symmetry

MDPI

Article

A Note on Generalized q-Difference Equations and Their Applications Involving q-Hypergeometric Functions

Hari M. Srivastava [1,2,3,*], Jian Cao [4] and Sama Arjika [5,6]

1 Department of Mathematics and Statistics, University of Victoria, Victoria, BC V8W 3R4, Canada
2 Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
3 Department of Mathematics and Informatics, Azerbaijan University, 71 Jeyhun Hajibeyli Street, Baku AZ1007, Azerbaijan
4 Department of Mathematics, Hangzhou Normal University, Hangzhou City 311121, China; 21caojian@hznu.edu.cn
5 Department of Mathematics and Informatics, University of Agadez, Post Office Box 199, Agadez 8000, Niger; rjksama2008@gmail.com
6 International Chair of Mathematical Physics and Applications (ICMPA-UNESCO Chair), University of Abomey-Calavi, Post Office Box 072, Cotonou 50, Benin
* Correspondence: harimsri@math.uvic.ca

Received:13 October 2020; Accepted: 30 October 2020 ; Published: 2 November 2020

check for updates

Abstract: Our investigation is motivated essentially by the demonstrated applications of the basic (or q-) series and basic (or q-) polynomials, especially the basic (or q-) hypergeometric functions and basic (or q-) hypergeometric polynomials, in many diverse areas. Here, in this paper, we use two q-operators $\mathbb{T}(a,b,c,d,e,yD_x)$ and $\mathbb{E}(a,b,c,d,e,y\theta_x)$ to derive two potentially useful generalizations of the q-binomial theorem, a set of two extensions of the q-Chu-Vandermonde summation formula and two new generalizations of the Andrews-Askey integral by means of the q-difference equations. We also briefly describe relevant connections of various special cases and consequences of our main results with a number of known results.

Keywords: q-difference operator; q-binomial theorem; q-hypergeometric functions; q-Chu-Vandermonde summation formula; Andrews-Askey integral; q-series and q-integral identities; q-difference equations; Sears transformation

MSC: Primary 05A30; 11B65; 33D15; 33D45; Secondary 33D60; 39A13; 39B32

1. Introduction, Definitions and Preliminaries

Throughout this paper, we refer to [1] for definitions and notations. We also suppose that $0 < q < 1$. For complex numbers a, the q-shifted factorials are defined by

$$(a;q)_0 := 1, \tag{1}$$

$$(a;q)_n := \prod_{k=0}^{n-1}(1-aq^k) = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}) \qquad (n \in \mathbb{N} := \{1,2,3,\cdots\}) \tag{2}$$

and

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k), \tag{3}$$

where (see, for example, [1,2])

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty} \qquad \text{and} \qquad (a;q)_{n+m} = (a;q)_n (aq^n;q)_m$$

and

$$\left(\frac{q}{a};q\right)_n = (-a)^{-n}\, q^{\binom{n+1}{2}} \frac{(aq^{-n};q)_\infty}{(a;q)_\infty}.$$

We adopt the following notation:

$$(a_1, a_2, \cdots, a_r; q)_m = (a_1;q)_m (a_2;q)_m \cdots (a_r;q)_m \qquad (m \in \mathbb{N} := \{1,2,3,\cdots\}).$$

Also, for m large, we have

$$(a_1, a_2, \cdots, a_r; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \cdots (a_r;q)_\infty.$$

The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}. \tag{4}$$

The basic (or q-) hypergeometric function of the variable z and with $\mathfrak{r}$ numerator and $\mathfrak{s}$ denominator parameters is defined as follows (see, for details, the monographs by Slater ([2], Chapter 3) and by Srivastava and Karlsson ([3], p. 347, equation (272)); see also [4–6]):

$${}_{\mathfrak{r}}\Phi_{\mathfrak{s}} \left[\begin{matrix} a_1, a_2, \cdots, a_{\mathfrak{r}}; \\ \\ b_1, b_2, \cdots, b_{\mathfrak{s}}; \end{matrix} \; q; z \right] := \sum_{n=0}^{\infty} \left[(-1)^n \, q^{\binom{n}{2}} \right]^{1+\mathfrak{s}-\mathfrak{r}} \frac{(a_1, a_2, \cdots, a_{\mathfrak{r}}; q)_n}{(b_1, b_2, \cdots, b_{\mathfrak{s}}; q)_n} \, \frac{z^n}{(q;q)_n},$$

where $q \neq 0$ when $\mathfrak{r} > \mathfrak{s} + 1$. We also note that

$${}_{\mathfrak{r}+1}\Phi_{\mathfrak{r}} \left[\begin{matrix} a_1, a_2, \cdots, a_{\mathfrak{r}+1} \\ \\ b_1, b_2, \cdots, b_{\mathfrak{r}}; \end{matrix} \; q; z \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \cdots, a_{\mathfrak{r}+1}; q)_n}{(b_1, b_2, \cdots, b_{\mathfrak{r}}; q)_n} \, \frac{z^n}{(q;q)_n}.$$

We remark in passing that, in a recently-published survey-cum-expository review article, the so-called (p,q)-calculus was exposed to be a rather trivial and inconsequential variation of the classical q-calculus, the additional parameter p being redundant or superfluous (see, for details, ([7], p. 340)).

Basic (or q-) series and basic (or q-) polynomials, especially the basic (or q-) hypergeometric functions and basic (or q-) hypergeometric polynomials, are applicable particularly in several areas of Number Theory such as the Theory of Partitions and are useful also in a wide variety of fields including, for example, Combinatorial Analysis, Finite Vector Spaces, Lie Theory, Particle Physics, Non-Linear Electric Circuit Theory (see [8,9]), Mechanical Engineering (see [10]), Theory of Heat Conduction, Quantum Mechanics, Cosmology, Computation of Fractional-Order Derivatives (see [11]) and Statistics [see also ([3], pp. 350–351), and the references cited therein]. Here, in our present investigation, we are mainly concerned with the Cauchy polynomials $p_n(x,y)$ as given below (see [1,12]):

$$p_n(x,y) := (x-y)(x-qy)\cdots(x-q^{n-1}y) = \left(\frac{y}{x};q\right)_n x^n, \tag{5}$$

together with the following Srivastava-Agarwal type generating function (see also [13]):

$$\sum_{n=0}^{\infty} p_n(x,y) \frac{(\lambda;q)_n \, t^n}{(q;q)_n} = {}_2\Phi_1 \left[\begin{matrix} \lambda, \frac{y}{x}; \\ \\ 0; \end{matrix} \; q; xt \right]. \tag{6}$$

In particular, for $\lambda = 0$ in (6), we get the following simpler generating function [12]:

$$\sum_{n=0}^{\infty} p_n(x,y)\, \frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(xt;q)_\infty}. \tag{7}$$

The generating function (7) is also the homogeneous version of the Cauchy identity or the following q-binomial theorem (see, for example, [1–3,14]):

$$\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k}\, z^k = {}_1\Phi_0 \left[\begin{array}{c} a; \\ \\ -; \end{array} q;z \right] = \frac{(az;q)_\infty}{(z;q)_\infty} \qquad (|z|<1). \tag{8}$$

Upon further setting $a = 0$, this last relation (8) becomes Euler's identity (see, for example, [1]):

$$\sum_{k=0}^{\infty} \frac{z^k}{(q;q)_k} = \frac{1}{(z;q)_\infty} \qquad (|z|<1) \tag{9}$$

and its inverse relation given below [1]:

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(q;q)_k}\, q^{\binom{k}{2}}\, z^k = (z;q)_\infty. \tag{10}$$

Based upon the q-binomial theorem (8) and Heine's transformations, Srivastava et al. [15] established a set of two presumably new theta-function identities (see, for details, [15]).

The following usual q-difference operators are defined by [16–18]

$$D_a\{f(a)\} := \frac{f(a) - f(qa)}{a}, \quad \theta_a = \{f(a)\} := \frac{f(q^{-1}a) - f(a)}{q^{-1}a}, \tag{11}$$

and their Leibniz rules are given by (see [19])

$$D_a^n\,\{f(a)g(a)\} = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q q^{k(k-n)} D_a^k\,\{f(a)\}\, D_a^{n-k}\left\{g(q^k a)\right\} \tag{12}$$

and

$$\theta_a^n\,\{f(a)g(a)\} = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q \theta_a^k\,\{f(a)\}\, \theta_a^{n-k}\left\{g(q^{-k} a)\right\}, \tag{13}$$

respectively. Here, and in what follows, D_a^0 and θ_a^0 are understood as the identity operators.

Recently, Chen and Liu [16,20] constructed the following pair of augmentation operators, which is of great significance for deriving identities by applying its various special cases:

$$\mathbb{T}(bD_x) = \sum_{n=0}^{\infty} \frac{(bD_x)^n}{(q;q)_n} \qquad \text{and} \qquad \mathbb{E}(b\theta_x) = \sum_{n=0}^{\infty} \frac{(b\theta_x)^n}{(q;q)_n}. \tag{14}$$

Subsequently, Chen and Gu [21] defined the Cauchy augmentation operators as follows:

$$\mathbb{T}(a, bD_x) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n}\, (bD_x)^n \tag{15}$$

and

$$\mathbb{E}(a, b\theta_x) = \sum_{n=0}^{\infty} \frac{(b;q)_n}{(q;q)_n}\, (-b\theta_x)^n. \tag{16}$$

On the other hand, Fang [22] and Zhang and Wang [23] considered the following finite generalized q-exponential operators with two parameters:

$$\mathbb{T}\left[\begin{matrix} q^{-N}, w \\ v \end{matrix}\ \middle|\ q; tD_x\right] = \sum_{n=0}^{N} \frac{(q^{-N}, w; q)_n}{(v, q; q)_n} (tD_x)^n \tag{17}$$

and

$$\mathbb{E}\left[\begin{matrix} q^{-N}, w \\ v \end{matrix}\ \middle|\ q; t\theta_x\right] = \sum_{n=0}^{N} \frac{(q^{-N}, w; q)_n}{(v, q; q)_n} (t\theta_x)^n. \tag{18}$$

Moreover, Li and Tan [24] constructed two generalized q-exponential operators with three parameters as follows:

$$\mathbb{T}\left[\begin{matrix} u, v \\ w \end{matrix}\ \middle|\ q; tD_x\right] = \sum_{n=0}^{\infty} \frac{(u, v; q)_n}{(w, q; q)_n} (tD_x)^n \tag{19}$$

and

$$\mathbb{E}\left[\begin{matrix} u, v \\ w \end{matrix}\ \middle|\ q; t\theta_x\right] = \sum_{n=0}^{\infty} \frac{(u, v; q)_n}{(w, q; q)_n} (t\theta_x)^n. \tag{20}$$

Finally, we recall that Cao et al. [25] constructed the following q-operators:

$$\mathbb{T}(a, b, c, d, e, yD_x) = \sum_{n=0}^{\infty} \frac{(a, b, c; q)_n}{(q, d, e; q)_n} (yD_x)^n \tag{21}$$

and

$$\mathbb{E}(a, b, c, d, e, y\theta_x) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} (a, b, c; q)_n}{(q, d, e; q)_n} (y\theta_x)^n \tag{22}$$

and thereby generalized Arjika's results in [26] by using the q-difference equations (see, for details, [25]).

We remark that the q-operator (21) is a particular case of the homogeneous q-difference operator $\mathbb{T}(\mathbf{a}, \mathbf{b}, cD_x)$ (see [27]) by taking

$$\mathbf{a} = (a, b, c), \quad \mathbf{b} = (d, e) \quad \text{and} \quad c = y.$$

Furthermore, for $b = c = d = e = 0$, the q-operator (22) reduces to the operator $\widetilde{L}(a, y; \theta_x)$ which was investigated by Srivastava et al. [28].

Cao et al. [25] used the q-operators (21) and (22) and gave the following results.

Proposition 1. *(see ([25], theorem 3)) Let $f(a, b, c, d, e, x, y)$ be a seven-variable analytic function in a neighborhood of $(a, b, c, d, e, x, y) = (0, 0, 0, 0, 0, 0, 0) \in \mathbb{C}^7$.*

(I) If $f(a, b, c, d, e, x, y)$ satisfies the following difference equation:

$$\begin{aligned} x\{f&(a,b,c,d,e,x,y) - f(a,b,c,d,e,x,yq) \\ &- (d+e)q^{-1}\left[f(a,b,c,d,e,x,yq) - f(a,b,c,d,e,x,yq^2)\right] \\ &+ deq^{-2}\left[f(a,b,c,d,e,x,yq^2) - f(a,b,c,d,e,x,yq^3)\right]\} \\ = y\{&[f(a,b,c,d,e,x,y) - f(a,b,c,d,e,xq,y)] \\ &- (a+b+c)[f(a,b,c,d,e,x,yq) - f(a,b,c,d,e,xq,yq)] \\ &+ (ab+ac+bc)[f(a,b,c,d,e,x,yq^2) - f(a,b,c,d,e,xq,yq^2)] \\ &- abc[f(a,b,c,d,e,x,yq^3) - f(a,b,c,d,e,xq,yq^3)]\}, \end{aligned} \tag{23}$$

then

$$f(a,b,c,d,e,x,y) = \mathbb{T}(a,b,c,d,e,yD_x)\{f(a,b,c,d,e,x,0)\}. \tag{24}$$

(II) If $f(a,b,c,d,e,x,y)$ satisfies the following difference equation:

$$\begin{aligned} &x\{f(a,b,c,d,e,xq,y) - f(a,b,c,d,e,xq,yq) \\ &\qquad - (d+e)q^{-1}\,[f(a,b,c,d,e,xq,yq) - f(a,b,c,d,e,xq,yq^2)] \\ &\qquad + deq^{-2}\,[f(a,b,c,d,e,xq,yq^2) - f(a,b,c,d,e,xq,yq^3)]\} \\ &= y\{[f(a,b,c,d,e,xq,yq) - f(a,b,c,d,e,x,yq)] \\ &\qquad - (a+b+c)[f(a,b,c,d,e,xq,yq^2) - f(a,b,c,d,e,x,yq^2)] \\ &\qquad + (ab+ac+bc)[f(a,b,c,d,e,xq,yq^3) - f(a,b,c,d,e,x,yq^3)] \\ &\qquad - abc[f(a,b,c,d,e,xq,yq^4) - f(a,b,c,d,e,x,yq^4)]\}, \end{aligned} \tag{25}$$

then

$$f(a,b,c,d,e,x,y) = \mathbb{E}(a,b,c,d,e,y\theta_x)\{f(a,b,c,d,e,x,0)\}. \tag{26}$$

Liu [29,30] initiated the method based upon q-difference equations and deduced several results involving Bailey's ${}_6\psi_6$, q-Mehler formulas for the Rogers-Szegö polynomials and q-integral version of the Sears transformation.

Lemma 1. *Each of the following q-identities holds true*:

$$D_a^k\left\{\frac{1}{(as;q)_\infty}\right\} = \frac{s^k}{(as;q)_\infty}, \tag{27}$$

$$\theta_a^k\left\{\frac{1}{(as;q)_\infty}\right\} = \frac{s^k q^{-\binom{k}{2}}}{(asq^{-k};q)_\infty}, \tag{28}$$

$$D_a^k\,\{(as;q)_\infty\} = (-s)^k\, q^{\binom{k}{2}}\,(asq^k;q)_\infty, \tag{29}$$

$$\theta_a^k\,\{(as;q)_\infty\} = (-s)^k\,(as;q)_\infty, \tag{30}$$

$$D_a^n\left\{\frac{(as;q)_\infty}{(a\omega;q)_\infty}\right\} = \omega^n\,\frac{\left(\frac{s}{\omega};q\right)_n}{(as;q)_n}\,\frac{(as;q)_\infty}{(a\omega;q)_\infty} \tag{31}$$

and

$$\theta_a^n\left\{\frac{(as;q)_\infty}{(a\omega;q)_\infty}\right\} = \left(-\frac{q}{a}\right)^n\,\frac{\left(\frac{s}{\omega};q\right)_n}{\left(\frac{q}{a\omega};q\right)_n}\,\frac{(as;q)_\infty}{(a\omega;q)_\infty}. \tag{32}$$

We now state and prove the q-difference formulas as Theorem 1 below.

Theorem 1. *Each of the following assertions holds true*:

$$\begin{aligned} &\mathbb{T}(r,f,g,v,w,uD_a)\left\{\frac{(as;q)_\infty}{(az,at;q)_\infty}\right\} \\ &= \frac{(as;q)_\infty}{(az,at;q)_\infty}\sum_{k=0}^{\infty}\frac{\left(r,f,g,\frac{s}{z},at;q\right)_k(zu)^k}{(v,w,as,q;q)_k}\,{}_3\Phi_2\left[\begin{matrix} rq^k,fq^k,gq^k; \\ vq^k,wq^k; \end{matrix}\; q;ut\right] \end{aligned} \tag{33}$$

and

$$\mathbb{E}(r,f,g,v,w,u\theta_a)\left\{\frac{(az,at;q)_\infty}{(as;q)_\infty}\right\}$$
$$=\frac{(az,at;q)_\infty}{(as;q)_\infty}\sum_{k=0}^{\infty}\frac{\left(r,f,g,\frac{z}{s},\frac{q}{at};q\right)_k(-ut)^k}{\left(v,w,\frac{q}{as},q;q\right)_k}\,{}_3\Phi_3\left[\begin{matrix}rq^k,fq^k,gq^k;\\ vq^k,wq^k,0;\end{matrix}\,q;-ut\right],\tag{34}$$

provided that $\max\{|az|,|as|,|at|,|ut|\}<1$.

Proof. By means of the definition (21) of the operator $\mathbb{T}(r,f,g,v,w,uD_a)$ and the Leibniz rule (12), we observe that

$$\mathbb{T}(r,f,g,v,w,uD_a)\left\{\frac{(as;q)_\infty}{(az,at;q)_\infty}\right\}$$
$$=\sum_{n=0}^{\infty}\frac{(r,f,g;q)_nu^n}{(v,w,q;q)_n}D_a^n\left\{\frac{(as;q)_\infty}{(az,at;q)_\infty}\right\}$$
$$=\sum_{n=0}^{\infty}\frac{(r,f,g;q)_nu^n}{(v,w,q;q)_n}\sum_{k=0}^{n}\left[\begin{matrix}n\\k\end{matrix}\right]_q q^{k(k-n)}\,D_a^k\left\{\frac{(as;q)_\infty}{(az;q)_\infty}\right\}$$
$$\cdot D_a^{n-k}\left\{\frac{1}{(atq^k;q)_\infty}\right\}.\tag{35}$$

Now, using the q-identities (27) and (31), we find that

$$\mathbb{T}(r,f,g,v,w,uD_a)\left\{\frac{(as;q)_\infty}{(az,at;q)_\infty}\right\}$$
$$=\sum_{n=0}^{\infty}\frac{(r,f,g;q)_nu^n}{(v,w,q;q)_n}\sum_{k=0}^{n}\left[\begin{matrix}n\\k\end{matrix}\right]_q q^{k(k-n)}\frac{\left(\frac{s}{z};q\right)_k z^k}{(as;q)_k}\frac{(as;q)_\infty}{(az;q)_\infty}\frac{(tq^k)^{n-k}}{(atq^k;q)_\infty}$$
$$=\frac{(as;q)_\infty}{(az,at;q)_\infty}\sum_{k=0}^{\infty}\frac{\left(\frac{s}{z},at;q\right)_k z^k}{(as,q;q)_k}\sum_{n=k}^{\infty}\frac{(r,f,g;q)_n\,u^n\,t^{n-k}}{(v,w;q)_n(q;q)_{n-k}}.\tag{36}$$

Upon setting $n-k=n$, the right-hand side of (36) takes the following form:

$$\frac{(as;q)_\infty}{(az,at;q)_\infty}\sum_{k=0}^{\infty}\frac{\left(r,f,g,\frac{s}{z},at;q\right)_k(uz)^k}{(v,w,as,q;q)_k}\sum_{n=0}^{\infty}\frac{(rq^k,fq^k,gq^k;q)_n(ut)^n}{(vq^k,wq^k,q;q)_n}.\tag{37}$$

Summarizing the above observations (35) to (37), we get the assertion (33) of Theorem 1.

Similarly, by means of the definition (22) of the operator $\mathbb{E}(r,f,g,v,w,u\theta_a)$ and the Leibniz rule (13), we get

$$\mathbb{E}(r,f,g,v,w,u\theta_a)\left\{\frac{(az,at;q)_\infty}{(as;q)_\infty}\right\}$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n\,q^{\binom{n}{2}}(r,f,g;q)_nu^n}{(v,w,q;q)_n}\theta_a^n\left\{\frac{(az,at;q)_\infty}{(as;q)_\infty}\right\}$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n\,q^{\binom{n}{2}}(r,f,g;q)_nu^n}{(v,w,q;q)_n}\sum_{k=0}^{n}\left[\begin{matrix}n\\k\end{matrix}\right]_q\theta_a^k\left\{\frac{(az;q)_\infty}{(as;q)_\infty}\right\}\theta_a^{n-k}\left\{(atq^{-k};q)_\infty\right\},\tag{38}$$

By using the q-identities (30) and (32), we obtain

$$\begin{aligned}
&\mathbb{E}(r,f,g,v,w,u\theta_a)\left\{\frac{(az,at;q)_\infty}{(as;q)_\infty}\right\} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n\, q^{\binom{n}{2}}(r,f,g;q)_n u^n}{(v,w,q;q)_n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{\left(-\frac{q}{a}\right)^k\left(\frac{z}{s};q\right)_k}{\left(\frac{q}{as};q\right)_k} \\
&\quad\cdot\frac{(az;q)_\infty}{(as;q)_\infty}(atq^{-k};q)_\infty\,(-tq^{-k})^{n-k} \\
&= \frac{(az,at;q)_\infty}{(as;q)_\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{-\binom{k}{2}}\left(\frac{z}{s},\frac{q}{at};q\right)_k t^k}{\left(\frac{q}{as},q;q\right)_k} \\
&\quad\cdot\sum_{n=k}^{\infty}\frac{q^{\binom{n}{2}-k(n-k)}(r,f,g;q)_n\, u^n\, t^{n-k}}{(v,w;q)_n(q;q)_{n-k}}.
\end{aligned}\tag{39}$$

Now, if we set $n-k=n$, the right-hand side of (39) becomes

$$\frac{(az,at;q)_\infty}{(as;q)_\infty}\sum_{k=0}^{\infty}\frac{\left(r,f,g,\frac{z}{s},\frac{q}{at};q\right)_k(-ut)^k}{\left(v,w,\frac{q}{as},q;q\right)_k}\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}(rq^k,fq^k,gq^k;q)_n\,(ut)^n}{(vq^k,wq^k,q;q)_n}.\tag{40}$$

Finally, by combining the above observations (38) to (40), we arrive at the assertion (34) of Theorem 1. This evidently completes the proof of Theorem 1. □

We remark that, when $g=w=0$, Theorem 1 reduces to the concluding result of Li and Tan [24].

Corollary 1. *It is asserted that*

$$\mathbb{T}(r,f,g,v,w,uD_s)\left\{\frac{1}{(xs;q)_\infty}\right\}=\frac{1}{(xs;q)_\infty}{}_3\Phi_2\left[\begin{matrix} r,f,g; \\ \\ v,w; \end{matrix}\; q;xu\right]\tag{41}$$

and

$$\mathbb{E}(r,f,g,v,w,-u\theta_s)\left\{(xs;q)_\infty\right\}=(xs;q)_\infty\,{}_3\Phi_3\left[\begin{matrix} r,f,g; \\ \\ v,w,0; \end{matrix}\; q;xu\right],\tag{42}$$

provided that $\max\{|xs|,|xu|\}<1$.

The goal in this paper is to give potentially useful generalizations of a number q-series and q-integral identities such as the q-binomial theorem or the q-Gauss sum, the q-Chu-Vandermonde summation formula and the Andrews-Askey integral.

Our paper is organized as follows. In Section 2, we give two formal generalizations of the q-binomial theorem or the q-Gauss sum by applying the q-difference equations. In Section 3, we derive a set of two extensions q-Chu-Vandermonde summation formulas by making use of the q-difference equations. Next, in Section 4, we derive two new generalizations of the Andrews-Askey integral by means of the q-difference equations. Finally, in our last section (Section 5), we present a number of concluding remarks and observations concerning the various results which we have considered in this investigation.

2. A Set of Formal Generalizations of the q-Binomial Theorem

We begin this section by recalling the following q-binomial theorem (see, for example, [1–3,14]):

$$\sum_{n=0}^{\infty}\frac{(a;q)_n\,x^n}{(q;q)_n}=\frac{(ax;q)_\infty}{(x;q)_\infty}\qquad(|x|<1).\tag{43}$$

In Theorem 2 below, we give two generalizations of the q-binomial theorem (43) by applying the q-difference equations.

Theorem 2. *Each of the following assertions holds true*:

$$\sum_{n=0}^{\infty} \frac{(a;q)_n\, a^{-n}}{(q;q)_n} \sum_{k=0}^{\infty} \frac{(q^{-n},ax;q)_k\, q^k}{(q;q)_k} \sum_{j,i\geqq 0} \frac{(r,f,g;q)_{j+i}\, u^{j+i}}{(q;q)_i\,(v,w;q)_{j+i}} \frac{\left(\frac{c}{b},axq^k;q\right)_j}{(cx,q;q)_j} (aq^k)^i\, b^j$$
$$= \frac{(ax;q)_\infty}{(x;q)_\infty} \sum_{j,i\geqq 0} \frac{(r,f,g;q)_{j+i}\, u^{j+i}}{(q;q)_i\,(v,w;q)_{j+i}} \frac{\left(\frac{c}{b},x;q\right)_j}{(cx,q;q)_j}\, b^j \qquad (|x|<1) \tag{44}$$

and

$$\sum_{n=0}^{\infty} \frac{(a;q)_n\, a^{-n}}{(q;q)_n} \sum_{k=0}^{\infty} \frac{(q^{-n},ax;q)_k\, q^k}{(q;q)_k} \sum_{i,j\geqq 0} \frac{(-1)^{j+i}\, q^{\binom{i}{2}} (r,f,g;q)_{j+i}}{(q;q)_i\,(v,w;q)_{j+i}}$$
$$\cdot \frac{\left(\frac{bq^{1-k}}{a},\frac{q}{cx};q\right)_j}{\left(\frac{q^{1-k}}{ax},q;q\right)_j} (uc)^{j+i}$$
$$= \frac{(ax;q)_\infty}{(x;q)_\infty} \sum_{i,j=0}^{\infty} \frac{(-1)^{j+i}\, q^{\binom{i}{2}}\, (r,f,g;q)_{j+i}}{(q;q)_i(v,w;q)_{j+i}} \frac{\left(\frac{q}{bx};q\right)_j}{\left(\frac{q}{x},q;q\right)_j} (bu)^{j+i}, \tag{45}$$

provided that both sides of (44) *and* (45) *exist.*

Remark 1. *For $u=0$ and by using the fact that*

$$\sum_{k=0}^{\infty} \frac{(q^{-n},ax;q)_k}{(q;q)_k}\, q^k = {}_2\Phi_1 \left[\begin{matrix} q^{-n},ax; & \\ & q;q \\ 0; & \end{matrix} \right] = (ax)^n,$$

the assertions (44) *and* (45) *reduce to* (43).

In our proof of Theorem 2, we shall need Theorem 3 and Corollary 2 below.

Theorem 3. *Each of the following assertions holds true*:

$$\mathbb{T}(r,f,g,v,w,uD_x) \left\{ \frac{p_n\left(x,\frac{y}{a}\right)\, (cx;q)_\infty}{(ax,bx;q)_\infty} \right\}$$
$$= \frac{(y;q)_n}{a^n} \frac{(cx;q)_\infty}{(ax,bx;q)_\infty} \sum_{k=0}^{\infty} \frac{(q^{-n},ax;q)_k\, q^k}{(y,q;q)_k}$$
$$\cdot \sum_{j,i\geqq 0} \frac{(r,f,g;q)_{j+i}\, u^{j+i}}{(q;q)_i(v,w;q)_{j+i}} \frac{\left(\frac{c}{b},axq^k;q\right)_j}{(cx,q;q)_j} (aq^k)^i\, b^j \tag{46}$$

and

$$\begin{aligned}&\mathbb{E}(r,f,g,v,w,u\theta_x)\left\{\frac{p_n\left(x,\frac{y}{a}\right)(bx,cx;q)_\infty}{(ax;q)_\infty}\right\}\\&=\frac{(y;q)_n}{a^n}\frac{(bx,cx;q)_\infty}{(ax;q)_\infty}\sum_{i,j\geqq 0}\frac{(-1)^{j+i}\,q^{\binom{i}{2}}\,(r,f,g;q)_{j+i}}{(q;q)_i\,(v,w;q)_{j+i}}\\&\quad\cdot\frac{\left(\frac{bq^{1-k}}{a},\frac{q}{cx};q\right)_j}{\left(\frac{q^{1-k}}{ax},q;q\right)_j}(uc)^{j+i},\end{aligned}\tag{47}$$

provided that $\max\{|ax|,|bx|,|cx|\}<1$.

Corollary 2. *Each of the following assertions holds true*:

$$\begin{aligned}&\mathbb{T}(r,f,g,v,w,uD_x)\left\{\frac{x^n(cx;q)_\infty}{(ax,bx;q)_\infty}\right\}\\&=\frac{1}{a^n}\frac{(cx;q)_\infty}{(ax,bx;q)_\infty}\sum_{k=0}^{\infty}\frac{(q^{-n},ax;q)_k\,q^k}{(q;q)_k}\\&\quad\cdot\sum_{j,i\geqq 0}\frac{(r,f,g;q)_{j+i}\,u^{j+i}}{(q;q)_i\,(v,w;q)_{j+i}}\frac{\left(\frac{c}{b},axq^k;\right)_j}{(cx,q;q)_j}(aq^k)^i\,b^j\end{aligned}\tag{48}$$

and

$$\begin{aligned}&\mathbb{E}(r,f,g,v,w,u\theta_x)\left\{\frac{x^n(cx,bx;q)_\infty}{(ax;q)_\infty}\right\}\\&=\frac{1}{a^n}\frac{(bx,cx;q)_\infty}{(ax;q)_\infty}\sum_{k=0}^{\infty}\frac{(q^{-n},ax;q)_k\,q^k}{(q;q)_k}\sum_{i,j\geqq 0}\frac{(-1)^{j+i}q^{\binom{i}{2}}\,(r,f,g;q)_{j+i}}{(q;q)_i\,(v,w;q)_{j+i}}\\&\quad\cdot\frac{\left(\frac{bq^{1-k}}{a},\frac{q}{cx};q\right)_j}{\left(\frac{q^{1-k}}{ax},q;q\right)_j}(cu)^{j+i},\end{aligned}\tag{49}$$

provided that $\max\{|ax|,|bx|,|cx|,|cu|\}<1$.

Remark 2. *For* $y=0$*, the assertions* (46) *and* (47) *reduce to* (48) *and* (49)*, respectively.*

Proof of Theorem 3. Upon first setting $x\to ax$ in (55) and then multiplying both sides of the resulting equation by $\frac{(cx;q)_\infty}{(bx;q)_\infty}$, we get

$$\sum_{n=0}^{\infty}\frac{(q^{-n};q)_k\,q^k}{(y,q;q)_k}\frac{(cx;q)_\infty}{(axq^k,bx;q)_\infty}=\frac{(ax)^n\left(\frac{y}{ax};q\right)_n\,(cx;q)_\infty}{(y;q)_n(ax,bx;q)_\infty}.\tag{50}$$

Now, by applying the operator $\mathbb{T}(r,f,g,v,w,uD_x)$ to both sides of (50), it is easy to see that

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(q^{-n};q)_k\,q^k}{(y,q;q)_k}\,\mathbb{T}(r,f,g,v,w,uD_x)\left\{\frac{(cx;q)_\infty}{(axq^k,bx;q)_\infty}\right\}\\ &=\frac{a^n}{(y;q)_n}\,\mathbb{T}(r,f,g,v,w,uD_x)\left\{\frac{x^n\left(\frac{y}{ax};q\right)_n(cx;q)_\infty}{(ax,bx;q)_\infty}\right\}\\ &=\frac{a^n}{(y;q)_n}\,\mathbb{T}(r,f,g,v,w,uD_x)\left\{\frac{p_n\left(x,\frac{y}{a}\right)(cx;q)_\infty}{(ax,bx;q)_\infty}\right\}.\end{aligned}\tag{51}$$

The proof of the first assertion (46) of Theorem 3 is completed by using the relation (33) in the left-hand side of (51).

The proof of the second assertion (47) of Theorem 3 is much akin to that of the first assertion (46). The details involved are, therefore, being omitted here. □

Proof of Theorem 2. Multiplying both sides of (43) by $\frac{(cx;q)_\infty}{(bx;q)_\infty}$, we find that

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\,\frac{x^n\,(cx;q)_\infty}{(ax,bx;q)_\infty}=\frac{(cx;q)_\infty}{(bx,x;q)_\infty}.\tag{52}$$

Equation (44) can be written equivalently as follows:

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\cdot\frac{a^{-n}(cx;q)_\infty}{(bx,ax;q)_\infty}\sum_{k=0}^{\infty}\frac{(q^{-n},ax;q)_k\,q^k}{(q;q)_k}\\ \cdot\sum_{j,i\geqq 0}\frac{(r,f,g;q)_{j+i}\,u^{j+i}}{(q;q)_i(v,w;q)_{j+i}}\,\frac{\left(\frac{c}{b},axq^k;q\right)_j}{(cx,q;q)_j}\,(aq^k)^i\,b^j\\ =\frac{(cx;q)_\infty}{(bx,x;q)_\infty}\sum_{j,i=0}\frac{(r,f,g;q)_{j+i}\,u^{j+i}}{(q;q)_i(v,w;q)_{j+i}}\cdot\frac{\left(\frac{c}{b},x;q\right)_j}{(cx,q;q)_j}\,b^j.\end{aligned}\tag{53}$$

If we use $F(r,f,g,v,w,a,u)$ to denote the right-hand side of (53), it is easy to verify that $F(r,f,g,v,w,a,u)$ satisfies (23). By applying (24), we thus find that

$$\begin{aligned}F(r,f,g,v,w,x,u)&=\mathbb{T}(r,f,g,v,w,uD_x)\Big\{F(r,f,g,v,w,x,0)\Big\}\\ &=\mathbb{T}(r,f,g,v,w,uD_x)\left\{\frac{(cx;q)_\infty}{(bx,x;q)_\infty}\right\}\\ &=\mathbb{T}(r,f,g,v,w,uD_x)\left\{\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\,\frac{x^n\,(cx;q)_\infty}{(ax,bx;q)_\infty}\right\}\\ &=\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\,\mathbb{T}(r,f,g,v,w,uD_x)\left\{\frac{x^n\,(cx;q)_\infty}{(ax,bx;q)_\infty}\right\}.\end{aligned}\tag{54}$$

The proof of the first assertion (44) of Theorem 2 can now be completed by making use of the relation (48).

The proof of the second assertion (45) of Theorem 2 is much akin to that of the first assertion (44). The details involved are, therefore, being omitted here. □

3. Two Generalizations of the q-Chu-Vandermonde Summation Formula

The q-Chu-Vandermonde summation formula is recalled here as follows (see, for example, [1,31]):

$${}_2\Phi_1\left[\begin{matrix}q^{-n},x;\\ y;\end{matrix}\;q;q\right]=\frac{\left(\frac{y}{x};q\right)_n}{(y;q)_n}\,x^n\qquad\left(n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}\right).\tag{55}$$

In this section, we give two generalizations of the q-Chu-Vandermonde summation Formula (55) by applying q-difference equations.

Theorem 4. *The following assertion holds true for $y \neq 0$:*

$$\sum_{k=0}^{n} \frac{(q^{-n},x;q)_k\, q^k}{(q,y;q)_k}\, {}_3\Phi_2 \left[\begin{array}{c} r,f,g; \\ \\ v,w; \end{array} q;uq^k \right]$$
$$= \frac{x^n\left(\frac{y}{x};q\right)_n}{(y;q)_n} \sum_{k,j\geqq 0} \frac{(r,f,g;q)_{k+j}}{(q;q)_j\,(v,w;q)_{k+j}} \frac{\left(\frac{q^{1-n}}{y},\frac{qx}{y};q\right)_k}{\left(\frac{xq^{1-n}}{y},q;q\right)_k} u^{k+j}\left(\frac{q}{y}\right)^j. \tag{56}$$

We next derive another generalization of the q-Chu-Vandermonde summation Formula (55) as follows.

Theorem 5. *For $m \in \mathbb{N}_0$ and $y \neq 0$, it is asserted that*

$${}_2\Phi_1 \left[\begin{array}{c} q^{-n},x; \\ \\ y; \end{array} q;q^{1+m} \right] = \frac{x^n\left(\frac{y}{x};q\right)_n}{(y;q)_n} \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_q \frac{\left(\frac{q^{1-n}}{y},\frac{qx}{y};q\right)_{m-j}}{\left(\frac{xq^{1-n}}{y};q\right)_{m-j}} \left(\frac{q}{y}\right)^j. \tag{57}$$

Remark 3. *For $u = 0$ or $m = 0$, the assertion (56) or (57) reduces to the q-Chu-Vandermonde summation Formula (55). Furthermore, if we first set $i + j = m$ and then extract the coefficients of $\frac{(r,f,g;q)_m}{(v,w;q)_m} u^m$ from the two members of the assertion (56) of Theorem 4, we obtain the transformation formula (57), which leads us to the q-Chu-Vandermonde summation Formula (55) when $m = 0$. Also, upon putting $n = 0$, the assertion (57) reduces to the following identity:*

$$\sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_q \left(\frac{q}{y};q\right)_{m-j} \left(\frac{q}{y}\right)^j = 1 \qquad (y \neq 0). \tag{58}$$

Proof of Theorem 4. We first write (55) in the following form:

$$\sum_{k=0}^{n} \frac{(q^{-n};q)_k\, q^k}{(y,q;q)_k} \frac{1}{(xq^k;q)_\infty} = \frac{(-1)^n\, y^n\, q^{\binom{n}{2}}}{(y;q)_n} \frac{\left(\frac{xq^{1-n}}{y};q\right)_\infty}{\left(x,\frac{qx}{y};q\right)_\infty}. \tag{59}$$

Equation (56) can be written equivalently as follows:

$$\sum_{k=0}^{\infty} \frac{(q^{-n};q)_k\, q^k}{(q,c;q)_k} \cdot \frac{1}{(xq^k;q)_\infty}\, {}_3\Phi_2 \left[\begin{array}{c} r,f,g; \\ \\ v,w; \end{array} q;uq^k \right]$$
$$= \frac{(-1)^n\, y^n\, q^{\binom{n}{2}}}{(y;q)_n} \frac{\left(\frac{xq^{1-n}}{y};q\right)_\infty}{\left(x,\frac{qx}{y};q\right)_\infty} \cdot \sum_{i,j\geqq 0} \frac{(r,f,g;q)_{j+i}\, u^{j+i}}{(q;q)_i\,(v,w;q)_{j+i}} \frac{\left(\frac{q^{1-n}}{y},\frac{qx}{y};q\right)_j}{\left(\frac{xq^{1-n}}{y},q;q\right)_j} \left(\frac{q}{y}\right)^i. \tag{60}$$

If we use $G(r, f, g, v, w, x, u)$ to denote the right-hand side of (60), it is easy to observe that $G(r, f, g, v, w, x, u)$ satisfies (23). By using (24), we obtain

$$\begin{aligned} G(r,f,g,v,w,x,u) &= \mathbb{T}(r,f,g,v,w,uD_x)\Big\{G(r,f,g,v,w,x,0)\Big\} \\ &= \mathbb{T}(r,f,g,v,w,uD_x)\left\{\frac{(-1)^n\, y^n\, q^{\binom{n}{2}}}{(y;q)_n}\,\frac{\left(\frac{xq^{1-n}}{y};q\right)_\infty}{\left(x,\frac{qx}{y};q\right)_\infty}\right\} \\ &= \mathbb{T}(r,f,g,v,w,uD_x)\left\{\sum_{k=0}^{\infty}\frac{(q^{-n};q)_k\, q^k}{(y,q;q)_k}\,\frac{1}{(xq^k;q)_\infty}\right\} \\ &= \sum_{k=0}^{n}\frac{(q^{-n};q)_k\, q^k}{(y,q;q)_k}\,\mathbb{T}(r,f,g,v,w,uD_x)\left\{\frac{1}{(xq^k;q)_\infty}\right\}. \end{aligned} \tag{61}$$

Finally, by using the fact that

$$\mathbb{T}(r,f,g,v,w,uD_x)\left\{\frac{1}{(xq^k;q)_\infty}\right\} = \frac{1}{(xq^k;q)_\infty}\,{}_3\Phi_2\left[\begin{matrix} r,f,g; \\ v,w; \end{matrix}\; q; uq^k\right], \tag{62}$$

and after some simplification involving $\frac{1}{(x;q)_\infty}$, we get the left-hand side of (56). □

4. New Generalizations of the Andrews-Askey Integral

The following famous formula is known as the Andrews-Askey integral (see, for details, [32]). It was derived from Ramanujan's celebrated ${}_1\Psi_1$-summation formula.

Proposition 2. *(see ([32], Equation (2.1))). For* $\max\{|ac|, |ad|, |bc|, |bd|\} < 1$, *it is asserted that*

$$\int_c^d \frac{\left(\frac{qt}{c},\frac{qt}{d};q\right)_\infty}{(at,bt;q)_\infty}\, \mathrm{d}_q t = \frac{d(1-q)\left(q,\frac{dq}{c},\frac{c}{d},abcd;q\right)_\infty}{(ac,ad,bc,bd;q)_\infty}. \tag{63}$$

The Andrews-Askey integral (63) is indeed an important formula in the theory of q-series (see [16]).

Recently, Cao [33] gave the following two generalizations of the Andrews-Askey integral (63) by the method based upon q-difference equations.

Proposition 3. *(see ([33], Theorems 14 and 15)) For* $N \in \mathbb{N}$ *and* $r = q^{-N}$, *suppose that*

$$\max\left\{|ac|, |ad|, |bc|, |bd|, \left|\frac{qwr}{v}\right|, \left|\frac{q}{v}\right|\right\} < 1.$$

Then

$$\begin{aligned} &\int_c^d \frac{\left(\frac{qt}{c},\frac{qt}{d};q\right)_\infty}{(at,bt;q)_\infty}\,{}_4\Phi_2\left[\begin{matrix} r,w,\frac{c}{t},abcd; \\ ac,\frac{qwr}{v}; \end{matrix}\; q; \frac{qt}{vbcd}\right] \mathrm{d}_q t \\ &= \frac{d(1-q)\left(q,\frac{dq}{c},\frac{c}{d},abcd,\frac{qw}{v},\frac{qr}{v};q\right)_\infty}{\left(ac,ad,bc,bd,\frac{qwr}{v},\frac{q}{v};q\right)_\infty}\,{}_2\Phi_1\left[\begin{matrix} w,r; \\ v; \end{matrix}\; q; \frac{q}{bc}\right]. \end{aligned} \tag{64}$$

Furthermore, for $N \in \mathbb{N}$ *and* $r = q^{-N}$, *suppose that*

$$\max\left\{|ac|, |ad|, |bc|, |bd|, \left|\frac{v}{w}\right|, \left|\frac{v}{r}\right|\right\} < 1.$$

Then

$$\int_c^d \frac{\left(\frac{qt}{c},\frac{qt}{d};q\right)_\infty}{(at,bt;q)_\infty}\, {}_4\Phi_2\left[\begin{matrix} r,w,\frac{c}{t},\frac{q}{ad}; \\ \frac{q}{at},\frac{qrw}{v}; \end{matrix}\; q;q\right] d_qt$$
$$= \frac{d(1-q)\left(q,\frac{dq}{c},\frac{c}{d},abcd,\frac{v}{wr},v;q\right)_\infty}{\left(ac,ad,bc,bd,\frac{v}{w},\frac{v}{r};q\right)_\infty}\, {}_2\Phi_1\left[\begin{matrix} w,r; \\ v; \end{matrix}\; q;\frac{vbc}{wr}\right]. \tag{65}$$

In this section, we give the following two generalizations of the Andrews-Askey integral (63) by using the method of q-difference equations.

Theorem 6. *For $M \in \mathbb{N}$ and $r = q^{-M}$, suppose that*

$$\max\left\{|ac|,|ad|,|bc|,|bd|,\left|\frac{q}{bc}\right|\right\} < 1.$$

Then

$$\int_c^d \frac{\left(\frac{qt}{c},\frac{qt}{d};q\right)_\infty}{(at,bt;q)_\infty} \sum_{k=0}^{\infty} \frac{\left(r,f,g,\frac{c}{t},abcd;q\right)_k \left(\frac{qt}{bcd}\right)^k}{(v,w,ac,q;q)_k}\, {}_3\Phi_2\left[\begin{matrix} rq^k,fq^k,gq^k; \\ vq^k,wq^k; \end{matrix}\; q;q\right] \mathrm{d}_qt$$
$$= \frac{d(1-q)\left(q,\frac{dq}{c},\frac{c}{d},abcd;q\right)_\infty}{(ac,ad,bc,bd;q)_\infty}\, {}_3\Phi_2\left[\begin{matrix} r,f,g; \\ v,w; \end{matrix}\; q;\frac{q}{bc}\right]. \tag{66}$$

Theorem 7. *For $M \in \mathbb{N}$ and $r = q^{-M}$, suppose that* $\max\{|ac|,|ad|,|bc|,|bd|\} < 1$. *Then*

$$\int_c^d \frac{\left(\frac{qt}{c},\frac{qt}{d};q\right)_\infty}{(at,bt;q)_\infty} \sum_{k=0}^{\infty} \frac{\left(r,f,g,\frac{c}{t},\frac{q}{ad};q\right)_k \left(\frac{vw}{rfg}\right)^k}{\left(v,w,\frac{q}{at},q;q\right)_k}\, {}_3\Phi_3\left[\begin{matrix} rq^k,fq^k,gq^k; \\ vq^k,wq^k,0; \end{matrix}\; q;-\frac{vw}{rfg}\right] \mathrm{d}_qt$$
$$= \frac{d(1-q)\left(q,\frac{dq}{c},\frac{c}{d},abcd;q\right)_\infty}{(ac,ad,bc,bd;q)_\infty}\, {}_3\Phi_3\left[\begin{matrix} r,f,g; \\ v,w,0; \end{matrix}\; q;\frac{vwbc}{rfg}\right]. \tag{67}$$

Remark 4. *For $r = 1$, both (66) and (67) reduce to (63). Moreover, for $r = q^{-N}$, $g = w = 0$ and $u = \frac{q}{bcd}$, the assertion (66) of Theorem 6 reduces to (64). For $r = q^{-N}$, $g = w = 0$ and $u = \frac{v}{rfbcd}$, the assertion (66) of Theorem 7 reduces to (67).*

Proof of Theorems 6 and 7. Equation (66) can be written equivalently as follows:

$$\int_c^d \frac{\left(\frac{qt}{c},\frac{qt}{d};q\right)_\infty}{(bt;q)_\infty} \cdot \frac{(ac;q)_\infty}{(at,abcd;q)_\infty} \sum_{k=0}^{\infty} \frac{\left(r,f,g,\frac{c}{t},abcd;q\right)_k \left(\frac{qt}{bcd}\right)^k}{(v,w,ac,q;q)_k}$$
$$\cdot\, {}_3\Phi_2\left[\begin{matrix} rq^k,fq^k,gq^k; \\ vq^k,wq^k; \end{matrix}\; q;q\right] \mathrm{d}_qt$$
$$= \frac{d(1-q)\left(q,\frac{dq}{c},\frac{c}{d};q\right)_\infty}{(bc,bd;q)_\infty} \cdot \frac{1}{(ad;q)_\infty}\, {}_3\Phi_2\left[\begin{matrix} r,f,g; \\ v,w; \end{matrix}\; q;\frac{q}{bc}\right]. \tag{68}$$

If we use $H(r,f,g,v,w,a,u)$ to denote the right-hand side of (68), it is easy to see that $H(r,f,g,v,w,a,u)$ satisfies (23) with $u=\frac{q}{bcd}$. By making use of (24), we thus find that

$$\begin{aligned}
H(r,f,g,v,w,a,u) &= \mathbb{T}(r,f,g,v,w,uD_a)\Big\{H(r,f,g,v,w,a,0)\Big\}\\
&= \mathbb{T}\left(r,f,g,v,w,\frac{q}{bcd}\,D_a\right)\Big\{H(1,f,g,v,w,a,u)\Big\}\\
&= \mathbb{T}\left(r,f,g,v,w,\frac{q}{bcd}\,D_a\right)\left\{\frac{d(1-q)\left(q,\frac{dq}{c},\frac{c}{d};q\right)_\infty}{(bc,bd;q)_\infty}\cdot\frac{1}{(ad;q)_\infty}\right\}\\
&= \mathbb{T}\left(r,f,g,v,w,\frac{q}{bcd}\,D_a\right)\left\{\int_c^d\frac{\left(\frac{qt}{c},\frac{qt}{d};q\right)_\infty}{(bt;q)_\infty}\cdot\frac{(ac;q)_\infty}{(at,abcd;q)_\infty}\,\mathrm{d}_qt\right\}\\
&= \int_c^d\frac{\left(\frac{qt}{c},\frac{qt}{d};q\right)_\infty}{(bt;q)_\infty}\cdot\mathbb{T}\left(r,f,g,v,w,\frac{q}{bcd}\,D_a\right)\left\{\frac{(ac;q)_\infty}{(at,abcd;q)_\infty}\right\}\mathrm{d}_qt.
\end{aligned}$$

Now, by applying the fact that

$$\begin{aligned}
&\mathbb{T}\left(r,f,g,v,w,\frac{q}{bcd}\,D_a\right)\left\{\frac{(ac;q)_\infty}{(at,abcd;q)_\infty}\right\}\\
&\quad=\frac{(ac;q)_\infty}{(at,abcd;q)_\infty}\sum_{k=0}^\infty\frac{\left(r,f,g,\frac{c}{t},abcd;q\right)_k\left(\frac{qt}{bcd}\right)^k}{(v,w,ac,q;q)_k}\,{}_3\Phi_2\left[\begin{matrix}rq^k,fq^k,gq^k;\\ vq^k,wq^k;\end{matrix}\;q;q\right],
\end{aligned}$$

we get the left-hand side of (66).

Similarly, Equation (67) can be written equivalently as follows:

$$\begin{aligned}
&\int_c^d\frac{\left(\frac{qt}{c},\frac{qt}{d};q\right)_\infty}{(bt;q)_\infty}\cdot\frac{(ac;q)_\infty}{(at,abcd;q)_\infty}\sum_{k=0}^\infty\frac{\left(r,f,g,\frac{c}{t},\frac{q}{ad};q\right)_k\left(\frac{vw}{rfg}\right)^k}{\left(v,w,\frac{q}{at},q;q\right)_k}\\
&\qquad\cdot{}_3\Phi_3\left[\begin{matrix}rq^k,fq^k,gq^k;\\ vq^k,wq^k,0;\end{matrix}\;q;-\frac{vw}{rfg}\right]\mathrm{d}_qt\\
&\quad=\frac{d(1-q)\left(q,\frac{dq}{c},\frac{c}{d};q\right)_\infty}{(bc,bd;q)_\infty}\cdot\frac{1}{(ad;q)_\infty}\,{}_3\Phi_3\left[\begin{matrix}r,f,g;\\ v,w,0;\end{matrix}\;q;\frac{vwbc}{rfg}\right].
\end{aligned}\tag{69}$$

The proof of the assertion (67) of Theorem 7 is much akin to that of the assertion (66) of Theorem 6 by using Proposition 1 (II). The analogous details involved are, therefore, being omitted here.

The proofs of Theorems 6 and 7 are thus completed. □

5. Concluding Remarks and Observations

In our present investigation, we have introduced a set of two q-operators $\mathbb{T}(a,b,c,d,e,yD_x)$ and $\mathbb{E}(a,b,c,d,e,y\theta_x)$ with to applying them to derive two potentially useful generalizations of the q-binomial theorem, two extensions of the q-Chu-Vandermonde summation formula and two new generalizations of the Andrews-Askey integral by means of the q-difference equations. We have also briefly described relevant connections of various special cases and consequences of our main results with several known results.

It is believed that the q-series and q-integral identities, which we have presented in this paper, as well as the various related recent works cited here, will provide encouragement and motivation for further researches on the topics that are dealt with and investigated in this paper.

Just as we mentioned in Section 1 above, basic (or q-) series and basic (or q-) polynomials, especially the basic (or q-) hypergeometric functions and basic (or q-) hypergeometric polynomials, are applicable particularly in several diverse areas [see also ([3], pp. 350–351)]. In particular, the celebrated

Chu-Vandermonde summation theorem and its known q-extensions, which we have considered in this paper, have already been demonstrated to be useful (see, for details, [1,14,31]). The new q-Chu-Vandermone summation theorems, which we have presented in this paper, are believed to be useful as well.

In conclusion, we find it to be worthwhile to remark that some potential further applications of the methodology and findings, which we have presented here by means of the q-analysis and the q-calculus, can be found in the study of the zeta and q-zeta functions as well as their related functions of Analytic Number Theory (see, for example, [34,35]; see also [14]) and also in the study of analytic and univalent functions of Geometric Function Theory *via* number-theoretic objects (see, for example, [36]).

Author Contributions: Conceptualization, S.A., H.M.S. and J.C.; Formal analysis, H.M.S. and S.A.; Investigation, S.A. and J.C.; Methodology, S.A., H.M.S. and J.C.; Validation, S.A. and H.M.S.; Visualization, S.A.; Writing—Review and Editing, H.M.S. and S.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare that they have no conflict of interest.

References

1. Gasper, G.; Rahman, M. *Basic Hypergeometric Series (with a Foreword by Richard Askey), Encyclopedia of Mathematics and Its Applications*, 2nd ed.; Cambridge University Press: Cambridge, UK; London, UK; New York, NY, USA, 2004; Volume 96.
2. Slater, L.J. *Generalized Hypergeometric Functions*; Cambridge University Press: Cambridge, UK; London, UK; New York, NY, USA, 1966.
3. Srivastava, H.M.; Karlsson, P.W. *Multiple Gaussian Hypergeometric Series*; Ellis Horwood Limited: Chichester, UK, 1985.
4. Koekock, R.; Swarttouw, R.F. *The Askey-Scheme of Hypergeometric Orthogonal Polynomials and its q-Analogue*; Report No. 98-17; Delft University of Technology: Delft, The Netherlands, 1998.
5. Srivastava, H.M. Certain q-polynomial expansions for functions of several variables. I. *IMA J. Appl. Math.* **1983**, *30*, 315–323. [CrossRef]
6. Srivastava, H.M. Certain q-polynomial expansions for functions of several variables. II. *IMA J. Appl. Math.* **1984**, *33*, 205–209. [CrossRef]
7. Srivastava, H.M. Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis. *Iran. J. Sci. Technol. Trans. A Sci.* **2020**, *44*, 327–344. [CrossRef]
8. Morozov, V.N. Penetration of nonstationary ionospheric electric fields into lower atmospheric layers in the global electric circuit model. *Geomagnet. Aeronom.* **2018**, *58*, 113–118. [CrossRef]
9. Suzuki, A.T. Correspondence between the one-loop three-point vertex and the Y and Δ electric resistor networks. *Canad. J. Phys.* **2014**, *92*, 131–135. [CrossRef]
10. Di Barba, P.; Fattorusso, L.; Versaci, M. Electrostatic field in terms of geometric curvature in membrane MEMS devices. *Commun. Appl. Industr. Math.* **2017**, *8*, 165–184. [CrossRef]
11. Włodarczyk, M.; Zawadzki, A. The application of hypergeometric functions to computing fractional order derivatives of sinusoidal functions. *Bull. Polish Acad. Sci. Tech. Sci.* **2016**, *64*, 243–248. [CrossRef]
12. Chen, W.Y.C.; Fu, A.M.; Zhang, B. The homogeneous q-difference operator. *Adv. Appl. Math.* **2003**, *31*, 659–668. [CrossRef]
13. Cao, J.; Srivastava, H.M. Some q-generating functions of the Carlitz and Srivastava-Agarwal types associated with the generalized Hahn polynomials and the generalized Rogers-Szegö polynomials. *Appl. Math. Comput.* **2013**, *219*, 8398–8406. [CrossRef]
14. Srivastava, H.M.; Choi, J. *Zeta and q-Zeta Functions and Associated Series and Integrals*; Elsevier Science Publishers: Amsterdam, The Netherlands; London, UK; New York, NY, USA, 2012.
15. Srivastava, H.M.; Chaudhary, M.P.; Wakene, F.K. A family of theta-function identities based upon q-binomial theorem and Heine's transformations. *Montes Taurus J. Pure Appl. Math.* **2020**, *8*, 918.
16. Chen, W.Y.C.; Liu, Z.-G. Parameter augmenting for basic hypergeometric series. II. *J. Combin. Theory Ser. A* **1997**, *80*, 175–195. [CrossRef]
17. Saad, H.L.; Sukhi, A.A. Another homogeneous q-difference operator. *Appl. Math. Comput.* **2010**, *215*, 4332–4339. [CrossRef]

18. Srivastava, H.M.; Abdlhusein, M.A. New forms of the Cauchy operator and some of their applications. *Russ. J. Math. Phys.* **2016**, *23*, 124–134. [CrossRef]
19. Roman, S. The theory of the umbral calculus. I. *J. Math. Anal. Appl.* **1982**, *87*, 58–115. [CrossRef]
20. Chen, W.Y.C.; Liu, Z.-G. Parameter augmentation for basic hypergeometric series. I. In *Mathematical Essays in Honor of Gian-Carlo Rota*; Sagan, B.E., Stanley, R.P., Eds.; Birkäuser: Basel, Switzerland; New York, NY, USA, 1998; pp. 111–129.
21. Chen, V.Y.B.; Gu, N.S.S. The Cauchy operator for basic hypergeometric series. *Adv. Appl. Math.* **2008**, *41*, 177–196. [CrossRef]
22. Fang, J.P. Some applications of q-differential operator. *J. Korean Math. Soc.* **2010**, *47*, 223–233. [CrossRef]
23. Zhang, Z.Z.; Wang, J.Z. Finite q-exponential operators with two parameters and their applications. *Acta Math. Sinica* **2010**, *53*, 1007–1018. (In Chinese)
24. Li, N.N.; Tan, W. Two generalized q-exponential operators and their applications. *Adv. Differ. Equ.* **2016**, *2016*, 53. [CrossRef]
25. Cao, J.; Xu, B.; Arjika, S. A note on generalized q-difference equations for general Al-Salam-Carlitz polynomials. *J. Differ. Equ. Appl.* **2020**, submitted.
26. Arjika, S. q-difference equations for homogeneous q-difference operators and their applications. *J. Differ. Equ. Appl.* **2020**, *26*, 987–999. [CrossRef]
27. Srivastava, H.M.; Arjika, S. Generating functions for some families of the generalized Al-Salam-Carlitz q-polynomials. *Adv. Differ. Equ.* **2020**, *2020*, 498. [CrossRef]
28. Srivastava, H.M.; Arjika, S.; Kelil, A.S. Some homogeneous q-difference operators and the associated generalized Hahn polynomials. *Appl. Set Valued Anal. Optim.* **2019**, *1*, 187–201.
29. Liu, Z.-G. Two q-difference equations and q-operator identities. *J. Differ. Equ. Appl.* **2010**, *16*, 1293–1307. [CrossRef]
30. Liu, Z.-G. An extension of the non-terminating ${}_6\phi_5$-summation and the Askey-Wilson polynomials. *J. Differ. Equ. Appl.* **2011**, *17*, 1401–1411. [CrossRef]
31. Andrews, G.E. *q-Series: Their Development and Applications in Analysis, Number Theory, Combinatorics, Physics and Computer Algebra*; CBMS Regional Conference Lecture Series; American Mathematical Society: Ann Arbor, MI, USA, 1986; Volume 66.
32. Andrews, G.E.; Askey, R. Another q-extension of the beta function. *Proc. Am. Math. Soc.* **1981**, *81*, 97–100.
33. Cao, J. A note on generalized q-difference equations for q-beta and Andrews-Askey integral. *J. Math. Anal. Appl.* **2014**, *412*, 841–851. [CrossRef]
34. Srivastava, H.M. The Zeta and related functions: Recent developments. *J. Adv. Engrg. Comput.* **2019**, *3*, 329–354. [CrossRef]
35. Srivastava, H.M. Some general families of the Hurwitz-Lerch Zeta functions and their applications: Recent developments and directions for further researches. *Proc. Inst. Math. Mech. Nat. Acad. Sci. Azerbaijan* **2019**, *45*, 234–269. [CrossRef]
36. Shafiq, M.; Srivastava, H.M.; Khan, N.; Ahmad, Q.Z.; Darus, M.; Kiran, S. An upper bound of the third Hankel determinant for a subclass of q-starlike functions associated with k-Fibonacci numbers. *Symmetry* **2020**, *12*, 1043. [CrossRef]

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

© 2020 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).