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Some Geometric Properties of a Family of Analytic Functions Involving a Generalized q-Operator

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Abstract: In analysis, the introduction of q-calculus has been a revelation. It has a deep impact on various concepts and applications of pure and applied sciences. In this article we investigate certain geometric properties relating to convolution of functions of a newly defined class of analytic functions. The important region of the lemniscate of Bernoulli is considered. Here we utilize concepts of q-calculus which enhances and generalizes the vitality of this research work. In the same context we study the Fekete–Szegö problem.

Keywords: analytic functions; subordinations; integral operator; lemniscate of Bernoulli

MSC: 30C45; 30C50

1. Introduction and Definitions

The generalization of calculus to q-calculus has impacted several branches of mathematics and physics due to its applications to various concepts. One such example would be its utilization in optimal control problems. Other areas such as ordinary fractional calculus, quantum physics and operator theory are important in this regard. For more details, see, e.g., [1,2]. This idea was first introduced by Jackson [3,4] by giving some applications of this field and introduced the q-derivative and q-integral. The work of Srivastava and Bansal [5], p. 62, which in the introduction contains a description of the q-analogue of derivative in the field of Geometric function theory is worth mentioning. They defined the family $\mathcal{S}_q^*$ which is the q-analogue of starlike functions and studied some of their useful geometric properties. For more details see [6] (p. 347 *et seq.*). Later on, the q-analogue of starlike functions was further generalized by Agrawal and Sahoo in [7] by introducing the family $\mathcal{S}_q^*(\gamma)$ with order γ $(0 \leq \gamma < 1)$. In 2014, Kanas and Răducanu [8] defined q-analogue of Ruscheweyh differential operator using the ideas of convolution and then studied some of its properties. Another source of information is [9]. In the same way many mathematicians explored this field and wrote some valuable articles which played important role in developing the field of Geometric function theory, for instance see the references [10–14]. The current article introduces a class of analytic functions with help of a generalized integral operator and discusses some useful convolution properties for this family in the lemniscate of Bernoulli domain. We start by giving some preliminaries for a better understanding of the research work to follow.

Let $\mathcal{A}$ represent the family of functions f that are analytic in the open unit disc $\mathcal{D} = \{z \in \mathbb{C} : |z| < 1\}$ and which have the following normalization

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ \ (z \in \mathcal{D}). \tag{1}$$

Let $\mathcal{S}$ denote the subclass of class $\mathcal{A}$ consisting of univalent functions. Let

$$\mathcal{S}^*(\varphi) = \{f \in \mathcal{S} : zf'(z)/f(z) \prec \varphi(z)\},$$

and

$$\mathcal{C}(\varphi) = \left\{f \in \mathcal{S} : (zf'(z))' / f(z) \prec \varphi(z)\right\},$$

where $\varphi(z) = \dfrac{1+z}{1-z}$ and $\prec$ refers to subordination.

These classes were introduced and studied by Ma and Minda [15]. They also obtained the Fekete–Szegö inequality for the class $\mathcal{C}(\varphi)$. Using the Alexander relation i.e., $f \in \mathcal{C}(\varphi)$ if and only if $zf' \in \mathcal{S}^*(\varphi)$, they evaluated the Fekete–Szegö inequality for functions in the class $\mathcal{S}^*(\varphi)$. The Fekete–Szegö problem for different classes is studied by Ravichandran et al., in [16–18] and by Shanmugam et al., in [19,20]. For a brief discussion of the Fekete–Szegö problem for the class of starlike, convex and close-to-convex functions see Srivastava and Owa [21].

Let $\mathcal{P}$ denote the class of analytic function $q(z)$ normalized by

$$q(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \tag{2}$$

such that $\Re q(z) > 0$.

For two functions f and g that are analytic in $\mathcal{D}$ and have the form in Equation (1). Of course, $f(z)$ has the coefficients a_n and $g(z)$ has the coefficients b_n, for $n \geq 2$, the convolution of these functions is defined as

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad (z \in \mathcal{D}).$$

One interesting subclass of analytic functions $f(z)$, which is defined as

$$f(z) \prec \sqrt{1+z},$$

known to be the functions in domain of the lemniscate of Bernoulli. The geometrical representation of such functions is that they lie in the region bounded by the right-half of the lemniscate of Bernoulli. Such functions satisfies the inequality

$$\left|(f(z))^2 - 1\right| < 1.$$

The class $\mathcal{SL}^*$ for analytic functions is defined as

$$\mathcal{SL}^* = \left\{f(z) \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \sqrt{1+z},\ \{z \in \mathbb{C} : |z| < 1\}\right\},$$

was defined and studied by Sokol and Stankiewicz [22]. Further improved work in this field were done by different authors in [23–25]. The Coefficient estimates of this class were evaluated in [26].

For $0 < q < 1$, the q-derivative of a function f is defined by

$$\partial_q f(z) = \frac{f(qz) - f(z)}{z(q-1)}, \quad (z \neq 0,\ q \neq 1).$$

It can easily be seen that for $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$ and $z \in \mathcal{D}$

$$\partial_q \left\{\sum_{n=1}^{\infty} a_n z^n\right\} = \sum_{n=1}^{\infty} [n, q]\, a_n z^{n-1},$$

where

$$[n,q]=\frac{1-q^n}{1-q}=1+\sum_{l=1}^{n-1}q^l,\ [0,q]=0.$$

For any non-negative integer n the q-number shift factorial is defined by

$$[n,q]!=\begin{cases}1,\ n=0,\\ [1,q]\,[2,q]\,[3,q]\cdots[n,q]\,,n\in\mathbb{N}.\end{cases}$$

Also the q-generalized Pochhammer symbol for $x>0$ is given by

$$[x,q]_m=\begin{cases}1,\ m=0,\\ [x,q][x+1,q]\cdots[x+m-1,q],\ m\in\mathbb{N},\end{cases}$$

and for $x>0$, let q-gamma function is defined as

$$\Gamma_q\,(x+1)=[x,q]\,\Gamma_q\,(t)\ \text{ and }\Gamma_q\,(1)=1.$$

Now for $\mu>-1$ we define the function $\mathcal{F}^{-1}_{q,\mu+1}(z)$ by

$$\mathcal{F}^{-1}_{q,\mu+1}(z)*\mathcal{F}_{q,\mu+1}(z)=z\partial_q f(z),$$

where the function $\mathcal{F}_{q,\mu+1}(z)$ is given by

$$\mathcal{F}_{q,\mu+1}(z)=z+\sum_{n=2}^{\infty}\frac{[\mu+1,q]_{n-1}}{[n-1,q]!}z^n,\ \ (z\in\mathcal{D})\,. \tag{3}$$

Clearly, the series defined by Equation (3) is absolutely convergent in $\mathcal{D}$. Using the notion of q-derivative along with the idea of convolution we now define the q-integral operator $\mathcal{I}^{\mu}_q:\mathcal{A}\to\mathcal{A}$ by the equality

$$\mathcal{I}^{\mu}_q f(z)=\mathcal{F}^{-1}_{q,\mu+1}(z)*f(z)=z+\sum_{n=2}^{\infty}\psi_{n-1}a_nz^n,(z\in\mathcal{D}) \tag{4}$$

where $f\in\mathcal{A}$ and ψ_{n-1} is given by

$$\psi_{n-1}=\frac{[n,q]!}{[\mu+1,q]_{n-1}}.$$

From Equation (4) the following identity is easily achieved

$$[\mu+1,q]\mathcal{I}^{\mu}_q f(z)=[\mu,q]\mathcal{I}^{\mu+1}_q f(z)+q^{\mu}z\partial_q\mathcal{I}^{\mu+1}_q f(z).$$

We note that

$$\mathcal{I}^{0}_q f(z)=z\partial_q f(z)\text{ and }\mathcal{I}^{1}_q f(z)=f(z)$$

and

$$\lim_{q\to1^-}\mathcal{I}^{\mu}_q f(z)=z+\sum_{n=2}^{\infty}\frac{n}{(\mu+1)_{n-1}}a_nz^n.$$

By taking $q\to1^-$, the operator defined in Equation (4) reduces to the familiar differential operator introduced in [27], see also [28]. For more details on the q-analogue of differential and integral operator see work in [29–31].

Motivated by the work studied in [8,29,31,32], we now define a subfamily $\mathcal{SL}^*\,(\mu,q)$ of $\mathcal{A}$ by using the operator $\mathcal{I}^{\mu}_q$ as follows.

Definition 1. *Let $q \in (0,1)$, then a function $f \in \mathcal{A}$ belongs to the class $\mathcal{SL}^*(\mu,q)$ if it satisfies*

$$\frac{z\partial_q \mathcal{I}_q^\mu f(z)}{\mathcal{I}_q^\mu f(z)} \prec \sqrt{1+z},\ (z \in \mathcal{D}), \tag{5}$$

where the notion " $\prec$ " denotes subordination.

Equivalently, a function $f \in \mathcal{A}$ is in the class $\mathcal{SL}^*(\mu,q)$ if and only if

$$\left| \left(\frac{z\partial_q \mathcal{I}_q^\mu f(z)}{\mathcal{I}_q^\mu f(z)} \right)^2 - 1 \right| < 1,\ (z \in \mathcal{D}). \tag{6}$$

Note that $\mathcal{SL}^*(0, \lim q \to 1^-) = \mathcal{SL}^*$, studied by Sokól [22].

2. Auxiliary Lemmas

In this section we give two important lemmas proved by Ma and Minda, see [15] for details. These results are used in our main results in the next sections.

Lemma 1. *If $q(z) \in \mathcal{P}$ is of the form in Equation (2), then*

$$\left|c_2 - \nu c_1^2\right| \le \begin{cases} -4\nu + 2 & (\nu \le 0), \\ 2 & (0 \le \nu \le 1) \\ 4\nu - 2 & (\nu \ge 1). \end{cases}$$

When $\nu < 0$ or $\nu > 0$, equality holds if and only if $q(z)$ is $(1+z)/(1-z)$ or one of its rotations. If $0 < \nu < 1$, then equality holds if and only if $q(z)$ is $(1+z^2)/(1-z^2)$ or one of its rotations. If $\nu = 0$, equality holds if and only if

$$q(z) = \left(\frac{1+\eta}{2}\right)\left(\frac{1+z}{1-z}\right) + \left(\frac{1-\eta}{2}\right)\left(\frac{1-z}{1+z}\right),\ 0 \le \eta \le 1,\ z \in \mathcal{D},$$

or one of its rotations. While for $\nu = 1$, equality holds if and only if $q(z)$ is the reciprocal of one of the functions such that equality holds in the case of $\nu = 0$. Although the above upper bound is sharp, it can be improved as follows when $0 < \nu < 1$;

$$\left|c_2 - \nu c_1^2\right| + \nu\left|c_1\right|^2 \le 2,\ \ 0 < \nu \le \frac{1}{2}$$

and

$$\left|c_2 - \nu c_1^2\right| + (1-\nu)\left|c_1\right|^2 \le 2,\ \ \frac{1}{2} \le \nu < 1.$$

Lemma 2. *If $q(z) \in \mathcal{P}$ is of the form in Equation (2), then*

$$\left|c_2 - \nu c_1^2\right| \le 2\ \max\{1; |2\nu - 1|\},\ \nu \in \mathbb{C},$$

and the result is sharp for the functions given by

$$q(z) = \frac{1+z^2}{1-z^2}\ \ \&\ \ q(z) = \frac{1+z}{1-z}.$$

3. Main Results

In this section we investigate this newly defined class by evaluating some of its nice properties like convolution property, sufficiency condition and integral representation.

Theorem 1. *Let $f \in \mathcal{A}$ be given by Equation (1). Then the function f belongs to the class $\mathcal{SL}^*(\mu, q)$ if and only if*

$$\frac{1}{z}\left[\mathcal{I}_q^{\mu} f(z) * \frac{Mz - Lqz^2}{(1-z)(1-qz)}\right] \neq 0, \quad (z \in \mathcal{D}), \tag{7}$$

where

$$L = L_\theta = (1+e^{i\theta})^{\frac{1}{2}}, \ \ M = M_\theta = 1 - (1+e^{i\theta})^{\frac{1}{2}}. \tag{8}$$

for $0 \leq \theta \leq 2\pi$ and also for $L = M = 1$.

Proof. Since $f \in \mathcal{SL}^*(\mu, q)$ is analytic in $\mathcal{D}$, it follows that $\frac{1}{z}\mathcal{I}_q^{\mu} f(z) \neq 0$ for all z in $\mathcal{D}^* = \mathcal{D} - \{0\}$ and is equivalent to Equation (7) for $L = M = 1$. According to the definition of subordination and by using Equation (5), there exists a Schwartz function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$, such that

$$\frac{z\partial_q \mathcal{I}_q^{\mu} f(z)}{\mathcal{I}_q^{\mu} f(z)} = \sqrt{1 + w(z)}, \ \ (z \in \mathcal{D}),$$

which is equivalent to

$$\frac{z\partial_q \mathcal{I}_q^{\mu} f(z)}{\mathcal{I}_q^{\mu} f(z)} \neq \sqrt{1 + e^{i\theta}}$$

$$\Rightarrow z\partial_q \mathcal{I}_q^{\mu} f(z) - \sqrt{1 + e^{i\theta}}\mathcal{I}_q^{\mu} f(z) \neq 0. \tag{9}$$

Now using the following basic convolution properties in Equation (9)

$$\mathcal{I}_q^{\mu} f(z) * \frac{z}{1-z} = \mathcal{I}_q^{\mu} f(z) \text{ and } \mathcal{I}_q^{\mu} f(z) * \frac{z}{(1-z)(1-qz)} = z\partial_q \mathcal{I}_q^{\mu} f(z),$$

we get

$$\mathcal{I}_q^{\mu} f(z) * \left(\frac{z}{(1-z)(1-qz)} - \frac{z\sqrt{1+e^{i\theta}}}{(1-z)}\right) \neq 0,$$

which gives

$$\mathcal{I}_q^{\mu} f(z) * \left(\frac{\left(1 - \sqrt{1+e^{i\theta}}\right) z \ + \ qz^2\sqrt{1+e^{i\theta}}}{(1-z)(1-qz)}\right) \neq 0,$$

$$\Rightarrow \mathcal{I}_q^{\mu} f(z) * \left(\frac{Mz \ + \ Lqz^2}{(1-z)(1-qz)}\right) \neq 0,$$

where $M = 1 - \sqrt{1+e^{i\theta}}$ and $L = \sqrt{1+e^{i\theta}}$.

Thus the necessary condition

$$\frac{1}{z}\left[\mathcal{I}_q^{\mu} f(z) * \left(\frac{Mz \ + \ Lqz^2}{(1-z)(1-qz)}\right)\right] \neq 0$$

follows.

Conversely, suppose that the condition in Equation (7) hold for $L = M = 1$, it follows that $\frac{1}{z}\mathcal{I}_q^{\mu} f(z) \neq 0$ for all $z \in \mathcal{D}$. Thus the function $h(z) = \frac{z\partial_q \mathcal{I}_q^{\mu} f(z)}{\mathcal{I}_q^{\mu} f(z)}$ is analytic in $\mathcal{D}$ and $h(0) = 1$. Since we have

$$\begin{aligned}
\frac{z\partial_q \mathcal{I}_q^{\mu} f(z)}{\mathcal{I}_q^{\mu} f(z)} &= \frac{z + \sum_{n=2}^{\infty} \psi_{n-1}[n,q]\, a_n\, z^n}{z + \sum_{n=2}^{\infty} \psi_{n-1}\, a_n\, z^n}, \\
\left[\frac{z\partial_q \mathcal{I}_q^{\mu} f(z)}{\mathcal{I}_q^{\mu} f(z)}\right]_{z=0} &= \left[\frac{1 + \sum_{n=2}^{\infty} \psi_{n-1}[n,q]\, a_n\, z^{n-1}}{1 + \sum_{n=2}^{\infty} \psi_{n-1}\, a_n\, z^{n-1}}\right]_{z=0} = 1, \\
\frac{z\partial_q \mathcal{I}_q^{\mu} f(z)}{\mathcal{I}_q^{\mu} f(z)} &\neq \sqrt{1+e^{i\theta}}. \qquad (10)
\end{aligned}$$

Suppose that

$$\mathcal{H}(z) = \sqrt{1+z},\ z \in \mathcal{D}.$$

Now from the relation in Equation (10) it is clear that $\mathcal{H}(\partial\mathcal{D}) \cap h(\mathcal{D}) = \emptyset$. Therefore the simply connected domain $h(\mathcal{D})$ is contained in connected component of $\mathbb{C}\backslash\mathcal{H}(\partial\mathcal{D})$. The univalence of "$h$" together with the fact $\mathcal{H}(0) = h(0) = 1$ shows that $h \prec \mathcal{H}$ which in turn implies that the function f belongs to $\mathcal{SL}(\mu, q)$. □

Theorem 2. *Let $f \in \mathcal{A}$ be given by Equation (1). Then a necessary and sufficient condition for $f \in \mathcal{SL}^*(\mu, q)$ is that*

$$1 - \sum_{n=2}^{\infty} \psi_{n-1} \left\{ [n,q] \left(\frac{L}{M} - 1 \right) - \frac{L}{M} \right\} a_n z^{n-1} \neq 0,\ z \in \mathcal{D},$$

where L and M are defined by Equation (8) above.

Proof. From the above Theorem 1, f is in the class $\mathcal{SL}^*(\mu, q)$ if and only if

$$\frac{1}{z}\left[\mathcal{I}_q^{\mu} f(z) * \frac{Mz - Lqz^2}{(1-z)(1-qz)}\right] \neq 0,$$

which is equivalent to

$$\frac{1}{z}\left[\mathcal{I}_q^{\mu} f(z) * \frac{Mz}{(1-z)(1-qz)} - \mathcal{I}_q^{\mu} f(z) * \frac{Lqz^2}{(1-z)(1-qz)}\right] \neq 0. \qquad (11)$$

By using convolution properties we have

$$\mathcal{I}_q^{\mu} f(z) * \frac{z}{(1-z)(1-qz)} = z\partial_q \mathcal{I}_q^{\mu} f(z) \text{ and } \mathcal{I}_q^{\mu} f(z) * \frac{z}{(1-z)} = \mathcal{I}_q^{\mu} f(z),$$

and so from Equation (11) we deduce

$$\frac{1}{z}\left[Mz\partial_q \mathcal{I}_q^{\mu} f(z) - L\left\{z\partial_q \mathcal{I}_q^{\mu} f(z) - \mathcal{I}_q^{\mu} f(z)\right\}\right] \neq 0.$$

Now using

$$\mathcal{I}_q^{\mu} f(z) = z + \sum_{n=2}^{\infty} \psi_{n-1} a_n z^n \text{ and } z\partial_q \mathcal{I}_q^{\mu} f(z) = z + \sum_{n=2}^{\infty} \psi_{n-1}[n,q] a_n z^n,$$

we have

$$\frac{1}{z}\left[M\left(z + \sum_{n=2}^{\infty} \psi_{n-1}[n,q] a_n z^n\right) - \right.$$

$$L\left(z+\sum_{n=2}^{\infty}\psi_{n-1}[n,q]a_nz^n-z-\sum_{n=2}^{\infty}\psi_{n-1}a_nz^n\right)\Bigg]\neq 0,$$

which implies that

$$\frac{1}{z}\left[Mz-\sum_{n=2}^{\infty}\psi_{n-1}\left\{[n,q](L-M)-L\right\}a_nz^n\right]\neq 0.$$

After some simplifications it is easily seen that the latter condition is equivalent to

$$1-\sum_{n=2}^{\infty}\psi_{n-1}\left\{[n,q]\left(\frac{L}{M}-1\right)-\frac{L}{M}\right\}a_nz^{n-1}\neq 0,$$

and hence the result follows. □

Theorem 3. *Let $f\in\mathcal{SL}^*(\mu,q)$. Then*

$$\mathcal{I}_q^{\mu}f(z)=\exp\int_0^z\frac{1}{t}\left(\varphi(t)+1\right)^{\frac{1}{2}}d_qt,$$

with $|\varphi(z)|<1$ and $z\in\mathcal{D}$.

Proof. Let $f\in\mathcal{SL}^*(\mu,q)$ and for simplicity take

$$\frac{z\partial_q\mathcal{I}_q^{\mu}f(z)}{\mathcal{I}_q^{\mu}f(z)}=u(z),$$

and hence

$$u(z)\prec\sqrt{1+z}.$$

Equivalently, we can write

$$\left|(u(z))^2-1\right|<1.$$

But on the other hand we also have

$$(u(z))^2-1=\varphi(z),\ \text{where}\ |\varphi(z)|<1\ \text{for}\ z\in\mathcal{D}.$$

Thus we can rewrite

$$\frac{z\partial_q\mathcal{I}_q^{\mu}f(z)}{\mathcal{I}_q^{\mu}f(z)}=\left(\varphi(z)+1\right)^{\frac{1}{2}},$$

and further by simple computation of integration, the proof is completed. □

4. Fekete–Szegö Problem

Fekete–Szegö inequality, for the class $\mathcal{P}$ has already been discussed earlier in the second section, has a key role in determining the the third Hankel determinant for the coefficients of functions belonging to various important classes e.g., see [14]. For many subclasses this problem has been investigated by various authors. Ma and Minda obtained the Fekete–Szegö inequality for the class $\mathcal{C}(\varphi)$ [15]. The work of Ravichandran and Shanmugam et al. [16–20] is worth mentioning in this regard. While a brief discussion of Fekete–Szegö problem for class of starlike, convex and close-to-convex functions were carried out by Srivastava and Owa [21]. In this section we evaluate the Fekete–Szegö inequality for our newly defined class.

Theorem 4. *Let $f(z) \in \mathcal{SL}^*(\mu, q)$ be of the form in Equation (1). Then*

$$\left|a_3 - \lambda a_2^2\right| \leq \begin{cases} \frac{-\psi_1^2(q-2)-2\lambda q(1+q)\psi_2}{8q^2(1+q)\psi_1^2\psi_2}, & \text{for} \quad \lambda < -\frac{\psi_1^2(5q-2)}{2q(1+q)\psi_2} \\ \frac{1}{2q(1+q)\psi_2}, & \text{for} \quad -\frac{\psi_1^2(5q-2)}{2q(1+q)\psi_2} \leq \lambda \leq \frac{\psi_1^2(3q+2)}{2q(1+q)\psi_2} \\ \frac{\psi_1^2(q-2)+2\lambda q(1+q)\psi_2}{8q^2(1+q)\psi_1^2\psi_2}, & \text{for} \quad \lambda > \frac{\psi_1^2(3q+2)}{2q(1+q)\psi_2}. \end{cases}$$

Proof. If $f(z) \in \mathcal{SL}^*(\mu, q)$, then by using Equation (5), it follows that

$$\frac{z\partial_q \mathcal{I}_q^\mu f(z)}{\mathcal{I}_q^\mu f(z)} \prec \Phi(z), \quad z \in \mathcal{D}, \tag{12}$$

where $\Phi(z) = \sqrt{1+z}$. From Equation (12), we have

$$\frac{z\partial_q \mathcal{I}_q^\mu f(z)}{\mathcal{I}_q^\mu f(z)} = \Phi(w(z)),$$

where $w(z)$ is the Schwartz function corresponding to subordination and

$$\Phi(w(z)) = \sqrt{\frac{2p(z)}{p(z)+1}}.$$

Now

$$\begin{aligned} \sqrt{\frac{2p(z)}{p(z)+1}} &= 1 + \frac{1}{4}c_1 z + \left(\frac{1}{4}c_2 - \frac{5}{32}c_1^2\right)z^2 + \\ &\quad \left(\frac{1}{4}c_3 - \frac{5}{16}c_1c_2 + \frac{13}{128}c_1^3\right)z^3 + \cdots. \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{z\partial_q \mathcal{I}_q^\mu f(z)}{\mathcal{I}_q^\mu f(z)} &= 1 + ([2,q]-1)a_2\psi_1 z + (([3,q]-1)a_3\psi_1 + \\ &\quad (1-[2,q])a_2^2\psi_1^2)z^2 + ((1-[2,q])a_2^2\psi_1^3 + (2-[2,q]- \\ &\quad [3,q])\psi_1\psi_2 a_2 a_3 + ([4,q]-1)a_4\psi_3)z^3 + \cdots. \end{aligned}$$

Equating the coefficients of z and z^2 we obtain

$$a_2 = \frac{1}{4([2,q]-1)\psi_1}c_1. \tag{13}$$

$$a_3 = \frac{1}{4([3,q]-1)\psi_2}c_2 - \frac{5q-2}{32q([3,q]-1)\psi_2}c_1^2. \tag{14}$$

Using Equations (13) and (14), we have

$$\left|a_3 - \lambda a_2^2\right| = \frac{1}{4q(1+q)\psi_2}\left|c_2 - \frac{\psi_1^2(5q-2) + 2q(1+q)\psi_2}{8q\psi_1^2}c_1^2\right|.$$

Using Lemma 1, we have

$$\left|a_3-\lambda a_2^2\right| \le \begin{cases} \frac{-\psi_1^2(q-2)-2\lambda q(1+q)\psi_2}{8q^2(1+q)\psi_1^2\psi_2}, & \text{if} \quad \lambda < -\frac{\psi_1^2(5q-2)}{2q(1+q)\psi_2} \\ \frac{1}{2q(1+q)\psi_2}, & \text{if} \ -\frac{\psi_1^2(5q-2)}{2q(1+q)\psi_2} \le \lambda \le \frac{\psi_1^2(3q+2)}{2q(1+q)\psi_2} \\ \frac{\psi_1^2(q-2)+2\lambda q(1+q)\psi_2}{8q^2(1+q)\psi_1^2\psi_2}, & \text{if} \quad \lambda > \frac{\psi_1^2(3q+2)}{2q(1+q)\psi_2}. \end{cases}$$

□

Corollary 1 ([14])**.** *Let $f(z) \in \mathcal{SL}^*$ be of the form in Equation (1). Then*

$$\left|a_3-\lambda a_2^2\right| \le \begin{cases} \frac{1-4\lambda}{16}, & \text{for} \quad \lambda < -\frac{3}{4} \\ \frac{1}{4}, & \text{for} \ -\frac{3}{4} \le \lambda \le \frac{5}{4} \\ \frac{4\lambda-1}{16}, & \text{for} \quad \lambda > \frac{5}{4}. \end{cases}$$

Theorem 5. *Let $f(z) \in \mathcal{SL}^*(\mu, q)$ be of the form in Equation (1). Then*

$$\left|a_3-\lambda a_2^2\right| \le \frac{1}{2q(1+q)\psi_2}\max\{1, |2v-1|\},$$

where v is the complex number given by

$$v = \frac{\psi_1^2(5q-2)+\lambda q(1+q)\psi_2}{8q\psi_1^2}.$$

Proof. Using Equations (13) and (14), we have

$$\left|a_3-\lambda a_2^2\right| = \frac{1}{4q(1+q)\psi_2}\left|c_2 - \frac{\psi_1^2(5q-2)+2\lambda q(1+q)\psi_2}{8q\psi_1^2}c_1^2\right|,$$

Using Lemma 2, we get

$$\left|a_3-\lambda a_2^2\right| \le \frac{1}{2q(1+q)\psi_2}\max\{1, |2v-1|\},$$

where v is given above. □

Corollary 2. *[14] Let $f(z) \in \mathcal{SL}^*$ be of the form in Equation (1). Then*

$$\left|a_3-\lambda a_2^2\right| \le \frac{1}{4}\max\left(1, \left|\lambda - \frac{1}{4}\right|\right).$$

5. Conclusions

Keeping in view the numerous applications of quantum and fractional calculus in the fields of mathematics and physics, we introduced a new class of analytic functions by using a $q-$operator in domain of lemniscate of Bernoulli. Various properties of this class were investigated via some analytical methods. These means and methods can be utilized along with this new class to investigate and connect functions in other domains like cardoid, the domain of sine function, the domain of exponential functions, etc. Also the operator can generalized to multivalent analytic and meromorphic functions, etc, which will contribute to the development of various fields of mathematics immensely.

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