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The Symmetry of Lower and Upper Approximations, Determined by a Cyclic Hypergroup, Applicable in Control Theory

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Abstract: In the first part of our paper, we construct a cyclic hypergroup of matrices using the Ends Lemma. Its properties are then, in the second part of the paper, used to describe the symmetry of lower and upper approximations in certain rough sets with respect to invertible subhypergroups of this cyclic hypergroup. Since our approach is widely used in autonomous robotic systems, we suggest an application of our results for the study of detection sensors, which are used especially in mobile robot mapping.

Keywords: single-power cyclic hypergroup; invertible subhypergroup; lower approximation; upper approximation; rough set

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1. Introduction

Our perception of the real world is never fully precise. Our decisions are always made with a certain level of uncertainty or lack of some pieces of information. Mathematical tools that make this decision making easier include *fuzzy sets* [1], *rough sets* [2,3] or *soft sets* [4]. In the *algebraic hyperstructure theory*, i.e., in the theory of *algebraic hypercompositional structures*, there are numerous constructions leading to such structures. One of these is the application of the *Ends lemma* [5–7] in which the hyperoperation is the principal end of a partially ordered semigroup. For some theoretical results regarding the construction, see for example Novák et al. [7–9]. In [9] the authors modify the construction in order to increase its applicability.

In our paper we develop an idea similar to [9–11]. With the help of matrix calculus, which we believe is a suitable tool, we construct cyclic hypergroups and their invertible subhypergroups. Notice that matrix calculus linked to the theory of algebraic hypercompositional structures has been used as a suitable tool in various contexts such as [12–14]. Since the notion of hypercompositional cyclicity has a rather complicated evolution, we recommend the reader to study [15], which gives a complex discussion of the topic, and [16] which is the source of our definition.

In [17] the authors consider the application of rough sets in various contexts based on establishing the set describing its upper and lower approximation. In a general case, this issue is discussed, e.g., in [11,18,19]. Suppose that we can see the Ends Lemma as a certain boundary used in various areas such as economics (the need to generate *at least* certain profit), electrical engineering (the transistor basis of a p-n junction needs *at least* certain current for the charge flow yet this must not be too great), etc. Motivated by these considerations we investigate hypercompositional structures constructed with

the help of the principal end (or beginning) and their subhyperstructures. We study their cyclicity and their generators. In the context of rough sets we use cyclic hypergroups to construct the universum with the help of indiscernibility relation and its subhyperstructures describing their upper and lower approximations. In order to describe these we make use of some natural relations between matrix characteristics. In the end of our paper we demonstrate how this type of rough sets, especially the upper approximations, can be used for description of an area monitored by sensors of autonomous robotic systems.

2. Notation and Context

In our paper, we work with matrices. In a general case these are matrices over a field. However, within this field we regard a subset (infinite such as \mathbb{N} or finite such as in Example 5), elements of which are actually used as entries. We denote such a set by \mathcal{E} , and the set of all $m \times n$ matrices with entries from a \mathcal{E} by $\mathbb{M}_{m,n}(\mathcal{E})$. We also suppose that there exists a total order on \mathcal{E} with the smallest element of \mathcal{E} denoted (if it exists) by e . Within the set \mathcal{E} we, in our hyperoperations, rely on the operation $\min\{a, b\}$, the result of which is, given our context, equal to the smaller element.

On $\mathbb{M}_{m,n}(\mathcal{E})$ we, for an arbitrary pair of matrices $\mathbf{A}, \mathbf{B} \in \mathbb{M}_{m,n}(\mathcal{E})$, define relation \leq_M by

$$\mathbf{A} \leq_M \mathbf{B} \text{ if } \|\mathbf{A}\|_\infty \leq_M \|\mathbf{B}\|_\infty \quad (1)$$

where $\|\mathbf{A}\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{ij}|$, i.e., $\|\mathbf{A}\|_\infty$ is the row norm.

Example 1. For matrices

$$\mathbf{A} = \begin{bmatrix} 8 & 1 & 2 \\ 4 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 0 & 3 \\ 5 & 1 & 0 \end{bmatrix},$$

there is

$$\|\mathbf{A}\|_\infty = 11 \quad \text{and} \quad \|\mathbf{B}\|_\infty = 6.$$

Therefore $\mathbf{B} \leq_M \mathbf{A}$.

Basic Notions of the Theory of Algebraic Hypercompositional Structures

Before we give our results, recall some basic notions of the algebraic hyperstructure theory (or theory of algebraic hypercompositional structures). For further reference see, for example, books [20,21]. A *hypergroupoid* is a pair $(H, *)$, where H is a nonempty set and the mapping $* : H \times H \longrightarrow \mathcal{P}^*(H)$ is a binary *hyperoperation* (or *hypercomposition*) on H (here $\mathcal{P}^*(H)$ denotes the system of all nonempty subsets of H). If $a * (b * c) = (a * b) * c$ holds for all $a, b, c \in H$, then $(H, *)$ is called a *semihypergroup*. If moreover the reproduction axiom, i.e., relation $a * H = H = H * a$ for all $a \in H$, is satisfied, then the semihypergroup $(H, *)$ is called *hypergroup*. Unlike in groups, in hypergroups neutral elements or inverses need not be unique. By a neutral element, or an *identity* (or *unit*) of $(H, *)$ we mean such an element $e \in H$ that $e \in x * e \cap e * x$ for all $x \in H$ while by an *inverse* of $a \in H$ we mean such an element $a' \in H$ that there exists an identity $e \in H$ such that $e \in a * a' \cap a' * a$. By *idempotence* in the sense of hypercompositional structures we mean that $a \in a * a$, i.e., that the element is included in its “second power” (which is, in general, a set).

Numerous notions of algebraic structures can be generalized for algebraic hypercompositional structures while some hypercompositional notions have no counterparts in algebraic structures. One of the key algebraic concepts, *cyclicity*, can be transferred to theory of algebraic hypercompositional structures in several ways. For a complex discussion of these approaches as well as their historical context and evolution and clarification of naming and notation, see Novák, Křehlík and Cristea [15].

In our paper we will work with the following definition introduced by Vougiouklis [16] (reworded as in [15]); for more results regarding cyclic hypergroups see Vougiouklis [22].

Definition 1 ([15,16]). A hypergroup (H, \circ) is called cyclic if, for some $h \in H$, there is

$$H = h^1 \cup h^2 \cup \dots \cup h^n \cup \dots, \quad (2)$$

where $h^1 = \{h\}$ and $h^m = \underbrace{h \circ \dots \circ h}_m$. If there exists $n \in \mathbb{N}$ such that Formula (2) is finite, we say that H is a cyclic hypergroup with finite period; otherwise, H is a cyclic hypergroup with infinite period. The element $h \in H$ in Formula (2) is called generator of H , the smallest power n for which Formula (2) is valid is called period of h . If all generators of H have the same period n , then H is called cyclic with period n . If, for a given generator h , Formula (2) is valid but no such n exists (i.e., Formula (2) cannot be finite), then H is called cyclic with infinite period. If we can, for some $h \in H$, write

$$H = h^n. \quad (3)$$

Then, the hypergroup H is called single-power cyclic with a generator h . If Formula (2) is valid and for all $n \in \mathbb{N}$ and, for a fixed $n_0 \in \mathbb{N}$, $n \geq n_0$ there is

$$h^1 \cup h^2 \cup \dots \cup h^{n-1} \subsetneq h^n, \quad (4)$$

then we say that H is a single-power cyclic hypergroup with an infinite period for h .

Apart from the notion of cyclicity we will work with the notion of *EL-hyperstructures*, i.e., hypercompositional structures constructed from ordered (or sometimes pre-ordered) semigroups by means of what is known as the “Ends Lemma”. For details and applications see, for example, [8–10,12,23].

3. Single-Power Cyclic Hypergroup of Matrices

In order to construct a single-power cyclic hypergroup of matrices, we first, for an arbitrary pair of matrices $\mathbf{A}, \mathbf{B} \in \mathbb{M}_{m,n}(\mathcal{E})$, define a hyperoperation by

$$\mathbf{A} * \mathbf{B} = \{\mathbf{A}, \mathbf{B}\} \cup [\mathbf{A} \circ_m \mathbf{B}]_{\leq_M}, \quad (5)$$

where $\mathbf{A} \circ_m \mathbf{B}$ is such a matrix \mathbf{D} that $\mathbf{D} = \{[d_{i,j}] \mid d_{i,j} = \min\{a_{ij}; b_{ij}\}, i \in \{1, \dots, m\}, j \in \{1, \dots, n\}\}$ and $[\mathbf{D}]_{\leq_M}$ is the set of all matrices greater than \mathbf{D} , i.e., $[\mathbf{D}]_{\leq_M} = \{\mathbf{X} \in \mathbb{M}_{m,n}(\mathcal{E}) \mid \mathbf{D} \leq_M \mathbf{X}\}$. Thus, using terminology of Chvalina [5], $(\mathbb{M}_{m,n}(\mathcal{E}), *)$ is an *extensive hypergroupoid*, i.e., for an arbitrary pair of matrices $\mathbf{A}, \mathbf{B} \in \mathbb{M}_{m,n}(\mathcal{E})$ there is $\{\mathbf{A}, \mathbf{B}\} \subseteq \mathbf{A} * \mathbf{B}$. Notice that some other authors, motivated by the geometrical meaning, call such hypergroupoids “closed” as contrasted to “open”.

Example 2. For matrices \mathbf{A} and \mathbf{B} from Example 1 we have:

$$\mathbf{A} * \mathbf{B} = \left\{ \begin{bmatrix} 8 & 1 & 2 \\ 4 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 1 \\ 3 & 0 & 3 \\ 5 & 1 & 0 \end{bmatrix} \right\} \cup \left(\begin{bmatrix} 2 & 1 & 1 \\ 3 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \right)_{\leq_M}$$

Remark 1. With the above example we not only demonstrate the meaning of the hyperoperation “*” but also provide an example to the forthcoming Lemma 3. In this respect notice that $\|\mathbf{A}\|_\infty = 11$, $\|\mathbf{B}\|_\infty = 6$ and $\|\mathbf{A} \circ_m \mathbf{B}\|_\infty = 4$, which means that $\mathbf{A}, \mathbf{B} \in [\mathbf{A} \circ_m \mathbf{B}]_{\leq_M}$. Since in our paper we regard \mathcal{E} as a part of $\mathbb{N} \cup \{0\}$, writing the hyperoperation (5) explicitly in the form of union is not necessary. However, for, as an

example, $\mathcal{E} = \{-6, -5, -4, \dots\}$ the hyperoperation would no longer be extensive (without explicitly including $\{\mathbf{A}, \mathbf{B}\}$). Indeed,

$$\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 4 & -6 \\ 2 & 1 \end{bmatrix}; \|\mathbf{A}\|_\infty = 4, \|\mathbf{B}\|_\infty = 10$$

$$\|\mathbf{A} \circ_m \mathbf{B}\|_\infty = \left\| \begin{bmatrix} 3 & -6 \\ 1 & 1 \end{bmatrix} \right\|_\infty = 9.$$

The result of the hyperoperation is influenced by the absolute value in the calculations of the matrix norm. Since in our paper we restrict ourselves to positive entries of matrices, especially in Section 4 we could omit the two-element set in the definition of the hyperoperation. Yet in our paper we prefer being more general, especially as far as the construction of the hyperoperation is concerned. Since negative values violate extensivity, we prefer including the two-element set in (5). Another way of preserving extensivity would be to modify the row norm by leaving out absolute values. In this respect also notice the result proved by Massouros [24] which says that adding $\{a, b\}$ to $a \cdot b$, where (H, \cdot) is a group or a hypergroup, i.e., defining $a \bullet b = \{a, b\} \cup a \cdot b$, results in the fact that (H, \bullet) is a hypergroup.

Example 3. Suppose we have a manufacturing company with two production lines L_1, L_2 , such that both lines produce products A and B. Consider some specific conditions for production under which the first line L_1 only produces 10 pieces of A and 5 pieces of B per week, and the line L_2 only produces 6 pieces of A and 7 pieces of B per week. This can be denoted by:

$$\mathbf{P}_1 = \begin{bmatrix} 10 & 5 \\ 6 & 7 \end{bmatrix},$$

In the following week, under the same specific conditions, the production can be described by $\mathbf{P}_2 = \begin{bmatrix} 8 & 6 \\ 4 & 8 \end{bmatrix}$. The norm of the matrix, i.e., 15 in case of \mathbf{P}_1 and 13 in case of \mathbf{P}_2 , describes the production of the better line. The result of the hyperoperation (5), i.e.,

$$\mathbf{P}_1 * \mathbf{P}_2 = \left[\begin{bmatrix} 8 & 5 \\ 4 & 7 \end{bmatrix} \right]_{\leq M}$$

describes all possibilities for the minimal guaranteed production in case that in future the same conditions repeat. The associativity of the hyperoperation means that if we have more conditions of the same type, their order is not important for the value of the guaranteed minimal production.

Now we include several lemmas which will simplify proofs of our forthcoming theorems. Recall that e stands for the smallest element of \mathcal{E} , i.e., the set of entries of matrices in $\mathbb{M}_{m,n}(\mathcal{E})$.

Lemma 1. The matrix $\mathbf{E}^* = \begin{bmatrix} e & \dots & e \\ \vdots & \ddots & \vdots \\ e & \dots & e \end{bmatrix}$ is a unit of $(\mathbb{M}_{m,n}(\mathcal{E}), *)$. Moreover, there is $\|\mathbf{E}^*\|_\infty = n \cdot e$.

Proof. Obvious because for all $\mathbf{A} \in \mathbb{M}_{m,n}(\mathcal{E})$ holds $\mathbf{A} \in \mathbf{A} * \mathbf{E}^* \cap \mathbf{E}^* * \mathbf{A} = [\mathbf{E}^*]_{\leq M}$ \square

Lemma 2. Every matrix $\mathbf{A} \in \mathbb{M}_{m,n}(\mathcal{E})$ is idempotent (with respect to $*$).

Proof. Obvious because for all $\mathbf{A} \in \mathbb{M}_{m,n}(\mathcal{E})$ holds

$$\begin{aligned} & \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} * \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} = \\ & \left[\begin{bmatrix} \min\{a_{11}, a_{11}\} & \dots & \min\{a_{1n}, a_{1n}\} \\ \vdots & \ddots & \vdots \\ \min\{a_{m1}, a_{m1}\} & \dots & \min\{a_{mn}, a_{mn}\} \end{bmatrix} \right]_{\leq M} = \left[\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \right]_{\leq M} \end{aligned}$$

which means that $\mathbf{A} \in \mathbf{A} * \mathbf{A}$. \square

Lemma 3. For an arbitrary pair of matrices $\mathbf{A}, \mathbf{B} \in \mathbb{M}_{m,n}(\mathcal{E})$, where $\mathcal{E} \subseteq \mathbb{N} \cup \{0\}$, there is

$$\|\mathbf{A} \circ_m \mathbf{B}\|_{\infty} \leq_M \min \{\|\mathbf{A}\|_{\infty}, \|\mathbf{B}\|_{\infty}\}.$$

Proof. We consider that $\|\mathbf{A}\|_{\infty} = \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{ij}|$, $\|\mathbf{B}\|_{\infty} = \max_{1 \leq i \leq m} \sum_{j=1}^n |b_{ij}|$ and $\|\mathbf{A} \circ_m \mathbf{B}\|_{\infty} = \max_{1 \leq i \leq m} \sum_{j=1}^n |\min\{a_{ij}, b_{ij}\}|$. Then we have

$$\begin{aligned} & \max_{1 \leq i \leq m} \sum_{j=1}^n |\min\{a_{ij}, b_{ij}\}| \leq_M \min \left\{ \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{ij}|, \max_{1 \leq i \leq m} \sum_{j=1}^n |b_{ij}| \right\} \\ & \max_{1 \leq i \leq m} \sum_{j=1}^n |\min\{a_{ij}, b_{ij}\}| \leq_M \max_{1 \leq i \leq m} \sum_{j=1}^n a_{ij} \wedge \max_{1 \leq i \leq m} \sum_{j=1}^n |\min\{a_{ij}, b_{ij}\}| \leq_M \max_{1 \leq i \leq m} \sum_{j=1}^n b_{ij} \end{aligned}$$

For every row of the matrix, i.e., for every $i \in \{1, \dots, m\}$, there is

$$\sum_{j=1}^n |\min\{a_{ij}, b_{ij}\}| \leq_M \sum_{j=1}^n |a_{ij}| \wedge \sum_{j=1}^n |\min\{a_{ij}, b_{ij}\}| \leq_M \sum_{j=1}^n |b_{ij}|.$$

Thus, it is obvious that $\|\mathbf{A} \circ_m \mathbf{B}\|_{\infty} \leq_M \min \{\|\mathbf{A}\|_{\infty}, \|\mathbf{B}\|_{\infty}\}$ \square

Theorem 1. The extensive hypergroupoid $(\mathbb{M}_{m,n}(\mathcal{E}), *)$ is a commutative hypergroup.

Proof. Commutativity of the hyperoperation is obvious because the operation \min is commutative. Next, we have to show that associativity axiom is satisfied, i.e., that there is $\mathbf{A} * (\mathbf{B} * \mathbf{C}) = (\mathbf{A} * \mathbf{B}) * \mathbf{C}$ for all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{M}_{m,n}(\mathcal{E})$.

We calculate left hand side:

$$\begin{aligned} \mathbf{A} * (\mathbf{B} * \mathbf{C}) &= \bigcup_{\mathbf{X} \in \{\mathbf{B}, \mathbf{C}\} \cup [\mathbf{B} \circ_m \mathbf{C}]} \mathbf{A} * \mathbf{X} = \bigcup_{\mathbf{X} \in \{\mathbf{B}, \mathbf{C}\} \cup [\mathbf{B} \circ_m \mathbf{C}]} \{\mathbf{A}, \mathbf{X}\} \cup [\mathbf{A} \circ_m \mathbf{X}]_{\leq M} = \\ &= \{\mathbf{A}, \mathbf{B}, \mathbf{C}\} \cup [\mathbf{B} \circ_m \mathbf{C}]_{\leq M} \cup [\mathbf{A} \circ_m \mathbf{B}]_{\leq M} \cup [\mathbf{A} \circ_m \mathbf{C}]_{\leq M} \cup \bigcup_{\mathbf{X} \in [\mathbf{B} \circ_m \mathbf{C}]} [\mathbf{A} \circ_m \mathbf{X}]_{\leq M}. \end{aligned}$$

For the right hand side we have:

$$(\mathbf{A} * \mathbf{B}) * \mathbf{C} = \bigcup_{\mathbf{Y} \in \{\mathbf{A}, \mathbf{B}\} \cup [\mathbf{A} \circ_m \mathbf{B}]_{\leq M}} \mathbf{Y} * \mathbf{C} = \bigcup_{\mathbf{Y} \in \{\mathbf{A}, \mathbf{B}\} \cup [\mathbf{A} \circ_m \mathbf{B}]_{\leq M}} \{\mathbf{Y}, \mathbf{C}\} \cup [\mathbf{Y} \circ_m \mathbf{C}]_{\leq M} = \\ \{\mathbf{A}, \mathbf{B}, \mathbf{C}\} \cup [\mathbf{A} \circ_m \mathbf{B}]_{\leq M} \cup [\mathbf{A} \circ_m \mathbf{C}]_{\leq M} \cup [\mathbf{B} \circ_m \mathbf{C}]_{\leq M} \cup \bigcup_{\mathbf{Y} \in [\mathbf{A} \circ_m \mathbf{B}]_{\leq M}} [\mathbf{Y} \circ_m \mathbf{C}]_{\leq M}.$$

The left hand side and the right hand side are the same except for the last part of the union. We are going to show that

$$\bigcup_{\mathbf{X} \in [\mathbf{B} \circ_m \mathbf{C}]_{\leq M}} [\mathbf{A} \circ_m \mathbf{X}]_{\leq M} = \bigcup_{\mathbf{Y} \in [\mathbf{A} \circ_m \mathbf{B}]_{\leq M}} [\mathbf{Y} \circ_m \mathbf{C}]_{\leq M}.$$

The following calculation holds for all $i = \{1, \dots, m\}$, $j = \{1, \dots, n\}$:

$$\bigcup_{\mathbf{X} \in [\mathbf{B} \circ_m \mathbf{C}]_{\leq M}} [\mathbf{A} \circ_m \mathbf{X}]_{\leq M} = \bigcup_{(x_{ij}) \in [\min\{b_{ij}; c_{ij}\}]_{\leq M}} [\mathbf{A} \circ_m \mathbf{X}]_{\leq M} = \\ [\min\{a_{ij}; \min\{b_{ij}; c_{ij}\}\}]_{\leq M} = [\min\{a_{ij}; b_{ij}; c_{ij}\}]_{\leq M} = [\min\{\min\{a_{ij}; b_{ij}\}; c_{ij}\}]_{\leq M} = \\ \bigcup_{(y_{ij}) \in [\min\{a_{ij}; b_{ij}\}]_{\leq M}} [\mathbf{Y} \circ_m \mathbf{C}]_{\leq M} = \bigcup_{\mathbf{Y} \in [\mathbf{A} \circ_m \mathbf{B}]_{\leq M}} [\mathbf{Y} \circ_m \mathbf{C}]_{\leq M}.$$

Thus the associativity axiom holds, which means that the hypergroupoid $(\mathbb{M}_{m,n}(\mathcal{E}), *)$ is a semihypergroup. Finally, because of extensivity of the hyperoperation (5) we immediately see that reproduction axiom holds as well, i.e., the semihypergroup $(\mathbb{M}_{m,n}(\mathcal{E}), *)$ is an extensive hypergroup. \square

Now we can include the result concerning cyclicity of the discussed hypergroup. Notice that since we use n to denote one of the dimensions of the matrices, we will denote period of Definition 1 by p instead of n .

Theorem 2. *If the set \mathcal{E} has the smallest element e , then the hypergroup $(\mathbb{M}_{m,n}(\mathcal{E}), *)$ is single-power cyclic and all matrices containing e (other than \mathbf{E}^*) are generators of $\mathbb{M}_{m,n}(\mathcal{E})$ with period $p = 3$.*

Proof. The proof is rather straightforward. Denote by \mathbf{A}_e an arbitrary matrix from $\mathbb{M}_{m,n}(\mathcal{E})$ such that at least one of its entries (e.g., $a_{1,2}$) is e . By definition, $\mathbf{A}_e^1 = \mathbf{A}_e$. By Lemma 2, we have that $\mathbf{A}_e^2 = \mathbf{A}_e * \mathbf{A}_e = [\mathbf{A}_e]_{\leq M}$. Now, consider such a matrix \mathbf{B} , elements of which are different from e

at least at those places where $a_{ij} = e$. For example, consider matrix $\mathbf{B} = \begin{bmatrix} e & b_{12} & \dots & e \\ \vdots & \vdots & \ddots & \vdots \\ e & e & \dots & e \end{bmatrix}$, where

$b_{12} = \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{ij}|$. Obviously, there is $\mathbf{B} \in [\mathbf{A}_e]_{\leq M}$. Now, we have

$$\mathbf{A}_e^3 = \mathbf{E}^* * \mathbf{E}^* * \mathbf{E}^* = \mathbf{E}^* * [\mathbf{E}^*]_{\leq M} = \bigcup_{\mathbf{X} \in [\mathbf{E}^*]_{\leq M}} \mathbf{E}^* * \mathbf{X}.$$

Since we know that $\mathbf{B} \in [\mathbf{E}^*)_{\leq_M}$, there is

$$\bigcup_{X \in [\mathbf{E}^*)_{\leq_M}} \mathbf{E}^* * X \supseteq \mathbf{E}^* * \mathbf{B} =$$

$$\begin{bmatrix} a_{11} & e & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} * \begin{bmatrix} e & b_{12} & \dots & e \\ \vdots & \vdots & \ddots & \vdots \\ e & e & \dots & e \end{bmatrix} = \left[\begin{bmatrix} e & e & \dots & e \\ \vdots & \vdots & \ddots & \vdots \\ e & e & \dots & e \end{bmatrix} \right]_{\leq_M} = [\mathbf{E}^*)_{\leq_M} = \mathbb{M}_{m,n}(\mathcal{E}).$$

Thus we have that $\mathbf{A}_e^3 = \mathbb{M}_{m,n}(\mathcal{E})$, which means that $\mathbb{M}_{m,n}(\mathcal{E})$ is single-power cyclic with period $p = 3$ with generators being all matrices of the form \mathbf{A}_e . \square

Example 4. Suppose $\mathcal{E} = \mathbb{N}_0$, i.e., consider the semiring of natural numbers including zero. Then all matrices containing 0 are generators of $\mathbb{M}_{m,n}(\mathbb{N}_0)$. For $m = 5$ and $n = 2$ e.g., matrix $\mathbf{M} = \begin{bmatrix} 50 & 15 & 400 & 3 & 45 \\ 10 & 0 & 89 & 17 & 80 \end{bmatrix}$ generates $\mathbb{M}_{m,n}(\mathbb{N}_0)$. Indeed,

$$\begin{bmatrix} 50 & 15 & 400 & 3 & 45 \\ 10 & 0 & 89 & 17 & 80 \end{bmatrix} * \begin{bmatrix} 50 & 15 & 400 & 3 & 45 \\ 10 & 0 & 89 & 17 & 80 \end{bmatrix} = \left[\begin{bmatrix} 50 & 15 & 400 & 3 & 45 \\ 10 & 0 & 89 & 17 & 80 \end{bmatrix} \right]_{\leq_M} \quad (6)$$

If we now denote by \mathbf{D} a matrix with $d_{22} = 50 + 15 + 400 + 3 + 45 = 513$ and all other elements zero, then $\mathbf{D} \in [\mathbf{M}]_{\leq_M}$. We can see that

$$\begin{bmatrix} 50 & 15 & 400 & 3 & 45 \\ 10 & 0 & 89 & 17 & 80 \end{bmatrix} * \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 513 & 0 & 0 & 0 \end{bmatrix} =$$

$$\left[\begin{bmatrix} \min\{51, 0\} & \min\{15, 0\} & \min\{400, 0\} & \min\{3, 0\} & \min\{45, 0\} \\ \min\{10, 0\} & \min\{0, 513\} & \min\{89, 0\} & \min\{17, 0\} & \min\{80, 0\} \end{bmatrix} \right]_{\leq_M} =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{\leq_M} = \mathbb{M}_{m,n}(\mathbb{N}).$$

Theorem 3. The unit matrix $\mathbf{E}^* \in \mathbb{M}_{m,n}(\mathcal{E})$ is a generator of $(\mathbb{M}_{m,n}(\mathcal{E}), *)$ with period 2.

Proof. Proof is obvious to thanks relation " \leq_M'' ", the fact that we regard total order on \mathcal{E} and given the proof of the Lemma 1. We have that $\mathbf{E}^{*2} = \mathbf{E}^* * \mathbf{E}^* = [\mathbf{E}^*)_{\leq_M} = \mathbb{M}_{m,n}(\mathcal{E})$. \square

It will be useful to investigate cyclicity of hypergroups $(\mathbb{M}_{m,n}(\mathcal{E}), *)$, where \mathcal{E} is a finite set. Notice that in the case of a finite set \mathcal{E} we cannot construct an analogue of matrix \mathbf{D} with entry d_{22} as we did in Example 4, simply because d_{22} need not be an element of \mathcal{E} . Also, since \mathcal{E} is finite and we suppose that it is a chain, its smallest element e always exists.

Theorem 4. If \mathcal{E} is finite, then the hypergroup $(\mathbb{M}_{m,n}(\mathcal{E}), *)$ is single-power cyclic and all matrices \mathbf{A}_e containing the smallest element of \mathcal{E} , denoted by e , are generators of $(\mathbb{M}_{m,n}(\mathcal{E}), *)$ with period $p = 2 + m$, where m is the number of columns of \mathbf{A}_e .

Proof. The proof is analogous to the proof of Theorem 2, except for the row of \mathbf{A}_e containing element e . We construct matrix \mathbf{B} in the following way: the row of \mathbf{B} which in \mathbf{A}_e contains e , will consist of m copies of the greatest elements of \mathcal{E} , denoted u , while all other entries of \mathbf{B} will be equal to e . In this way we have $\|\mathbf{A}\|_\infty \leq_M \|\mathbf{B}\|_\infty$, i.e., $\mathbf{B} \in [\mathbf{A}_e]_{\leq_M}$. We calculate once again, now $\mathbf{A} * \mathbf{B} = [\mathbf{C}]_{\leq_M}$:

$$\begin{bmatrix} a_{11} & e & \dots & a_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix} * \begin{bmatrix} u & u & \dots & u \\ \vdots & \vdots & \ddots & \vdots \\ e & e & \dots & e \end{bmatrix} = \left[\begin{bmatrix} u & e & \dots & u \\ \vdots & \vdots & \ddots & \vdots \\ e & e & \dots & e \end{bmatrix} \right]_{\leq M} .$$

Next, we reorganize entries in the first row, which does not affect the row norm. We get

$$\begin{bmatrix} a_{11} & e & \dots & a_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix} * \begin{bmatrix} e & u & \dots & u \\ \vdots & \vdots & \ddots & \vdots \\ e & e & \dots & e \end{bmatrix} = \left[\begin{bmatrix} e & e & \dots & u \\ \vdots & \vdots & \ddots & \vdots \\ e & e & \dots & e \end{bmatrix} \right]_{\leq M} .$$

Now it is obvious that after we do this procedure $2 + m$ times, we obtain matrix \mathbf{E}^* , for which there is $(\mathbf{E}^*)_{\leq M} = \mathbb{M}_{m,n}(\mathcal{E})$. \square

Example 5. Consider $\mathcal{E} = \{1, 2, \dots, 10\}$ and $n = 2, m = 3$. In this case matrix $\mathbf{A} = \begin{bmatrix} 10 & 1 & 10 \\ 10 & 10 & 10 \end{bmatrix}$ is a generator of $(\mathbb{M}_{m,n}(\mathcal{E}), *)$ with period $p = 5$. Indeed, we calculate:

$$\mathbf{A}^2 = \begin{bmatrix} 10 & 1 & 10 \\ 10 & 10 & 10 \end{bmatrix} * \begin{bmatrix} 10 & 1 & 10 \\ 10 & 10 & 10 \end{bmatrix} = \left[\begin{bmatrix} 10 & 1 & 10 \\ 10 & 10 & 10 \end{bmatrix} \right]_{\leq M} \ni \begin{bmatrix} 10 & 10 & 10 \\ 1 & 1 & 1 \end{bmatrix} = \mathbf{B}$$

It is obvious, following from the use of the row norm, that $\mathbf{B} \in \mathbf{A}^2$. We get

$$\begin{aligned} \mathbf{A}^3 &= \bigcup_{\mathbf{X} \in \mathbf{A}^2} \mathbf{X} * \mathbf{A} \supseteq \begin{bmatrix} 10 & 1 & 10 \\ 10 & 10 & 10 \end{bmatrix} * \begin{bmatrix} 10 & 10 & 10 \\ 1 & 1 & 1 \end{bmatrix} = \left[\begin{bmatrix} 10 & 1 & 10 \\ 1 & 1 & 1 \end{bmatrix} \right]_{\leq M} \ni \begin{bmatrix} 1 & 10 & 10 \\ 1 & 1 & 1 \end{bmatrix} \\ \mathbf{A}^4 &= \bigcup_{\mathbf{Y} \in \mathbf{A}^3} \mathbf{Y} * \mathbf{A} \supseteq \begin{bmatrix} 10 & 1 & 10 \\ 10 & 10 & 10 \end{bmatrix} * \begin{bmatrix} 10 & 1 & 10 \\ 10 & 10 & 10 \end{bmatrix} = \left[\begin{bmatrix} 1 & 1 & 10 \\ 1 & 1 & 1 \end{bmatrix} \right]_{\leq M} \ni \begin{bmatrix} 1 & 10 & 1 \\ 1 & 1 & 1 \end{bmatrix} \\ \mathbf{A}^5 &= \bigcup_{\mathbf{Z} \in \mathbf{A}^4} \mathbf{Z} * \mathbf{A} \supseteq \begin{bmatrix} 10 & 1 & 10 \\ 10 & 10 & 10 \end{bmatrix} * \begin{bmatrix} 1 & 10 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \left[\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \right]_{\leq M} \end{aligned}$$

And we see that $\mathbf{A}^5 = \mathbb{M}_{m,n}(\mathcal{E})$.

Remark 2. The generators described by Theorem 4 are neither only ones nor with the smallest period. Indeed, if in Example 5 we consider matrix $\mathbf{C} = \begin{bmatrix} 1 & 2 & 10 \\ 1 & 1 & 1 \end{bmatrix}$, there is $\mathbf{D} = \begin{bmatrix} 10 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \in \mathbf{C}^2$. Then $\mathbf{C}^3 \supseteq \mathbf{C} * \mathbf{D} = \left[\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \right]_{\leq M} \ni \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $\mathbf{C}^4 \supseteq \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \left[\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \right]_{\leq M} = \mathbb{M}_{m,n}(\mathcal{E})$ and we see that \mathbf{C} is a generator of $(\mathbb{M}_{m,n}(\mathcal{E}), *)$ with period $p = 4$.

4. Approximation Space Determined by the Cyclic Hypergroup

Now we rewrite some basic terminology of the rough set theory introduced by Pawlak [3] into our notation.

Let $\mathbb{M}_{m,n}(\mathcal{E})$ be a certain set called the universe, and let R_M be an equivalence relation on $\mathbb{M}_{m,n}(\mathcal{E})$. The pair $\mathfrak{A} = (\mathbb{M}_{m,n}(\mathcal{E}), R_M)$ will be called an *approximation space*. We will call R_M an *indisernibility relation*. If $\mathbf{A}, \mathbf{B} \in \mathbb{M}_{m,n}(\mathcal{E})$ and $(\mathbf{A}, \mathbf{B}) \in R_M$, we will say that \mathbf{A} and \mathbf{B} are *indistinguishable* in \mathfrak{A} . Subsets of $\mathbb{M}_{m,n}(\mathcal{E})$ will be denoted by X, Y, Z , possibly with indices. The empty set will be denoted by 0, and the universe \mathfrak{U} will also be denoted by 1. Equivalence classes of the relation R_M will be called

elementary sets (*atoms*) in \mathfrak{A} or, briefly, *elementary sets*. The set of all atoms in \mathfrak{A} will be denoted by $(\mathbb{M}_{m,n}(\mathcal{E}), R_M)$. We assume that the empty set is also elementary in every \mathfrak{A} . Every finite union of elementary sets in \mathfrak{A} will be called a composed set in \mathfrak{A} , or in short, a *composed set*. The family of all composed sets in \mathfrak{A} will be denoted as $Com(\mathfrak{A})$. Obviously, $Com(\mathfrak{A})$ is a Boolean algebra, i.e., the family of all composed set is closed under intersection, union, and complement of sets.

Now, let X be a certain subset of $\mathbb{M}_{m,n}(\mathcal{E})$. The least composed set in \mathfrak{A} containing X will be called the *best upper approximation* of X in \mathfrak{A} , in symbols $\overline{Apr}_A(X)$; the greatest composed set in \mathfrak{A} contained in X will be called the *best lower approximation* of X in \mathfrak{A} , in symbols $\underline{Apr}_A(X)$. If \mathfrak{A} is known, instead of $\overline{Apr}_A(X)$ ($\underline{Apr}_A(X)$) we will write $\overline{Apr}(X)$ ($\underline{Apr}(X)$), respectively. The set $Bnd_A(X) = \overline{Apr}_A(X) - \underline{Apr}_A(X)$ (in short $Bnd(X)$) will be called the *boundary* of X in \mathfrak{A} .

Definition 2 ([20]). Let H be a set and R be an equivalence relation on H . Let A be subset of H . A rough set is a pair of subsets $(\overline{R}(A), \underline{R}(A))$ of H which approximates A as closer as possible from outside and inside, respectively:

$$\begin{aligned}\overline{R}(A) &= \bigcup_{R(x) \cap A \neq \emptyset} R(x) \\ \underline{R}(A) &= \bigcup_{R(x) \subseteq A} R(x)\end{aligned}$$

Example 6. Let $S = \{-12, \dots, -2, -1, 0, 1, 2, \dots, 12\}$ and R be defined on S by:

$$aRb \text{ iff } a \equiv b \pmod{6}.$$

In this way we obtain the following decomposition of S :

$$\begin{aligned}R(0) &= \{-12, -6, 0, 6, 12, \} \\ R(1) &= \{-7, -1, 1, 7, \} \\ R(2) &= \{-8, -2, 2, 8, \} \\ R(3) &= \{-9, -3, 3, 9, \} \\ R(4) &= \{-10, -4, 4, 10, \} \\ R(5) &= \{-11, -5, 5, 11, \}\end{aligned}$$

Now, consider $A = \{-8, -3, -2, 1, 2, 8, 9\} \subset S$. Then

$$\begin{aligned}\overline{R}(A) &= \bigcup_{R(x) \cap A \neq \emptyset} R(x) = R(1) \cup R(2) \cup R(3), \\ \underline{R}(A) &= \bigcup_{R(x) \subseteq A} R(x) = R(2).\end{aligned}$$

In what follows we consider square matrices, i.e., $\mathbb{M}_{n,n}(\mathcal{E})$ only.

In order to study links between rough sets and the above cyclic hypergroup $(\mathbb{M}_{m,n}(\mathcal{E}), *)$, or rather its special case $(\mathbb{M}_{n,n}(\mathcal{E}), *)$, we need to define a new relation R_M on $\mathbb{M}_{n,n}(\mathcal{E})$. For all $\mathbf{A}, \mathbf{B} \in \mathbb{M}_{n,n}(\mathcal{E})$ we define:

$$\mathbf{A}R_M \mathbf{B} \text{ if } \|\mathbf{A}\|_\infty = \|\mathbf{B}\|_\infty \text{ and } \text{tr}(\mathbf{A}) = \text{tr}(\mathbf{B}). \quad (7)$$

It is obvious that such a relation is reflexive, transitive and symmetric. In this way we obtain a decomposition of $\mathbb{M}_{n,n}(\mathcal{E})$ into equivalence classes by row norm of matrices and their traces.

We denote by $\mathbb{M}_{n,n}(\mathcal{SE}_E)$ an arbitrary subset of $\mathbb{M}_{n,n}(\mathcal{E})$ with entries from \mathcal{SE}_{E^*} , where \mathcal{SE}_{E^*} is a set generated by the principal beginning $\leq(x)$, where $x \in \mathcal{E}$ and \leq is the total order defined on \mathcal{E} which we had already regarded. To sum up,

$$\mathcal{SE}_E = \leq(x) = \{y \in \mathcal{E} : y \leq x\}. \quad (8)$$

Theorem 5. Every hypergroup $(\mathbb{M}_{n,n}(\mathcal{SE}_E), *)$, where \mathcal{SE}_{E^*} is defined by (8), is an invertible subhypergroup of the hypergroup $(\mathbb{M}_{n,n}(\mathcal{E}), *)$.

Proof. Recall that for and invertible subhypergroup A of a hypergroup H there holds $y \in A \circ x \Rightarrow x \in A \circ y$ for every $x, y \in H$.

Consider now an arbitrary set $\mathbb{M}_{n,n}(\mathcal{SE}_E)$, where \mathcal{SE}_{E^*} is defined by (8). It is obvious that $E^* \in \mathbb{M}_{n,n}(\mathcal{SE}_E)$. Then we have that $\mathbf{A} \in \mathbb{M}_{n,n}(\mathcal{SE}_E) * \mathbf{B}$ for all $\mathbf{A}, \mathbf{B} \in \mathbb{M}_{n,n}(\mathcal{E})$, this is because $\mathbb{M}_{n,n}(\mathcal{SE}_E) * \mathbf{B} = \bigcup_{\mathbf{X} \in \mathbb{M}_{n,n}(\mathcal{SE}_E)} \mathbf{X} * \mathbf{B} = [E^*]_{\leq_M}$. By the proof of Theorem 3 we have that $\mathbf{A} \in [E^*]_{\leq_M}$.

For $\mathbf{B} \in \mathbb{M}_{n,n}(\mathcal{SE}_E) * \mathbf{A}$ the proof is the same. Thus we obtain that $(\mathbb{M}_{n,n}(\mathcal{SE}_E), *)$ is an invertible subhypergroup of $(\mathbb{M}_{n,n}(\mathcal{E}), *)$. \square

By applying Theorems 4 and 5 we immediately obtain the following corollary.

Corollary 1. Every $(\mathbb{M}_{n,n}(\mathcal{SE}_E), *)$ is single power cyclic.

In the paper we assume that \mathcal{E} is a chain and an equivalence R_M . As a result we can consider the set $\mathbb{M}_{n,n}(\mathcal{SE}_E)$ as a suitable set for constructing lower and upper approximations, i.e., $\underline{R}_M(\mathbb{M}_{n,n}(\mathcal{SE}_E))$ and $\overline{R}_M(\mathbb{M}_{n,n}(\mathcal{SE}_E))$. When discussing our system $(\mathbb{M}_{n,n}(\mathcal{E}), R_M)$, we can see that every subset $\mathbb{M}_{n,n}(\mathcal{SE}_E)$ is in the beginning of the system, i.e., the class with the smallest trace and the smallest row norm of the matrix is included in the lower and upper approximations.

Notation 1. Our results regarding rough sets are visualised by means of figures. In all figures, a white square means that there does not exist any matrix with the given properties created by the approximation set, a coloured square means that all matrices with the given properties belong to the approximation set, and a partially coloured square that there exists at least one matrix with the given properties which belong to the approximation set and at least one matrix which does not belong to it.

Example 7. Consider $\mathcal{E} = (\mathbb{N} \cup \{0\})$ and $\mathcal{SE}_{E^*} = \{0, 1\}$, i.e., $(\mathbb{M}_{2,2}(\{0, 1\}))$ is the set of 2×2 Boolean matrices. Then $(\mathbb{M}_{2,2}(\{0, 1\}), *)$ is invertible in $(\mathbb{M}_{2,2}(\mathcal{E}), *)$. Since $E^* \in \mathbb{M}_{2,2}(\{0, 1\})$, we have, for an arbitrary pair of matrices $\mathbf{A}, \mathbf{B} \in \mathbb{M}_{2,2}(\mathcal{E})$ that $\mathbf{B} * \mathbb{M}_{2,2}(\{0, 1\}) = \mathbb{M}_{2,2}(\mathcal{E}) \ni \mathbf{A}$ and $\mathbf{A} * \mathbb{M}_{2,2}(\{0, 1\}) = \mathbb{M}_{2,2}(\mathcal{E}) \ni \mathbf{B}$. Moreover,

$$\begin{aligned} \overline{R}_M((\mathbb{M}_{2,2}(\{0, 1\}))) &= R_{[0,0]} \cup R_{[1,0]} \cup R_{[1,1]} \cup R_{[1,2]} \cup R_{[2,1]} \cup R_{[2,2]} \\ \underline{R}_M((\mathbb{M}_{2,2}(\{0, 1\}))) &= R_{[0,0]} \cup R_{[1,0]} \cup R_{[1,1]} \cup R_{[1,2]} \end{aligned}$$

The upper approximation, i.e., the set $\overline{R}_M((\mathbb{M}_{n,n}(\{0, 1\}))$, is in Figure 1 visualized as the union of all squares which include some coloured parts. The lower approximation, i.e., the set $\underline{R}_M((\mathbb{M}_{n,n}(\{0, 1\}))$, is the union of all square which are fully coloured.

| | | | | | | | |
|-------------------------------|---|---|---|---|---|---|---|
| ⋮ | $R_{[2,0]}$ | $R_{[2,1]}$ | $R_{[2,2]}$ | ⋮ | ⋮ | ⋮ | ⋮ |
| $\ \mathbf{A}\ _\infty^m = 2$ | $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}; \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}; \dots$ | $\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}; \dots$ | $\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}; \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}; \dots$ | ⋮ | ⋮ | ⋮ | ⋮ |
| $\ \mathbf{A}\ _\infty^m = 1$ | $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}; \dots$ | $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}; \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}; \dots$ | $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}; \dots$ | ⋮ | ⋮ | ⋮ | ⋮ |
| $\ \mathbf{A}\ _\infty^m = 0$ | $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ | $R_{[0,0]}$ | | ⋮ | ⋮ | ⋮ | ⋮ |

Tr = 0 Tr = 1 Tr = 2 ⋯

Figure 1. The lower and upper approximation with relation R_M for matrices 2×2 .

Theorem 6. Let $\mathcal{E} = \mathbb{N}_0$ and $\mathbb{M}_{n,n}(\mathcal{SE}_E)$ be invertible hypergroups in $\mathbb{M}_{n,n}(\mathcal{E})$, where \mathcal{SE}_{E^*} is defined by $\mathcal{SE}_{E^*} = (x]_≤ = \{y \in \mathcal{E} : y \leq x\}$ for $x = 1, n \in \mathbb{N}$, with relation R_M . Then there is

$$\underline{R}_M(\mathbb{M}_{n,n}(\mathcal{SE}_E)) = \bigcup_{i=1}^n R_{[1,i]} \cup R_{[0,0]}$$

$$\overline{R}_M(\mathbb{M}_{n,n}(\mathcal{SE}_E)) = \bigcup_{i=1}^n \bigcup_{j=1}^n R_{[i,j]} \cup \bigcup_{j=0}^{n-1} R_{[0,j]}$$

where n is the size of the matrix.

Proof. For the proof we use the idea of Example 7 and Figure 1, in which the lower approximation is the union of all squares which are fully coloured while the upper approximation is the union of all squares which contain some coloured parts. Notice that in Figure 1 rows indicate matrices with the same row norm (counted from the bottom) while columns (counted from the left) indicate matrices with the same trace. When one realizes how such a scheme is constructed, the proof becomes obvious. Indeed, take e.g., $n = 3$ and focus on Figure 2 and (9).

$$\begin{aligned} \overline{R}_M((\mathbb{M}_{3,3}(\{0,1\})) = & R_{[0,0]} \cup R_{[1,0]} \cup R_{[1,1]} \cup R_{[1,2]} \cup R_{[1,3]} \cup R_{[2,0]} \cup R_{[2,1]} \\ & \cup R_{[2,2]} \cup R_{[2,3]} \cup R_{[3,1]} \cup R_{[3,2]} \cup R_{[3,3]} \end{aligned} \quad (9)$$

$$\underline{R}_M((\mathbb{M}_{3,3}(\{0,1\})) = R_{[0,0]} \cup R_{[1,0]} \cup R_{[1,1]} \cup R_{[1,2]} \cup R_{[1,3]} \quad (10)$$

□

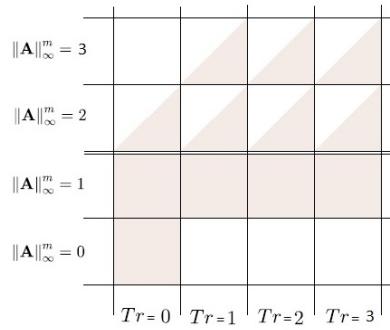


Figure 2. The lower and upper approximation with relation R_M , for matrices 3×3 .

Remark 3. Note that the boundary, defined as the difference between the upper and the lower approximation, expands across the universe so that it has only two classes for $n = 2$ and 7 classes for $n = 3$ and 14 classes for $n = 4$ and 23 classes for $n = 5$, i.e., $n^2 - 2$ classes for an $n \times n$ matrix. Also, the upper approximation has $n \cdot (n + 1)$ classes and the lower approximation has $n + 2$ classes. All these formulas can be easily seen in Figures 1 and 2.

In what follows, we will consider $\overline{R_M}(A) = \overline{R_M}(\mathbf{A} * \mathbf{B})$ and $\underline{R_M}(A) = \underline{R_M}(\mathbf{A} * \mathbf{B})$, where $\mathbf{A}, \mathbf{B} \in \mathbb{M}_{n,n}(\mathcal{E})$. In this way the construction can be considered as a dynamic system, where the set X is the hyperoperproduct defined by (5).

Corollary 2. For $X = \mathbf{A} * \mathbf{B}$ there is $\overline{R_M}(X) = \underline{R_M}(X)$. Moreover,

$$\overline{R_M}(X) = \bigcup_{j \geq \|\mathbf{A} * \mathbf{B}\|_\infty} \bigcup_{i \in \{0, 1, 2, \dots, \infty\}} R_{ij}.$$

Now we will use the relation R_M to present some basic and natural properties of the lower and upper approximations with respect to the above mentioned theorem. Recall that matrices \mathbf{A}_e are generators of the cyclic hypergroup $\mathbb{M}_{m,n}(\mathcal{E})$.

Theorem 7. The following properties hold:

- (1) $\overline{R_M}(\mathbf{E}^* * \mathbf{E}^*) = \mathbb{M}_{m,n}(\mathcal{E}) = \underline{R_M}(\mathbf{E}^* * \mathbf{E}^*)$
- (2) $\underline{R_M}(\mathbf{A}_e) \subseteq \underline{R_M}(\mathbf{A}_e^2) \subseteq \underline{R_M}(\mathbf{A}_e^3) = \mathbb{M}_{m,n}(\mathcal{E})$
- (3) $\overline{R_M}(\mathbf{A}_e) \subseteq \overline{R_M}(\mathbf{A}_e^2) \subseteq \overline{R_M}(\mathbf{A}_e^3) = \mathbb{M}_{m,n}(\mathcal{E})$

Recall that when defining the relation R_M in (7) we used the row norm and trace. Let us now modify the definition to make use of the row norm and determinant. We define:

$$\mathbf{A}R_D\mathbf{B} \text{ if } \|\mathbf{A}\|_\infty = \|\mathbf{B}\|_\infty \text{ and } |\mathbf{A}| = |\mathbf{B}|. \quad (11)$$

If we use the relation R_D instead of R_M , Figure 1 changes to Figure 3. This results in the following theorem.

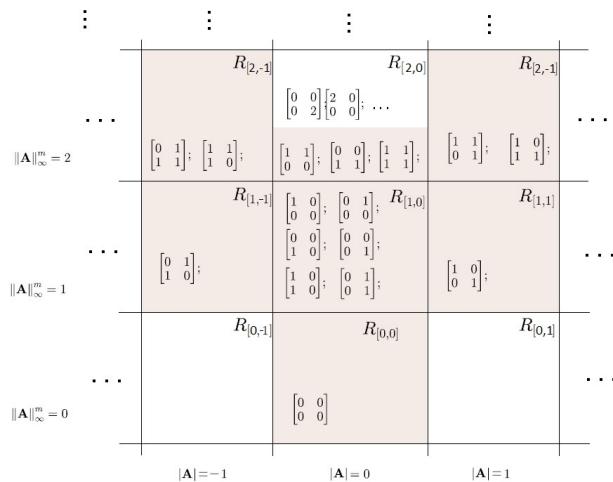


Figure 3. The lower and upper approximation with relation R_D for matrices 2×2 .

Theorem 8. Let $\mathcal{E} = \mathbb{N}_0$ and $\mathbb{M}_{n,n}(\mathcal{SE}_E)$ be invertible hypergroups in $\mathbb{M}_{n,n}(\mathcal{E})$, where \mathcal{SE}_{E^*} is defined by $\mathcal{SE}_{E^*} = [x]_{\leq} = \{y \in \mathcal{E} : y \leq x\}$ for $x = 1, n \in \mathbb{N}$, with relation R_D . Then there is

$$\underline{R_D}(\mathbb{M}_{n,n}(\mathcal{SE}_E)) = R_{[0,0]} \cup R_{[1,-1]} \cup R_{[1,0]} \cup R_{[1,1]} \cup R_{[-2,1]} \cup R_{[2,1]}$$

where n is the size of the matrix.

Proof. We use Figures 3 and 4 in the proof. For an arbitrary size of the matrix, n , the lower approximation always consists of 6 classes of equivalence. Obviously, for an arbitrary n and $\mathcal{E} = \{0, 1\}$ there exists only one matrix (the null one), which is in $R_{[0,0]}$, see the coloured square. For norm equal to 1, the proof is again obvious because thanks to the norm in every matrix of size n there can be maximum one 1 in every row, which means that the determinant of such a matrix can only be $-1, 0$ or 1. For norm 2 there exists, for an arbitrary size n a matrix with element 2 and zero determinant. Also, there exists a matrix which has one row with two 1's. If we repeat such a row, the determinant is zero, see the partially coloured square. At the same time no matrix with 2 as an entry can be in a class with norm 2 and determinant 1 or -1 because in that case the respective row must contain only zeros as other entries. If we now expand the determinant with respect to that row, the calculation will include 2s, where s is a subdeterminant and $s \in \mathbb{N}$, which means that the determinant can never be 1 or -1 . For norm 3 we get that the numbers of matrices can include 2 and 1, which means that the calculation of the determinant will include a difference, i.e., we can get an arbitrary number. In other words, for norm greater than 2 we cannot obtain fully coloured squares, i.e., classes of equivalence which consist only of matrices with entries 0 and 1. \square

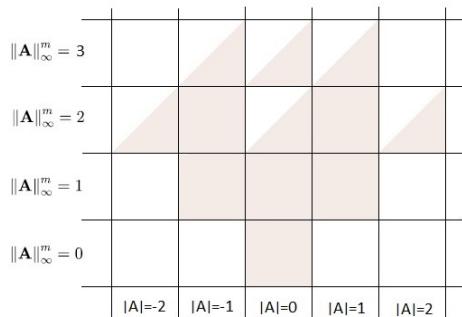


Figure 4. The lower and upper approximation with relation R_D for matrices 3×3 .

Finding a general rule for the upper approximation when using the determinant is not as easy as it may seem. Even though it seems that the whole universum “behaves accordingly”, it is not easy to find an algorithm which would easily define the upper approximation. Therefore we at least include a theorem which shows an important property of the upper approximation. At the same time, its proof describes the upper approximation for $n \in \{2, 3, 4, 5\}$.

Theorem 9. For $\mathcal{E} = \mathbb{N}_0, \mathbb{M}_{n,n}(\mathcal{SE}_E)$ and $n \in \{2, 3, 4, 5\}$, the following holds:

$$\overline{R_D}(\mathbb{M}_{2,2}(\mathcal{SE}_{E^*})) \subset \overline{R_D}(\mathbb{M}_{3,3}(\mathcal{SE}_{E^*})) \subset \overline{R_D}(\mathbb{M}_{4,4}(\mathcal{SE}_{E^*})) \subset \overline{R_D}(\mathbb{M}_{5,5}(\mathcal{SE}_{E^*}))$$

Proof. For the proof we use Figures 3 and 5. Results for $n = 4$ and $n = 5$, which had been computed by software means, are included below. \square

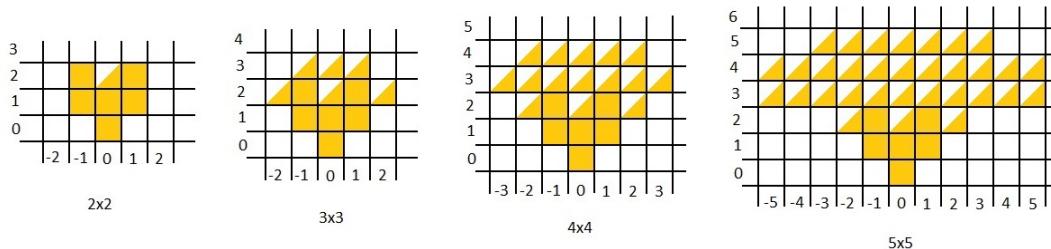


Figure 5. The lower and upper approximation with relation R_D for matrices 2×2 – 5×5 .

Application in the Control Theory

Application of rough set theory is widely used in information technologies. This approach has fundamental importance in knowledge acquisition, cognitive science, pattern recognition, machine learning, database systems, etc. Rough sets are also used in sensors mapping in robotics.

If we examine how an autonomous mobile robot can get from point A to point B, we realize that it must have information about obstacles in front of itself to avoid collision. To find this out it uses sensors, mostly *camera* and *LIDAR* (an abbreviation of “Light Detection And Ranging”) for environment mapping [25,26].

We can easily find surjective function from a set of $\mathbb{M}_{n,n}(\mathcal{SE}_E)$ matrices (our model) to coordinates $[x, y]$, where $x, y \in \mathbb{Z}$. These are coordinates of occupancy grid [26] cells (robotics usage), which must meet the following condition:

$$[|M|, ||M||] = [\text{floor}(x_r), \text{floor}(y_r)], \quad (12)$$

where $x_r, y_r \in \mathbb{R}$ are real coordinates that represents the interval/size of the cells in occupancy grid (see Figure 6). This process is called *quantization*. We can mark the occupancy grid as *equivalence classes*.

Both sensors can be attached to specific places on the top of robot. (Notice that all visualised information from sensors what we are working with are projected from 3D to 2D plane because of assumption that the robot can move only in 2 directions.) In our figures (for $n = 4$, for example) the *lower approximation* describes the robot’s shape and pose. We can be sure for 100% occupation of these cells because of the fusion of sensor data with the known relative position between the sensors and the robot (its shape) in particular. As we can see, this approximation will never change depending on change of n (matrix size). This lower approximation is typical for an industrial warehouse robot.

Next part of LIDAR/camera scan is the *upper approximation*. This gives us information about all cells which were hit by the sensors. However, we can only say that these cells are occupied with probability between 0 and 1 (never 0 or 1) because this is how all current sensor work. The bigger the n , the bigger the range of sensors. Notice that the sensor scanning angle is *symmetric* by vertical axis of the sensor.

The above example makes use of a single sensor scan. However, we can process these scans into a whole simple mapping algorithm described in [12,26] or improved algorithm via particle filter described in [27].

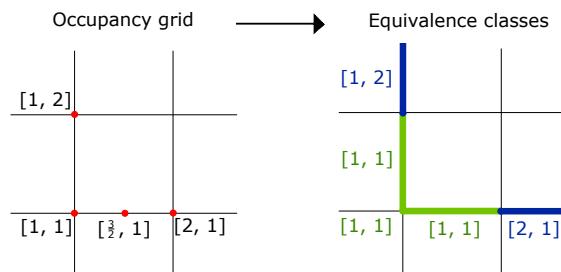


Figure 6. Quantization from real coordinates to equivalence classes.

5. Conclusions

In our paper we deal with a construction of cyclic hypergroups and present their links to rough sets. We construct subsets of such hypergroups which determine the approximation for finding the lower and upper approximation of the rough set. For this we use various types of equivalence relations. In our paper we choose the input of the matrix subhypergroups out of a two element set. Generalizing our results for arbitrary more-element sets is a topic of our further research. The constructions of the lower and upper approximation described in this paper for various n are very interesting, which can be, in case of norm and trace and $\mathcal{E} \in \{0, 1, 2\}$, visualised in a simple figure; see Figure 7. Further research can also focus on finding the approximation set. For this we can look for inspiration in the intersection of the principal end and beginning of the hyperoperation in [10]. As a result we can form an approximation set which need not include the origin of the universe, i.e., $R_{[0,0]}$.

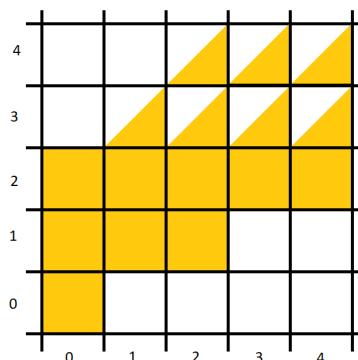


Figure 7. The lower and upper approximation with relation R_M for matrices 2×2 and $\mathcal{E} = \{0, 1, 2\}$.

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