The graphical representations of some obtained solutions are demonstrated to better understand their physical features, including bell-shaped solitary wave solutions, singular soliton solutions, solitary wave solutions of kink type, and so on. The method is very efficient, powerful, and reliable for solving the proposed equations and other nonlinear fractional partial differential equations with the aid of a symbolic software package.

Keywords: time-fractional (2+1)-dimensional extended quantum Zakharov-Kuznetsov equation; space-time-fractional generalized Hirota-Satsuma coupled Korteweg-de Vries system; $(G'/G, 1/G)$-expansion method; conformable fractional derivative

1. Introduction

Nonlinear evolution equations (NLEEs), which can be described using partial differential equations (PDEs), play a significant role for understanding qualitative behaviors of many real-world phenomena. Obtaining exact solutions of a complicated nonlinear evolution system makes it possible to visually comprehend the mechanism of the system considered. Nonlinear wave phenomena occur in various scientific and engineering fields, such as quantum mechanics [1], fluid mechanics [2], optical fibers [3], chemical physics and geochemistry [4], solid-state physics [5], and biology [6]. With the advanced development of symbolically computational packages, such as Maple or Mathematica, constructing for the exact traveling wave solutions of NLEEs has become one of the important themes of challenging interest in mathematical physics and the applied sciences.

Over the last few decades, many kinds of solutions of NLEEs, including exact solutions, analytical approximate solutions, and numerical solutions, have been successfully obtained using various and efficient methods. Examples of the methods for obtaining analytical approximate solutions of NLEEs are the Adomian decomposition method (ADM) [7,8], the revised variational iteration method (RVIM) [9], the reduced differential transform method [10], and the homotopy perturbation method (HPM) [11,12]. Useful methods for solving NLEEs numerically are those such as the finite element method [13], the finite volume method [14], and the finite-difference predictor–corrector method [15]. Several efficient and reliable methods which have recently been developed to obtain exact explicit solutions for NLEEs are, for instance, the Jacobi elliptic function method [16], the (G'/G)-expansion method and its various modifications [17–20], the Exp-function method [21], the sub-equation method [22], the first integral method [23], the modified trial equation method [24,25], and the simplest equation method [26].

Recently, fractional differential equations (FDEs) has been able to be used extensively as the generalized type of integer-order differential equations, including ordinary differential equations and partial differential equations. FDEs have attracted the researchers' attention for modeling real-world phenomena, such as modeling anomalous diffusion using a nonlinear fractional Fokker–Planck equation with fractional velocity derivatives and Langevin dynamics to elucidate the effect of non-local transport in the plasma turbulence [27]. More examples of applications of FDEs for real-world problems can be found in [28–30] and the references therein. In general, systems actually may not rely only on the local time but also on the former time in history. Hence, the memory and hereditary properties of materials and processes can be described using the theory of fractional derivatives and integrals [31–33]. In consequence, nonlinear fractional evolution equations (NLFEEs) have been widely investigated in many aspects, for example, solving the equations for solutions and establishing conditions for which their solutions are asymptotically stable. The exploration for exact solutions of NLFEEs is currently of high interest in applied mathematics and engineering research [19,34,35]. However, the objective of our work is to use the $(G'/G, 1/G)$-expansion method to construct exact traveling wave solutions of the following two NLFEEs in the sense of the conformable fractional derivative.

1. The time-fractional (2+1)-dimensional extended quantum Zakharov-Kuznetsov equation [36] is written as:

$$D_t^{\alpha} u + auu_x + b(u_{xxx} + u_{yyy}) + c(u_{xyy} + u_{yxx}) = 0, \tag{1}$$

where $D_t^{\alpha} u$ denotes the conformable fractional derivative of u with respect to t of order α and a, b, c are real constants, and the solution $u(x, y, t)$, which is a function of the time variable t and space variables x and y, represents the potential of electrostatic wave in space. Solutions of the equation elucidates the spreading of optical pulse in fiber optics [36]. Some articles involved in finding exact solutions of Equation (1) are as follows. Raza et al., [36] found the exact solutions of Equation (1), consisting of the trigonometric function, Jacobi elliptic sine-cosine functions, and hyperbolic function solutions, using the trial equation method. Conversely, Ali et al., [37] obtained the exact solutions of Equation (1) using the (G'/G^2)-expansion method and the modified Kudryashov method. Their exact solutions include the trigonometric, hyperbolic, and rational solutions.

2. The space-time-fractional generalized Hirota-Satsuma coupled Korteweg de Vries (KdV) system [38] can be expressed as:

$$\begin{aligned} D_t^{\rho} u &= \frac{1}{4} D_x^{3\eta} u + 3u D_x^{\eta} u + 3 D_x^{\eta}(-v^2 + w), \\ D_t^{\rho} v &= -\frac{1}{2} D_x^{3\eta} v - 3u D_x^{\eta} v, \\ D_t^{\rho} w &= -\frac{1}{2} D_x^{3\eta} w - 3u D_x^{\eta} w, \end{aligned} \tag{2}$$

where $D_t^{\rho} \varphi$ and $D_x^{\eta} \varphi$ denote the conformable fractional derivative of φ with respect to t of order ρ and to x of order η, respectively. The first-order Hirota-Satsuma coupled KdV system [39], which was first proposed by Satsuma and Hirota in 1981 and obtained from the four reductions of Kadomtsev-Petviashvili (KP) hierarchy [40], describes interactions of two long waves with different dispersion relations, while the generalized first-order Hirota-Satsuma coupled KdV system [40] is one of the essential nonlinear equations in applied mathematics and physics. The system emerges as a special case of the Toda lattice equation, which is used to describe the interaction of neighboring particles of equal mass in a lattice formation with a crystal [41]. The interesting applications of the generalized Hirota-Satsuma coupled KdV system are as follows [41–43]. Firstly, it can be used to explain generic properties of string dynamics for strings and multi-strings in constant curvature space. Secondly, the system is associated with most types of long waves with weak dispersion, internal, acoustic, and planetary waves in geophysical hydrodynamics. Therefore, finding solutions of Equation (2) is potentially useful to describing the physical behaviors of the applications, as mentioned above. The associated equations of the generalized Hirota-Satsuma coupled KdV system have been solved using different methods as follows. In 2007, Zhang [44] used the direct algebraic method to construct the exact solutions for the first-order generalized Hirota-Satsuma coupled KdV systems. In 2010, Zigao et al., [45] applied the improved F-expansion method to the variable-coefficient first-order generalized Hirota-Satsuma coupled KdV system for obtaining the new exact solutions. In 2017, Khater et al., [46] found the exact traveling wave solutions of the system using the modified simple equation method, while the time-fractional generalized Hirota-Satsuma coupled KdV system was solved using the direct algebraic method by Neirameh [41] in 2015.

The rest of this article is organized as follows. In Section 2, the description of the conformable fractional derivative and its important properties are presented. In Section 3, the main steps of the $(G'/G, 1/G)$-expansion method is provided. The applications of the method for solving the two problems mentioned are given in Section 4. Finally, the conclusions of this paper are discussed in Section 5.

2. Conformable Fractional Derivative and Its Properties

In this section, the definition of the conformable fractional derivative and its important properties are given as follows.

Definition 1. *Given a function $f : [0, \infty) \to \mathbb{R}$, the conformable fractional derivative of f of order α is defined by [47,48]*

$$D_t^\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \text{ for all } t > 0,\ 0 < \alpha \le 1. \tag{3}$$

If the limit in Equation (3) exists, then we say that f is α-conformable differentiable at a point $t > 0$.

Theorem 1. *Let $\alpha \in (0, 1]$, and $f(t)$, $g(t)$ be α-conformable differentiable at a point $t > 0$, then*

$$\begin{aligned}
D_t^\alpha(\lambda) &= 0, \text{ where } \lambda = \text{constant}, \\
D_t^\alpha(t^\mu) &= \mu t^{\mu-\alpha}, \text{ for all } \mu \in \mathbb{R}, \\
D_t^\alpha(af(t) + bg(t)) &= aD_t^\alpha f(t) + bD_t^\alpha g(t), \text{ for all } a, b \in \mathbb{R}, \\
D_t^\alpha(f(t)g(t)) &= f(t)D_t^\alpha g(t) + g(t)D_t^\alpha f(t), \\
D_t^\alpha\left(\frac{f(t)}{g(t)}\right) &= \frac{g(t)D_t^\alpha f(t) - f(t)D_t^\alpha g(t)}{g(t)^2}.
\end{aligned}$$

Remark 1. *Conformable fractional derivative of some functions are as follows [47].*

(1) $D_t^\alpha(e^{ct}) = ct^{1-\alpha}e^{ct}$, $c \in \mathbb{R}$.
(2) $D_t^\alpha(\sin bt) = bt^{1-\alpha}\cos bt$, $b \in \mathbb{R}$.
(3) $D_t^\alpha(\cos bt) = -bt^{1-\alpha}\sin bt$, $b \in \mathbb{R}$.
(4) $D_t^\alpha(\frac{1}{\alpha}t^\alpha) = 1$.
(5) $D_t^\alpha(f(t)) = t^{1-\alpha}\dfrac{df(t)}{dt}$, *provided that $f(t)$ is differentiable.*

The following chain rule is very useful for transforming a partial differential equation into an ordinary differential equation, which is required for the methods in finding exact solutions of the equations.

Theorem 2. *Let $f : (0, \infty) \to \mathbb{R}$ be a function such that f is differentiable and α-conformable differentiable. Also, let g be a differentiable function defined in the range of f. Then,*

$$D_t^\alpha (f \circ g)(t) = t^{1-\alpha} f'(g(t)) g'(t),$$

where the prime notation ($'$) represents the ordinary derivative.

3. Algorithm of the $(G'/G, 1/G)$-Expansion Method

In this section, the description of the $(G'/G, 1/G)$-expansion method [19,49–51] is concisely provided. Consider a nonlinear fractional evolution partial differential equation in three independent variables, x, y, and t, as follows:

$$F(u, D_t^\alpha u, D_x^\beta u, D_y^\gamma u, D_t^{2\alpha} u, D_t^\alpha D_x^\beta u, D_t^\alpha D_y^\gamma u, \ldots) = 0, \ \ 0 < \alpha, \beta, \gamma \le 1, \tag{4}$$

where $D_t^\alpha u$, $D_x^\beta u$, and $D_y^\gamma u$ are the conformable derivatives of a dependent variable u with respect to independent variables t, x, and y when F is a polynomial of unknown function $u = u(x, y, t)$, and its various partial derivatives are those in which the highest order derivatives and nonlinear terms are involved.

Using the following traveling wave transformation

$$u(x, y, t) = U(\xi), \ \ \xi = \frac{kx^\beta}{\beta} + \frac{ly^\gamma}{\gamma} + \frac{ct^\alpha}{\alpha}, \tag{5}$$

where k, l, and c are constants to be determined later, then Equation (4) is reduced to an ODE in $U = U(\xi)$ as

$$P(U, U', U'', \ldots) = 0, \tag{6}$$

where P is a polynomial of $U(\xi)$ and its various derivatives. The prime notation ($'$) in the above equation denotes the derivative with respect to ξ.

The following necessary concepts are introduced before providing the main steps of the $(G'/G, 1/G)$-expansion method. Consider the following second-order linear ODE:

$$G''(\xi) + \lambda G(\xi) = \mu, \tag{7}$$

where the prime notation ($'$) denotes the derivative with respect to ξ and where λ, μ, are constants. Next, we set

$$\phi(\xi) = \frac{G'(\xi)}{G(\xi)} \ \text{ and } \ \psi(\xi) = \frac{1}{G(\xi)}. \tag{8}$$

Equations (7) and (8) can be transformed into the system of two nonlinear ODEs, as follows:

$$\phi' = -\phi^2 + \mu\psi - \lambda, \ \ \psi' = -\phi\psi. \tag{9}$$

The solutions of Equation (7) can be classified into the following three cases.

Case 1: If $\lambda < 0$, then the general solution of Equation (7) is written as

$$G(\xi) = A_1 \sinh\left(\xi\sqrt{-\lambda}\right) + A_2 \cosh\left(\xi\sqrt{-\lambda}\right) + \frac{\mu}{\lambda}, \tag{10}$$

and we have

$$\psi^2 = \frac{-\lambda}{\lambda^2\sigma_1 + \mu^2}\left(\phi^2 - 2\mu\psi + \lambda\right), \tag{11}$$

where A_1 and A_2 are arbitrary constants and $\sigma_1 = A_1^2 - A_2^2$.

Case 2: If $\lambda > 0$, then the general solution of Equation (7) can be given as

$$G(\xi) = A_1 \sin\left(\xi\sqrt{\lambda}\right) + A_2 \cos\left(\xi\sqrt{\lambda}\right) + \frac{\mu}{\lambda}, \tag{12}$$

and we have the following relation

$$\psi^2 = \frac{\lambda}{\lambda^2\sigma_2 - \mu^2}\left(\phi^2 - 2\mu\psi + \lambda\right), \tag{13}$$

where A_1 and A_2 are arbitrary constants and $\sigma_2 = A_1^2 + A_2^2$.

Case 3: If $\lambda = 0$, then the general solution of Equation (7) can be provided as

$$G(\xi) = \frac{\mu}{2}\xi^2 + A_1\xi + A_2, \tag{14}$$

and the corresponding relation is

$$\psi^2 = \frac{1}{A_1^2 - 2\mu A_2}\left(\phi^2 - 2\mu\psi\right), \tag{15}$$

where A_1 and A_2 are arbitrary constants.

The main steps of the $(G'/G, 1/G)$-expansion method are described as follows.

Step 1: Suppose that the solution to Equation (6) can be expressed by a polynomial of the two variables ϕ and ψ, as follows:

$$U(\xi) = a_0 + \sum_{j=1}^{N} a_j\phi^j + \sum_{j=1}^{N} b_j\phi^{j-1}\psi, \tag{16}$$

where a_0, a_j and b_j $(j = 1, 2, ..., N)$ are constants to be determined later with $a_N^2 + b_N^2 \neq 0$ and where the functions $\phi = \phi(\xi)$ and $\psi = \psi(\xi)$ are implicitly associated with Equation (7) using the relations in Equation (8).

Step 2: Determine the positive integer N in Equation (16) by inserting Equation (16) into Equation (6), and then using the homogeneous balance between the highest-order derivatives and the nonlinear terms in Equation (6). If the degree of $U(\xi)$ is $\mathrm{Deg}[U(\xi)] = N$, then the degree of other terms will be formulated as follows:

$$\mathrm{Deg}\left[\frac{d^qU(\xi)}{d\xi^q}\right] = N + q, \quad \mathrm{Deg}\left[(U(\xi))^p\left(\frac{d^qU(\xi)}{d\xi^q}\right)^s\right] = Np + s(N+q). \tag{17}$$

In particular, if the balance number N of some nonlinear equations is not a positive integer (e.g., a fraction and a negative integer), then the special transformations are applied for $U(\xi)$ in Equation (6) to have a new equation in terms of the new function $W(\xi)$ with a positive integer balance number (see details in [51,52]).

Step 3: Substituting the resulting equation of Equation (16) into Equation (6) with the aid of Equations (9) and (11), the function P in Equation (6) can be transformed into a polynomial in ϕ and ψ, in which the degree of ψ is not larger than one. Equating each coefficient of the resulting polynomial to zero, we obtain a system of algebraic equations, which can be solved using symbolic computational packages, such as Maple or Mathematica, for the following unknowns: a_0, a_j, b_j $(j = 1, 2, \ldots, N)$, k, l, c, μ, $\lambda(< 0)$, A_1, and A_2. The resulting traveling wave solutions generated by this step with the transformation in Equation (5) are expressed in terms of hyperbolic functions.

Step 4: In the same manner as Step 3, substituting the resulting equation of Equation (16) into Equation (6) with the aid of Equations (9) and (13) for the case $\lambda > 0$, we can obtain the exact solutions of Equation (4) by using the transformation in Equation (5). They are written as trigonometric functions.

Step 5: Similarly to Step 3, substituting the resulting equation of Equation (16) into Equation (6) with the aid of Equations (9) and (15) for the case $\lambda = 0$, we can obtain the traveling wave solutions of Equation (4) by using the transformation in Equation (5). The resulting exact solutions are expressed as rational functions.

Remark 2. *The two-variable $(G'/G, 1/G)$-expansion method reduces to the (G'/G)-expansion method when $\mu = 0$ and $b_j = 0$ in Equations (7) and (16), respectively. In consequence, the $(G'/G, 1/G)$-expansion method is an extension of the (G'/G)-expansion method. Hence, the strength of the $(G'/G, 1/G)$-expansion method beyond the (G'/G)-expansion method is that the solutions obtained using the second method can be drawn from the solutions obtained using the first one. This is the reason why the $(G'/G, 1/G)$-expansion method is used in our work instead of the (G'/G)-expansion method.*

4. Applications of the $(G'/G, 1/G)$-Expansion Method

4.1. The Time-Fractional (2+1)-Dimensional Extended Quantum Zakharov-Kuznetsov Equation

Applying the transformation $\xi = -k\frac{t^\alpha}{\alpha} + x + y$ to Equation (1), we attain the following ordinary differential equation

$$-kU' + aUU' + 2(b + c)U''' = 0. \tag{18}$$

Integrating (18) with respect to ξ, it gives

$$-kU + \frac{a}{2}U^2 + 2(b + c)U'' + p = 0, \tag{19}$$

where p is a constant of integration. Applying the homogeneous balance principle to the terms U^2 and U'' in Equation (18), we obtain $N = 2$. Hence, the specific form of the solution in Equation (16) is written as

$$U(\xi) = a_0 + a_1\phi(\xi) + a_2\phi(\xi)^2 + b_1\psi(\xi) + b_2\psi(\xi)\phi(\xi), \tag{20}$$

where the constant coefficients a_0, a_1, a_2, b_1, and b_2 are determined at a later step, provided that $a_2^2 + b_2^2 \neq 0$. Using the $(G'/G, 1/G)$-expansion method, there are three cases of the function $G(\xi)$ associated with the functions $\phi(\xi)$ and $\psi(\xi)$, depending on the sign of λ in Equation (7) as described above.

Case 1: Hyperbolic function solutions ($\lambda < 0$)

If $\lambda < 0$, we substitute Equation (20) into Equation (19) along with the use of Equation (9) and Equation (11). Then, the left-hand side of (19) turns out to be a polynomial in $\phi(\xi)$ and $\psi(\xi)$. Equating all the coefficients of the resulting polynomial to be zero, we obtain the following system of nonlinear algebraic equations in $a_0,\ a_1,\ a_2,\ b_1,\ b_2,\ A_1,\ A_2,\ \lambda,\ \mu,\ k$, and p, provided that $\lambda^2\left(A_1^2-A_2^2\right)+\mu^2\neq 0$.

$$
\begin{aligned}
\phi^4(\xi):\ & 2\,a\lambda^2A_1^2a_1b_2+2\,a\lambda^2A_1^2a_2b_1-2a\lambda^2A_2^2a_1b_2-2a\lambda^2A_2^2a_2b_1-40b\lambda^2\mu A_1^2a_2+8c\mu^2b_1\\
&+40b\lambda^2\mu\,A_2^2a_2-40c\lambda^2\mu\,A_1^2a_2+40\,c\lambda^2\mu\,A_2^2a_2+8\,b\lambda^2A_1^2b_1-8\,b\lambda^2A_2^2b_1+8\,c\lambda^2A_1^2b_1\\
&-8\,c\lambda^2A_2^2b_1+2\,a\lambda\,\mu\,b_2^2+2a\mu^2a_1b_2+2a\mu^2a_2b_1-40b\mu^3a_2-40c\mu^3a_2+8b\mu^2b_1=0,\\
\phi^3(\xi):\ & 2a\lambda^2A_1^2a_1a_2-2a\lambda^2A_2^2a_1a_2+8b\lambda^2A_1^2a_1-8b\lambda^2A_2^2a_1+8c\lambda^2A_1^2a_1-8c\lambda^2A_2^2a_1\\
&+2a\mu^2a_1a_2-2a\lambda\,b_1b_2+24b\lambda\mu\,b_2+8b\mu^2a_1+24c\lambda\mu b_2+8c\mu^2a_1=0,\\
\phi^3(\xi)\psi(\xi):\ & 2\,a\lambda^2A_1^2a_2b_2-2a\lambda^2A_2^2a_2b_2+24b\lambda^2A_1^2b_2-24b\lambda^2A_2^2b_2+24c\lambda^2A_1^2b_2-24c\lambda^2A_2^2b_2\\
&+2a\mu^2a_2b_2+24b\mu^2b_2+24c\mu^2b_2=0,\\
\phi^2(\xi):\ & 2\,a\lambda^2A_1^2a_0a_2+a\lambda^2A_1^2a_1^2-2a\lambda^2A_2^2a_0a_2-a\lambda^2A_2^2a_1^2+32b\lambda^3A_1^2a_2-32b\lambda^3A_2^2a_2\\
&+32c\lambda^3A_1^2a_2-32c\lambda^3A_2^2a_2-2k\lambda^2A_1^2a_2+2k\lambda^2A_2^2a_2-a\lambda^2b_2^2+2a\mu^2a_0a_2+a\mu^2a_1^2\\
&+24b\lambda\mu^2a_2+24c\lambda\mu^2a_2-a\lambda\,b_1^2+4b\lambda\mu\,b_1+4c\lambda\mu b_1-2k\mu^2a_2=0,\\
\phi^2(\xi)\psi(\xi):\ & 2\,a\lambda^2A_1^2a_1b_2+2a\lambda^2A_1^2a_2b_1-2a\lambda^2A_2^2a_1b_2-2a\lambda^2A_2^2a_2b_1-40b\lambda^2\mu\,A_1^2a_2+2a\lambda\mu b_2^2\\
&-40\,c\lambda^2\mu\,A_1^2a_2+40\,c\lambda^2\mu\,A_2^2a_2+8\,b\lambda^2A_1^2b_1-8\,b\lambda^2A_2^2b_1+8\,c\lambda^2A_1^2b_1-8\,c\lambda^2A_2^2b_1\\
&+2\,a\mu^2a_1b_2+2\,a\mu^2a_2b_1-40\,b\mu^3a_2-40\,c\mu^3a_2+8\,b\mu^2b_1+8\,c\mu^2b_1+40\,b\lambda^2\mu\,A_2^2a_2=0,\\
\phi(\xi):\ & 2\,a\lambda^2A_1^2a_0a_1-2\,a\lambda^2A_2^2a_0a_1+8\,b\lambda^3A_1^2a_1-8\,b\lambda^3A_2^2a_1+8\,c\lambda^3A_1^2a_1-8\,c\lambda^3A_2^2a_1\\
&-2k\lambda^2A_1^2a_1+2k\lambda^2A_2^2a_1-2\,a\lambda^2b_1b_2+2a\mu^2a_0a_1+24\,b\lambda^2\mu\,b_2+8b\lambda\,\mu^2a_1+24\,c\lambda^2\mu\,b_2\\
&+8\,c\lambda\,\mu^2a_1-2\,k\mu^2a_1=0,\\
\phi(\xi)\psi(\xi):\ & 2\,a\lambda^2A_1^2a_0b_2+2\,a\lambda^2A_1^2a_1b_1-2\,a\lambda^2A_2^2a_0b_2-2\,a\lambda^2A_2^2a_1b_1+20\,b\lambda^3A_1^2b_2-20\,b\lambda^3A_2^2b_2\\
&-12\,b\lambda^2\mu\,A_1^2a_1+12\,b\lambda^2\mu\,A_2^2a_1+20\,c\lambda^3A_1^2b_2-20c\lambda^3A_2^2b_2-12c\lambda^2\mu\,A_1^2a_1-2\,k\mu^2b_2\\
&-2\,k\lambda^2A_1^2b_2+2\,k\lambda^2A_2^2b_2+4a\lambda\mu\,b_1b_2+2\,a\mu^2a_0b_2+2\,a\mu^2a_1b_1-28\,b\lambda\,\mu^2b_2-12\,b\mu^3a_1\\
&-28c\lambda\mu^2b_2-12c\mu^3a_1+12c\lambda^2\mu\,A_2^2a_1=0,\\
\psi(\xi):\ & -16\,b\lambda^3\mu\,A_1^2a_2+16\,b\lambda^3\mu\,A_2^2a_2-16\,c\lambda^3\mu\,A_1^2a_2+16\,c\lambda^3\mu\,A_2^2a_2+2\,a\lambda^2A_1^2a_0b_1\\
&-2\,a\lambda^2A_2^2a_0b_1+4\,b\lambda^3A_1^2b_1-4\,b\lambda^3A_2^2b_1+4\,c\lambda^3A_1^2b_1-4\,c\lambda^3A_2^2b_1-2\,k\lambda^2A_1^2b_1\\
&+2\,k\lambda^2A_2^2b_1+2\,a\lambda\,\mu\,b_1^2+2\,a\mu^2a_0b_1-4\,b\lambda\,\mu^2b_1-4\,c\lambda\,\mu^2b_1-2\,k\mu^2b_1=0,\\
\phi(\xi)^0:\ & 8\,b\lambda^4A_1^2a_2-8b\lambda^4A_2^2a_2+8c\lambda^4A_1^2a_2-8c\lambda^4A_2^2a_2+a\lambda^2A_1^2a_0^2-a\lambda^2A_2^2a_0^2-2\,\lambda^2pA_2^2\\
&-2k\lambda^2A_1^2a_0+2k\lambda^2A_2^2a_0-a\lambda^2b_1^2+a\mu^2a_0^2+4\,b\lambda^2\mu\,b_1+4c\lambda^2\mu\,b_1+2\lambda^2pA_1^2+2\mu^2p\\
&-2k\mu^2a_0=0.
\end{aligned}
\tag{21}
$$

Using the Maple package program to solve the above algebraic system, we obtain the following results.

Result 1:

$$
\begin{aligned}
&a_0 = -\frac{16(b+c)-k}{a},\ a_1 = 0,\ a_2 = -\frac{24(b+c)}{a},\ b_1 = 0,\ b_2 = 0, \\
&p = -\frac{64\,(b+c)^2\,\lambda^2 - k^2}{2a},\ \mu = 0,\ \lambda = \lambda,\ k = k,
\end{aligned} \tag{22}
$$

where $a \neq 0,\ b,\ c,\ \lambda(<0),\ \mu,\ k$ are arbitrary constants. From Equations (10), (20), and (22), we obtain the traveling wave solution of Equation (1) as follows:

$$
u(x,y,t) = -\frac{16(b+c)+k}{a} + \frac{24\lambda\,\left(\cosh\left(\xi\sqrt{-\lambda}\right)A_1 + \sinh\left(\xi\sqrt{-\lambda}\right)A_2\right)^2(b+c)}{\left(A_1\sinh\left(\xi\sqrt{-\lambda}\right) + A_2\cosh\left(\xi\sqrt{-\lambda}\right)\right)^2 a}, \tag{23}
$$

where $\xi = -k\frac{t^\alpha}{\alpha} + x + y$ and $A_1,\ A_2$ are arbitrary constants.

Result 2:

$$
\begin{aligned}
&a_0 = a_0,\ a_1 = 0,\ a_2 = -\frac{12\,(b+c)}{a},\ b_1 = \frac{12\mu\,(b+c)}{a},\ b_2 = \pm\frac{12(b+c)}{a}\sqrt{\frac{-\sigma_1\,\lambda^2-\mu^2}{\lambda}}, \\
&p = \frac{(12\lambda\,(b+c)+aa_0)\,(8\lambda\,(b+c)+aa_0)}{2a},\ \mu = \mu,\ \lambda = \lambda,\ k = aa_0 + 10\,b\lambda + 10\,c\lambda,
\end{aligned} \tag{24}
$$

where $a_0,\ a \neq 0,\ b,\ c,\ \lambda(<0),\ \mu$ are arbitrary constants and $\sigma_1 = A_1^2 - A_2^2$, where $A_1,\ A_2$ are arbitrary constants. From Equations (12), (20), and (24), we obtain the exact solution of Equation (1) as follows:

$$
\begin{aligned}
u(x,y,t) = a_0 &+ \frac{12\lambda^3\left(\cosh\left(\xi\sqrt{-\lambda}\right)A_1 + \sinh\left(\xi\sqrt{-\lambda}\right)A_2\right)^2(b+c)}{\left(A_2\cosh\left(\xi\sqrt{-\lambda}\right)\lambda + A_1\sinh\left(\xi\sqrt{-\lambda}\right)\lambda + \mu\right)^2 a} \\
&+ \frac{12\lambda\,\mu\,(b+c)}{\left(A_2\cosh\left(\xi\sqrt{-\lambda}\right)\lambda + A_1\sinh\left(\xi\sqrt{-\lambda}\right)\lambda + \mu\right)a} \\
&\mp \frac{12\,(-\lambda)^{3/2}\sqrt{\lambda}\left(\cosh\left(\xi\sqrt{-\lambda}\right)A_1 + \sinh\left(\xi\sqrt{-\lambda}\right)A_2\right)\sqrt{-\sigma_1\lambda^2-\mu^2}\,(b+c)}{\left(A_2\cosh\left(\xi\sqrt{-\lambda}\right)\lambda + A_1\sinh\left(\xi\sqrt{-\lambda}\right)\lambda + \mu\right)^2 a},
\end{aligned} \tag{25}
$$

where $\xi = -k\frac{t^\alpha}{\alpha} + x + y$ with k defined in Equation (24).

Case 2: Trigonometric function solutions ($\lambda > 0$)

If $\lambda > 0$, we insert Equation (20) into Equation (19) along with the use of Equations (9) and (13). Then, the left-hand side of (19) becomes a polynomial in $\phi(\xi)$ and $\psi(\xi)$. Setting all of coefficients of this resulting polynomial to be zero, we have the following system of nonlinear algebraic equations in $a_0,\ a_1,\ a_2,\ b_1,\ b_2,\ A_1,\ A_2,\ \lambda,\ \mu,\ k,$ and p, provided that $\lambda^2\left(A_1^2+A_2^2\right)-\mu^2\neq 0$.

$$
\begin{aligned}
\phi^4(\xi) :\ & a\lambda^2A_1^2a_2^2+a\lambda^2A_2^2a_2^2+24\,b\lambda^2A_1^2a_2+24\,b\lambda^2A_2^2a_2+24c\lambda^2A_1^2a_2+24c\lambda^2A_2^2a_2\\
&-a\mu^2a_2^2+a\lambda\,b_2^2-24\,b\mu^2a_2-24\,c\mu^2a_2=0,\\
\phi^3(\xi) :\ & 2\,a\lambda^2A_1^2a_1a_2+2\,a\lambda^2A_2^2a_1a_2+8\,b\lambda^2A_1^2a_1+8\,b\lambda^2A_2^2a_1+8\,c\lambda^2A_1^2a_1+8\,c\lambda^2A_2^2a_1\\
&-2\,a\mu^2a_1a_2+2\,a\lambda\,b_1b_2-24\,b\lambda\,\mu\,b_2-8\,b\mu^2a_1-24\,c\lambda\,\mu\,b_2-8\,c\mu^2a_1=0,\\
\phi^3(\xi)\psi(\xi) :\ & 2\,a\lambda^2A_1^2a_2b_2+2\,a\lambda^2A_2^2a_2b_2+24\,b\lambda^2A_1^2b_2+24\,b\lambda^2A_2^2b_2+24\,c\lambda^2A_1^2b_2+24\,c\lambda^2A_2^2b_2\\
&-2\,a\mu^2a_2b_2-24\,b\mu^2b_2-24\,c\mu^2b_2=0,\\
\phi^2(\xi) :\ & 2\,a\lambda^2A_1^2a_0a_2+a\lambda^2A_1^2a_1^2+2\,a\lambda^2A_2^2a_0a_2+a\lambda^2A_2^2a_1^2+32\,b\lambda^3A_1^2a_2+32\,b\lambda^3A_2^2a_2\\
&+32\,c\lambda^3A_1^2a_2+32\,c\lambda^3A_2^2a_2-2\,k\lambda^2A_1^2a_2-2\,k\lambda^2A_2^2a_2+a\lambda^2b_2^2-2\,a\mu^2a_0a_2-a\mu^2a_1^2\\
&-24\,b\lambda\,\mu^2a_2-24\,c\lambda\,\mu^2a_2+a\lambda\,b_1^2-4\,b\lambda\,\mu\,b_1-4\,c\lambda\,\mu\,b_1+2\,k\mu^2a_2=0,\\
\phi^2(\xi)\psi(\xi) :\ & 2\,a\lambda^2A_1^2a_1b_2+2\,a\lambda^2A_1^2a_2b_1+2\,a\lambda^2A_2^2a_1b_2+2\,a\lambda^2A_2^2a_2b_1-40\,b\lambda^2\mu\,A_1^2a_2-8\,c\mu^2b_1\\
&-40\,b\lambda^2\mu\,A_2^2a_2-40\,c\lambda^2\mu\,A_1^2a_2-40\,c\lambda^2\mu\,A_2^2a_2+8\,b\lambda^2A_1^2b_1+8\,b\lambda^2A_2^2b_1+8\,c\lambda^2A_1^2b_1\\
&+8\,c\lambda^2A_2^2b_1-2\,a\lambda\,\mu\,b_2^2-2\,a\mu^2a_1b_2-2\,a\mu^2a_2b_1+40\,b\mu^3a_2+40\,c\mu^3a_2-8\,b\mu^2b_1=0,\\
\phi(\xi) :\ & 2\,a\lambda^2A_1^2a_0a_1+2\,a\lambda^2A_2^2a_0a_1+8\,b\lambda^3A_1^2a_1+8\,b\lambda^3A_2^2a_1+8\,c\lambda^3A_1^2a_1+8\,c\lambda^3A_2^2a_1\\
&-2\,k\lambda^2A_1^2a_1-2\,k\lambda^2A_2^2a_1+2\,a\lambda^2b_1b_2-2\,a\mu^2a_0a_1-24\,b\lambda^2\mu\,b_2-8\,b\lambda\,\mu^2a_1-24\,c\lambda^2\mu\,b_2\\
&-8\,c\lambda\,\mu^2a_1+2\,k\mu^2a_1=0,\\
\phi(\xi)\psi(\xi) :\ & 2\,a\lambda^2A_1^2a_0b_2+2\,a\lambda^2A_1^2a_1b_1+2\,a\lambda^2A_2^2a_0b_2+2\,a\lambda^2A_2^2a_1b_1+20\,b\lambda^3A_1^2b_2+2\,k\mu^2b_2\\
&+20\,b\lambda^3A_2^2b_2-12\,b\lambda^2\mu\,A_1^2a_1-12\,b\lambda^2\mu\,A_2^2a_1+20\,c\lambda^3A_1^2b_2+20\,c\lambda^3A_2^2b_2+12\,c\mu^3a_1\\
&-12\,c\lambda^2\mu\,A_1^2a_1-12\,c\lambda^2\mu\,A_2^2a_1-2\,k\lambda^2A_1^2b_2-2\,k\lambda^2A_2^2b_2-4\,a\lambda\,\mu\,b_1b_2-2\,a\mu^2a_0b_2\\
&-2\,a\mu^2a_1b_1+28\,b\lambda\,\mu^2b_2+12\,b\mu^3a_1+28\,c\lambda\,\mu^2b_2=0,\\
\psi(\xi) :\ & -16\,b\lambda^3\mu\,A_1^2a_2-16\,b\lambda^3\mu\,A_2^2a_2-16\,c\lambda^3\mu\,A_1^2a_2-16\,c\lambda^3\mu\,A_2^2a_2+2\,a\lambda^2A_1^2a_0b_1\\
&+2\,a\lambda^2A_2^2a_0b_1+4\,b\lambda^3A_1^2b_1+4\,b\lambda^3A_2^2b_1+4\,c\lambda^3A_1^2b_1+4\,c\lambda^3A_2^2b_1-2\,k\lambda^2A_1^2b_1\\
&-2\,k\lambda^2A_2^2b_1-2\,a\lambda\,\mu\,b_1^2-2\,a\mu^2a_0b_1+4\,b\lambda\,\mu^2b_1+4\,c\lambda\,\mu^2b_1+2\,k\mu^2b_1=0,\\
\phi(\xi)^0 :\ & 8\,b\lambda^4A_1^2a_2+8\,b\lambda^4A_2^2a_2+8\,c\lambda^4A_1^2a_2+8\,c\lambda^4A_2^2a_2+a\lambda^2A_1^2a_0^2+a\lambda^2A_2^2a_0^2-2\,\mu^2p\\
&-2\,k\lambda^2A_1^2a_0-2\,k\lambda^2A_2^2a_0+a\lambda^2b_1^2-a\mu^2a_0^2-4\,b\lambda^2\mu\,b_1-4\,c\lambda^2\mu\,b_1+2\,\lambda^2pA_1^2\\
&+2\,k\mu^2a_0+2\,\lambda^2pA_2^2=0.
\end{aligned}
\tag{26}
$$

By solving the above algebraic system using the Maple package program, we obtain the following results.

Result 1:

$$
\begin{aligned}
&a_0=-\frac{16(b+c)-k}{a},\ a_1=0,\ a_2=-\frac{24(b+c)}{a},\ b_1=0,\ b_2=0,\\
&p=-\frac{64\,(b+c)^2\,\lambda^2-k^2}{2a},\ \mu=0,\ \lambda=\lambda,\ k=k,
\end{aligned}
\tag{27}
$$

where $a\neq 0,\ b,\ c,\ \lambda(>0),\ k$ are arbitrary constants. From Equations (12), (20) and (27), we obtain the exact solution of Equation (1) as follows:

$$
u(x,y,t)=-\frac{16(b+c)-k}{a}-\frac{24\lambda\,\left(\cos\left(\xi\sqrt{\lambda}\right)A_1-\sin\left(\xi\sqrt{\lambda}\right)A_2\right)^2(b+c)}{\left(A_1\sin\left(\xi\sqrt{\lambda}\right)+A_2\cos\left(\xi\sqrt{\lambda}\right)\right)^2a},
\tag{28}
$$

where $\xi=-k\frac{t^{\alpha}}{\alpha}+x+y$ and $A_1,\ A_2$ are arbitrary constants.

Result 2:

$$
\begin{aligned}
&a_0 = a_0,\ a_1 = 0,\ a_2 = -\frac{12(b+c)}{a},\ b_1 = \frac{12\mu\,(b+c)}{a},\ b_2 = \pm\frac{12(b+c)}{a}\sqrt{\frac{\sigma_2\lambda^2-\mu^2}{\lambda}},\\
&p = \frac{(12\lambda\,(b+c)+aa_0)\,(8\lambda\,(b+c)+aa_0)}{2a},\ \mu=\mu,\ \lambda=\lambda, k = aa_0 + 10\,b\lambda + 10\,c\lambda,
\end{aligned}
\tag{29}
$$

where a_0, $a \neq 0$, b, c, $\lambda(>0)$, μ are arbitrary constants and $\sigma_2 = A_1^2 + A_2^2$, where A_1, A_2 are arbitrary constants. From Equations (12), (20), and (29), we obtain the exact solution of Equation (1) as follows:

$$
\begin{aligned}
u(x,y,t) = a_0 &- \frac{12\lambda^3\left(\cos\left(\xi\sqrt{\lambda}\right)A_1 - \sin\left(\xi\sqrt{\lambda}\right)A_2\right)^2(b+c)}{\left(A_2\cos\left(\xi\sqrt{\lambda}\right)\lambda + A_1\sin\left(\xi\sqrt{\lambda}\right)\lambda + \mu\right)^2 a}\\
&+ \frac{12\lambda\,\mu\,(b+c)}{\left(A_2\cos\left(\xi\sqrt{\lambda}\right)\lambda + A_1\sin\left(\xi\sqrt{\lambda}\right)\lambda + \mu\right)a}\\
&\pm \frac{12\lambda^{5/2}\left(\cos\left(\xi\sqrt{\lambda}\right)A_1 - \sin\left(\xi\sqrt{\lambda}\right)A_2\right)\sqrt{\sigma_2\lambda^2-\mu^2}\,(b+c)}{\left(A_2\cos\left(\xi\sqrt{\lambda}\right)\lambda + A_1\sin\left(\xi\sqrt{\lambda}\right)\lambda + \mu\right)^2 a},
\end{aligned}
\tag{30}
$$

where $\xi = -k\frac{t^\alpha}{\alpha} + x + y$ with k defined in Equation (29).

Case 3: Rational function solutions ($\lambda = 0$)

If $\lambda = 0$, we substitute Equation (20) into Equation (19) along with the use of Equations (9) and (15). Then, the left-hand side of (19) becomes a polynomial in variables $\phi(\xi)$ and $\psi(\xi)$. Setting all of the coefficients of the resulting polynomial to be zero, we have the following system of nonlinear algebraic equations in a_0, a_1, a_2, b_1, b_2, A_1, A_2, λ, μ, k, and p, provided that $A_1^2 - 2\mu A_2 \neq 0$.

$$
\begin{aligned}
\phi^4(\xi) :\ & -2\,a\mu\,A_2a_2^2 + aA_1^2a_2^2 - 48\,b\mu\,A_2a_2 + 24\,bA_1^2a_2 - 48\,c\mu\,A_2a_2 + 24\,cA_1^2a_2 + ab_2^2 = 0,\\
\phi^3(\xi) :\ & -4\,a\mu\,A_2a_1a_2 + 2\,aA_1^2a_1a_2 - 16\,b\mu\,A_2a_1 + 8\,bA_1^2a_1 - 16\,c\mu\,A_2a_1 + 8\,cA_1^2a_1 + 2\,ab_1b_2\\
& - 24\,b\mu\,b_2 - 24\,c\mu\,b_2 = 0,\\
\phi^3(\xi)\psi(\xi) :\ & -4\,a\mu\,A_2a_2b_2 + 2\,a{A_1}^2a_2b_2 - 48\,b\mu\,A_2b_2 + 24\,bA_1^2b_2 - 48\,c\mu\,A_2b_2 + 24\,cA_1^2b_2 = 0,\\
\phi^2(\xi) :\ & -4\,a\mu\,A_2a_0a_2 - 2\,a\mu\,A_2a_1^2 + 2\,aA_1^2a_0a_2 + aA_1^2a_1^2 + 8\,b\mu^2a_2 + 8\,c\mu^2a_2 + 4\,k\mu\,A_2a_2\\
& - 2\,kA_1^2a_2 + ab_1^2 - 4\,b\mu\,b_1 - 4\,c\mu\,b_1 = 0,\\
\phi^2(\xi)\psi(\xi) :\ & -4\,a\mu\,A_2a_1b_2 - 4\,a\mu\,A_2a_2b_1 + 2\,aA_1^2a_1b_2 + 2\,aA_1^2a_2b_1 + 80\,b\mu^2A_2a_2 - 40\,b\mu\,A_1^2a_2\\
& + 80\,c\mu^2A_2a_2 - 40\,c\mu\,A_1^2a_2 - 2\,a\mu\,b_2^2 - 16\,b\mu\,A_2b_1 + 8\,bA_1^2b_1 - 16\,c\mu\,A_2b_1 + 8\,c{A_1}^2b_1 = 0,\\
\phi(\xi) :\ & -4\,a\mu\,A_2a_0a_1 + 2\,aA_1^2a_0a_1 + 4\,k\mu\,A_2a_1 - 2\,kA_1^2a_1 = 0,\\
\phi(\xi)\psi(\xi) :\ & -4\,a\mu\,A_2a_0b_2 - 4\,a\mu\,A_2a_1b_1 + 2\,aA_1^2a_0b_2 + 2\,aA_1^2a_1b_1 + 24\,b\mu^2A_2a_1 - 12\,b\mu\,A_1^2a_1\\
& + 24\,c\mu^2A_2a_1 - 12\,c\mu\,A_1^2a_1 - 4\,a\mu\,b_1b_2 + 48\,b\mu^2b_2 + 48\,c\mu^2b_2 + 4\,k\mu\,A_2b_2 - 2\,kA_1^2b_2 = 0,\\
\psi(\xi) :\ & -4\,a\mu\,A_2a_0b_1 + 2\,aA_1^2a_0b_1 - 16\,b\mu^3a_2 - 16\,c\mu^3a_2 - 2\,a\mu\,b_1^2 + 8\,b\mu^2b_1 + 8\,c\mu^2b_1 + 4\,k\mu\,A_2b_1\\
& - 2\,kA_1^2b_1 = 0,\\
\phi(\xi)^0 :\ & -2\,a\mu\,A_2a_0^2 + aA_1^2a_0^2 + 4\,k\mu\,A_2a_0 - 2\,kA_1^2a_0 - 4\,\mu\,pA_2 + 2\,pA_1^2 = 0.
\end{aligned}
\tag{31}
$$

On solving the above algrebraic system using the Maple package program, we obtain the following results.

Result 1:

$$a_0 = a_0,\ a_1 = 0,\ a_2 = -\frac{24(b+c)}{a},\ b_1 = 0,\ b_2 = 0,\ p = \frac{aa_0^2}{2},\ \mu = 0, k = aa_0, \tag{32}$$

where $a_0,\ a \neq 0,\ b,\ c$ are arbitrary constants. From Equations (14), (20) and (32), we obtain the traveling wave solution of Equation (1) as follows:

$$u(x,y,t) = a_0 - \frac{A_1^2\,(24\,b + 24\,c)}{(A_1\xi + A_2)^2\,a}, \tag{33}$$

where $\xi = -k\frac{t^\alpha}{\alpha} + x + y$ with k defined in Equation (32) and $A_1,\ A_2$ are arbitrary constants.

Result 2:

$$\begin{aligned} &a_0 = a_0,\ a_1 = 0,\ a_2 = -\frac{12(b+c)}{a},\ b_1 = \frac{144\,(b+c)^2\,A_1^2 - a^2 b_2^2}{24aA_2\,(b+c)},\ b_2 = b_2,\ p = \frac{aa_0^2}{2}, \\ &\mu = \frac{b_1 a}{12(b+c)},\ k = aa_0, \end{aligned} \tag{34}$$

where $a_0,\ b_2,\ a,\ b,\ c,\ A_1,\ A_2$ are arbitrary constants such that $aA_2\,(b+c) \neq 0$. From Equations (14), (20), and (34), we obtain the traveling wave solution of Equation (1) as follows:

$$\begin{aligned} u(x,y,t) = &-\frac{48\,(b+c)\,(12\,(b+c)\,A_1 + b_1 a\xi)^2}{(24\,A_1\xi\,\,(b+c) + b_1 a\xi^2 + 24\,A_2\,(b+c))^2\,a} + \frac{24b_1\,(b+c)}{24\,A_1\xi\,\,(b+c) + b_1 a\xi^2 + 24\,A_2\,(b+c)} \\ &+ \frac{48b_2\,(b+c)\,(12\,(b+c)\,A_1 + b_1 a\xi)}{(24\,A_1\xi\,\,(b+c) + b_1 a\xi^2 + 24\,A_2\,(b+c))^2}, \end{aligned} \tag{35}$$

where $\xi = -k\frac{t^\alpha}{\alpha} + x + y$ with k defined in Equation (34).

In the following part, the selected exact solutions of Equation (1), which are expressed in Equations (25), (28) and (35), are plotted for the three-dimensional representations. They will be portrayed on $-10 \leq x,\ t \leq 10$ by varying the fractional order $\alpha \in \{1,\ 0.9,\ 0.8\}$. The graphical results are as follows.

The following fixed values $a_0 = 1,\ \mu = 1,\ \lambda = -0.1,\ A_1 = 2,\ A_2 = 1,\ a = 1,\ b = 1,\ c = 1$ and the variation of $\alpha \in \{1,\ 0.9,\ 0.8\}$ are utilized to plot associated graphs of $u(x,y,t)$ expressed in Equation (25). In Figure 1a, the solution $u(x,y,t)$ with $\alpha = 1$ is plotted to describe the bell-shaped solitary wave solution. The graphs of the solution $u(x,y,t)$ for $\alpha = 0.9$ and $\alpha = 0.8$ are shown in Figure 1b,c, respectively. The graph of $|u(x,y,t)|$ for $\alpha = 0.8$ is depicted in Figure 1d. Figure 1b,c cannot show a graphical representation for $-10 < t < 0$ since $u(x,y,t)$ is a complex-valued function on this interval.

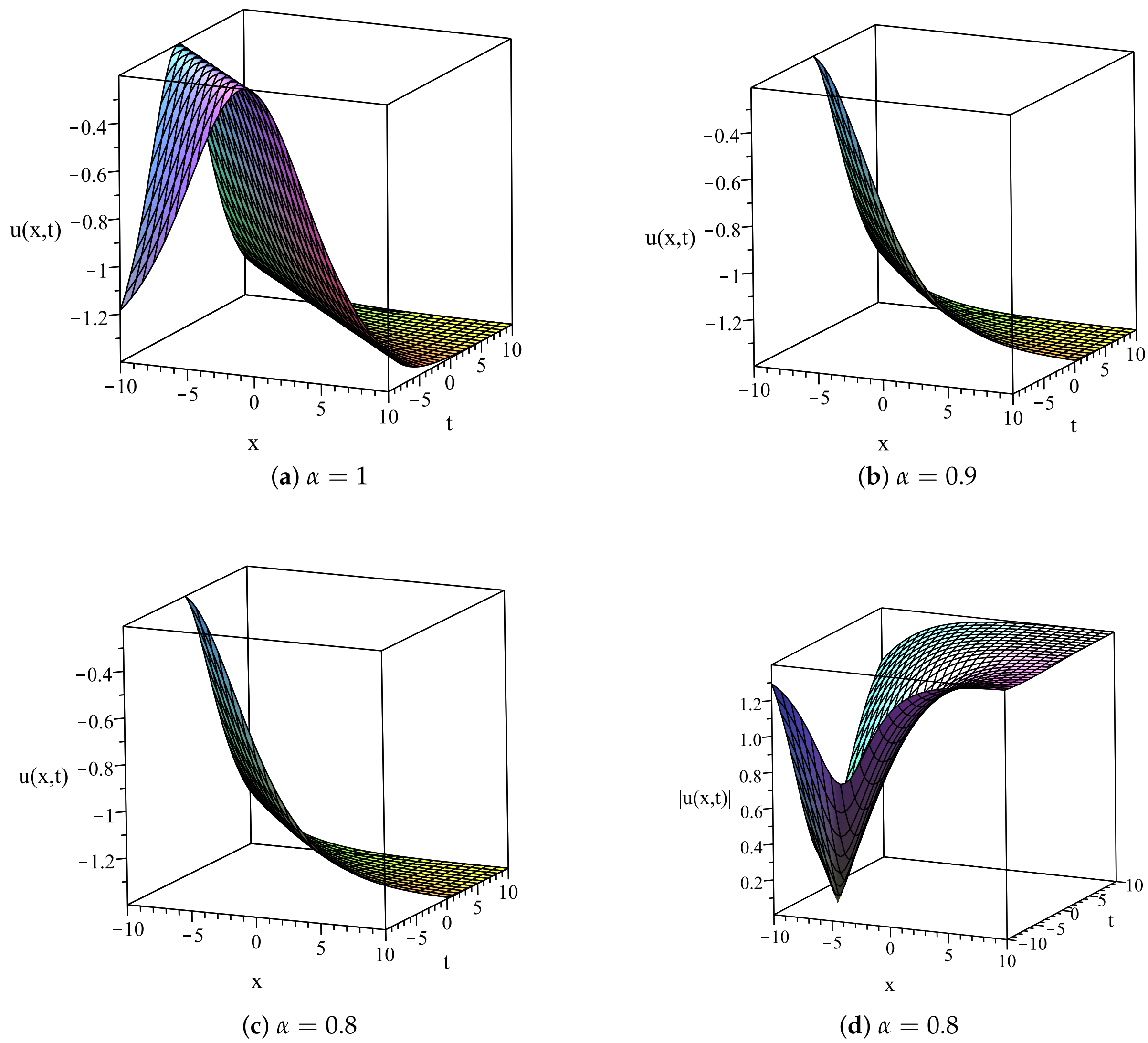
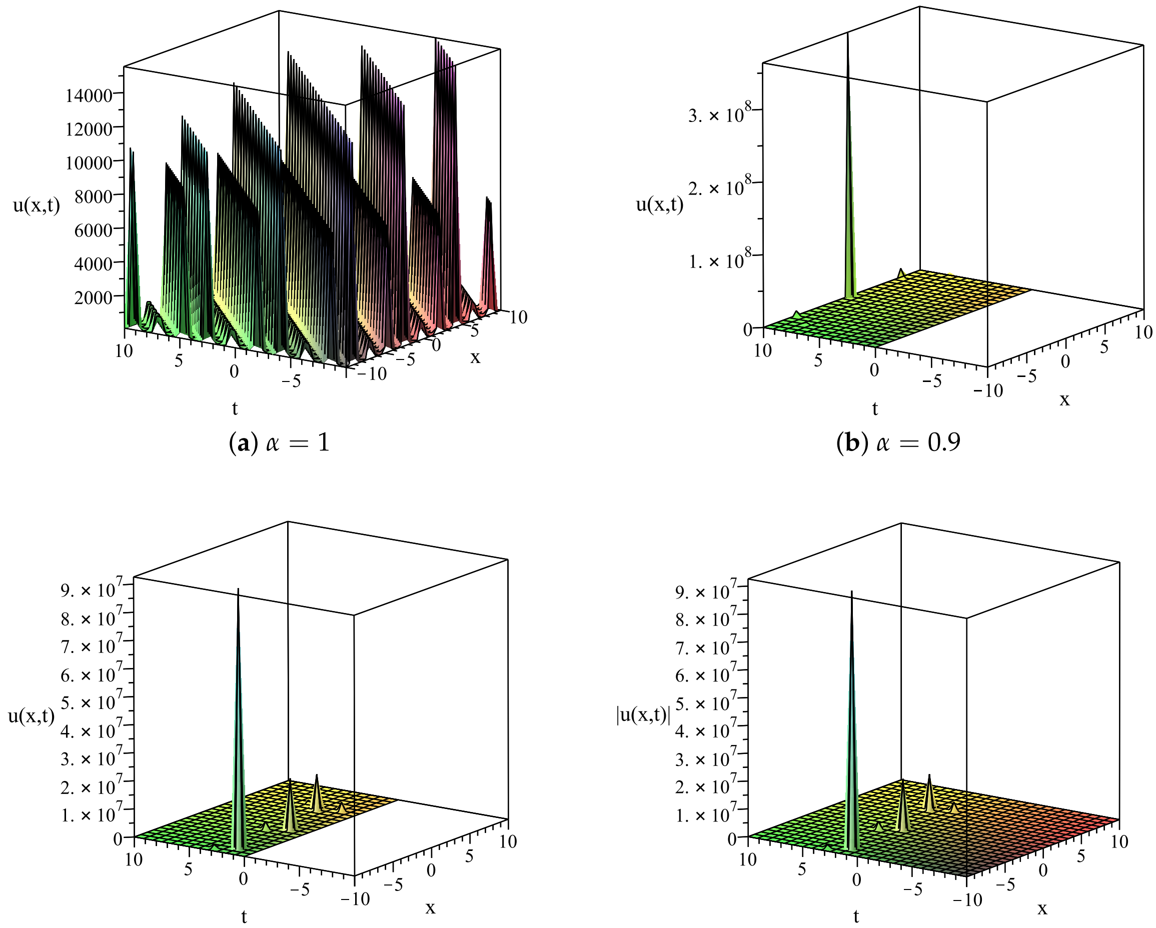


Figure 1. Associated plots of $u(x,y,t)$ in Equation (25) of Equation (1) on $-10 \leq x,\ t \leq 10$ using the $(G'/G, 1/G)$-expansion method.

In Figure 2a, the periodic traveling wave solution, obtained using the solution $u(x,y,t)$ in Equation (28), is displayed using the parameter values $k = 1,\ \lambda = 2,\ A_1 = 2,\ A_2 = 1,\ a = -1,$ $b = 1,\ c = 1$, and the fractional orders $\alpha = 1$. Using the above parameter values, Figure 2b,c, represent the solution $u(x,y,t)$ describing singular soliton solutions for $\alpha = 0.9$ and $\alpha = 0.8$, respectively. The graph of $|u(x,y,t)|$ with $\alpha = 0.8$ is portrayed in Figure 2d. We can observe that Figure 2b,c cannot give a graphical representation for $-10 < t < 0$, since $u(x,y,t)$ is a complex-valued function on this interval.

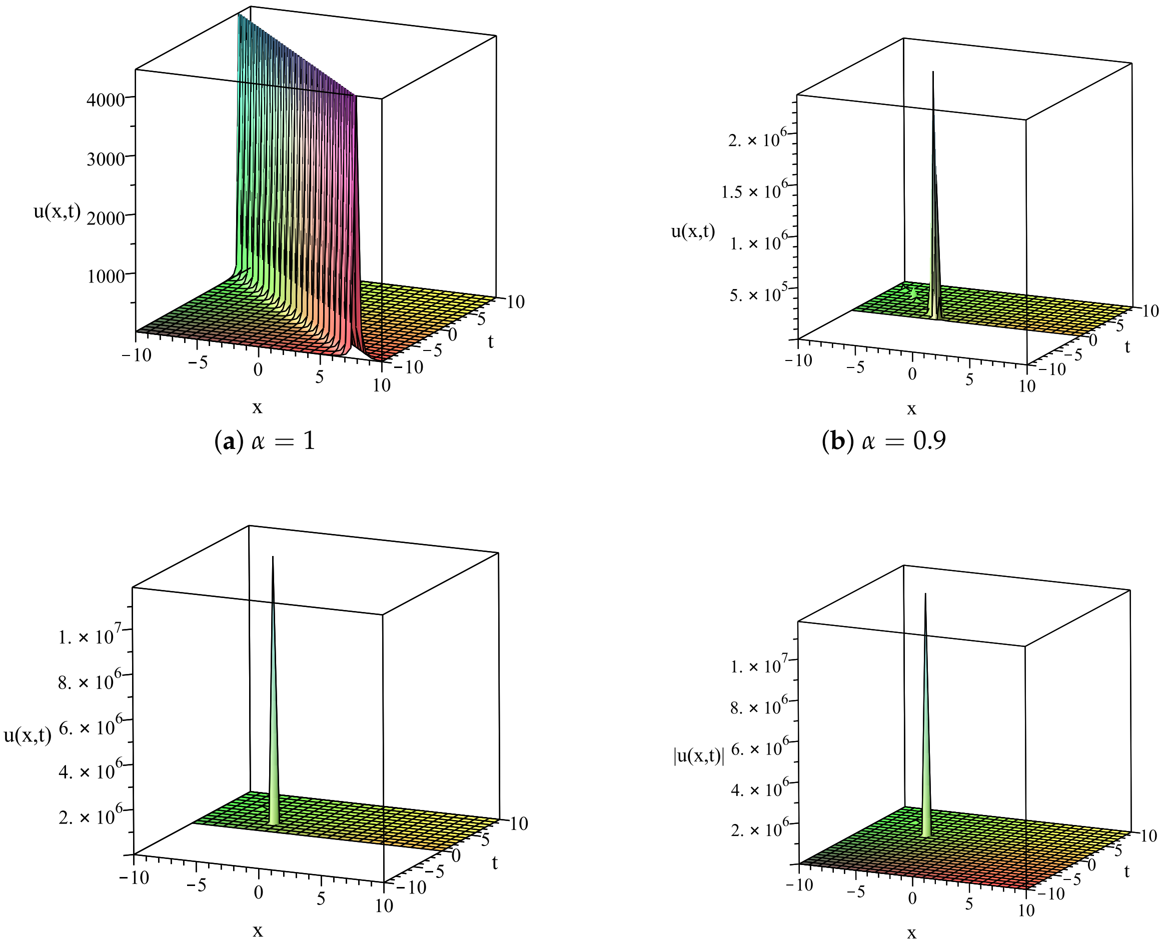


(**a**) $\alpha = 1$ (**b**) $\alpha = 0.9$

(**c**) $\alpha = 0.8$ (**d**) $\alpha = 0.8$

Figure 2. Associated plots of $u(x, y, t)$ in Equation (28) of Equation (1) on $-10 \leq x,\ t \leq 10$ using the $(G'/G, 1/G)$-expansion method.

For the fixed values $a_0 = 1,\ b_2 = 1,\ A_1 = 2,\ A_2 = 1,\ a = -1,\ b = 1,\ c = 1$, the graphs of the exact solutions $u(x, y, t)$ in Equation (35) of Equation (1) corresponding to the given variation of α are investigated. The solution $u(x, y, t)$ with $\alpha = 1$, describing the solitary wave solution of singular soliton type, is depicted in Figure 3a. The solutions $u(x, y, t)$ with $\alpha = 0.9$ and $\alpha = 0.8$, showing the discontinuous singular single-soliton solution, are presented in Figure 3b,c, respectively. Since $u(x, y, t)$ is a complex-valued function on $-10 < t < 0$, then these figures do not present any graph for this interval. The graph of $|u(x, y, t)|$ with $\alpha = 0.8$ is plotted in Figure 3d.



(**a**) $\alpha = 1$ (**b**) $\alpha = 0.9$

(**c**) $\alpha = 0.8$ (**d**) $\alpha = 0.8$

Figure 3. Associated plots of $u(x, y, t)$ in Equation (35) of Equation (1) on $-10 \leq x,\ t \leq 10$ using the $(G'/G, 1/G)$-expansion method.

Next, we compare our exact solutions of Equation (1), achieved using the $(G'/G, 1/G)$-expansion method to the ones obtained using the different methods, which were reported before. In 2019, Ali et al., [37] analytically solved Equation (1) using the modified Kudryashov method and the (G'/G^2)-expansion method. They found that the former method provided the two exact solutions written in terms of the reciprocal of exponential function solutions. The latter method, which they employed, released six sets of the coefficients and parameter values in which each set generated three classes of the solutions, including trigonometric, hyperbolic, and rational function solutions, while our results generated using the $(G'/G, 1/G)$-expansion method included two hyperbolic function solutions, two trigonometric function solutions, and two rational function solutions. When comparing the number of solution classes obtained using the (G'/G^2)-expansion method and the $(G'/G, 1/G)$-expansion method, they are the same number. However, their solutions and our solutions are not exactly the same. Applying the $(G'/G, 1/G)$-expansion method to Equation (1), our solutions are new and distinct from the results in [37].

4.2. The Space-Time-Fractional Generalized Hirota-Satsuma Coupled KdV System

Before finding exact traveling wave solutions of the space-time-fractional generalized Hirota-Satsuma coupled KdV system in Equation (2) by using the $(G'/G, 1/G)$-expansion method, we must convert it to a system of ordinary differential equations using the following transformations

$$u(x,t) = U(\xi),\ v(x,t) = V(\xi),\ w(x,t) = W(\xi),\ \xi = k\left(\frac{x^\eta}{\eta} - c\frac{t^\rho}{\rho}\right), \tag{36}$$

where k and c are non-zero arbitrary constants to be determined later. Substituting Equation (36) into Equation (2), we yield a system of ODEs, as follows:

$$-ckU' = \frac{1}{4}k^3U''' + 3kUU' + 3k(-V^2 + W)', \tag{37}$$

$$-ckV' = -\frac{1}{2}k^3V''' - 3kUV', \tag{38}$$

$$-ckW' = -\frac{1}{2}k^3W''' - 3kUW'.. \tag{39}$$

Let [53]

$$U = \alpha V^2 + \beta V + \gamma,\ \ W = AV + B, \tag{40}$$

where $\alpha,\ \beta,\ \gamma,\ A$ and B are constants to be determined later.

Substituting Equation (40) into Equations (38) and (39), and then integrating once, we know that Equations (38) and (39) give the same resulting equation as follows:

$$k^2V'' = -2\alpha V^3 - 3\beta V^2 + 2(c - 3\gamma)V + c_1, \tag{41}$$

where c_1 is a constant of integration. Multiplying Equation (41) by V' and then integrating the resulting equation with respect to ξ, we obtain

$$k^2(V')^2 = -\alpha V^4 - 2\beta V^3 + 2(c - 3\gamma)V^2 + 2c_1V + c_2, \tag{42}$$

where c_2 is also a constant of integration.

Differentiating Equation (40) with respect to ξ and then using Equations (41) and (42), we obtain

$$\begin{aligned} k^2U'' &= 2\alpha k^2(V')^2 + k^2(2\alpha V + \beta)V'', \\ &= 2\alpha\left[-\alpha V^4 - 2\beta V^3 + 2(c - 3\gamma)V^2 + 2c_1V + c_2\right] \\ &\quad +(2\alpha V + \beta)\left[-2\alpha V^3 - 3\beta V^2 + 2(c - 3\gamma)V + c_1\right]. \end{aligned} \tag{43}$$

Integrating Equation (37) once, we get

$$\frac{1}{4}k^2U'' + \frac{3}{2}U^2 + cU + 3(-V^2 + W) + c_3 = 0, \tag{44}$$

where c_3 is a constant of integration. Substituting Equations (40) and (43) into Equation (44), we obtain that the following coefficients of the resulting polynomial are zero, as follows:

$$\begin{aligned} 3\alpha c - 3\alpha\gamma + \frac{3}{4}\beta^2 - 3 &= 0, \\ \frac{1}{2}(\alpha c_1 + \beta c + \gamma\beta) + A &= 0, \\ \frac{1}{4}(2\alpha c_2 + \beta c_1) + \frac{3}{2}\gamma^2 + c\gamma + 3B + c_3 &= 0. \end{aligned} \tag{45}$$

Let

$$c_1 = \frac{1}{2\alpha^2}(\beta^3 + 2c\alpha\beta - 6\alpha\beta\gamma), \ \ V(\xi) = aP(\xi) - \frac{\beta}{2\alpha}. \tag{46}$$

We find from (45) that

$$\begin{aligned}
\alpha &= \frac{\beta^2-4}{4(\gamma-c)}, \ \ A = \frac{4\beta(c-\gamma)}{\beta^2-4},\\
B &= \frac{1}{6\,(-\gamma+c)\,(\beta^2-4)^2}\Big(16\,c_3c\,\beta^2 - 2\,c_3c\,\beta^4 - 16\,c_3\gamma\,\beta^2 + 2\,c_3\gamma\,\beta^4\\
&+ 56\,c^2\gamma\,\beta^2 - 48\,\gamma^2c\,\beta^2 - 16\,c_2 + \frac{1}{4}c_2\beta^6 - 3\,c_2\beta^4 + 12\,c_2\beta^2 - 16\,\gamma^2c\\
&- 32\,c^2\gamma - 8\,c^3\beta^2 + \beta^4\gamma^3 - 2\,\beta^4c^3 + 32\,c_3\gamma - 32\,c_3c + 48\,\gamma^3 + \beta^4\gamma^2c\Big).
\end{aligned} \tag{47}$$

From (41), we hence acquire

$$ak^2P'' - a\left(2c - 6\gamma + \frac{3\beta^2}{2\alpha}\right)P + 2\alpha a^3P^3 = 0. \tag{48}$$

Applying the homogeneous balance principle and the formulas in Equation (17) mentioned in Step 3 to the terms P'' and P^3, we then have that

$$\text{Deg}\left[P''\right] = N + 2 = \text{Deg}\left[P^3\right] = 3N, \tag{49}$$

which leads to $N = 1$. Hence, the form of exact solutions of the ODE in Equation (48) using the method is

$$P(\xi) = a_0 + a_1\phi(\xi) + b_1\psi(\xi), \tag{50}$$

where the constant coefficients a_0, a_1 and b_1 are determined at a later step, provided that $a_1^2 + b_1^2 \neq 0$. Using the $(G'/G, 1/G)$-expansion method, the following three cases of the obtained exact traveling solutions of Equation (2), depending on the function $G(\xi)$ which is a solution of the auxiliary Equation (7), are as follows.

Case 1: Hyperbolic function solutions ($\lambda < 0$)

If $\lambda < 0$, we substitute Equation (50) into Equation (48), along with the use of Equations (9) and (11). Then, the left-hand side of Equation (48) becomes a polynomial in $\phi(\xi)$ and $\psi(\xi)$. Setting all of the coefficients of this resulting polynomial to be zero, we obtain the following system of nonlinear algebraic equations in $a_0,\ a_1,\ b_1,\ A_1,\ A_2,\ \lambda,\ \mu,\ k$, and c, provided that $\lambda^2\left(A_1^2-A_2^2\right)+\mu^2\neq 0$.

$$
\begin{aligned}
\phi^3(\xi):\ & 4a^3\alpha^2\lambda^4A_1^4a_1^3-8a^3\alpha^2\lambda^4A_1^2A_2^2a_1^3+4a^3\alpha^2\lambda^4A_2^4a_1^3+8a^3\alpha^2\lambda^2\mu^2A_1^2a_1^3-8a^3\alpha^2\lambda^2\mu^2A_2^2a_1^3\\
&-12a^3\alpha^2\lambda^3A_1^2a_1b_1^2+12a^3\alpha^2\lambda^3A_2^2a_1b_1^2+4a\alpha k^2\lambda^4A_1^4a_1-8a\alpha k^2\lambda^4A_1^2A_2^2a_1+4a\alpha k^2\lambda^4A_2^4a_1\\
&+4a^3\alpha^2\mu^4a_1^3-12a^3\alpha^2\lambda\mu^2a_1b_1^2+8\,a\alpha\,k^2\lambda^2\mu^2A_1^2a_1-8\,a\alpha\,k^2\lambda^2\mu^2A_2^2a_1+4\,a\alpha\,k^2\mu^4a_1=0,\\
\phi^2(\xi):\ & 12\,a^3\alpha^2\lambda^4A_1^4a_0a_1^2-24\,a^3\alpha^2\lambda^4A_1^2A_2^2a_0a_1^2+12\,a^3\alpha^2\lambda^4A_2^4a_0a_1^2+24a^3\alpha^2\lambda^2\mu^2A_1^2a_0a_1^2\\
&-24a^3\alpha^2\lambda^2\mu^2A_2^2a_0a_1^2-12a^3\alpha^2\lambda^3A_1^2a_0b_1^2+12a^3\alpha^2\lambda^3A_2^2a_0b_1^2+12a^3\alpha^2\mu^4a_0a_1^2-8a^3\alpha^2\lambda^2\mu b_1^3\\
&-12a^3\alpha^2\lambda\mu^2a_0b_1^2+2a\alpha k^2\lambda^3\mu A_1^2b_1-2a\alpha k^2\lambda^3\mu A_2^2b_1+2a\alpha\,k^2\lambda\mu^3b_1=0,\\
\phi^2(\xi)\psi(\xi):\ & 12a^3\alpha^2\lambda^4A_1^4a_1^2b_1-24\,a^3\alpha^2\lambda^4A_1^2A_2^2a_1^2b_1+12\,a^3\alpha^2\lambda^4A_2^4a_1^2b_1+24\,a^3\alpha^2\lambda^2\mu^2A_1^2a_1^2b_1\\
&-24\,a^3\alpha^2\lambda^2\mu^2A_2^2a_1^2b_1-4\,a^3\alpha^2\lambda^3A_1^2b_1^3+4\,a^3\alpha^2\lambda^3A_2^2b_1^3+4\,a\alpha\,k^2\lambda^4A_1^4b_1-8\,a\alpha\,k^2\lambda^4A_1^2A_2^2b_1\\
&+4\,a\alpha\,k^2\lambda^4A_2^4b_1+12\,a^3\alpha^2\mu^4a_1^2b_1-4\,a^3\alpha^2\lambda\,\mu^2b_1^3+8\,a\alpha\,k^2\lambda^2\mu^2A_1^2b_1-8\,a\alpha\,k^2\lambda^2\mu^2A_2^2b_1\\
&+4a\alpha\,k^2\mu^4b_1=0,\\
\phi(\xi):\ & 12\,a^3\alpha^2\lambda^4A_1^4a_0^2a_1-24\,a^3\alpha^2\lambda^4A_1^2A_2^2a_0^2a_1+12\,a^3\alpha^2\lambda^4A_2^4a_0^2a_1-12\,a^3\alpha^2\lambda^4A_1^2a_1b_1^2\\
&+12\,a^3\alpha^2\lambda^4A_2^2a_1b_1^2+24\,a^3\alpha^2\lambda^2\mu^2A_1^2a_0^2a_1-24\,a^3\alpha^2\lambda^2\mu^2A_2^2a_0^2a_1+4\,a\alpha\,k^2\lambda^5A_1^4a_1\\
&-8\,a\alpha\,k^2\lambda^5A_1^2A_2^2a_1+4\,a\alpha\,k^2\lambda^5A_2^4a_1-12\,a^3\alpha^2\lambda^2\mu^2a_1b_1^2+12\,a^3\alpha^2\mu^4a_0^2a_1-4\,a\alpha\,c\lambda^4A_1^4a_1\\
&+8\,a\alpha\,c\lambda^4A_1^2A_2^2a_1-4\,a\alpha\,c\lambda^4A_2^4a_1+12\,a\alpha\,\gamma\,\lambda^4A_1^4a_1-24\,a\alpha\,\gamma\,\lambda^4A_1^2A_2^2a_1+12\,a\alpha\,\gamma\,\lambda^4A_2^4a_1\\
&+8\,a\alpha\,k^2\lambda^3\mu^2A_1^2a_1-8\,a\alpha\,k^2\lambda^3\mu^2A_2^2a_1-3a\beta^2\lambda^4A_1^4a_1+6\,a\beta^2\lambda^4A_1^2A_2^2a_1-3\,a\beta^2\lambda^4A_2^4a_1\\
&-8a\alpha c\lambda^2\mu^2A_1^2a_1+8\,a\alpha\,c\lambda^2\mu^2A_2^2a_1+24\,a\alpha\,\gamma\,\lambda^2\mu^2A_1^2a_1-24\,a\alpha\,\gamma\,\lambda^2\mu^2A_2^2a_1+4\,a\alpha\,k^2\lambda\,\mu^4a_1\\
&-6\,a\beta^2\lambda^2\mu^2A_1^2a_1+6\,a\beta^2\lambda^2\mu^2A_2^2a_1-4\,a\alpha\,c\mu^4a_1+12\,a\alpha\,\gamma\,\mu^4a_1-3\,a\beta^2\mu^4a_1=0,\\
\phi(\xi)\psi(\xi):\ & 24\,a^3\alpha^2\lambda^4A_1^4a_0a_1b_1-48\,a^3\alpha^2\lambda^4A_1^2A_2^2a_0a_1b_1+24\,a^3\alpha^2\lambda^4A_2^4a_0a_1b_1+24\,a^3\alpha^2\lambda^3\mu\,A_1^2a_1b_1^2\\
&-24\,a^3\alpha^2\lambda^3\mu\,A_2^2a_1b_1^2+48\,a^3\alpha^2\lambda^2\mu^2A_1^2a_0a_1b_1-48\,a^3\alpha^2\lambda^2\mu^2A_2^2a_0a_1b_1-6\,a\alpha\,k^2\lambda^4\mu\,A_1^4a_1\\
&+12\,a\alpha\,k^2\lambda^4\mu\,A_1^2A_2^2a_1-6\,a\alpha\,k^2\lambda^4\mu\,A_2^4a_1+24\,a^3\alpha^2\lambda\mu^3a_1b_1{}^2+24a^3\alpha^2\mu^4a_0a_1b_1\\
&-12a\alpha\,k^2\lambda^2\mu^3A_1^2a_1+12\,a\alpha\,k^2\lambda^2\mu^3A_2^2a_1-6\,a\alpha\,k^2\mu^5a_1=0,\\
\psi(\xi):\ & 12\,a^3\alpha^2\lambda^4A_1^4a_0^2b_1-24\,a^3\alpha^2\lambda^4A_1^2A_2^2a_0^2b_1+12\,a^3\alpha^2\lambda^4A_2^4a_0^2b_1-4\,a^3\alpha^2\lambda^4A_1^2b_1^3+4\,a^3\alpha^2\lambda^4A_2^2b_1^3\\
&+24\,a^3\alpha^2\lambda^3\mu\,A_1^2a_0b_1^2-24\,a^3\alpha^2\lambda^3\mu\,A_2^2a_0b_1^2+24\,a^3\alpha^2\lambda^2\mu^2A_1^2a_0^2b_1-24\,a^3\alpha^2\lambda^2\mu^2A_2^2a_0^2b_1\\
&+2\,a\alpha\,k^2\lambda^5A_1^4b_1-4a\alpha k^2\lambda^5A_1^2A_2^2b_1+2\,a\alpha\,k^2\lambda^5A_2^4b_1+12a^3\alpha^2\lambda^2\mu^2b_1^3+24a^3\alpha^2\lambda\mu^3a_0b_1^2\\
&+12a^3\alpha^2\mu^4a_0^2b_1-4a\alpha\,c\lambda^4A_1^4b_1+8\,a\alpha\,c\lambda^4A_1^2A_2^2b_1-4a\alpha\,c\lambda^4A_2^4b_1+12a\alpha\gamma\lambda^4A_1^4b_1\\
&-24a\alpha\gamma\lambda^4A_1^2A_2^2b_1+12a\alpha\gamma\lambda^4A_2^4b_1-3a\beta^2\lambda^4A_1^4b_1+6a\beta^2\lambda^4A_1^2A_2^2b_1-3a\beta^2\lambda^4A_2^4b_1\\
&-8a\alpha c\lambda^2\mu^2A_1^2b_1+8a\alpha c\lambda^2\mu^2A_2^2b_1+24a\alpha\gamma\lambda^2\mu^2A_1^2b_1-24a\alpha\gamma\lambda^2\mu^2A_2^2b_1-2a\alpha k^2\lambda\mu^4b_1\\
&-6a\beta^2\lambda^2\mu^2A_1^2b_1+6a\beta^2\lambda^2\mu^2A_2^2b_1-4a\alpha c\mu^4b_1+12a\alpha\gamma\mu^4b_1-3a\beta^2\mu^4b_1=0,\\
\phi^0(\xi):\ & 4a^3\alpha^2\lambda^4A_1^4a_0^3-8a^3\alpha^2\lambda^4A_1^2A_2^2a_0^3+4a^3\alpha^2\lambda^4A_2^4a_0^3-12a^3\alpha^2\lambda^4A_1^2a_0b_1^2+12a^3\alpha^2\lambda^4A_2^2a_0b_1^2\\
&+8a^3\alpha^2\lambda^2\mu^2A_1^2a_0^3-8a^3\alpha^2\lambda^2\mu^2A_2^2a_0^3-8a^3\alpha^2\lambda^3\mu b_1^3-12a^3\alpha^2\lambda^2\mu^2a_0b_1^2+4a^3\alpha^2\mu^4a_0^3\\
&-4a\alpha c\lambda^4A_1^4a_0+8a\alpha c\lambda^4A_1^2A_2^2a_0-4a\alpha c\lambda^4A_2^4a_0+12a\alpha\gamma\lambda^4A_1^4a_0-24a\alpha\gamma\lambda^4A_1^2A_2^2a_0\\
&+12a\alpha\gamma\lambda^4A_2^4a_0+2a\alpha k^2\lambda^4\mu\,A_1^2b_1-2a\alpha\,k^2\lambda^4\mu\,A_2^2b_1-3a\beta^2\lambda^4A_1^4a_0+6\,a\beta^2\lambda^4A_1^2A_2^2a_0\\
&-3\,a\beta^2\lambda^4A_2^4a_0-8\,a\alpha\,c\lambda^2\mu^2A_1^2a_0+8a\alpha c\lambda^2\mu^2A_2^2a_0+24a\alpha\gamma\lambda^2\mu^2A_1^2a_0-24a\alpha\gamma\lambda^2\mu^2A_2^2a_0\\
&+2a\alpha k^2\lambda^2\mu^3b_1-6\,a\beta^2\lambda^2\mu^2A_1^2a_0+6\,a\beta^2\lambda^2\mu^2A_2^2a_0-4a\alpha c\mu^4a_0+12a\alpha\gamma\mu^4a_0-3a\beta^2\mu^4a_0=0.
\end{aligned}
\tag{51}
$$

Solving the above algebraic system using the Maple package program, we have the following results.

Result 1:

$$
a_0=0,\ a_1=\pm\frac{k\sqrt{-\frac{1}{\alpha}}}{a},\ b_1=0,\ k=k,\ c=\frac{12\gamma-\lambda k^2(\beta^2-4)}{2(\beta^2+2)},\ \mu=0,\ \lambda=\lambda,
\tag{52}
$$

where k, $a \neq 0$, β, γ, $\lambda(<0)$ are arbitrary constants and α is defined in Equation (47). From Equations (10), (50) and (52), we obtain the traveling wave solutions of Equation (2), as follows:

$$\begin{aligned} v_1^1(x,t) &= \pm k\sqrt{-\frac{1}{\alpha}}\,\frac{\left(A_1\sqrt{-\lambda}\cosh\left(\xi\sqrt{-\lambda}\right)+A_2\sqrt{-\lambda}\sinh\left(\xi\sqrt{-\lambda}\right)\right)}{A_1\sinh\left(\xi\sqrt{-\lambda}\right)+A_2\cosh\left(\xi\sqrt{-\lambda}\right)}-\frac{\beta}{2\alpha}, \\ u_1^1(x,t) &= \alpha\left(v_1^1(x,t)\right)+\beta\left(v_1^1(x,t)\right)+\gamma, \quad w_1^1(x,t)=A\left(v_1^1(x,t)\right)+B, \end{aligned} \tag{53}$$

where ξ is defined in Equation (36) with k, c defined in Equation (52), A_1, A_2 are arbitrary constants and A, B are defined in Equation (47).

Result 2:

$$\begin{aligned} &a_0=0,\ a_1=\pm\frac{k\sqrt{-\frac{1}{\alpha}}}{2a},\ b_1=b_1,\ k=k, \\ &c=\frac{\left(\beta^2-4\right)\sqrt{4\gamma^2\sigma_1^2+2a^2b_1^2\left(\beta^2+2\right)\sigma_1}+2\,\gamma\sigma_1\left(\beta^2+8\right)}{4\,\sigma_1\left(\beta^2+2\right)}, \\ &\mu=0,\ \lambda=\frac{4\alpha a^2b_1^2}{k^2\sigma_1}<0, \end{aligned} \tag{54}$$

where b_1, k, $a \neq 0$, β, γ are arbitrary constants, α is defined in Equation (47) and $\sigma_1 = A_1^2 - A_2^2$, where A_1, A_2 are arbitrary constants. From Equations (10), (50) and (54), we obtain the traveling wave solutions of Equation (2) as follows:

$$\begin{aligned} v_2^1(x,t) &= \pm k\sqrt{-\frac{1}{\alpha}}\,\frac{\left(A_1\sqrt{-\frac{\alpha a^2b_1^2}{k^2\sigma_1}}\cosh\left(2\xi\sqrt{-\frac{\alpha a^2b_1^2}{k^2\sigma_1}}\right)+A_2\sqrt{-\frac{\alpha a^2b_1^2}{k^2\sigma_1}}\sinh\left(2\xi\sqrt{-\frac{\alpha a^2b_1^2}{k^2\sigma_1}}\right)\right)}{\left(A_1\sinh\left(2\xi\sqrt{-\frac{\alpha a^2b_1^2}{k^2\sigma_1}}\right)+A_2\cosh\left(2\xi\sqrt{-\frac{\alpha a^2b_1^2}{k^2\sigma_1}}\right)\right)} \\ &\quad+\frac{ab_1}{\left(A_1\sinh\left(2\,\xi\sqrt{-\frac{\alpha a^2b_1^2}{k^2\sigma_1}}\right)+A_2\cosh\left(2\xi\sqrt{-\frac{\alpha a^2b_1^2}{k^2\sigma_1}}\right)\right)}-\frac{\beta}{2\alpha}, \\ u_2^1(x,t) &= \alpha\left(v_2^1(x,t)\right)+\beta\left(v_2^1(x,t)\right)+\gamma,\ w_2^1(x,t)=A\left(v_2^1(x,t)\right)+B, \end{aligned} \tag{55}$$

where ξ is defined in Equation (36) with k, c defined in Equation (54) and A, B are defined in Equation (47).

Result 3:

Result 3.1

$$\begin{aligned} &a_0=0,\ a_1=\frac{k\sqrt{-\frac{1}{\alpha}}}{2a},\ b_1=\pm\frac{k}{2a}\sqrt{\frac{\lambda^2\sigma_1+\mu^2}{\alpha\lambda}},\ k=k,\ c=\frac{48\,\gamma-\lambda k^2(\beta^2-4)}{8(\beta^2+2)}, \\ &\mu=\mu,\ \lambda=\lambda, \end{aligned} \tag{56}$$

Result 3.2

$$\begin{aligned} &a_0=0,\ a_1=-\frac{k\sqrt{-\frac{1}{\alpha}}}{2a},\ b_1=\pm\frac{k}{2a}\sqrt{\frac{\lambda^2\sigma_1+\mu^2}{\alpha\lambda}},\ k=k,\ c=\frac{48\,\gamma-\lambda k^2(\beta^2-4)}{8(\beta^2+2)}, \\ &\mu=\mu,\ \lambda=\lambda, \end{aligned} \tag{57}$$

where k, $a \neq 0$, β, γ, μ, $\lambda(<0)$ are arbitrary constants, α is defined in Equation (47) and $\sigma_1 = A_1^2 - A_2^2$, where A_1, A_2 are arbitrary constants. From Equations (10), (50) and (56), we obtain the traveling wave solutions of Equation (2) as follows:

$$\begin{aligned}
v_3^1(x,t) &= k\sqrt{-\frac{1}{\alpha}}\frac{\left(A_1\sqrt{-\lambda}\cosh\left(\xi\sqrt{-\lambda}\right) + A_2\sqrt{-\lambda}\sinh\left(\xi\sqrt{-\lambda}\right)\right)}{2\left(A_1\sinh\left(\xi\sqrt{-\lambda}\right) + A_2\cosh\left(\xi\sqrt{-\lambda}\right) + \frac{\mu}{\lambda}\right)} \\
&\quad \pm \frac{k\sqrt{\frac{\lambda^2\sigma_1+\mu^2}{\alpha\lambda}}}{2\left(A_1\sinh\left(\xi\sqrt{-\lambda}\right) + A_2\cosh\left(\xi\sqrt{-\lambda}\right) + \frac{\mu}{2\lambda}\right)} - \frac{\beta}{2\alpha}, \\
u_3^1(x,t) &= \alpha\left(v_3^1(x,t)\right) + \beta\left(v_3^1(x,t)\right) + \gamma,\ w_3^1(x,t) = A\left(v_3^1(x,t)\right) + B,
\end{aligned} \tag{58}$$

where ξ is defined in Equation (36) with k, c defined in Equation (56) and A, B are defined in Equation (47). Similarly, we can use Equations (10), (50) and (57) to obtain the traveling wave solutions of Equation (2), but they are omitted here.

Case 2: Trigonometric function solutions ($\lambda > 0$)

If $\lambda > 0$, we substitute Equation (50) into Equation (48), along with the use of Equations (9) and (13). Then, the left-hand side of Equation (48) becomes a polynomial in $\phi(\xi)$ and $\psi(\xi)$. Setting all of the coefficients of the resulting polynomial to be zero, we obtain the following system of nonlinear algebraic equations in a_0, a_1, $b_1, A_1, A_2, \lambda, \mu, k, c$, provided that $\lambda^2\left(A_1^2 + A_2^2\right) - \mu^2 \neq 0$.

$$\begin{aligned}
\phi^3(\xi) &: 4a^3\alpha^2\lambda^4A_1^4a_1^3 + 8a^3\alpha^2\lambda^4A_1^2A_2^2a_1^3 + 4a^3\alpha^2\lambda^4A_2^4a_1^3 - 8a^3\alpha^2\lambda^2\mu^2A_1^2a_1^3 - 8a^3\alpha^2\lambda^2\mu^2A_2^2a_1^3 \\
&\quad + 12a^3\alpha^2\lambda^3A_1^2a_1b_1^2 + 12a^3\alpha^2\lambda^3A_2^2a_1b_1^2 + 4a\alpha k^2\lambda^4A_1^4a_1 + 8a\alpha k^2\lambda^4A_1^2A_2^2a_1 + 4a\alpha k^2\lambda^4A_2^4a_1 \\
&\quad + 4a^3\alpha^2\mu^4a_1^3 - 12a^3\alpha^2\lambda\mu^2a_1b_1^2 - 8a\alpha k^2\lambda^2\mu^2A_1^2a_1 - 8a\alpha k^2\lambda^2\mu^2A_2^2a_1 + 4a\alpha k^2\mu^4a_1 = 0, \\
\phi^2(\xi) &: 12a^3\alpha^2\lambda^4A_1^4a_0a_1^2 + 24a^3\alpha^2\lambda^4A_1^2A_2^2a_0a_1^2 + 12a^3\alpha^2\lambda^4A_2^4a_0a_1^2 - 24a^3\alpha^2\lambda^2\mu^2A_1^2a_0a_1^2 \\
&\quad - 24a^3\alpha^2\lambda^2\mu^2A_2^2a_0a_1^2 + 12a^3\alpha^2\lambda^3A_1^2a_0b_1^2 + 12a^3\alpha^2\lambda^3A_2^2a_0b_1^2 + 12a^3\alpha^2\mu^4a_0a_1^2 - 8a^3\alpha^2\lambda^2\mu b_1^3 \\
&\quad - 12a^3\alpha^2\lambda\mu^2a_0b_1^2 - 2a\alpha k^2\lambda^3\mu A_1^2b_1 - 2a\alpha k^2\lambda^3\mu A_2^2b_1 + 2a\alpha k^2\lambda\mu^3b_1 = 0, \\
\phi^2(\xi)\psi(\xi) &: 12a^3\alpha^2\lambda^4A_1^4a_1^2b_1 + 24a^3\alpha^2\lambda^4A_1^2A_2^2a_1^2b_1 + 12a^3\alpha^2\lambda^4A_2^4a_1^2b_1 - 24a^3\alpha^2\lambda^2\mu^2A_1^2a_1^2b_1 \\
&\quad - 24a^3\alpha^2\lambda^2\mu^2A_2^2a_1^2b_1 + 4a^3\alpha^2\lambda^3A_1^2b_1^3 + 4a^3\alpha^2\lambda^3A_2^2b_1^3 + 4a\alpha k^2\lambda^4A_1^4b_1 + 8a\alpha k^2\lambda^4A_1^2A_2^2b_1 \\
&\quad + 4a\alpha k^2\lambda^4A_2^4b_1 + 12a^3\alpha^2\mu^4a_1^2b_1 - 4a^3\alpha^2\lambda\mu^2b_1^3 - 8a\alpha k^2\lambda^2\mu^2A_1^2b_1 - 8a\alpha k^2\lambda^2\mu^2A_2^2b_1 \\
&\quad + 4a\alpha k^2\mu^4b_1 = 0, \\
\phi(\xi) &: -3a\beta^2\mu^4a_1 - 24a^3\alpha^2\lambda^2\mu^2A_2^2a_0^2a_1 + 12a\alpha\gamma\lambda^4A_1^4a_1 + 12a^3\alpha^2\mu^4a_0^2a_1 - 12a^3\alpha^2\lambda^2\mu^2a_1b_1^2 \\
&\quad - 8a\alpha k^2\lambda^3\mu^2A_2^2a_1 + 12a\alpha\gamma\mu^4a_1 - 24a\alpha\gamma\lambda^2\mu^2A_1^2a_1 + 8a\alpha c\lambda^2\mu^2A_1^2a_1 + 12a^3\alpha^2\lambda^4A_1^4a_0^2a_1 \\
&\quad + 12a^3\alpha^2\lambda^4A_2^4a_0^2a_1 - 4a\alpha c\mu^4a_1 - 8a\alpha k^2\lambda^3\mu^2A_1^2a_1 + 24a^3\alpha^2\lambda^4A_1^2A_2^2a_0^2a_1 + 4a\alpha k^2\lambda\mu^4a_1 \\
&\quad + 8a\alpha c\lambda^2\mu^2A_2^2a_1 - 4a\alpha c\lambda^4A_1^4a_1 + 6a\beta^2\lambda^2\mu^2A_1^2a_1 - 24a^3\alpha^2\lambda^2\mu^2A_1^2a_0^2a_1 + 12a^3\alpha^2\lambda^4A_1^2a_1b_1^2 \\
&\quad + 12a^3\alpha^2\lambda^4A_2^2a_1b_1^2 + 12a\alpha\gamma\lambda^4A_2^4a_1 - 8a\alpha c\lambda^4A_1^2A_2^2a_1 - 3a\beta^2\lambda^4A_2^4a_1 - 6a\beta^2\lambda^4A_1^2A_2^2a_1 \\
&\quad + 4a\alpha k^2\lambda^5A_2^4a_1 + 24a\alpha\gamma\lambda^4A_1^2A_2^2a_1 - 4a\alpha c\lambda^4A_2^4a_1 + 6a\beta^2\lambda^2\mu^2A_2^2a_1 - 3a\beta^2\lambda^4A_1^4a_1 \\
&\quad + 8a\alpha k^2\lambda^5A_1^2A_2^2a_1 + 4a\alpha k^2\lambda^5A_1^4a_1 - 24a\alpha\gamma\lambda^2\mu^2A_2^2a_1 = 0,
\end{aligned}$$

(59)

$$\begin{aligned}
\phi(\xi)\psi(\xi): &\, 24a^3\alpha^2\lambda^4A_1^4a_0a_1b_1 + 48a^3\alpha^2\lambda^4A_1^2A_2^2a_0a_1b_1 + 24a^3\alpha^2\lambda^4A_2^4a_0a_1b_1 - 24a^3\alpha^2\lambda^3\mu A_1^2a_1b_1^2 \\
&- 24a^3\alpha^2\lambda^3\mu A_2^2a_1b_1^2 - 48a^3\alpha^2\lambda^2\mu^2A_1^2a_0a_1b_1 - 48a^3\alpha^2\lambda^2\mu^2A_2^2a_0a_1b_1 - 6a\alpha k^2\lambda^4\mu A_1^4a_1 \\
&- 12a\alpha k^2\lambda^4\mu A_1^2A_2^2a_1 - 6a\alpha k^2\lambda^4\mu A_2^4a_1 + 24a^3\alpha^2\lambda\mu^3a_1b_1^2 + 24a^3\alpha^2\mu^4a_0a_1b_1 \\
&+ 12a\alpha k^2\lambda^2\mu^3A_1^2a_1 + 12a\alpha k^2\lambda^2\mu^3A_2^2a_1 - 6a\alpha k^2\mu^5a_1 = 0, \\
\psi(\xi): &\, 12a\alpha\gamma\mu^4b_1 - 2a\alpha k^2\lambda\mu^4b_1 - 24a^3\alpha^2\lambda^3\mu A_1^2a_0b_1^2 - 3a\beta^2\mu^4b_1 + 24a^3\alpha^2\lambda^4A_1^2A_2^2a_0^2b_1 \\
&- 24a^3\alpha^2\lambda^3\mu A_2^2a_0b_1^2 - 24a^3\alpha^2\lambda^2\mu^2A_2^2a_0^2b_1 + 4a^3\alpha^2\lambda^4A_2^2b_1^3 + 12a^3\alpha^2\lambda^4A_1^4a_0^2b_1 + 2a\alpha k^2\lambda^5A_1^4b_1 \\
&+ 12a^3\alpha^2\lambda^4A_2^4a_0^2b_1 + 2a\alpha k^2\lambda^5A_2^4b_1 + 12a\alpha\gamma\lambda^4A_2^4b_1 + 6a\beta^2\lambda^2\mu^2A_1^2b_1 + 4a^3\alpha^2\lambda^4A_1^2b_1^3 \\
&+ 12a^3\alpha^2\mu^4a_0^2b_1 - 4a\alpha c\lambda^4A_1^4b_1 + 4a\alpha k^2\lambda^5A_1^2A_2^2b_1 - 24a\alpha\gamma\lambda^2\mu^2A_2^2b_1 + 24a^3\alpha^2\lambda\mu^3a_0b_1^2 \\
&- 8a\alpha c\lambda^4A_1^2A_2^2b_1 - 6a\beta^2\lambda^4A_1^2A_2^2b_1 + 6a\beta^2\lambda^2\mu^2A_2^2b_1 + 8a\alpha c\lambda^2\mu^2A_1^2b_1 + 12\alpha^2a^3b_1^3\lambda^2\mu^2 \\
&+ 12a\alpha\gamma\lambda^4A_1^4b_1 - 3a\beta^2\lambda^4A_1^4b_1 - 4a\alpha c\lambda^4A_2^4b_1 - 4a\alpha c\mu^4b_1 - 24a\alpha\gamma\lambda^2\mu^2A_1^2b_1 \\
&- 24a^3\alpha^2\lambda^2\mu^2A_1^2a_0^2b_1 - 3a\beta^2\lambda^4A_2^4b_1 + 24a\alpha\gamma\lambda^4A_1^2A_2^2b_1 + 8a\alpha c\lambda^2\mu^2A_2^2b_1 = 0, \\
\phi^0(\xi): &\, -24a\alpha\gamma\lambda^2\mu^2A_2^2a_0 - 8a^3\alpha^2\lambda^2\mu^2A_1^2a_0^3 + 12a^3\alpha^2\lambda^4A_2^2a_0b_1^2 + 12a^3\alpha^2\lambda^4A_1^2a_0b_1^2 - 8a^3\alpha^2\lambda^2\mu^2A_2^2a_0^3 \\
&- 4a\alpha c\lambda^4A_2^4a_0 - 8\alpha^2a^3b_1^3\lambda^3\mu + 24a\alpha\gamma\lambda^4A_1^2A_2^2a_0 + 12a\alpha\gamma\lambda^4A_1^4a_0 - 4a\alpha c\mu^4a_0 - 4a\alpha c\lambda^4A_1^4a_0 \\
&- 3a\beta^2\lambda^4A_1^4a_0 - 12a^3\alpha^2\lambda^2\mu^2a_0b_1^2 + 4a^3\alpha^2\lambda^4A_2^4a_0^3 + 8a^3\alpha^2\lambda^4A_1^2A_2^2a_0^3 + 4a^3\alpha^2\lambda^4A_1^4a_0^3 \\
&- 8a\alpha c\lambda^4A_1^2A_2^2a_0 + 6a\beta^2\lambda^2\mu^2A_2^2a_0 - 3a\beta^2\lambda^4A_2^4a_0 + 12a\alpha\gamma\mu^4a_0 + 12a\alpha\gamma\lambda^4A_2^4a_0 \\
&- 2a\alpha k^2\lambda^4\mu A_1^2b_1 + 2a\alpha k^2\lambda^2\mu^3b_1 - 2a\alpha k^2\lambda^4\mu A_2^2b_1 - 24a\alpha\gamma\lambda^2\mu^2A_1^2a_0 + 6a\beta^2\lambda^2\mu^2A_1^2a_0 \\
&+ 8a\alpha c\lambda^2\mu^2A_1^2a_0 + 8a\alpha c\lambda^2\mu^2A_2^2a_0 - 6a\beta^2\lambda^4A_1^2A_2^2a_0 - 3a\beta^2\mu^4a_0 + 4a^3\alpha^2\mu^4a_0^3 = 0.
\end{aligned}$$

On solving the above algebraic system using the Maple package program, we obtain the following results.

Result 1:

$$a_0 = 0,\ a_1 = \pm\frac{k\sqrt{-\frac{1}{\alpha}}}{a},\ b_1 = 0,\ k = k,\ c = \frac{12\gamma - \lambda k^2(\beta^2-4)}{2(\beta^2+2)},\ \mu = 0,\ \lambda = \lambda, \tag{60}$$

where k, $a \neq 0$, β, γ, $\lambda(> 0)$ are arbitrary constants, α is defined in Equation (47). From Equations (12), (50) and (60), we obtain the traveling wave solutions of Equation (2) as follows:

$$\begin{aligned}
v_1^2(x,t) &= \pm k\sqrt{-\frac{1}{\alpha}}\frac{\left(A_1\sqrt{\lambda}\cos\left(\xi\sqrt{\lambda}\right) - A_2\sqrt{\lambda}\sin\left(\xi\sqrt{\lambda}\right)\right)}{A_1\sin\left(\xi\sqrt{\lambda}\right) + A_2\cos\left(\xi\sqrt{\lambda}\right)} - \frac{\beta}{2\alpha}, \\
u_1^2(x,t) &= \alpha\left(v_1^2(x,t)\right) + \beta\left(v_1^2(x,t)\right) + \gamma,\ w_1^2(x,t) = A\left(v_1^2(x,t)\right) + B,
\end{aligned} \tag{61}$$

where ξ is defined in Equation (36) with k, c defined in Equation (60), A_1, A_2 are arbitrary constants and A, B are defined in Equation (47).

Result 2:

$$\begin{aligned}
&a_0 = 0,\ a_1 = \pm\frac{k\sqrt{-\frac{1}{\alpha}}}{2a},\ b_1 = b_1,\ k = k, \\
&c = \frac{\left(\beta^2-4\right)\sqrt{\gamma^2\sigma_2^2 + \left(\beta^2-4\right)b_1^2a^2\sigma_2} - 2\gamma\sigma_2\left(\beta^2+8\right)}{2\sigma_2\left(\beta^2-4\right)}, \\
&\mu = 0,\ \lambda = -\frac{4\alpha a^2b_1^2}{k^2\sigma_2} > 0,
\end{aligned} \tag{62}$$

where b_1, k, $a \neq 0$, β, γ are arbitrary constants, α is defined in Equation (47) and $\sigma_2 = A_1^2 + A_2^2$, where A_1, A_2 are arbitrary constants. From Equations (12), (50) and (62), we obtain the traveling wave solutions of Equation (2) as follows:

$$\begin{aligned} v_2^2(x,t) &= \pm k\sqrt{-\frac{1}{\alpha}}\,\frac{\left(A_1\sqrt{-\frac{\alpha a^2 b_1^2}{k^2\sigma_2}}\cos\left(2\,\xi\sqrt{-\frac{\alpha a^2 b_1^2}{k^2\sigma_2}}\right) - A_2\sqrt{-\frac{\alpha a^2 b_1^2}{k^2\sigma_2}}\sin\left(2\,\xi\sqrt{-\frac{\alpha a^2 b_1^2}{k^2\sigma_2}}\right)\right)}{\left(A_1\sin\left(2\,\xi\sqrt{-\frac{\alpha a^2 b_1^2}{k^2\sigma_2}}\right) + A_2\cos\left(2\xi\sqrt{-\frac{\alpha a^2 b_1^2}{k^2\sigma_2}}\right)\right)} \\ &\quad + \frac{a b_1}{\left(A_1\sin\left(2\,\xi\sqrt{-\frac{\alpha a^2 b_1^2}{k^2\sigma_2}}\right) + A_2\cos\left(2\xi\sqrt{-\frac{\alpha a^2 b_1^2}{k^2\sigma_2}}\right)\right)} - \frac{\beta}{2\alpha}, \\ u_2^2(x,t) &= \alpha\left(v_2^2(x,t)\right) + \beta\left(v_2^2(x,t)\right) + \gamma,\; w_2^2(x,t) = A\left(v_2^2(x,t)\right) + B, \end{aligned} \tag{63}$$

where ξ is defined in Equation (36) with k, c defined in Equation (62) and A, B are defined in Equation (47).

Result 3:

Result 3.1

$$\begin{aligned} &a_0 = 0,\; a_1 = \frac{k\sqrt{-\frac{1}{\alpha}}}{2a}, b_1 = \pm\frac{k}{2a}\sqrt{\frac{\mu^2 - \lambda^2\sigma_2}{\alpha\lambda}},\; k = k,\;\; c = \frac{48\,\gamma - \lambda k^2(\beta^2 - 4)}{8(\beta^2 + 2)}, \\ &\mu = \mu,\; \lambda = \lambda, \end{aligned} \tag{64}$$

Result 3.2

$$\begin{aligned} &a_0 = 0,\; a_1 = -\frac{k\sqrt{-\frac{1}{\alpha}}}{2a}, b_1 = \pm\frac{k}{2a}\sqrt{\frac{\mu^2 - \lambda^2\sigma_2}{\alpha\lambda}},\; k = k,\;\; c = \frac{48\,\gamma - \lambda k^2(\beta^2 - 4)}{8(\beta^2 + 2)}, \\ &\mu = \mu,\; \lambda = \lambda, \end{aligned} \tag{65}$$

where k, $a \neq 0$, β, γ, μ, $\lambda(> 0)$ are arbitrary constants, α is defined in Equation (47) and $\sigma_2 = A_1^2 + A_2^2$, where A_1, A_2 are arbitrary constants. From Equations (12), (50) and (64), we obtain the traveling wave solutions of Equation (2), as follows:

$$\begin{aligned} v_3^2(x,t) &= k\sqrt{-\frac{1}{\alpha}}\,\frac{\left(A_1\sqrt{\lambda}\cos\left(\xi\sqrt{\lambda}\right) - A_2\sqrt{\lambda}\sin\left(\xi\sqrt{\lambda}\right)\right)}{\left(2\,A_1\sin\left(\xi\sqrt{\lambda}\right) + 2\,A_2\cos\left(\xi\sqrt{\lambda}\right) + 2\,\frac{\mu}{\lambda}\right)} \\ &\quad \pm \frac{k\sqrt{\frac{\mu^2 - \lambda^2\sigma_2}{\alpha\lambda}}}{\left(2A_1\sin\left(\xi\sqrt{\lambda}\right) + 2\,A_2\cos\left(\xi\sqrt{\lambda}\right) + 2\frac{\mu}{\lambda}\right)} - \frac{\beta}{2\alpha}, \\ u_3^2(x,t) &= \alpha\left(v_3^2(x,t)\right) + \beta\left(v_3^2(x,t)\right) + \gamma,\; w_3^2(x,t) = A\left(v_3^2(x,t)\right) + B, \end{aligned} \tag{66}$$

where ξ is defined in Equation (36) with k, c defined in Equation (64) and A, B are defined in Equation (47). Similarly, we can use Equations (12), (50) and (65) to construct the traveling wave solutions of Equation (2), but they are omitted here.

Case 3: Rational function solutions ($\lambda = 0$)

If $\lambda = 0$, we substitute Equation (50) into Equation (48), along with the use of Equations (9) and (15). Then, the left-hand side of Equation (48) becomes a polynomial in $\phi(\xi)$ and $\psi(\xi)$. Setting all of the coefficients of this polynomial to be zero, we obtain the following system of nonlinear algebraic equations in $a_0,\ a_1,\ b_1, A_1, A_2, \lambda, \mu, k, c$, provided that $A_1^2 - 2\mu A_2 \neq 0$.

$$
\begin{aligned}
\phi^3(\xi) :\ & 16a^3\alpha^2\mu^2A_2^2a_1^3 - 16a^3\alpha^2\mu A_1^2A_2a_1^3 + 4a^3\alpha^2A_1^4a_1^3 - 24a^3\alpha^2\mu A_2a_1b_1^2 + 12a^3\alpha^2A_1^2a_1b_1^2 \\
& + 16a\alpha k^2\mu^2A_2^2a_1 - 16a\alpha k^2\mu A_1^2A_2a_1 + 4a\alpha k^2A_1^4a_1 = 0, \\
\phi^2(\xi) :\ & 48a^3\alpha^2\mu^2A_2^2a_0a_1^2 - 48a^3\alpha^2\mu\, A_1^2A_2a_0a_1^2 + 12a^3\alpha^2A_1^4a_0a_1^2 - 24a^3\alpha^2\mu A_2a_0b_1^2 + 12a^3\alpha^2A_1^2a_0b_1^2 \\
& - 8a^3\alpha^2\mu b_1^3 + 4a\alpha\, k^2\mu^2A_2b_1 - 2a\alpha k^2\mu A_1^2b_1 = 0, \\
\phi^2(\xi)\psi(\xi) :\ & 48a^3\alpha^2\mu^2A_2^2a_1^2b_1 - 48a^3\alpha^2\mu\, A_1^2A_2a_1^2b_1 + 12a^3\alpha^2A_1^4a_1^2b_1 - 8a^3\alpha^2\mu A_2b_1^3 + 4a^3\alpha^2A_1^2b_1^3 \\
& + 16a\alpha k^2\mu^2A_2^2b_1 - 16a\alpha k^2\mu A_1^2A_2b_1 + 4a\alpha k^2A_1^4b_1 = 0, \\
\phi(\xi) :\ & 48a^3\alpha^2\mu^2A_2^2a_0^2a_1 - 48a^3\alpha^2\mu A_1^2A_2a_0^2a_1 + 12a^3\alpha^2A_1^4a_0^2a_1 - 16a\alpha c\mu^2A_2^2a_1 + 16a\alpha c\mu A_1^2A_2a_1 \\
& - 4a\alpha cA_1^4a_1 + 48a\alpha\gamma\mu^2A_2^2a_1 - 48a\alpha\gamma\mu A_1^2A_2a_1 + 12a\alpha\gamma A_1^4a_1 - 12a\beta^2\mu^2A_2^2a_1 \\
& + 12a\beta^2\mu A_1^2A_2a_1 - 3a\beta^2A_1^4a_1 = 0, \\
\phi(\xi)\psi(\xi) :\ & 96a^3\alpha^2\mu^2A_2^2a_0a_1b_1 - 96a^3\alpha^2\mu A_1^2A_2a_0a_1b_1 + 24a^3\alpha^2A_1^4a_0a_1b_1 + 48a^3\alpha^2\mu^2A_2a_1b_1^2 \\
& - 24a^3\alpha^2\mu A_1^2a_1b_1^2 - 24a\alpha k^2\mu^3A_2^2a_1 + 24a\alpha k^2\mu^2A_1^2A_2a_1 - 6a\alpha k^2\mu A_1^4a_1 = 0, \\
\psi(\xi) :\ & 48a^3\alpha^2\mu^2A_2^2a_0^2b_1 - 48a^3\alpha^2\mu A_1^2A_2a_0^2b_1 + 12a^3\alpha^2A_1^4a_0^2b_1 + 48a^3\alpha^2\mu^2A_2a_0b_1^2 - 24a^3\alpha^2\mu A_1^2a_0b_1^2 \\
& + 16a^3\alpha^2\mu^2b_1^3 - 8a\alpha k^2\mu^3A_2b_1 + 4a\alpha k^2\mu^2A_1^2b_1 - 16a\alpha\, c\mu^2A_2^2b_1 + 16a\alpha c\mu\, A_1^2A_2b_1 - 4a\alpha\, cA_1^4b_1 \\
& + 48a\alpha\gamma\mu^2A_2^2b_1 - 48a\alpha\gamma\mu A_1^2A_2b_1 + 12a\alpha\gamma\, A_1^4b_1 - 12a\beta^2\mu^2A_2^2b_1 + 12a\beta^2\mu\, A_1^2A_2b_1 \\
& - 3a\beta^2A_1^4b_1 = 0, \\
\phi^0(\xi) :\ & 16a^3\alpha^2\mu^2A_2^2a_0^3 - 16a^3\alpha^2\mu A_1^2A_2a_0^3 + 4a^3\alpha^2A_1^4a_0^3 - 16a\alpha c\mu^2A_2^2a_0 + 16a\alpha c\mu A_1^2A_2a_0 - 4a\alpha cA_1^4a_0 \\
& + 48\gamma a\alpha\mu^2A_2^2a_0 - 48\gamma a\alpha\mu A_1^2A_2a_0 + 12\gamma a\alpha A_1^4a_0 - 12a\beta^2\mu^2A_2^2a_0 + 12a\beta^2\mu A_1^2A_2a_0 \\
& - 3a\beta^2A_1^4a_0 = 0.
\end{aligned} \tag{67}
$$

On solving the above algebraic system using the Maple package program, we obtain the following results.

Result 1:

$$
a_0 = 0,\ a_1 = \pm\frac{k\sqrt{-\frac{1}{\alpha}}}{a},\ b_1 = 0,\ k = k,\ c = \frac{6\gamma}{\beta^2 + 2},\ \mu = 0, \tag{68}
$$

where k, $a \neq 0$, β, γ are arbitrary constants, α is defined in Equation (47). From Equations (14), (50) and (68), we obtain the traveling wave solutions of Equation (2) as follows:

$$
\begin{aligned}
& v_1^3(x,t) = \pm k\sqrt{-\frac{1}{\alpha}}\left(\frac{A_1}{A_1\xi + A_2}\right) - \frac{\beta}{2\alpha}, \\
& u_1^3(x,t) = \alpha\left(v_1^3(x,t)\right) + \beta\left(v_1^3(x,t)\right) + \gamma,\ w_1^3(x,t) = A\left(v_1^3(x,t)\right) + B,
\end{aligned} \tag{69}
$$

where ξ is defined in Equation (36) with k, c defined in Equation (68), A_1, A_2 are arbitrary constants, and A, B are defined in Equation (47).

Result 2:

$$
a_0 = 0,\ a_1 = 0,\ b_1 = \pm\frac{A_1k\sqrt{-\frac{1}{\alpha}}}{a},\ k = k,\ c = \frac{6\gamma}{\beta^2 + 2},\ \mu = 0, \tag{70}
$$

where k, $a \neq 0$, β, γ, A_1 are arbitrary constants, and α is defined in Equation (47). From Equations (14), (50) and (70), we obtain the traveling wave solutions of Equation (2), as follows:

$$\begin{aligned} v_2^3(x,t) &= \pm k\sqrt{-\frac{1}{\alpha}}\left(\frac{A_1}{A_1\xi + A_2}\right) - \frac{\beta}{2\alpha}, \\ u_2^3(x,t) &= \alpha\left(v_2^3(x,t)\right) + \beta\left(v_2^3(x,t)\right) + \gamma,\ w_2^3(x,t) = A\left(v_2^3(x,t)\right) + B, \end{aligned} \tag{71}$$

where ξ is defined in Equation (36) with k, c defined in Equation (70), A_2 is an arbitrary constant, and A, B are defined in Equation (47).

Result 3:

Result 3.1

$$a_0 = 0,\ a_1 = \pm\frac{k\sqrt{-\frac{1}{\alpha}}}{2a},\ b_1 = \frac{A_1 k\sqrt{-\frac{1}{\alpha}}}{2a},\ k = k,\ c = \frac{6\gamma}{\beta^2 + 2},\ \mu = 0, \tag{72}$$

Result 3.2

$$a_0 = 0,\ a_1 = \pm\frac{k\sqrt{-\frac{1}{\alpha}}}{2a},\ b_1 = -\frac{A_1 k\sqrt{-\frac{1}{\alpha}}}{2a},\ k = k,\ c = \frac{6\gamma}{\beta^2 + 2},\ \mu = 0, \tag{73}$$

where k, $a \neq 0$, β, γ, A_1 are arbitrary constants, and α is defined in Equation (47). From Equations (14), (50) and (72), we obtain the traveling wave solutions of Equation (2) as follows:

$$\begin{aligned} v_3^3(x,t) &= \pm k\sqrt{-\frac{1}{\alpha}}\left(\frac{A_1}{A_1\xi + A_2}\right) - \frac{\beta}{2\alpha}, \\ u_3^3(x,t) &= \alpha\left(v_3^3(x,t)\right) + \beta\left(v_3^3(x,t)\right) + \gamma,\ w_3^3(x,t) = A\left(v_3^3(x,t)\right) + B, \end{aligned} \tag{74}$$

where ξ is defined in Equation (36) with k, c defined in Equation (72), A_2 is an arbitrary constant and A, B are defined in Equation (47). Similarly, we can utilize Equations (14), (50) and (73) to construct the traveling wave solutions of Equation (2), but they are omitted here.

Result 4:

$$a_0 = 0,\ a_1 = \pm\frac{k\sqrt{-\frac{1}{\alpha}}}{2a},\ b_1 = b_1,\ k = k,\ c = \frac{6\gamma}{\beta^2 + 2},\ \mu = \frac{4a^2\alpha b_1^2 + k^2 A_1^2}{2k^2 A_2}, \tag{75}$$

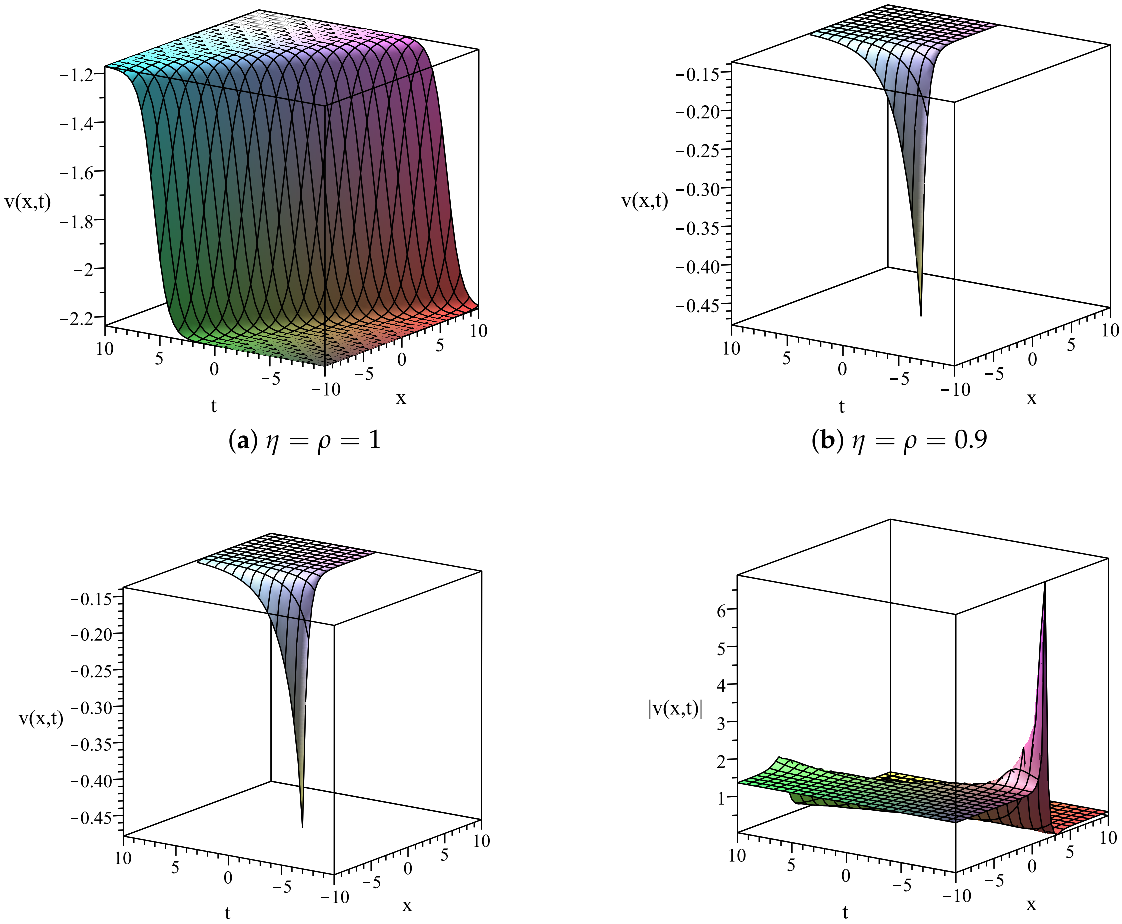
where $b_1, k, a \neq 0, \beta, \gamma, A_1, A_2$ are arbitrary constants, α is defined in Equation (47). From Equations (14), (50) and (75), we obtain the traveling wave solutions of Equation (2), as follows:

$$\begin{aligned} v_4^3(x,t) &= \pm k\sqrt{-\frac{1}{\alpha}}\,\frac{\left(\frac{(4a^2\alpha b_1^2 + k^2 A_1^2)\xi}{k^2 A_2} + A_1\right)}{\left(\frac{2(4a^2\alpha b_1^2 + k^2 A_1^2)\xi^2}{k^2 A_2} + 2A_1\xi + 2A_2\right)} + \frac{a b_1}{\frac{(4a^2\alpha b_1^2 + k^2 A_1^2)\xi^2}{4k^2 A_2} + A_1\xi + A_2} - \frac{\beta}{2\alpha}, \\ u_4^3(x,t) &= \alpha\left(v_4^3(x,t)\right) + \beta\left(v_4^3(x,t)\right) + \gamma,\ w_4^3(x,t) = A\left(v_4^3(x,t)\right) + B, \end{aligned} \tag{76}$$

where ξ is defined in Equation (36) with k, c defined in Equation (75) and A, B are defined in Equation (47).

Next, we show the three-dimensional plots of some selected exact solutions of Equation (2). The three exact solutions selected to provide graphical representation are $v_3^1(x,t)$ in Equation (58), $v_1^2(x,t)$ in Equation (61), and $v_3^3(x,t)$ in Equation (74). They will be drawn on $-10 \leq x, t \leq 10$ with the varied fractional orders η ,ρ among 1, 0.9, and 0.8. The graphical results of the selected solutions are described below.

The following fixed values $k = 1,\ a = 1,\ \beta = -3,\ \gamma = -3,\ \mu = 1,\ \lambda = -1,\ A_1 = 3,\ A_2 = 2$ and the variation of $\eta\ ,\rho \in \{1,\ 0.9,\ 0.8\}$ are used to plot associated graphs of $v_3^1(x,t)$ in Equation (58). In Figure 4a, the solution $v_3^1(x,t)$ with $\eta = \rho = 1$ was plotted to describe the kink-type solitary wave solution. The graphs of the solution $v_3^1(x,t)$ with $\eta = \rho = 0.9$ and $\eta = 0.9\ ,\rho = 0.8$ are presented in Figure 4b,c, respectively. The graph of $|v_3^1(x,t)|$ with $\eta = 0.9\ ,\rho = 0.8$ is shown in Figure 4d.

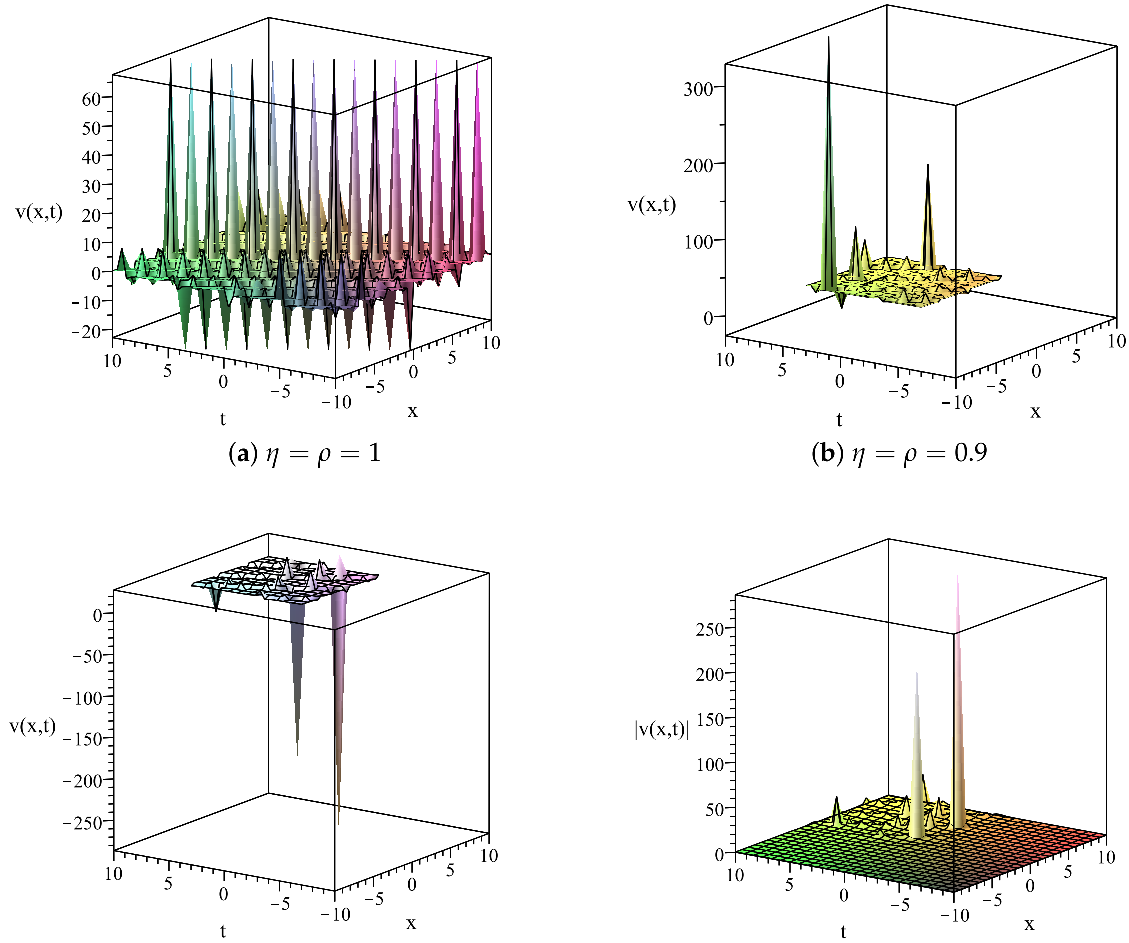


(**a**) $\eta = \rho = 1$ (**b**) $\eta = \rho = 0.9$

(**c**) $\eta = 0.9,\ \rho = 0.8$ (**d**) $\eta = 0.9,\ \rho = 0.8$

Figure 4. Associated plots of $v_3^1(x,t)$ in Equation (58) of Equation (2) on $-10 \le x,\ t \le 10$ using the $(G'/G, 1/G)$-expansion method.

In Figure 5a, the graphical representation of the periodic wave solutions obtained using the solution $v_1^2(x,t)$ in Equation (61), the parameter values $k = 1,\ a = 1,\ \beta = -4,\ \gamma = -1,\ \lambda = 1,$ $A_1 = 1,\ A_2 = 1$, and the fractional orders $\eta = \rho = 1$. Using the above parameter values, Figure 5b,c describe the singular multiple-soliton solutions for $v_1^2(x,t)$ with $\eta = 0.9\ ,\rho = 0.9$, and $\eta = 0.9\ ,\rho = 0.8$, respectively. The graph of $|v_1^2(x,t)|$ with $\eta = 0.9\ ,\rho = 0.8$ is portrayed in Figure 5d.

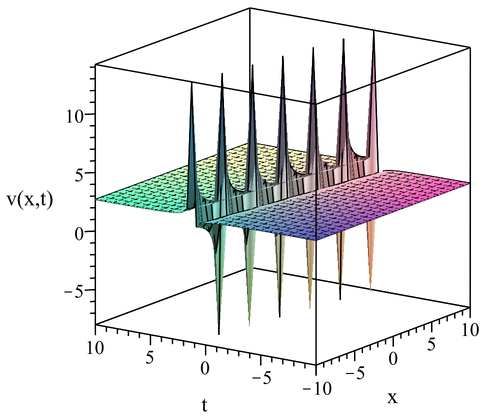


(**a**) $\eta = \rho = 1$ (**b**) $\eta = \rho = 0.9$

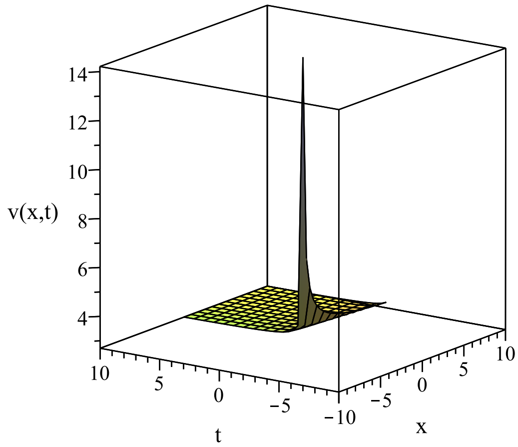
(**c**) $\eta = 0.9,\ \rho = 0.8$ (**d**) $\eta = 0.9,\ \rho = 0.8$

Figure 5. Associated plots of $v_1^2(x,t)$ in Equation (61) of Equation (2) on $-10 \leq x,\ t \leq 10$ using the $(G'/G, 1/G)$-expansion method.

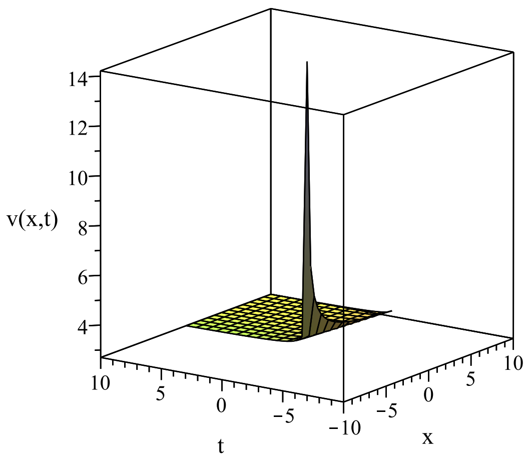
For the fixed values $k = 5,\ a = 5,\ \beta = 1,\ \gamma = -4,\ A_1 = 1,\ A_2 = 1$, the graphs of the exact solutions $v_3^3(x,t)$ in Equation (74) of Equation (2) corresponding to the variation of $\eta,\ \rho$ are investigated. The solution $v_3^3(x,t)$ with $\eta = \rho = 1$, describing the solitary wave solution of singular kink type, is depicted in Figure 6a. The solutions $v_3^3(x,t)$ with $\eta = \rho = 0.9$ and $\eta = 0.9,\ \rho = 0.8$, describing the 1-soliton solitary wave solution, are presented in Figure 6b,c, respectively. The graph of $|v_3^3(x,t)|$ with $\eta = 0.9,\ \rho = 0.8$ is plotted in Figure 6d.



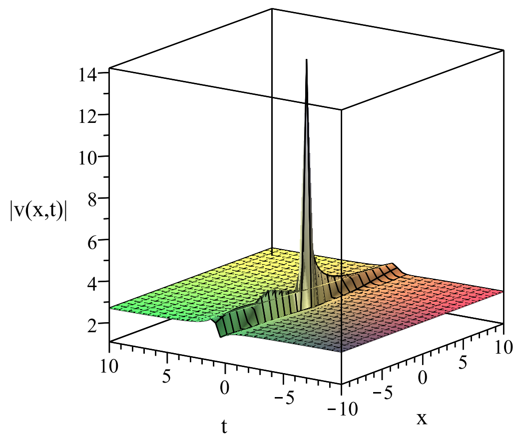
(**a**) $\eta = \rho = 1$



(**b**) $\eta = \rho = 0.9$



(**c**) $\eta = 0.9,\ \rho = 0.8$



(**d**) $\eta = 0.9,\ \rho = 0.8$

Figure 6. Associated plots of $v_3^3(x,t)$ in Equation (74) of Equation (2) on $-10 \leq x,\ t \leq 10$ using the $(G'/G, 1/G)$-expansion method.

Our results of the space-time-fractional generalized Hirota-Satsuma coupled KdV system in Equation (2) obtained using the $(G'/G, 1/G)$-expansion method are the generalization of the exact solutions reported in [54]. There are two reasons for this: (1) The equation, which was solved in [54], is the fractional generalized Hirota-Satsuma coupled KdV system, with only the time-conformable fractional derivative; and (2) the method, which was used to solve the equation in [54], is the (G'/G)-expansion method which is the particular case of the $(G'/G, 1/G)$-expansion method [51] by setting the parameter μ in Equation (7) and the coefficient b_j in Equation (16) to be zero. Particularly comparing our solutions with the ones in [54], the common solutions obtained using both methods consist of the hyperbolic, trigonometric, and rational function solutions. However, the number of our exact solutions is more than the number of solutions obtained in [54].

5. Conclusions

In this article, the two-variable $(G'/G, 1/G)$-expansion method has been used to obtain some novel exact solutions of the time-fractional (2+1)-dimensional extended quantum Zakharov-Kuznetsov equation and the space-time-fractional generalized Hirota-Satsuma coupled KdV system, as given in Equations (1) and (2), respectively. The method employed provided a variety of solutions for both problems, including the hyperbolic, trigonometric, and rational function solutions. Some of the solutions of (1) have been characterized in distinct physical structures, such as a bell-shaped solitary wave solution, a periodic traveling wave solution, and a singular soliton solution. The kink-type solitary wave solution and the singular multiple-soliton solution were found from the exact solutions of (2), which are depicted in Section 4.2. All solutions obtained in our work have been checked with the Maple package program by substituting them back into the original equations. To the best of our knowledge, these new solutions have not been constructed in previous literature—hence, they may be of vital importance for explaining some relevant physical phenomena of the mentioned equations. In summary, the $(G'/G, 1/G)$-expansion method equipped with the fractional complex transform is very powerful, reliable, and efficient in its application for obtaining exact traveling solutions for a wide class of NLFEEs.

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