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Operator Symbols and Operator Indices

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Abstract: We suggest a certain variant of symbolic calculus for special classes of linear bounded operators acting in Banach spaces. According to the calculus we formulate an index theorem and give applications to elliptic pseudo-differential operators on smooth manifolds with non-smooth boundaries.

Keywords: local operator; symbol; index; pseudo-differential operator; Fredholm property

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1. Introduction

In this paper, we consider some abstract operators acting in some functional spaces. These considerations were inspired by studies of I.B. Simonenko [1] related to special operators of a local type (we say here local operators). Such operators and corresponding equations play an important role in the theory of pseudo-differential operators and equations [2–4]. There are a lot of books in mathematics devoted to the theory of pseudo-differential operators and equations on non-smooth manifolds and manifolds with non-smooth boundaries [5–10], but it seems that the suggested abstract variant is very close to this theory. Some first steps were done in the author's paper [11], and here we develop this abstract variant and give some applications. We think this approach can be useful for similar problems related to concrete operators.

This way for elliptic pseudo-differential equations was suggested by the author earlier and partially described in his works of that period: at least two-dimensional situation was desxribed exactly (such results and review can be found in [4]). The main difference of the author's approach from previous works is systematic using the concept of wave factorization of the elliptic symbol. In other words, we deal with a multi-dimensional version of the Wiener–Hopf method [12] or one of analogues of the Riemann boundary value problem [13,14].

2. Operator Symbols

2.1. Local Operators

Here we give some constructions and definitions from [1,11]. Here, we consider such functional spaces which include smooth functions and corresponding multipliers and only local operators. Additionally, all considered operators are defined up to compact operators.

Let M be a compact m-dimensional manifold with a boundary. Below, we will consider the case of piecewise smooth boundary, and all singularities will be described. Here, we will try to develop certain general statements.

Let B_1, B_2 be Banach spaces consisting of functions defined on compact m-dimensional manifold M. We assume that smooth functions with compact support are dense in such spaces. Let $A : B_1 \to B_2$ be a linear bounded operator (We remind that an index of the operator A is called the following number $dim\ Ker A - dim\ Coker A$ [2–4]). We will denote by letter f the function f and the operator of multiplication by f, so that the notation $A \cdot f$ denotes the following operator:

$$(A \cdot f)u = A(fu), \quad u \in B_1.r.$$

Definition 1. *An operator A is called a local operator if the operator*

$$f \cdot A \cdot g$$

is a compact operator for arbitrary smooth functions f, g defined on M with non-intersecting supports.

A typical example of local operator is a pseudo-differential operator in the Sobolev–Slobodetski space (see below).

2.2. Operators on a Compact Manifold

On the manifold M, we fix a finite open covering and a partitions of unity corresponding to this covering $\{U_j, f_j\}_{j=1}^n$ and choose smooth functions $\{g_j\}_{j=1}^n$ so that $supp\ g_j \subset V_j$, $\overline{U_j} \subset V_j$, and $g_j(x) \equiv 1$ for $x \in supp\ f_j, supp\ f_j \cap (1 - g_j) = \emptyset$.

Proposition 1. *The operator A on the manifold M can be represented in the form*

$$A = \sum_{j=1}^{n} f_j \cdot A \cdot g_j + T,$$

where $T : B_1 \to B_2$ is a compact operator.

Proof. It is very simple. Since

$$\sum_{j=1}^{n} f_j \equiv 1,$$

then we can write

$$\left(\sum_{j=1}^{n} f_j\right) \cdot A = \sum_{j=1}^{n} f_j \cdot A = \sum_{j=1}^{n} f_j \cdot A \cdot g_j + \sum_{j=1}^{n} f_j \cdot A \cdot (1 - g_j),$$

so we have the conclusion needed. □

Remark 1. *The operator A is defined uniquely up to a compact operators that do not have an influence on an index.*

By definition, for an arbitrary operator $A : B_1 \to B_2$

$$|||A||| \equiv \inf ||A + T||,$$

where *infimum* is taken over all compact operators $T : B_1 \to B_2$.

Let B_1', B_2' be Banach spaces consisting of functions defined on $\mathbf{R}^m$, $\widetilde{A} : B_1' \to B_2'$ be a linear bounded operator.

Since M is a compact manifold, then, for every point $x \in M$, there exists a neighborhood $U \ni x$ and diffeomorphism $\omega : U \to D \subset \mathbf{R}^m, \omega(x) \equiv y$. We denote by S_ω the following operator (Really, this operator is defined locally; in general, it may be unbounded $B_k \to B_k'$ (see [15])) acting from B_k to $B_k', k = 1, 2$. For every function $u \in B_k$ vanishing out of U,

$$(S_\omega u)(y) = u(\omega^{-1}(y)), \quad y \in D, \quad (S_\omega u)(y) = 0, \quad y \notin D.$$

Of course, for every function $v \in B'_k$ vanishing out of D, we can define

$$(S_\omega^{-1} v)(x) = v(\omega(x)), \quad x \in U, \quad (S_\omega^{-1} v)(x) - 0, \quad x \notin U.$$

Definition 2. *A local representative of the operator $A : B_1 \to B_2$ at the point $x \in M$ is called the operator $\widetilde{A} : B'_1 \to B'_2$ such that, $\forall \varepsilon > 0$, there exists the neighborhood U_j of the point $x \in U_j \subset M$ and diffeomorphism $\omega'_j : U_j \to D_j \subset \mathbf{R}^m$ with the property*

$$|||g_j \cdot A \cdot f_j - S_{\omega_j}^{-1} \hat{g}_j \cdot \widetilde{A} \cdot \hat{f}_j S_{\omega_j}||| < \varepsilon,$$

where $\hat{f}_j, \hat{g}_j$ are the functions f_j, g_j for other local coordinates.

3. Generating Operator

Let M be a compact m-dimensional manifold with a boundary ∂M, and $A(x)$ be a certain operator-function defined on M. Let $M_k, k = 0, 1, \ldots, m-1$, be smooth k-dimensional sub-manifolds on ∂M so that, by definition, $M_{m-1} \equiv \partial M, M_0$ consists of isolated points on ∂M. Furthermore, we introduce a set of operator classes $\mathrm{T}_k, k = 0, 1, \ldots, m$, so that, for $x \in M_k$, $A(x) : H_k^{(1)} \to H_k^{(2)}$ is a linear bounded operator, where $H_k^{(j)}, k = 0, 1, \ldots, m, j = 1, 2$, are some Banach spaces.

We say that sub-manifold M_k is a singular k-sub-manifold if, $\forall x \in M_k$, we have $A(x) \in \mathrm{T}_k$. Additionally, we will assume that, if $x \in M_r \cap M_{k-1} \neq \emptyset$, then $A(x) \in \mathrm{T}_{k-1}$.

Definition 3. *If the family $A(x)$ consists of local Fredholm operators and this family is continuous on each component $\overline{M_k \setminus \cup_{i=0}^{k-1} M_i}, k = 0, 1, \ldots, m,$, then it generates a unique Fredholm operator A acting in the spaces $\sum\limits_{k=0}^{m} \oplus H_k^{(1)} \to \sum\limits_{k=0}^{m} \oplus H_k^{(2)}$.*

Proof. First, we construct such an operator in the following way. Let $\varepsilon > 0$ be small enough. We take a covering for M by balls as follows. We take a covering for M_0; it consists of a finite number of open sets and denote this covering by $\mathcal{U}_0$. Furthermore, we compose $M \setminus \mathcal{U}_0$. For every point $x \in M_1 \cap (M \setminus \mathcal{U}_0)$, we take a ball with the center x of radius ε. The union of such balls is covering for the set $x \in M_1 \cap (M \setminus \mathcal{U}_0)$. According to compactness of the set, we extract a finite sub-covering that will be denoted by $\mathcal{U}_1$. Then, we compose the set $M_2 \cap (M \setminus (\mathcal{U}_0 \cup \mathcal{U}_1))$, repeat the procedure mentioned above and obtain the sub-covering $\mathcal{U}_2$. Continuing the process, we obtain the finite covering for M of the following type:

$$M \subset \bigcup_{k=0}^{m} \mathcal{U}_k \equiv \mathcal{U}.$$

without loss of generality, we can mean that elements of the covering are balls with centers at points $x_j^{(k)} \in M_k, j = 0, 1, \cdots, n_k, k = 0, 1, \cdots, m$.

Since the set M_0 consists of isolated points only, we have a finite number of operators acting $H_0^{(1)} \to H_0^{(2)}$. We construct a partition of unity $f_j^{(k)}$ for every sub-covering $\mathcal{U}_k$ and associated set of functions $g_j^{(k)}, j = 0, 1, \cdots, n_k, k = 1, 2, \cdots, m$. Let us consider the kth component.

Using a piece of the operator-function $A(x)$ related to M_k, we construct the following sequence of operators acting $H_k^{(1)} \to H_k^{(2)}$. Let us denote

$$A_{n_k} = \sum_{j-1}^{n_k} f_j^{(k)} \cdot A(x_j^{(k)}) \cdot g_j^{(k)}$$

and consider another sub-covering $\mathcal{V}_k$ for the set $M \setminus (\bigcup_{l=0}^{k-1} \mathcal{U}_l)$. Let us suppose that this covering consists of balls with centers in $y_i^{(k)} \in M_k, i = 1, 2, \cdots, r_k$ of a small enough radius. We can construct the operator

$$A_{r_k} = \sum_{i-1}^{r_k} f_i^{(k)} \cdot A(y_i^{(k)}) \cdot g_i^{(k)}.$$

We would like to prove the following sentence:

$$|||A_{n_k} - A_{r_k}||| \to 0, \quad \text{if} \quad n_k, r_k \to \infty. \tag{1}$$

under an appropriate choice of coverings $\mathcal{U}_k, \mathcal{V}_k$.

As soon as the formula (1) is proved, we conclude that the sequence $\{A_{n_k}\}$ is a Cauchy sequence with respect to the norm $||| \cdot |||$. Therefore, there exists the operator limit $A^{(k)} = \lim_{n_k \to \infty} A_{n_k}$.

The rest of the proof repeats, in general, arguments from [11], but, for reader's convenience, we give these reasonings here in view of their values.

We will construct the kth component for the operator A in the following way. Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence such that $\varepsilon_n > 0, \forall n \in \mathbf{N}, \lim_{n \to \infty} \varepsilon_n = 0$. Given ε_n, we choose coverings $\{U_j^{(k)}\}_{j=1}^{n_k} \equiv \mathcal{U}_k$ as above with partition of unity $\{f_j^{(k)}\}$ and corresponding functions $\{g_j^{(k)}\}$ such that

$$|||f_j^{(k)} \cdot (A(x) - A(x_j^{(k)})) \cdot g_j^{(k)}||| < \varepsilon_{n_k}, \quad \forall x \in U_j^{(k)},$$

and $\{V_i^{(k)}\}_{i=1}^{r_k} \equiv \mathcal{V}_k$ with partition of unity $\{F_i^{(k)}\}$ and corresponding functions $\{G_i^{(k)}\}$ such that

$$|||F_i^{(k)} \cdot (A(x) - A(y_i^{(k)})) \cdot G_i^{(k)}||| < \varepsilon_{r_k}, \quad \forall x \in V_i^{(k)};$$

we remind readers that $U_j^{(k)}, V_i^{(k)}$ are balls with centers at $x_j^{(k)}, y_i^{(k)} \in \overline{M_k}$ of radius ε and 2ε. This requirement is possible according to continuity of the operator family $A(x)$ with respect to the norm $||| \cdot |||$ on the sub-manifold $\overline{M_k}$.

We can write

$$A_{n_k} = \sum_{j=1}^{n_k} f_j^{(k)} \cdot A(x_j^{(k)}) \cdot g_j^{(k)} = \sum_{i=1}^{r_k} F_i^{(k)} \cdot \sum_{j=1}^{n_k} f_j^{(k)} \cdot A(x_j^{(k)}) \cdot g_j^{(k)} =$$

$$\sum_{i=1}^{r_k} \sum_{j=1}^{n_k} F_i^{(k)} \cdot f_j^{(k)} \cdot A(x_j^{(k)}) \cdot g_j^{(k)} = \sum_{i=1}^{r_k} \sum_{j=1}^{n_k} F_i^{(k)} \cdot f_j^{(k)} \cdot A(x_j^{(k)}) \cdot g_j^{(k)} \cdot G_i^{(k)} + T_1,$$

and we can write the same for A_{r_k}

$$A_{r_k} = \sum_{i=1}^{r_k} F_i^{(k)} \cdot A(y_i^{(k)}) \cdot G_i^{(k)} = \sum_{j=1}^{n_k} f_j^{(k)} \cdot \sum_{i=1}^{r_k} F_i^{(k)} \cdot A(y_i^{(k)}) \cdot G_i^{(k)} =$$

$$\sum_{j=1}^{n_k} \sum_{i=1}^{r_k} f_j^{(k)} \cdot F_i^{(k)} \cdot A(y_i^{(k)}) \cdot G_i^{(k)} = \sum_{j=1}^{n_k} \sum_{i=1}^{r_k} f_j^{(k)} \cdot F_i^{(k)} \cdot A(y_i^{(k)}) \cdot G_i^{(k)} \cdot g_j^{(k)} + T_2.$$

Let us consider the difference

$$|||A_{n_k} - A_{r_k}||| = ||| \sum_{j=1}^{n_k} \sum_{i=1}^{r_k} f_j^{(k)} \cdot F_i^{(k)} \cdot (A(x_j^{(k)}) - A(y_i^{(k)})) \cdot G_i^{(k)} \cdot g_j^{(k)} |||.$$

We take into account only summands with non-vanishing supplements to the formula (4) such that $U_j^{(k)} \cap V_i^{(k)} \neq \emptyset$. A number of such neighborhoods are finite always for arbitrary finite coverings, hence we obtain

$$|||A_{n_k} - A_{r_k}||| \leq \sum_{j=1}^{n_k}\sum_{i=1}^{r_k} |||f_j^{(k)} \cdot F_i^{(k)} \cdot (A(x_j^{(k)}) - A(y_i^{(k)})) \cdot G_i^{(k)} \cdot g_j^{(k)}||| \leq$$

$$\sum_{x \in U_j^{(k)} \cap V_i^{(k)} \neq \emptyset} |||f_j^{(k)} \cdot F_i^{(k)} \cdot (A(x_j^{(k)}) - A(x)) \cdot G_i^{(k)} \cdot g_j^{(k)}|||+$$

$$\sum_{x \in U_j^{(k)} \cap V_i^{(k)} \neq \emptyset} |||f_j^{(k)} \cdot F_i^{(k)} \cdot (A(x) - A(y_i^{(k)})) \cdot G_i^{(k)} \cdot g_j^{(k)}||| \leq 2K \max[\varepsilon_{n_k}, \varepsilon_{r_k}],$$

where K is a universal constant.

Thus, we have proved that the sequence $\{A_{n_k}\}$ is a Cauchy sequence, hence there exists $\lim\limits_{n_k \to \infty} A_{n_k} = A^{(k)}$.

Using the same process, we can construct all operators $A^{(k)}$ for every $k = 0, 1, \cdots, m$. Let us note that all operators $A^{(k)} : H_k^{(1)} \to H_k^{(2)}$ act in different spaces. Finally, it is easy to compose the resulting operator A acting in direct sums of such spaces. Indeed, if

$$u = \oplus \sum_{k-1}^{m} u_k,$$

then we define

$$Au = \sum_{k-1}^{m} A^{(k)} u_k.$$

This operator A will be a generating operator. $\square$

Such operator A is called an elliptic operator if the operator-function $A(x)$ consists of Fredholm operators $\forall x \in M$. In a certain sense, we can obtain the inverse result.

4. The Index Theorem

4.1. Auxiliaries

We introduce some definitions for an index of a linear bounded operator and describe its principal properties [16,17].

Let B_1, B_2 be Banach spaces, $A : B_1 \to B_2$ be a linear bounded operator. By definition

$$Ker\ A = \{x \in B_1 : Ax = 0\}.$$

$$ImA = \{y \in B_2 : \exists x \in B_1, Ax = y\}.$$

The factor space B_2/ImA is called $Coker\ A$. The operator A is called Fredholm operator if

$$\dim\ Ker\ A < +\infty, \quad \dim\ Coker\ A < +\infty.$$

The difference

$$Ind\ A \equiv \dim\ Ker\ A - \dim\ Coker\ A$$

is called an index of the linear bounded operator A.

Basic properties of an index are as follows:

1. A Stability with Respect to Small and Compact Perturbations;

2. Homotopical Invariance of an Index.

   In other words, it means that, if $C : B_1 \to B_2$ is linear bounded operator with a small enough norm, then
$$Ind\ (A + C) = Ind\ A.$$
   If $T : B_1 \to B_2$ is a compact operator, then
$$Ind\ (A + T) = Ind\ A.$$
   If $\mathcal{L}(B_1, B_2)$ is the space of bounded linear operators acting from B_1 into B_2, A_t is a continuous map $[0;1] \to \mathcal{L}(B_1, B_2)$ (a homotopy), and the operator A_0 has Fredholm property, then all operators $A_t, t \in [0,1]$ have Fredholm property and
$$Ind\ A_0 = Ind\ A_1.$$

4.2. Indices

Here, we will give an index theorem for our operators. It seems that it does not give a real instrument for calculating indices, but it shows us what kinds of operators we need to study for obtaining good index formulas.

The family $A(x)$ from a previous section we call an operator symbol of the operator A [11] because the operator $A(x)$ is a local representative of the operator A at the point x..

We can see (cf [1,18]) this definition preserves basic properties of a symbolic calculus. Up to compact summands we have

- product and sum of two symbols correspond to product and sum of operators;
- adjoint symbol corresponds to adjoint operator;
- Fredholm property of symbol corresponds to Fredholm property of operator;
- homotopies of symbols correspond to homotopies of operators.

Theorem 1. *The index of the operator A on the manifold M is a sum of corresponding indices*
$$Ind\ A = \sum_{k=0}^{m} Ind\ A^{(k)}. \tag{2}$$

Proof. Indeed, all operators $A^{(k)}, k = 0, 1, \cdots, m$ act in different spaces. Therefore, the generating operator A has the following kernel and co-kernel:
$$Ker\ A = \sum_{k=0}^{m} Ker\ A^{(k)},$$
$$Coker\ A = \sum_{k=0}^{m} Coker\ A^{(k)}.$$
According to definition for an index, we obtain formula (2). □

5. Example: Pseudo-Differential Constructions

5.1. Local Situations

Here we consider a pseudo-differential operator A on compact manifold M with a boundary. This operators is defined by the function $A(x,\xi), (x,\xi) \in \mathbf{R}^{2m}$. We will suppose that the symbol has the order $\alpha \in \mathbf{R}$, i.e.,
$$c_1(1 + |\xi|)^\alpha \le |A(x,\xi)| \le c_2(1 + |\xi|)^\alpha,$$

for all admissible x, ξ with universal positive constants c_1, c_2.

We consider such a compact manifold M with a boundary that there are some smooth compact sub-manifolds M_k of dimension $0 \leq k \leq m-1$ on the boundary ∂M of manifold M that are singularities of a boundary. These singular manifolds are introduced by a local representative of operator A in a point $x_0 \in M$ on the map $U \ni x_0$ in the following way:

$$(A_{x_0}u)(x) = \int\limits_{D_{x_0}} \int\limits_{\mathbf{R}^m} e^{i\xi\cdot(x-y)} A(\varphi(x_0), \xi) u(y) d\xi dy, \quad x \in D_{x_0}, \tag{3}$$

where $\varphi : U \to D_{x_0}$ is a diffeomorphism, and the canonical domain D_{x_0} has a distinct form depending on a placement of the point x_0 on manifold M. We consider following canonical domains D_{x_0}: $\mathbf{R}^m, \mathbf{R}^m_+ = \{x \in \mathbf{R}^m : x = (x', x_m), x_m > 0\}, W^k = \mathbf{R}^k \times C^{m-k}$, where C^{m-k} is a convex cone in $\mathbf{R}^{m-k}$. For instance, if we consider a cube Q in three-dimensional space, then we have four canonical domains: $\mathbf{R}^3$ for inner points, $\mathbf{R}^3_+ = \{x \in \mathbf{R}^3 : x = (x_1, x_2, x_3), x_3 > 0\}$ for six 2-faces, $\mathbf{R} \times C^2 = \{x \in \mathbf{R}^3 : x = (x_1, x_2, x_3), x_> 0, x_3 > 0\}$ for twelve one-dimensional, edges, and $C^3 = \{x \in \mathbf{R}^3 : x = (x_1, x_2, x_3), x_1 > 0, x_2 > 0, x_3 > 0\}$ for eight vertices.

Such an operator A will be considered in Sobolev–Slobodetskii spaces $H^s(M)$, and local variants of such spaces will be spaces $H^s(D_{x_0})$.

Definition 4. *The symbol of an operator A is called the operator-function $A(x) : M \to \{A_x\}_{x\in M}$, which is defined by local representatives of the operator A.*

For a simple case when we consider a pseudo-differential operator in $\mathbf{R}^m$, its classical symbol can be treated as a multiplier.

Under some additional assumptions on smoothness properties of the function $A(x, \xi)$, one has the following:

Theorem 2. *The operator A has a Fredholm property iff its symbol is composed by Fredholm operators.*

Simplest variant of this theorem was proved in [1,4]. For general local operators in Lebesgue spaces, Theorem 3 was proved in [18].

Definition 5. *An operator A is said to be an elliptic operator if its symbol consists of invertible operators.*

Ellipticity property in our sense can be disappeared for example in boundary points. To avoid this, they usually add some boundary conditions to obtain *elliptic boundary value problem* (see [2–4] and below).

Corollary 1. *Elliptic operator is a Fredholm operator.*

Remark 2. *If an ellipticity property does not hold on sub-manifolds, the M_k one needs to modify local representatives of the operator A adding a special boundary or co-boundary operators.*

Using a special partition of a unity on the manifold M, elliptic symbol $A(x)$ for each $x \in \overline{M_k}$ which is given by formula (3) and the above constructions from Theorem 1, we obtain $m+1$ operators $A^{(k)}$ according to a number of singular sub-manifolds including whole boundary ∂M and the manifold M.

Theorem 3. *Index of the Fredholm pseudo-differential operator A is given by the formula*

$$Ind\ A = \sum_{k=0}^{m} Ind\ A^{(k)}.$$

Proof. Really, this is a simply corollary from Theorem 2. Indeed, we need to show exactly what spaces we choose as $H_k^{(j)}, j = 1,2; k = 0,1,\cdots,m$. We enumerate:

$$A^{(m)} : H^s(\mathbf{R}^m) \to H^{s-\alpha}(\mathbf{R}^m);$$

$$A^{(m-1)} : H^s(\mathbf{R}^m_+) \to H^{s-\alpha}(\mathbf{R}^m_+)'$$

$$A^{(k)} : H^s(W^k) \to H^{s-\alpha}(W^k), k = 0,1,\cdots m-2,$$

so that $H_m^{(1)} = H^s(\mathbf{R}^m), H_m^{(2)} = H^{s-\alpha}(\mathbf{R}^m), H_{m-1}^{(1)} = H^s(\mathbf{R}^m_+), H_{m-1}^{(2)} = H^{s-\alpha}(\mathbf{R}^m_+), H_k^{(1)} = H^s(W^k), H_k^{(2)} = H^{s-\alpha}(W^k)$. Then, we compose the direct sum of such spaces and the operator A' acting in these direct sums

$$A' : H^s(\mathbf{R}^m) \oplus H^s(\mathbf{R}^m_+) \oplus \sum_{k=0}^{m-2} H^s(W^k) \longrightarrow H^{s-\alpha}(\mathbf{R}^m) \oplus H^{s-\alpha}(\mathbf{R}^m_+) \oplus \sum_{k=0}^{m-2} H^{s-\alpha}(W^k).$$

Let us note that the operator A' doesn't coincide with the operator A, but these operators have the same local representatives, i.e., the same symbols. We call the operator A' *virtual representative* of the operator A. Since homotopies of symbols one-to-one correspond to homotopies of operators, we complete the index theorem. □

Of course, Theorem 4 does not give effective index formulas, but it shows what kinds of operators we need to consider from an index theory viewpoint.

Remark 3. *If we consider an elliptic pseudo-differential operator in $H^s(\mathbf{R}^m_+)$ [2] with the smooth symbol $A(x,\xi)$, we have two decomposition operators: $A^{(m)}$ related to closure of inner points of $\mathbf{R}^m_+$ and $A^{(m-1)}$ related to boundary points $\mathbf{R}^{m-1}$. Operator symbols are a distinct nature for inner and boundary points. For the first case, such a symbol is represented by integral over the whole $\mathbf{R}^m$, but, for the second case, this integral is taken for a half-space. The index of $A^{(m)}$ will be zero according to a classical Atiyah–Singer theorem, but the index of $A^{(m-1)}$ depends on the so-called index of factorization for the symbol $A(x,\xi)$ at boundary point $x \in \mathbf{R}^{m-1}$.*

5.2. The Wave Factorization: Harmonic Analysis and Complex Variables

To obtain invertibility conditions for local operators, we need some additional characteristics for the classical symbol of elliptic pseudo-differential operators. The studying invertibility of a local operator in W^k, or in other words the unique solvability of the equation

$$(A_{x_0}u)(x) = v(x), \quad x \in W^k$$

in Sobolev–Slobodetskii space $H^s(W^k)$, is equivalent to a unique solvability for a so-called paired equation

$$(A_{x_0}P_+U)(x) + (IP_-U)(x) = V(x), \quad x \in \mathbf{R}^m, \tag{4}$$

in the space $H^s(\mathbf{R}^m)$, where P_+, P_- are projectors on $W^k, \mathbf{R}^m \setminus W^k$, and it can be easily proved. In addition, now, if we apply the Fourier transform, then we will come to complex spaces [4].

We denote by $\overset{*}{C}{}^{m-k}$ the conjugate cone for the C^{m-k}:

$$\overset{*}{C}{}^{m-k} = \{x \in \mathbf{R}^m : x \cdot y > 0, \forall y \in C^{m-k}\}.$$

$T(\pm \overset{*}{C}{}^{m-k})$ denotes a radial tube domains over the cone $\pm \overset{*}{C}{}^{m-k}$ [19,20], i.e., a domain of multidimensional complex space $\mathbf{C}^m$ of the type $\mathbf{R}^m \pm \overset{*}{C}{}^{m-k}$.

Let the classical symbol $a(\xi), \xi \in \mathbf{R}^m$, in local coordinates satisfy the condition

$$c_1(1+|\xi|)^{\alpha} \le |a(\xi)| \le c_1(1+|\xi|)^{\alpha}.$$

Let us denote $\xi = (\xi'', \xi'), \xi'' = (\xi_1, \cdots, \xi_k), \xi' = (\xi_{k+1}, \cdots, \xi_m)$.

Definition 6. *k-wave factorization of the symbol $a(\xi)$ with respect to the cone C^{m-k} is called its representation in the form*

$$a(\xi) = a_{\neq}(\xi) a_{=}(\xi),$$

where the factors $a_{\neq}(\xi), a_{=}(\xi)$ must have the following properties:

(1) $a_{\neq}(\xi), a_{=}(\xi)$ are defined for all $\xi \in \mathbf{R}^m$ excluding may be the points $\mathbf{R}^k \times \partial \left(\overset{*}{C}{}^{m-k} \cup (-\overset{*}{C}{}^{m-k}) \right)$;

(2) $a_{\neq}(\xi), a_{=}(\xi)$ admit analytical continuation into radial tube domains $T(\overset{*}{C}{}^{m-k}), T(-\overset{*}{C}{}^{m-k})$ for almost all $\xi'' \in \mathbf{R}^k$ respectively with estimates

$$|a_{\neq}^{\pm 1}(\xi'', \xi' + i\tau)| \le c_1(1+|\xi|+|\tau|)^{\pm {}_k},$$

$$|a_{=}^{\pm 1}(\xi'', \xi' - i\tau)| \le c_2(1+|\xi|+|\tau|)^{\pm(\alpha - {}_k)}, \ \forall \tau \in \overset{*}{C}{}^{m-k}.$$

The number ${}_k \in \mathbf{R}$ is called an index of k-wave factorization.

5.3. Fredholm Properties

For simplicity, we consider here the case when M is a bounded domain in $\mathbf{R}^m$ and its classical symbol looks like $A(x, \xi)$. Here, we assume additionally that a symbol of the operator A is continuous on $M_k, k = 0, 1, \ldots, m$, family of operators (of course with respect to the norm $|||\cdot|||$). This property holds, for example, if the function $A(x, \xi), (x, \xi) \in M \times \mathbf{R}^m$ is continuous differentiable up to boundary. Then, according to enveloping theorem [1] using an operator symbol, one can construct n operators A_k. If these operators have a Fredholm property, then the general operator will have a Fredholm property with the index according to Theorem 4.

Let ${}_{n-1}(x)$ be the index of factorization [2] of the function $A(x, \xi)$ in the point $x \in \partial M \setminus \cup_{k=0}^{m-2} M_k$, ${}_k(x)$ be indices of k-wave factorization with respect to the cone C_x^{m-k} at points $x \in M_k, k = 0, 1, \cdots, m-2$, and we assume that the functions ${}_k(x), k = 0, 1, \cdots, m-1$, are continuously continued in $\overline{M_k}$.

Remark 4. *Similarly, [2] using a uniqueness result for the wave factorization [4], one can verify that the functions ${}_k(x), k = 0, 1, \cdots, m-1$, do not depend on local coordinates.*

Theorem 4. *If the classical elliptic symbol $A(x, \xi)$ admits k-wave factorization with respect to the cones C^{m-k} with indices ${}_k(x), k = 0, 1, \cdots, m-2$, satisfying the condition*

$$|{}_k(x) - s| < 1/2, \quad \forall x \in \overline{M_k}, \quad k = 0, 1, \cdots, m-1, \tag{5}$$

then the operator $A : H^s(M) \to H^{s-\alpha}(M)$ has a Fredholm property.

Proof. To prove the theorem, we need to verify invertibility properties for all local representatives for our pseudo-differential operator A.

A whole space. This case was historically the first in the theory of pseudo-differential equations. If $x_0 \in \overset{\circ}{M}$ is an inner point, then the local representative in the formula (3) has the following form (in local coordinates φ)

$$(A_{x_0}u)(x) = \int\limits_{\mathbf{R}^m}\int\limits_{\mathbf{R}^m} e^{i\xi\cdot(x-y)}A(\varphi(x_0),\xi)u(y)d\xi dy, \quad x \in \mathbf{R}^m,$$

and this is classical pseudo-differential operator. [2–4]. The ellipticity condition for the classical symbol

$$A(x,\xi) \neq 0$$

for all admissible x,ξ is necessary and the sufficient condition for invertibility of every such operator.

Unfortunately, if we have a piece of the space $\mathbf{R}^m$, we need to study invertibility properties for the operator on the left-hand side of Equation (4).

A half-space. If $x_0 \in \partial M$ is smoothness point of ∂M, then a local representative for the operator A has the following form:

$$(A_{x_0}u)(x) = \int\limits_{\mathbf{R}^m_+}\int\limits_{\mathbf{R}^m} e^{i\xi\cdot(x-y)}A(\varphi(x_0),\xi)u(y)d\xi dy, \quad x \in \mathbf{R}^m_+.$$

To study solvability for a corresponding paired equation (4), a factorization theory and one-dimensional, singular integral operators were used [2,13,14]. Full solvability theory for such equations was constructed in Vishik–Eskin papers (see [2]). A principal role takes the index of factorization, in our notation $_{m-1}$, if the condition (5) holds, then the operator$H^s(\mathbf{R}^m_+) \to H^{s-\alpha}(\mathbf{R}^m_+)$ is invertible.

A k-wedge. Here, we have more complicated local representative

$$(A_{x_0}u)(x) = \int\limits_{W^k}\int\limits_{\mathbf{R}^m} e^{i\xi\cdot(x-y)}A(\varphi(x_0),\xi)u(y)d\xi dy, \quad x \in W^k,$$

but the factorization idea works here also in a multidimensional context; if k-wave factorization exists, then condition (5) is sufficient for invertibility of such operator [4].

Basic components of the proof are the following [4]:

(1) if x_0 is a k-wedge point and $a(x_0,\xi)$ is symbol of the operator A_{x_0} in local coordinates, then the equation with such operator A_{x_0} in the space $H^s(W^k)$ is equivalent to the paired equation (4) in the space $H^s(\mathbf{R}^m)$;

(2) after applying the Fourier transform to Equation (4), we obtain the so-called multidimensional Riemann problem with parameter $\xi'' \in \mathbf{R}^k$. If the symbol $a(x_0,\xi)$ admits the k-wave factorization with respect to C^{m-k} with the index $_k(x_0)$, then we can describe solvability conditions for the problem;

(3) these solvability conditions depend on the index $_k(x_0)$, particularly unique solvability is possible only if $|_k(x_0) - s| < 1/2$ (for all points $x_0 \in \overline{M_k}$). For other cases, we have either a formula for a general solution $(- s = n + \delta, n \in \mathbf{N}, |\delta| < 1/2)$ or solvability conditions for the right-hand side of the equation (4) $(- s = -n + \delta, n \in \mathbf{N}, |\delta| < 1/2)$. Thus, additional conditions related to a k-wedge are needed for two latter cases only [4]. □

Remark 5. *If the ellipticity property does not hold on sub-manifold M_k, then we need to modify the operator A adding boundary or co-boundary operators [4]. For example, if one of conditions (5) does not hold we must use such constructions.*

Some considerations related to this paper are given in [15,21–23], particularly these are related to more complicated singularities and more general spaces.

6. Conclusions

We have described a new abstract approach to the theory of a wide class of operators. This approach is based on general principles for special local operators. It will possibly be useful for studying new classes of pseudo-differential operators and related problems.

In our opinion, such considerations will also be useful for discrete situations in which pseudo-differential operators are defined in functional spaces of discrete variables. Some first considerations in this direction were done, for example, in [24]. Moreover, a discrete situation is more practical, since it permits applying computer calculations. We hope to develop these studies in this direction including a comparison between discrete and continuous cases.

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