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Note on the Type 2 Degenerate Multi-Poly-Euler Polynomials

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Abstract: Kim and Kim (Russ. J. Math. Phys. 26, 2019, 40-49) introduced polyexponential function as an inverse to the polylogarithm function and by this, constructed a new type poly-Bernoulli polynomials. Recently, by using the polyexponential function, a number of generalizations of some polynomials and numbers have been presented and investigated. Motivated by these researches, in this paper, multi-poly-Euler polynomials are considered utilizing the degenerate multiple polyexponential functions and then, their properties and relations are investigated and studied. That the type 2 degenerate multi-poly-Euler polynomials equal a linear combination of the degenerate Euler polynomials of higher order and the degenerate Stirling numbers of the first kind is proved. Moreover, an addition formula and a derivative formula are derived. Furthermore, in a special case, a correlation between the type 2 degenerate multi-poly-Euler polynomials and degenerate Whitney numbers is shown.

Keywords: Euler polynomials; degenerate multiple polyexponential function; degenerate multi-poly-Euler polynomials; degenerate Stirling numbers; degenerate Whitney numbers

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1. Introduction and Preliminaries

Special functions have recently been applied in numerous fields of applied and pure mathematics besides in such other disciplines as physics, economics, statistics, probability theory, biology and engineering, cf. [1–26], and see also the references cited therein. One of the most important families of special functions is the family of special polynomials, cf. [1,3–8,10,21,23–26]. Intense research activities in such an area as the family of special polynomials are principally motivated by their importance in both pure and applied mathematics and other disciplines. The degenerate forms of special polynomials are firstly considered by Leonard Carlitz [2] by defining the degenerate forms of the Bernoulli, Stirling, and Eulerian numbers. Despite there being more than 60 years old, these studies are still a hot topic and today enveloped in an aura of mystery within the scientific community, cf. [6–8,14–21,23,25,26]. For instance, Duran and Acikgoz [6] considered the degenerate truncated exponential polynomials and gave their several properties. After that, degenerate truncated forms of various special polynomials including Genocchi, Bell, Bernstein, Fubini, Euler, and Bernoulli polynomials were introduced via the degenerate truncated exponential polynomials and their various properties and relationships were derived in [6]. Kim and Kim [15] considered degenerate poly-Bernoulli polynomials by means of the degenerate polylogarithm function and investigated several properties and relations. Kim et al. [16] defined a new type of the degenerate poly-Genocchi polynomials and numbers constructed from the modified polyexponential function and the degenerate unipoly Genocchi polynomials and derived several combinatorial identities and some explicit expressions. Kim [17] introduced a degenerate form of the Stirling polynomials of the second kind and proved some novel relations and identities for these polynomials. Kim and Kim [18] considered a new type degenerate Bell polynomials via degenerate polyexponential functions and then gave some of their properties. Kim et al. [20] introduced degenerate multiple polyexponential functions whereby the degenerate multi-poly-Genocchi polynomials are considered and multifarious explicit expressions and some properties were investigated. Lee et al. [25] studied a new type of type 2 poly-Euler polynomials and its degenerate form by utilizing the modified polyexponential function.

In this paper, we introduce a novel class of degenerate multi-poly-Euler polynomials and numbers utilizing the degenerate multi-polyexponential function and studied their main explicit relations and identities. This work is organized as follows:

- Section 2 includes several known definitions and notations.
- In Section 3, we consider a novel class of degenerate multi-poly-Euler polynomials and numbers and investigate their diverse properties and relations.
- The last section outlines finding gains and the conclusions in this work and mentions recommendations for future studies.

2. Definitions

Let $\mathbb{Z}$ denotes the set of all integers, $\mathbb{R}$ denotes the set of all real numbers and $\mathbb{C}$ denotes the set of all complex numbers. Let $\lambda \in \mathbb{R}/\ \{0\}$ (or $\mathbb{C}/\{0\}$). The degenerate exponential function $e_{\lambda}^{x}(t)$ is defined as follows

$$e_{\lambda}^{x}(t) := (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \tag{1}$$

where $(x)_{0,\lambda} = 1$ and $(x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda)$ for $n \geq 1$, cf. [1,2,6–8,14–25] and see also the references cited therein.

It is easily observed that $\lim_{\lambda\to 0} e_{\lambda}^{x}(t) = e^{xt}$. Notice that $e_{\lambda}^{1}(t) := e_{\lambda}(t)$.

The usual Bernoulli $B_n(x)$ and Euler $E_n(x)$ polynomials (cf. [3]), and the degenerate Bernoulli $B_{n,\lambda}(x)$ and Euler $E_{n,\lambda}(x)$ (cf. [1,6,8,14–21,23,25,26]) polynomials are defined by the following generating functions:

$$\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \frac{t}{e^t-1} e^{xt} \text{ and } \sum_{n=0}^{\infty} B_{n,\lambda}(x) \frac{t^n}{n!} = \frac{t}{e_{\lambda}(t)-1} e_{\lambda}^{x}(t) \tag{2}$$

and

$$\sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} = \frac{2}{e^t+1} e^{xt} \text{ and } \sum_{n=0}^{\infty} E_{n,\lambda}(x) \frac{t^n}{n!} = \frac{2}{e_{\lambda}(t)+1} e_{\lambda}^{x}(t). \tag{3}$$

The polyexponential function $\mathrm{Ei}_k(x)$ is defined by (cf. [13])

$$\mathrm{Ei}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!n^k}, \ (k \in \mathbb{Z}). \tag{4}$$

For $k=1$ in (4), it yields $\mathrm{Ei}_1(x) = e^x - 1$.

The modified degenerate polyexponential function $\mathrm{Ei}_{k,\lambda}(x)$ is defined by (cf. [14])

$$\mathrm{Ei}_{k,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda}}{(n-1)!n^k} x^n. \tag{5}$$

It is noted that for $k=1$, $\mathrm{Ei}_{1,\lambda}(x) = e_{\lambda}(x) - 1$.

Let $k \in \mathbb{Z}$ and $\lambda \in \mathbb{R}$. The degenerate version of the logarithm function $\log_\lambda(1+t)$ given by (cf. [15])

$$\log_\lambda(1+t) = \sum_{n=1}^{\infty} \lambda^{n-1} (1)_{n,1/\lambda} \frac{t^n}{n!}.$$

which is also the inverse function of the degenerate exponential function $e_\lambda(t)$ as shown below

$$e_\lambda(\log_\lambda(1+t)) = \log_\lambda (e_\lambda(1+t)) = 1+t.$$

In [25], the type 2 poly-Euler polynomials $E_n^{(k)}(x)$ and the type 2 degenerate poly-Euler polynomials $E_{n,\lambda}^{(k)}(x)$ are introduced using the following generating functions to be

$$\sum_{n=0}^{\infty} E_n^{(k)}(x)\frac{t^n}{n!} = \frac{\mathrm{Ei}_k\left(\log(1+2t)\right)}{t(e^t+1)}e^{xt} \text{ and } \sum_{n=0}^{\infty} E_{n,\lambda}^{(k)}(x)\frac{t^n}{n!} = \frac{\mathrm{Ei}_{k,\lambda}\left(\log(1+2t)\right)}{t(e_\lambda(t)+1)}e_\lambda^x(t). \tag{6}$$

Multifarious relations and identities for these polynomials are investigated intensely in [25].

The degenerate Stirling numbers of the first kind (cf. [15,16]) and second kind (cf. [1,6,8,15–21,23,25,26]) are defined, respectively, by

$$\sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!} = \frac{(\log_\lambda(1+t))^k}{k!} \quad (k \geq 0) \tag{7}$$

and

$$\sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!} = \frac{(e_\lambda(t)-1)^k}{k!} \quad (k \geq 0). \tag{8}$$

Noting here that as $\lambda \to 0$, the degenerate Stirling numbers of the first kind $S_{1,\lambda}(n,k)$ and the second kind $S_{2,\lambda}(n,k)$ reduce to the usual Stirling numbers of the first kind $S_1(n,k)$ and the second kind $S_2(n,k)$ as follows (cf. [1,6,8,15–21,23,25,26])

$$\sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!} = \frac{(\log(1+t))^k}{k!} \quad (k \geq 0),$$

and

$$\sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!} = \frac{(e^t-1)^k}{k!} \quad (k \geq 0).$$

3. Type 2 Degenerate Multi-Poly-Euler Polynomials

Let $k_1, k_2, \cdots, k_r \in \mathbb{Z}$. The degenerate multi-polyexponential function is given by, (cf. [20])

$$\mathrm{Ei}_{k_1,k_2,\cdots,k_r,\lambda}(x) = \sum_{0<n_1<n_2<\cdots<n_r} \frac{(1)_{n_1,\lambda}\cdots(1)_{n_r,\lambda}x^{n_r}}{(n_1-1)!\cdots(n_r-1)!n_1^{k_1}\cdots n_r^{k_r}}, \tag{9}$$

where the sum is over all integers $n_1, n_2, \cdots, n_r$ satisfying $0 < n_1 < n_2 < \cdots < n_r$. By means of this function, Kim et al. [20] defined and investigated the degenerate multi-poly-Genocchi polynomials $g_{n,\lambda}^{(k_1,k_2,\cdots k_r)}(x)$ given by the following generating function to be

$$\sum_{n=0}^{\infty} g_{n,\lambda}^{(k_1,k_2,\cdots k_r)}(x)\frac{t^n}{n!} = \frac{r!\mathrm{Ei}_{k_1,k_2,\cdots,k_r,\lambda}(\log_\lambda(1+t))}{(e_\lambda(t)+1)^r}e_\lambda^x(t). \tag{10}$$

Motivated by the definition of degenerate multi-poly-Genocchi polynomials, utilizing the degenerate multi-polyexponential function (9), we consider the following definition.

Definition 1. *Let $k \in \mathbb{Z}$. Type 2 degenerate multi-poly-Euler polynomials $\mathfrak{E}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)$ are defined by the following Taylor expansion about $t = 0$:*

$$\sum_{n=0}^{\infty} \mathfrak{E}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\frac{t^n}{n!} = \frac{\mathrm{Ei}_{k_1,k_2,\cdots,k_r,\lambda}\left(\log_\lambda(1+2t)\right)}{t^r(e_\lambda(t)+1)^r}e_\lambda^x(t). \tag{11}$$

In the case when $x = 0$ in (11), the type 2 degenerate multi-poly-Euler polynomials $\mathfrak{E}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)$ reduce to the corresponding numbers, that is the type 2 degenerate multi-poly-Euler numbers denoted by $\mathfrak{E}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}$.

Remark 1. *Letting $\lambda \to 0$, the type 2 degenerate multi-poly-Euler polynomials $\mathfrak{E}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)$ reduce to a new type multi-poly-Euler polynomials which we denote $\mathfrak{E}_{n}^{(k_1,k_2,\cdots,k_r)}(x)$, which are different from the polynomials $E_{n}^{(k_1,k_2,\cdots,k_r)}(x)$ introduced by Jolany et al. [10], as follows:*

$$\sum_{n=0}^{\infty} E_{n}^{(k_1,k_2,\cdots,k_r)}(x)\frac{t^n}{n!} = \frac{\mathrm{Ei}_{k_1,k_2,\cdots,k_r}\left(\log(1+2t)\right)}{t^r(e^t+1)^r}e^{xt}.$$

Remark 2. *In the case when $r = 1$, the type 2 degenerate multi-poly-Euler polynomials reduce to a new type degenerate poly-Euler polynomials that we denote $\mathfrak{E}_{n,\lambda}^{(k)}(x)$, which are different from the polynomials $E_{n,\lambda}^{(k)}(x)$ in (6) defined by Lee et al. [25], as follows:*

$$\sum_{n=0}^{\infty} \mathfrak{E}_{n,\lambda}^{(k)}(x)\frac{t^n}{n!} = \frac{\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+2t)\right)}{t(e_\lambda(t)+1)}e_\lambda^x(t). \tag{12}$$

Also, when $x = 0$ in (12), these new type degenerate poly-Euler polynomials $\mathfrak{E}_{n,\lambda}^{(k)}(x)$ reduce to the corresponding numbers $\mathfrak{E}_{n,\lambda}^{(k)}$, that is, $\mathfrak{E}_{n,\lambda}^{(k)}(0) := \mathfrak{E}_{n,\lambda}^{(k)}$.

Now, we investigate some properties of the type 2 degenerate multi-poly-Euler polynomials.

Theorem 1. *The following relation*

$$\mathfrak{E}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x) = \sum_{m=0}^{n}\binom{n}{m}\mathfrak{E}_{n-m,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)_{m,\lambda} \tag{13}$$

holds true for $n \geq 0$.

Proof. From Definition 1, we observe that

$$\begin{aligned}\sum_{n=0}^{\infty} \mathfrak{E}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\frac{t^n}{n!} &= \frac{\mathrm{Ei}_{k_1,k_2,\cdots,k_r,\lambda}\left(\log_\lambda(1+2t)\right)}{t^r(e_\lambda(t)+1)^r}e_\lambda^x(t) \\ &= \sum_{n=0}^{\infty} \mathfrak{E}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}\frac{t^n}{n!}\sum_{m=0}^{\infty}(x)_{m,\lambda}\frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\mathfrak{E}_{n-m,\lambda}^{(k_1,k_2,\cdots k_r)}(x)_{m,\lambda}\right)\frac{t^n}{n!},\end{aligned}$$

which gives the desired result (13). □

Remark 3. *When λ approaches to 0, we get the following known relation for the multi-poly-Euler polynomials (cf. [4,10])*

$$E_n^{(k_1,k_2,\cdots,k_r)}(x) = \sum_{m=0}^{n} \binom{n}{m} E_{n-m}^{(k_1,k_2,\cdots,k_r)} x^m.$$

The degenerate Euler polynomials of higher order are given by the following Maclaurin series:

$$\sum_{n=0}^{\infty} E_n^{(r)}(x;\lambda)\frac{t^n}{n!} = \left(\frac{2}{e_\lambda(t)+1}\right)^r e_\lambda^x(t),$$

cf. [1,6,25], and also see the references cited therein. We also notice that when $r=1$, the degenerate Euler polynomials of higher order reduce to the degenerate Euler polynomials in (3), namely, $E_n^{(1)}(x;\lambda) := E_{n,\lambda}(x)$.

A summation formula for the type 2 degenerate multi-poly-Euler polynomials is stated in the following theorem.

Theorem 2. *For $k_1,k_2,\cdots k_r \in \mathbb{Z}$, and $n,r \in \mathbb{N}$ with $n \geq r$, we have*

$$\mathfrak{E}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x) = \sum_{m=0}^{n+r} \frac{n!E_{n+r-m}^{(r)}(x;\lambda)}{(n+r-m)!m!} \sum_{0<n_1<n_2<\cdots<n_r} \frac{n_r!(1)_{n_1,\lambda}\cdots(1)_{n_r,\lambda}S_{1,\lambda}(n+r-m,n_r)}{(n_1-1)!\cdots(n_r-1)!n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}n_r^{k_r}}2^{m-r}. \tag{14}$$

Proof. From Definition 1 and (9), we see that

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathfrak{E}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\frac{t^n}{n!} &= \frac{e_\lambda^x(t)}{t^r(e_\lambda(t)+1)^r} \sum_{0<n_1<n_2<\cdots<n_r} \frac{(1)_{n_1,\lambda}\cdots(1)_{n_r,\lambda}(\log_\lambda(1+2t))^{n_r}}{(n_1-1)!\cdots(n_r-1)!n_1^{k_1}\cdots n_r^{k_r}} \\
&= \frac{e_\lambda^x(t)}{t^r(e_\lambda(t)+1)^r} \sum_{0<n_1<n_2<\cdots<n_r} \frac{(1)_{n_1,\lambda}\cdots(1)_{n_r,\lambda}n_r!}{(n_1-1)!\cdots(n_r-1)!n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}n_r^{k_r}} \sum_{m=n_r}^{\infty} S_{1,\lambda}(m,n_r)2^m\frac{t^m}{m!} \\
&= \frac{1}{2^r t^r}\left(\frac{2^r e_\lambda^x(t)}{(e_\lambda(t)-1)^r}\right) \sum_{m=n_r}^{\infty}\left(\sum_{0<n_1<n_2<\cdots<n_r} \frac{(1)_{n_1,\lambda}\cdots(1)_{n_r,\lambda}S_{1,\lambda}(m,n_r)n_r!}{(n_1-1)!\cdots(n_r-1)!n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}n_r^{k_r}}2^m\right)\frac{t^m}{m!} \\
&= \frac{1}{2^r t^r}\sum_{l=0}^{\infty} E_n^{(r)}(x;\lambda)\frac{t^l}{l!} \sum_{m=n_r}^{\infty}\left(\sum_{0<n_1<n_2<\cdots<n_r} \frac{(1)_{n_1,\lambda}\cdots(1)_{n_r,\lambda}S_{1,\lambda}(m,n_r)n_r!}{(n_1-1)!\cdots(n_r-1)!n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}n_r^{k_r}}2^m\right)\frac{t^m}{m!} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{0<n_1<n_2<\cdots<n_r} \frac{\binom{n}{m}n_r!(1)_{n_1,\lambda}\cdots(1)_{n_r,\lambda}E_{n-m}^{(r)}(x;\lambda)S_{1,\lambda}(n-m,n_r)}{(n_1-1)!\cdots(n_r-1)!n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}n_r^{k_r}}2^{m-r}\frac{t^{n-r}}{n!},
\end{aligned}$$

which means the asserted result (14). □

Remark 4. *When $r=1$, we have*

$$\mathfrak{E}_{n,\lambda}^{(k)}(x) = n!\sum_{m=0}^{n+1}\sum_{l=1}^{\infty} \frac{(1)_{l,\lambda}}{(n+1-m)!m!}\frac{2^{m-1}}{l^{k-1}}E_{n+1-m,\lambda}(x)S_{1,\lambda}(n+1-m,l),$$

which is a new relation including a new type degenerate poly-Euler polynomials (12), degenerate Euler polynomials (3), and degenerate Stirling numbers of the first kind (7).

An addition formula for the type 2 degenerate multi-poly-Euler polynomials is given by the following theorem.

Theorem 3. *The following addition formula*

$$\mathfrak{E}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x+y)=\sum_{m=0}^{n}\binom{n}{m}(y)_{m,\lambda}\mathfrak{E}_{n-m,\lambda}^{(k_1,k_2,\cdots,k_r)}(x) \tag{15}$$

is valid for $k_1,k_2,\cdots,k_r\in\mathbb{Z}$ *and* $n\geq 0$.

Proof. Given Definition 1, we see that

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{E}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x+y)\frac{t^n}{n!}&=\frac{\mathrm{Ei}_{k_1,k_2,\cdots,k_r,\lambda}\left(\log_\lambda(1+2t)\right)}{t^r(e_\lambda(t)+1)^r}e_\lambda^{x+y}(t)\\&=\sum_{n=0}^{\infty}\mathfrak{E}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\frac{t^n}{n!}\sum_{m=0}^{\infty}(y)_{m,\lambda}\frac{t^m}{m!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}(y)_{m,\lambda}\mathfrak{E}_{n-m,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\right)\frac{t^n}{n!},\end{aligned}$$

which implies the claimed result (15). □

The derivative property of the type 2 degenerate multi-poly-Euler polynomials is provided below.

Theorem 4. *The following relation*

$$\frac{d}{dx}\mathfrak{E}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)=n!\sum_{l=1}^{\infty}\mathfrak{E}_{n-l,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\frac{(-1)^{l+1}}{(n-l)!l}\lambda^{l-1} \tag{16}$$

is valid for $k_1,k_2,\cdots,k_r\in\mathbb{Z}$ *and* $n\geq 0$.

Proof. By Definition 1, we observe that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{d}{dx}\mathfrak{E}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\frac{t^n}{n!}&=\frac{\mathrm{Ei}_{k_1,k_2,\cdots,k_r,\lambda}\left(\log_\lambda(1+2t)\right)}{t^r(e_\lambda(t)+1)^r}\frac{d}{dx}e_\lambda^{x}(t)\\&=\sum_{n=0}^{\infty}\mathfrak{E}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\frac{t^n}{n!}\frac{1}{\lambda}\ln(1+\lambda t)\\&=\sum_{n=0}^{\infty}\mathfrak{E}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\frac{t^n}{n!}\sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l}\lambda^{l-1}t^l\\&=\sum_{n=0}^{\infty}\sum_{l=1}^{\infty}\mathfrak{E}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\frac{(-1)^{l+1}}{l}\lambda^{l-1}\frac{t^{n+l}}{n!},\end{aligned}$$

which provides the asserted result (16). □

Remark 5. *Upon setting* $r=1$, *we acquire*

$$\frac{d}{dx}\mathfrak{E}_{n,\lambda}^{(k)}(x)=n!\sum_{l=1}^{\infty}\mathfrak{E}_{n-l,\lambda}^{(k)}(x)\frac{(-1)^{l+1}}{(n-l)!l}\lambda^{l-1},$$

which is the derivative formula for the new type degenerate poly-Euler polynomials (12).

Remark 6. *Taking* $r=k=1$, *we attain*

$$\frac{d}{dx}E_{n,\lambda}(x)=n!\sum_{l=1}^{\infty}E_{n-l,\lambda}(x)\frac{(-1)^{l+1}}{(n-l)!l}\lambda^{l-1},$$

which is the derivative formula for the degenerate Euler polynomials, cf. [6].

Theorem 5. *The following correlation*

$$\mathfrak{E}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)=\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}(x)_l S_{2,\lambda}(m,l)\,\mathfrak{E}_{n-m,\lambda}^{(k_1,k_2,\cdots,k_r)}. \tag{17}$$

is valid for $k_1,k_2,\cdots,k_r\in\mathbb{Z}$ and $n\geq 0$.

Proof. By means of Definition 1, we attain that

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{E}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\frac{t^n}{n!}&=\frac{\mathrm{Ei}_{k_1,k_2,\cdots,k_r,\lambda}\left(\log_\lambda(1+2t)\right)}{t^r(e_\lambda(t)+1)^r}e_\lambda^x(t)\\&=\frac{\mathrm{Ei}_{k_1,k_2,\cdots,k_r,\lambda}\left(\log_\lambda(1+2t)\right)}{t^r(e_\lambda(t)+1)^r}(e_\lambda(t)-1+1)^x\\&=\frac{\mathrm{Ei}_{k_1,k_2,\cdots,k_r,\lambda}\left(\log_\lambda(1+2t)\right)}{t^r(e_\lambda(t)+1)^r}\sum_{l=0}^{\infty}\binom{x}{l}(e_\lambda(t)-1)^l\\&=\sum_{n=0}^{\infty}\mathfrak{E}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}\frac{t^n}{n!}\sum_{l=0}^{\infty}(x)_l\sum_{m=l}^{\infty}S_{2,\lambda}(m,l)\frac{t^m}{m!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}(x)_l S_{2,\lambda}(m,l)\,\mathfrak{E}_{n-m,\lambda}^{(k_1,k_2,\cdots,k_r)}\right)\frac{t^n}{n!},\end{aligned}$$

where the notation $(x)_l$ is a falling factorial and is defined by $(x)_0=1$ and $(x)_n=x(x-1)\cdots(x-(n-1))$ for $n\geq 1$, cf. [1,2,6–8,14–22,25]. Therefore, we arrive at the asserted Formula (17). □

Remark 7. *In the case when $r=1$, we acquire*

$$\mathfrak{E}_{n,\lambda}^{(k)}(x)=\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}(x)_l S_{2,\lambda}(m,l)\,\mathfrak{E}_{n-m,\lambda}^{(k)},$$

which is a relation for the new type degenerate poly-Euler polynomials (12) and the degenerate Stirling numbers of the second kind (8).

Kim [17] introduced the degenerate Whitney numbers which are defined by the generating function to be

$$\frac{(e_\lambda^m(t)-1)^k}{m^k k!}e_\lambda^\alpha(t)=\sum_{n=k}^{\infty}W_{m,\alpha}(n,k\,|\,\lambda)\frac{t^n}{n!},\quad(k\geq 0).$$

Remark 8. *In the special case $m=1$ and $\alpha=0$, the degenerate Whitney numbers $W_{m,\alpha}(n,k\,|\,\lambda)$ reduce to the degenerate Stirling numbers $S_{2,\lambda}(n,k)$ of the second kind in (8), that is, $W_{1,0}(n,k\,|\,\lambda):=S_{2,\lambda}(n,k)$.*

A correlation including both the type 2 degenerate multi-poly-Euler numbers and polynomials and the degenerate Whitney numbers.

Theorem 6. *For $k_1,k_2,\cdots k_r\in\mathbb{Z}$ and $n\geq 0$, we have*

$$\mathfrak{E}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(xu+\alpha)=\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}u^l(x)_l W_{u,\alpha}(m,l\,|\,\lambda)\mathfrak{E}_{n-m,\lambda}^{(k_1,k_2,\cdots,k_r)}. \tag{18}$$

Proof. Utilizing Definition 1, we attain that

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathfrak{E}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(xu+\alpha)\frac{t^n}{n!} &= \frac{\mathrm{Ei}_{k_1,k_2,\cdots,k_r,\lambda}\left(\log_\lambda(1+2t)\right)}{t^r(e_\lambda(t)+1)^r}e_\lambda^\alpha(t)e_\lambda^{xu}(t)\\
&= \frac{\mathrm{Ei}_{k_1,k_2,\cdots,k_r,\lambda}\left(\log_\lambda(1+2t)\right)}{t^r(e_\lambda(t)+1)^r}e_\lambda^\alpha(t)\left(e_\lambda^u(t)-1+1\right)^x\\
&= \frac{\mathrm{Ei}_{k_1,k_2,\cdots,k_r,\lambda}\left(\log_\lambda(1+2t)\right)}{t^r(e_\lambda(t)+1)^r}e_\lambda^\alpha(t)\sum_{l=0}^{\infty}\binom{x}{l}\left(e_\lambda^u(t)-1\right)^l\\
&= \frac{\mathrm{Ei}_{k_1,k_2,\cdots,k_r,\lambda}\left(\log_\lambda(1+2t)\right)}{t^r(e_\lambda(t)+1)^r}\sum_{l=0}^{\infty}u^l(x)_l\frac{\left(e_\lambda^u(t)-1\right)^l}{l!u^l}e_\lambda^\alpha(t)\\
&= \frac{\mathrm{Ei}_{k_1,k_2,\cdots,k_r,\lambda}\left(\log_\lambda(1+2t)\right)}{t^r(e_\lambda(t)+1)^r}\sum_{l=0}^{\infty}u^l(x)_l\frac{\left(e_\lambda^u(t)-1\right)^l}{l!u^l}e_\lambda^\alpha(t)\\
&= \sum_{n=0}^{\infty}\mathfrak{E}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}\frac{t^n}{n!}\sum_{n=0}^{\infty}\sum_{l=0}^{n}u^l(x)_l W_{u,\alpha}(n,l\,|\,\lambda)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}u^l(x)_l W_{u,\alpha}(m,l\,|\,\lambda)\mathfrak{E}_{n-m,\lambda}^{(k_1,k_2,\cdots,k_r)}\right)\frac{t^n}{n!},
\end{aligned}$$

which provides the claimed Formula (18). □

Remark 9. *Upon setting $r=1$, we get*

$$\mathfrak{E}_{n,\lambda}^{(k)}(xu+\alpha)=\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}u^l(x)_l W_{u,\alpha}(m,l\,|\,\lambda)\mathfrak{E}_{n-m,\lambda}^{(k)},$$

which is a relation between the degenerate Whitney numbers and the new type degenerate poly-Euler polynomials (12).

4. Conclusions

As is known, for $k\in\mathbb{Z}$, the polylogarithm function is defined by (cf. [4,10])

$$\mathrm{Li}_k(x)=\sum_{0<n}\frac{x^n}{n^k}.$$

It is easily seen that $\mathrm{Li}_1(x)=-\log(1-x)$.

For $k_1,k_2,\cdots k_r\in\mathbb{Z}$, the multiple polylogarithm function [4,10,19] is given by

$$\mathrm{Li}_{k_1,k_2,\cdots,k_r}(x)=\sum_{0<n_1<n_2<\cdots<n_r}\frac{x^{n_r}}{n_1^{k_1}n_2^{k_2}\cdots n_r^{k_r}},$$

where the sum is over all integers $n_1,n_2,\cdots,n_r$ satisfying $0<n_1<n_2<\cdots<n_r$.

By means of the multiple polylogarithm function, the degenerate multi-poly-Bernoulli polynomials are introduced (cf. [4,10,19]) as follows

$$\sum_{n=0}^{\infty}\beta_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\frac{t^n}{n!}=\frac{r!\mathrm{Li}_{k_1,k_2,\cdots,k_r}\left(1-e^{-t}\right)}{(e_\lambda(t)-1)^r}e_\lambda^x(t). \tag{19}$$

Then, several properties for those polynomials are investigated.

A slightly different version of the polylogarithm function, the polyexponential function $\mathrm{Ei}_k(x)$ is defined as an inverse to polylogarithm function as follows (cf. [13])

$$\mathrm{Ei}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!n^k}, \ (k \in \mathbb{Z}).$$

For $k = 1$ in (4), it yields $\mathrm{Ei}_1(x) = e^x - 1$.

The modified degenerate polyexponential function $\mathrm{Ei}_{k,\lambda}(x)$ is defined by (cf. [14])

$$\mathrm{Ei}_{k,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda}}{(n-1)!n^k} x^n.$$

It is noted that for $k = 1$, $\mathrm{Ei}_{1,\lambda}(x) = e_\lambda(x) - 1$.

Let $k_1, k_2, \cdots, k_r \in \mathbb{Z}$. The degenerate multi-polyexponential function is given by, (cf. [20])

$$\mathrm{Ei}_{k_1,k_2,\cdots,k_r,\lambda}(x) = \sum_{0<n_1<n_2<\cdots<n_r} \frac{(1)_{n_1,\lambda} \cdots (1)_{n_r,\lambda} x^{n_r}}{(n_1-1)! \cdots (n_r-1)! n_1^{k_1} \cdots n_r^{k_r}},$$

where the sum is over all integers $n_1, n_2, \cdots, n_r$ satisfying $0 < n_1 < n_2 < \cdots < n_r$. By means of this function, Kim et al. [20] defined and investigated the degenerate multi-poly-Genocchi polynomials $g_{n,\lambda}^{(k_1,k_2,\cdots k_r)}(x)$ given by the following generating function to be

$$\sum_{n=0}^{\infty} g_{n,\lambda}^{(k_1,k_2,\cdots k_r)}(x) \frac{t^n}{n!} = \frac{r!\mathrm{Ei}_{k_1,k_2,\cdots,k_r,\lambda}(\log_\lambda(1+t))}{(e_\lambda(t)+1)^r} e_\lambda^x(t).$$

Motivated and inspired by the definitions of the degenerate multi-poly-Bernoulli polynomials and the degenerate multi-poly-Genocchi polynomials introduced by Kim et al. [20], in this paper, we have introduced a new generating function for the degenerate multi-poly-Euler polynomials, called the type 2 degenerate multi-poly-Euler polynomials, by means of the degenerate multi-polyexponential function as follows:

$$\sum_{n=0}^{\infty} \mathfrak{E}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x) \frac{t^n}{n!} = \frac{\mathrm{Ei}_{k_1,k_2,\cdots,k_r,\lambda}\left(\log_\lambda(1+2t)\right)}{t^r(e_\lambda(t)+1)^r} e_\lambda^x(t).$$

Then, we have derived some useful relations and properties. In a special case, we have investigated a correlation including the type 2 degenerate multi-poly-Euler polynomials and numbers, and degenerate Whitney numbers. We have also analyzed several special circumstances of the results derived in this paper.

In the plans, we will continue to study degenerate versions of certain special polynomials and numbers and their applications to probability, physics and engineering in addition to mathematics.

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