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Some Improvements of the Hermite–Hadamard Integral Inequality

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Abstract: We propose several improvements of the Hermite–Hadamard inequality in the form of linear combination of its end-points and establish best possible constants. Improvements of a second order for the class $\Phi(I)$ with applications in Analysis and Theory of Means are also given.

Keywords: Convex function; Simpson’s rule; differentiable function

1. Introduction

A function $h : I \subset \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex on a non-empty interval I if the inequality

$$h\left(\frac{x+y}{2}\right) \leq \frac{h(x) + h(y)}{2} \quad (1)$$

holds for all $x, y \in I$.

If the inequality (1) reverses, then h is said to be concave on I [1].

Let $h : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex function on an interval I and $a, b \in I$ with $a < b$. Then

$$h\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b h(t) dt \leq \frac{h(a) + h(b)}{2}. \quad (2)$$

This double inequality is well known in the literature as the Hermite–Hadamard (HH) integral inequality for convex functions. It has a plenty of applications in different parts of Mathematics; see [2,3] and references therein.

If h is a concave function on I then both inequalities in (2) hold in the reversed direction.

Our task in this paper is to improve the inequality (2) in a simple manner, i.e., to find some constants $p, q; p + q = 1$ such that the relations

$$\frac{1}{b-a} \int_a^b h(t) dt \leq p \frac{h(a) + h(b)}{2} + q h\left(\frac{a+b}{2}\right), \quad (3)$$

hold for any convex h .

It can be easily seen that the condition

$$p + q = 1, \quad (4)$$

is necessary for (3) to hold for an arbitrary convex function.

Take, for example, $f(t) = Ct$, $C \in \mathbb{R}$.

Since

$$p\left(\frac{h(a)+h(b)}{2}\right) + qh\left(\frac{a+b}{2}\right) \leq \max\left\{\frac{h(a)+h(b)}{2}, h\left(\frac{a+b}{2}\right)\right\} = \frac{h(a)+h(b)}{2},$$

and, analogously,

$$p\left(\frac{h(a)+h(b)}{2}\right) + qh\left(\frac{a+b}{2}\right) \geq \min\left\{\frac{h(a)+h(b)}{2}, h\left(\frac{a+b}{2}\right)\right\} = h\left(\frac{a+b}{2}\right),$$

it follows that the inequality of the form (3) represents a refinement of Hermite–Hadamard inequality (2) for each $p, q > 0$, $p + q = 1$.

Note also that the linear form $p\frac{h(a)+h(b)}{2} + qh(\frac{a+b}{2})$ is monotone increasing in p . Therefore, if the inequality

$$\frac{1}{b-a} \int_a^b h(t) dt \leq p\frac{h(a)+h(b)}{2} + qh\left(\frac{a+b}{2}\right),$$

holds for some $p = p_0$, then it also holds for each $p \in [p_0, 1]$.

In the sequel we shall prove that the value $p_0 = 1/2$ is best possible for above inequality to hold for an arbitrary convex function on I .

Also, it will be shown that convexity/concavity of the second derivative is a proper condition for inequalities of the form (3) to hold (see Proposition 5 below).

This condition enables us to give refinements of second order and to increase interval of validity to $p_0 = 1/3$ as the best possible constant. In this case, coefficients $p_0 = 1/3, q_0 = 2/3$ are involved in the well-known form of Simpson's rule, which is of great importance in Numerical Analysis. Our results sharply improve Simpson's rule for this class of functions (Proposition 4).

Finally, we give some applications in Analysis and Numerical Analysis. Also, new and precise inequalities between generalized arithmetic means and power-difference means will be proved.

2. Results and Proofs

We shall begin with the basic contribution to the problem defined above.

Theorem 1. Let $h : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex function on an interval I and $a, b \in I$. Then

$$\frac{1}{b-a} \int_a^b h(t) dt \leq \frac{1}{2} \frac{h(a)+h(b)}{2} + \frac{1}{2} h\left(\frac{a+b}{2}\right). \quad (5)$$

The constants $p_0 = q_0 = 1/2$ are best possible.

If h is a concave function on I then the inequality is reversed.

Proof. We shall derive the proof by Hermite–Hadamard inequality itself. Indeed, applying twice the right part of this inequality, we get

$$\frac{2}{b-a} \int_a^{\frac{a+b}{2}} h(t) dt \leq \frac{1}{2} (h(a) + h(\frac{a+b}{2})),$$

and

$$\frac{2}{b-a} \int_{\frac{a+b}{2}}^b h(t) dt \leq \frac{1}{2} (h(\frac{a+b}{2}) + h(b)).$$

Summing up those inequalities the result appears. Therefore, HH inequality has this self-improving property.

That the constants $p_0 = q_0 = 1/2$ are best possible becomes evident by the example $f(t) = |t|, t \in [-a, a]$.

For the second part, note that concavity of f implies convexity of $-f$ on I . Hence, applying (5) we get the result. $\square$

For the sake of further refinements, we shall consider in the sequel functions from the class $C^{(m)}(I), m \in \mathbb{N}$ i.e., functions which are continuously differentiable up to m -th order on an interval $I \subset \mathbb{R}$.

Of utmost importance here is the class $\Phi(I)$ of functions which second derivative is convex on I . For this class we have the following

Theorem 2. Let $\phi \in \Phi(I)$ and the inequality

$$\frac{1}{b-a} \int_a^b \phi(t) dt \leq p \frac{\phi(a) + \phi(b)}{2} + q \phi\left(\frac{a+b}{2}\right), \quad (6)$$

holds for $a, b \in I$. Then $p \geq p_0 = 1/3$.

Proof. From (6) we have

$$p \geq \frac{\frac{1}{b-a} \int_a^b \phi(t) dt - \phi\left(\frac{a+b}{2}\right)}{\frac{\phi(a) + \phi(b)}{2} - \phi\left(\frac{a+b}{2}\right)} =: D_\phi(a, b).$$

Since this inequality should be valid for each $a, b \in I, a < b$, let $b \rightarrow a$. We obtain that $\lim_{b \rightarrow a} D_\phi(a, b) = 1/3$ almost everywhere on I i.e., whenever $\phi''(a) \neq 0$ or $\phi''(a) = 0, \phi'''(a) \neq 0$.

Indeed, applying L'Hospital's rule 3 and 4 times to the above quotient, we get

$$\lim_{b \rightarrow a} D_\phi(a, b) = \lim_{b \rightarrow a} \frac{\phi''(b) - \frac{3}{4}\phi''\left(\frac{a+b}{2}\right) - \frac{b-a}{8}\phi'''(\frac{a+b}{2})}{\frac{3}{2}\phi''(b) - \frac{3}{4}\phi''(\frac{a+b}{2}) + (b-a)(\frac{1}{2}\phi'''(b) - \frac{1}{8}\phi'''(\frac{a+b}{2}))},$$

and

$$\lim_{b \rightarrow a} D_\phi(a, b) = \lim_{b \rightarrow a} \frac{\phi'''(b) - \frac{1}{2}\phi'''(\frac{a+b}{2}) - \frac{b-a}{16}\phi^{(4)}(\frac{a+b}{2})}{2\phi'''(b) - \frac{1}{2}\phi'''(\frac{a+b}{2}) + (b-a)(\frac{1}{2}\phi^{(4)}(b) - \frac{1}{16}\phi^{(4)}(\frac{a+b}{2}))}.$$

Therefore, the result follows. $\square$

In the sequel we shall give sharp two-sided bounds of second order for inequalities of the type (3) involving functions from the class Φ with $p \geq 1/3$.

Main tool in all proofs will be the following relation.

Lemma 1. For an integrable function $\phi : I \rightarrow \mathbb{R}$ and arbitrary real numbers $p, q; p + q = 1$, we have the identity

$$p \frac{\phi(a) + \phi(b)}{2} + q \phi\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b \phi(t) dt = \frac{(b-a)^2}{16} \int_0^1 t(2p-t)(\phi''(x) + \phi''(y)) dt,$$

where $x := a\frac{t}{2} + b(1 - \frac{t}{2}), y := b\frac{t}{2} + a(1 - \frac{t}{2})$.

Proof. It is not difficult to prove this identity by double partial integration of its right-hand side. $\square$

For $t \in [0, 1]; a, b \in I, a < b$, denote

$$\begin{aligned}\xi(a, b; t) &:= \phi''\left(a\frac{t}{2} + b\left(1 - \frac{t}{2}\right)\right) + \phi''\left(b\frac{t}{2} + a\left(1 - \frac{t}{2}\right)\right) \\ &= \phi''(x) + \phi''(y).\end{aligned}$$

Lemma 2. If $\phi \in \Phi$ then the function $\xi(a, b; t)$ is monotone decreasing in t .

Hence,

$$2\phi''\left(\frac{a+b}{2}\right) \leq \phi''(x) + \phi''(y) \leq \phi''(a) + \phi''(b), \quad (7)$$

for all $t \in [0, 1]$.

Proof. Since $\phi''(\cdot)$ is convex, it follows that $\phi'''(\cdot)$ is increasing on I .

Also, $x \geq y$ for $t \in [0, 1]$ because $x - y = (b - a)(1 - t) \geq 0$.

Hence,

$$\xi'(a, b; t) = -\frac{b-a}{2}(\phi'''(x) - \phi'''(y)) \leq 0,$$

and $\xi(a, b; t)$ is decreasing in $t \in [0, 1]$.

Therefore,

$$2\phi''\left(\frac{a+b}{2}\right) = \xi(a, b; 1) \leq \xi(a, b; t) \leq \xi(a, b; 0) = \phi''(a) + \phi''(b),$$

which is equivalent with (7).

Note that, if ϕ is concave on I , then the function $\xi(a, b; t)$ is monotone increasing and the inequality (7) is reversed. $\square$

Remark 1. More general assertion than (7) is contained in [4].

Main results of this paper are given in the next two assertions.

Theorem 3. Let $\phi \in \Phi(I)$. Then

$$p\frac{\phi(a) + \phi(b)}{2} + q\phi\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b \phi(t)dt \leq \frac{(b-a)^2}{16} T_\phi(a, b; p),$$

where

$$T_\phi(a, b; p) = \begin{cases} \frac{4}{3}p^3(\phi''(a) + \phi''(b)) - \frac{2}{3}(1+p)(2p-1)^2\phi''\left(\frac{a+b}{2}\right) & , \frac{1}{3} \leq p \leq \frac{1}{2}; \\ (p - \frac{1}{3})(\phi''(a) + \phi''(b)) & , p \geq \frac{1}{2}. \end{cases}$$

Also, if $p \leq 0$, we have

$$p\frac{\phi(a) + \phi(b)}{2} + q\phi\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b \phi(t)dt \leq (p - \frac{1}{3})\frac{(b-a)^2}{8}\phi''\left(\frac{a+b}{2}\right).$$

Proof. If $p \geq 1/2$ we have that $2p - t \geq 0$. Therefore, applying Lemma 1 and the second part of Lemma 2, we obtain

$$\begin{aligned}p\frac{\phi(a) + \phi(b)}{2} + q\phi\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b \phi(t)dt &= \frac{(b-a)^2}{16} \int_0^1 t(2p-t)(\phi''(x) + \phi''(y))dt \\ &\leq \frac{(b-a)^2}{16}(\phi''(a) + \phi''(b)) \int_0^1 t(2p-t)dt = (p - 1/3)\frac{(b-a)^2}{16}(\phi''(a) + \phi''(b)).\end{aligned}$$

In the case $1/3 \leq p < 1/2$, write

$$\int_0^1 t(2p-t)(\phi''(x) + \phi''(y))dt = \int_0^{2p} t(2p-t)(\cdot)dt - \int_{2p}^1 t(t-2p)(\cdot)dt,$$

and apply Lemma 2 to each integral separately.

It follows that

$$\begin{aligned} \int_0^1 t(2p-t)(\phi''(x) + \phi''(y))dt &\leq (\phi''(a) + \phi''(b)) \int_0^{2p} t(2p-t)dt - 2\phi''\left(\frac{a+b}{2}\right) \int_{2p}^1 t(t-2p)dt \\ &= \frac{4p^3}{3}(\phi''(a) + \phi''(b)) - 2\left(\frac{1}{3} - p + \frac{4p^3}{3}\right)\phi''\left(\frac{a+b}{2}\right), \end{aligned}$$

which is equivalent to the stated assertion.

For $p \leq 0$ we have that $2p - t \leq 0$ and the proof develops in the same manner. $\square$

Theorem 4. If $\phi \in \Phi(I)$, then for $p \geq 1/3$ we get

$$p \frac{\phi(a) + \phi(b)}{2} + q\phi\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b \phi(t)dt \geq (p-1/3) \frac{(b-a)^2}{8} \phi''\left(\frac{a+b}{2}\right),$$

and

$$p \frac{\phi(a) + \phi(b)}{2} + q\phi\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b \phi(t)dt \geq (p-1/3) \frac{(b-a)^2}{16} (\phi''(a) + \phi''(b)),$$

for $p \leq 0$.

Proof. By Lemma 1, in terms of Lemma 2, we have

$$p \frac{\phi(a) + \phi(b)}{2} + q\phi\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b \phi(t)dt = \frac{(b-a)^2}{16} \int_0^1 t(2p-t) \xi(a, b; t) dt.$$

By partial integration, we obtain

$$\begin{aligned} \int_0^1 t(2p-t) \xi(a, b; t) dt &= (pt^2 - t^3/3) \xi(a, b; t) \Big|_0^1 - \int_0^1 t^2(p-t/3) \xi'(a, b; t) dt \\ &\geq 2(p-1/3) \phi''\left(\frac{a+b}{2}\right), \end{aligned}$$

since $p - t/3 \geq 0$ for $p \geq 1/3$ and, by Lemma 2, $\xi'(a, b; t) \leq 0$ for $t \in [0, 1]$.

If $p \leq 0$ then $2p - t \leq 0$ and, applying Lemmas 1 and 2, the result follows. $\square$

Above theorems are the source of a plenty of important inequalities which sharply refine Hermite–Hadamard inequality for this class of functions.

Some of them are listed in the sequel.

Proposition 1. Let $\phi \in \Phi(I)$. Then

$$\frac{(b-a)^2}{24} \phi''\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b \phi(t)dt - \phi\left(\frac{a+b}{2}\right) \leq \frac{(b-a)^2}{24} \frac{\phi''(a) + \phi''(b)}{2}.$$

Proof. Put $p = 0$ in the above theorems. $\square$

Proposition 2. Let $\phi \in \Phi(I)$. Then

$$\frac{(b-a)^2}{12} \phi''\left(\frac{a+b}{2}\right) \leq \frac{\phi(a) + \phi(b)}{2} - \frac{1}{b-a} \int_a^b \phi(t) dt \leq \frac{(b-a)^2}{12} \frac{\phi''(a) + \phi''(b)}{2}.$$

Proof. This proposition is obtained for $p = 1$. $\square$

The next assertion represents a refinement of Theorem 1 in the case of convex functions.

Proposition 3. Let $\phi \in \Phi(I)$. Then for each $a, b \in I, a < b$,

$$\frac{(b-a)^2}{48} \phi''\left(\frac{a+b}{2}\right) \leq \frac{1}{2} \frac{\phi(a) + \phi(b)}{2} + \frac{1}{2} \phi\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b \phi(t) dt \leq \frac{(b-a)^2}{48} \frac{\phi''(a) + \phi''(b)}{2}.$$

If ϕ'' is concave on I , then

$$\frac{(b-a)^2}{48} \frac{\phi''(a) + \phi''(b)}{2} \leq \frac{1}{2} \frac{\phi(a) + \phi(b)}{2} + \frac{1}{2} \phi\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b \phi(t) dt \leq \frac{(b-a)^2}{48} \phi''\left(\frac{a+b}{2}\right).$$

Proof. Put $p = 1/2$ in Theorems 3 and 4.

The second part follows from a variant of Lemma 2 for concave functions. $\square$

Note that the coefficients $p = 1/3$ and $q = 2/3$ are involved in well-known Simpson's rule which is of importance in numerical integration [5].

The next assertion sharply refines Simpson's rule for this class of functions.

Proposition 4. For $\phi \in \Phi(I)$, we have

$$\begin{aligned} 0 &\leq \frac{1}{3} \frac{\phi(a) + \phi(b)}{2} + \frac{2}{3} \phi\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b \phi(t) dt \\ &\leq \frac{(b-a)^2}{162} \left[\frac{\phi''(a) + \phi''(b)}{2} - \phi''\left(\frac{a+b}{2}\right) \right]. \end{aligned}$$

If ϕ'' is concave on I , then

$$\begin{aligned} 0 &\leq \frac{1}{b-a} \int_a^b \phi(t) dt - \frac{1}{6} [\phi(a) + \phi(b) + 4\phi\left(\frac{a+b}{2}\right)] \\ &\leq \frac{(b-a)^2}{162} \left[\phi''\left(\frac{a+b}{2}\right) - \frac{\phi''(a) + \phi''(b)}{2} \right]. \end{aligned}$$

Proof. Applying Theorems 3 and 4 with both parts of Lemma 2 for $p = 1/3$, the proof follows. $\square$

The next assertion gives a proper answer to the problem posed in Introduction.

Proposition 5. If ϕ is a convex and ϕ'' is a concave function on I , then

$$\frac{1}{3} \frac{\phi(a) + \phi(b)}{2} + \frac{2}{3} \phi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b \phi(t) dt \leq \frac{1}{2} \frac{\phi(a) + \phi(b)}{2} + \frac{1}{2} \phi\left(\frac{a+b}{2}\right).$$

Analogously, let ϕ be concave and ϕ'' a convex function on I , then

$$\frac{1}{2} \frac{\phi(a) + \phi(b)}{2} + \frac{1}{2} \phi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b \phi(t) dt \leq \frac{1}{3} \frac{\phi(a) + \phi(b)}{2} + \frac{2}{3} \phi\left(\frac{a+b}{2}\right).$$

Proof. Combining Proposition 4 with the results of Theorem 1, we obtain the proof. $\square$

3. Applications in Analysis

Theorems proved above are the source of interesting inequalities from Classical Analysis. As an illustration we shall give here a couple of Cusa-type inequalities.

Theorem 5. *The inequality*

$$\frac{1}{2} \cos x + \frac{1}{2} \leq \frac{\sin x}{x} \leq \frac{1}{3} \cos x + \frac{2}{3},$$

holds for $|x| \leq \pi/2$.

Also,

$$\frac{1}{4} \cosh x + \frac{3}{4} \leq \frac{\sinh x}{x} \leq \frac{1}{3} \cosh x + \frac{2}{3},$$

holds for $|x| \leq (3/2)^{3/2}$.

Proof. For the first part one should apply Proposition 5 to the function $\phi(t) = \cos t$ on a symmetric interval $t \in [-x, x] \subset [-\pi/2, \pi/2]$.

For the second part, applying Proposition 4 with $\phi(t) = e^t, t \in [-x, x]$, we get

$$0 \leq \frac{1}{3} \cosh x + \frac{2}{3} - \frac{\sinh x}{x} \leq \frac{2x^2}{81} (\cosh x - 1).$$

Hence,

$$\frac{\sinh x}{x} \leq \frac{1}{3} \cosh x + \frac{2}{3},$$

and

$$\begin{aligned} \frac{\sinh x}{x} &\geq \frac{1}{3} \cosh x + \frac{2}{3} - \frac{2x^2}{81} (\cosh x - 1) \\ &= \left(\frac{1}{12} - \frac{2x^2}{81}\right) \cosh x + \frac{1}{4} \cosh x + \frac{2}{3} + \frac{2x^2}{81} \geq \frac{1}{4} \cosh x + \frac{3}{4}, \end{aligned}$$

since $\cosh x \geq 1$ and $1/12 - 2x^2/81 \geq 0$ for $|x| \leq (3/2)^{3/2} \approx 1.8371$. $\square$

We give now some numerical examples of the above inequality

$$\frac{1}{2} \cos x + \frac{1}{2} \leq \frac{\sin x}{x} \leq \frac{1}{3} \cos x + \frac{2}{3}, \quad |x| \leq \pi/2. \quad (8)$$

Namely, using known formulae

$$\sin \frac{\pi}{2} = 1; \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}; \sin \frac{\pi}{6} = \frac{1}{2}; \sin \frac{\pi}{12} = \frac{\sqrt{2}}{4}(\sqrt{3} - 1) \approx 0.25882;$$

$$\sin \frac{\pi}{24} = \frac{1}{2} \sqrt{2 - \sqrt{2 + \sqrt{3}}} \approx 0.13053; \sin \frac{\pi}{60} = \frac{1}{16} [\sqrt{2}(\sqrt{3} + 1)(\sqrt{5} - 1) - 2(\sqrt{3} - 1)\sqrt{5 + \sqrt{5}}] \approx 0.052336,$$

and applying inequalities (8), we obtain bounds for the transcendental number π , as follows

$$x = \frac{\pi}{2} : 3 < \pi < 4; x = \frac{\pi}{4} : 3.1344 < \pi < 3.3137; x = \frac{\pi}{6} : 3.1402 < \pi < 3.2154;$$

$$x = \frac{\pi}{12} : 3.1415 < \pi < 3.1597; x = \frac{\pi}{24} : 3.1416 < \pi < 3.1461; x = \frac{\pi}{60} : 3.1416 < \pi < 3.1423.$$

Another application can be obtained by integrating both sides of (8) on the range $x \in [0, a]$, $0 < a < \pi/2$.

We get

$$\frac{1}{2} \sin a + \frac{1}{2} a \leq \int_0^a \frac{\sin x}{x} dx \leq \frac{1}{3} \sin a + \frac{2}{3} a,$$

that is,

$$\frac{a - \sin a}{3} \leq a - \int_0^a \frac{\sin x}{x} dx \leq \frac{a - \sin a}{2}.$$

By the power series expansion, we know that

$$a - \sin a = \frac{a^3}{3!} - \frac{a^5}{5!} + \frac{a^7}{7!} - \dots$$

Hence,

$$\frac{a^3}{18} - \frac{a^5}{360} \leq a - \int_0^a \frac{\sin x}{x} dx \leq \frac{a^3}{12}.$$

This estimation is effective for small values of a .

For example,

$$5.5528 \times 10^{-5} \leq \frac{1}{10} - \int_0^{1/10} \frac{\sin x}{x} dx \leq 8.3333 \times 10^{-5}.$$

4. Applications in Theory of Means

A mean $M(a, b)$ is a map $M : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, with the property

$$\min\{a, b\} \leq M(a, b) \leq \max\{a, b\},$$

for each $a, b \in \mathbb{R}_+$.

Some refinements of HH inequality by arbitrary means is given in [6].

An ordered set of elementary means is the following family,

$$H \leq G \leq L \leq I \leq A \leq S,$$

where

$$H = H(a, b) =: 2(1/a + 1/b)^{-1}; \quad G = G(a, b) =: \sqrt{ab}; \quad L = L(a, b) =: \frac{b - a}{\log b - \log a};$$

$$I = I(a, b) =: \frac{1}{e} (b^b / a^a)^{1/(b-a)}; \quad A = A(a, b) =: \frac{a+b}{2}; \quad S = S(a, b) =: a^{\frac{a}{a+b}} b^{\frac{b}{a+b}},$$

are the harmonic, geometric, logarithmic, identric, arithmetic and Gini mean, respectively.

Generalized arithmetic mean A_α is defined by

$$A_\alpha = A_\alpha(a, b) =: \begin{cases} \left(\frac{a^\alpha + b^\alpha}{2} \right)^{1/\alpha}, & \alpha \neq 0; \\ A_0 = G. \end{cases}$$

Power-difference mean K_α is defined by

$$K_\alpha = K_\alpha(a, b) =: \begin{cases} \frac{\alpha}{\alpha+1} \frac{a^{\alpha+1} - b^{\alpha+1}}{a^\alpha - b^\alpha} & , \alpha \neq 0, -1; \\ K_0(a, b) = L(a, b); \\ K_{-1}(a, b) = ab/L(a, b). \end{cases}$$

It is well known that both means are monotone increasing with α and, evidently,

$$A_{-1} = H, A_1 = A, K_{-2} = H, K_{-1/2} = G, K_1 = A.$$

As an illustration of our results, we shall give firstly some sharp bounds of power-difference means in terms of the generalized arithmetic mean.

Theorem 6. For $a, b \in \mathbb{R}^+$ and $\alpha \geq 1$, we have

$$\frac{1}{2}(A(a, b) + A_\alpha(a, b)) \leq K_\alpha(a, b) \leq A_\alpha(a, b). \quad (9)$$

For $\alpha < 1$ the inequality (9) is reversed.

Proof. Let $g_\alpha(t) = t^{1/\alpha}$, $\alpha \neq 0$. Since g_α is concave for $\alpha \geq 1$, Theorem 1 combined with the HH inequality gives

$$\begin{aligned} & \frac{1}{2} \left(\frac{x+y}{2} \right)^{1/\alpha} + \frac{1}{4} (x^{1/\alpha} + y^{1/\alpha}) \\ & \leq \frac{\alpha}{\alpha+1} \frac{x^{1+1/\alpha} - y^{1+1/\alpha}}{x - y} \leq \left(\frac{x+y}{2} \right)^{1/\alpha}. \end{aligned}$$

Now, simple change of variables $x = a^\alpha$, $y = b^\alpha$ yields the result.

For the second part, note that g_α is convex for $\alpha < 1$ and repeat the procedure. $\square$

The above inequality is refined by the following

Theorem 7. We have,

$$A_\alpha \leq K_\alpha \leq \frac{1}{3}(A + 2A_\alpha), \quad \alpha \in (-\infty, 1/3) \cup (1/2, 1);$$

$$\frac{1}{3}(A + 2A_\alpha) \leq K_\alpha \leq A_\alpha, \quad \alpha \in [1, \infty);$$

$$\frac{1}{3}(A + 2A_\alpha) \leq K_\alpha \leq \frac{1}{2}(A + A_\alpha), \quad \alpha \in [1/3, 1/2].$$

Proof. Observe that g_α'' is convex for $\alpha \in (-\infty, 1/3) \cup (1/2, 1)$ and concave for $\alpha \in (1/3, 1/2) \cup (1, \infty)$. Hence, applying Proposition 5 together with the HH inequality, we obtain the result. $\square$

Remark 2. Note that the above inequalities are so precise that in critical points for $\alpha = 1/3, 1/2, 1$ we have equality sign.

An inequality for the reciprocals follows.

Theorem 8. For $\beta \geq -2$ we have

$$\frac{1}{A_{\beta+1}} \leq \frac{1}{K_\beta} \leq \frac{1}{2} \left(\frac{1}{H} + \frac{1}{A_{\beta+1}} \right).$$

For $\beta < -2$ the inequality is reversed.

Proof. This is a consequence of Theorem 6. Indeed, putting there $\alpha = -\beta - 1$ and using identities

$$K_\alpha = \frac{ab}{K_\beta}, A_\alpha = \frac{ab}{A_{\beta+1}}, A = \frac{ab}{H},$$

the proof appears. $\square$

Finally, we give a new and precise double inequality for the identric mean $I(a, b)$.

Theorem 9. For arbitrary positive a, b we have

$$A^{4/3}S^{-1/3}\exp\left(-\frac{4}{81}\frac{(A-H)^2}{AH}\right) \leq I \leq A^{4/3}S^{-1/3}.$$

Proof. We need firstly an auxiliary result.

Lemma 3. For $a, b \in \mathbb{R}^+$, we have

$$A^{4/3}(a, b)S^{2/3}(a, b)\exp\left(-\frac{4}{81}\frac{(A(a, b) - H(a, b))^2}{A(a, b)H(a, b)}\right) \leq I(a^2, b^2) \leq A^{4/3}(a, b)S^{2/3}(a, b).$$

Proof. Indeed, for $\phi(t) = t \log t$ we get

$$\frac{1}{b-a} \int_a^b \phi(t) dt = \frac{1}{4} \left(\frac{b^2 \log b^2 - a^2 \log a^2}{b-a} - (a+b) \right) = \frac{a+b}{4} \log I(a^2, b^2).$$

Since $\phi''(t) = 1/t$, Proposition 5 yields

$$\begin{aligned} & \frac{1}{6}(a \log a + b \log b) + \frac{2}{3}A \log A - \frac{(b-a)^2}{324} \left(\frac{1}{a} + \frac{1}{b} - \frac{2}{A} \right) \\ & \leq \frac{a+b}{4} \log I(a^2, b^2) \leq \frac{1}{6}(a \log a + b \log b) + \frac{2}{3}A \log A, \end{aligned}$$

and the proof follows by dividing the last expression with $(a+b)/4 = A/2$. $\square$

Now, combining this assertion with the identity $I(a^2, b^2) = I(a, b)S(a, b)$, we obtain the desired inequality. $\square$

Remark 3. An equivalent form of the above result is

$$I^{3/4}S^{1/4} \leq A \leq I^{3/4}S^{1/4}\exp\left(\frac{(A-H)^2}{27AH}\right),$$

which refines well-known inequality $I \leq A \leq S$.

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