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Estimation for the Discretely Observed Cox–Ingersoll–Ross Model Driven by Small Symmetrical Stable Noises

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Abstract: This paper is concerned with the least squares estimation of drift parameters for the Cox–Ingersoll–Ross (CIR) model driven by small symmetrical α -stable noises from discrete observations. The contrast function is introduced to obtain the explicit formula of the estimators and the error of estimation is given. The consistency and the rate of convergence of the estimators are proved. The asymptotic distribution of the estimators is studied as well. Finally, some numerical calculus examples and simulations are given.

Keywords: least squares estimator; Cox–Ingersoll–Ross model; small symmetrical α -stable noises; consistency; asymptotic distribution

1. Introduction

Stochastic differential equations driven by Brownian motion are always used to model the phenomena influenced by stochastic factors such as molecular thermal motion and short-term interest rate [1,2]. When establishing a pricing formula, the parameters in stochastic models describe the relevant asset dynamics. Nevertheless, in most cases, parameters are always unknown. Over the past few decades, many authors have studied the parameter estimation problem by maximum likelihood estimation [3–5], least squares estimation [6–8], and Bayes estimation [9,10]. However, non-Gaussian noise such as α -stable noise can more accurately reflect the practical random perturbation. Therefore, stochastic differential equations driven by α -stable noise have been investigated by many authors in recent years. Particularly, the parameter estimation problem has been discussed as well [11,12].

The Cox–Ingersoll–Ross (CIR) model [13,14] introduced in 1985 is an extension of the Vasicek model [15]; it is mean-reverting and remains non-negative. As we all know, the parameter estimation problem for the CIR model has been well studied [16,17]. However, many financial processes exhibit discontinuous sample paths and heavy-tailed properties (e.g., certain moments are infinite). These features cannot be captured by the CIR model. Therefore, it is natural to replace the Brownian motion by an α -stable process. In recent years, parameter estimation problems for the Levy-type CIR model have been discussed in some literature studies. For example, Ma and Yang [18] used least squares methods to study the parameter estimation problem for the CIR model driven by α -stable noises. Li and Ma [19] derived the conditional least squares estimators for a stable CIR model. However, the asymptotic distribution of the estimators has not been discussed in the literature. Asymptotic properties of estimators such as consistency, asymptotic distribution of estimation errors, and hypothesis tests can reflect the effectiveness of estimators and estimation methods, which helps to obtain a more reasonable economic model structure and more accurately grasp the dynamics of related assets. Therefore, it is of great important to study these topics.

The parameter estimation problem for the discretely observed CIR model with small symmetrical α -stable noises is studied in this article. The contrast function is introduced to obtain the least squares

estimators. The consistency and asymptotic distribution of the estimators are derived by Markov inequality, Cauchy–Schwarz inequality and Gronwall’s inequality. Some numerical calculus examples and simulations are given as well.

The structure of this paper is as follows. In Section 2, we introduce the CIR model driven by small symmetrical α -stable noises and obtain the explicit formula of the least squares estimators. In Section 3, the consistency and asymptotic distribution of the estimators are studied. In Section 4, some simulation results are made. The conclusions are given in Section 5.

2. Problem Formulation and Preliminaries

In this paper, notation “ \xrightarrow{P} ” is used to denote “convergence in probability” and notation “ \Rightarrow ” is used to denote “convergence in distribution”. We write “ $\stackrel{d}{=}$ ” for equality in distribution.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a basic probability space equipped with a right continuous and increasing family of σ -algebras $\{\mathcal{F}_t\}_{t \geq 0}$ and $Z = \{Z_t, t \geq 0\}$ be a strictly symmetric α -stable Levy motion.

A random variable η is said to have a stable distribution with index of stability $\alpha \in (0, 2]$, scale parameter $\sigma \in (0, \infty)$, skewness parameter $\beta \in [-1, 1]$, and location parameter $\mu \in (-\infty, \infty)$, if it has the following characteristic function:

$$\phi_\eta(u) = \mathbb{E} \exp\{iu\eta\} = \begin{cases} \exp\{-\sigma^\alpha |u|^\alpha (1 - i\beta \operatorname{sgn}(u) \tan \frac{\alpha\pi}{2}) + i\mu u\}, & \text{if } \alpha \neq 1 \\ \exp\{-\sigma |u|(1 + i\beta \frac{2}{\pi} \operatorname{sgn}(u) \log |u|) + i\mu u\}, & \text{if } \alpha = 1. \end{cases}$$

We denote $\eta \sim S_\alpha(\sigma, \beta, \mu)$. When $\mu = 0$, we say η is strictly α -stable, if in addition $\beta = 0$, we call η symmetrical α -stable. Throughout this paper, it is assumed that α -stable motion is strictly symmetrical and $\alpha \in (1, 2)$.

In this paper, we study the parametric estimation problem for the Cox–Ingersoll–Ross Model driven by small α -stable noises described by the following stochastic differential equation:

$$\begin{cases} dX_t = (\theta_1 - \theta_2 X_t)dt + \varepsilon \sqrt{X_t} dZ_t, t \in [0, 1] \\ X_0 = x_0, \end{cases} \tag{1}$$

where θ_1 and θ_2 are unknown parameters. We assume that $\varepsilon \in (0, 1]$.

To get the least squares estimators, we introduce the following contrast function:

$$\rho_{n,\varepsilon}(\theta_1, \theta_2) = \sum_{i=1}^n \frac{|X_{t_i} - X_{t_{i-1}} - (\theta_1 - \theta_2 X_{t_{i-1}})\Delta t_{i-1}|^2}{\varepsilon^2 X_{t_{i-1}} \Delta t_{i-1}}, \tag{2}$$

where $\Delta t_{i-1} = t_i - t_{i-1} = \frac{1}{n}$. Then, the least squares estimators $\hat{\theta}_{1,n,\varepsilon}$ and $\hat{\theta}_{2,n,\varepsilon}$ are defined as follows:

$$\rho_{n,\varepsilon}(\hat{\theta}_{1,n,\varepsilon}, \hat{\theta}_{2,n,\varepsilon}) = \min \rho_{n,\varepsilon}(\theta_1, \theta_2)$$

It is easy to obtain the least squares estimators:

$$\begin{cases} \hat{\theta}_{1,n,\varepsilon} = \frac{n^2 \sum_{i=1}^n X_{t_{i-1}} - n \sum_{i=1}^n \frac{X_{t_i}}{X_{t_{i-1}}} \sum_{i=1}^n X_{t_{i-1}} + n^2 \sum_{i=1}^n (X_{t_i} - X_{t_{i-1}})}{n^2 - \sum_{i=1}^n X_{t_{i-1}} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}} \\ \hat{\theta}_{2,n,\varepsilon} = \frac{n^3 - n^2 \sum_{i=1}^n \frac{X_{t_i}}{X_{t_{i-1}}} + n \sum_{i=1}^n (X_{t_i} - X_{t_{i-1}}) \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}}{n^2 - \sum_{i=1}^n X_{t_{i-1}} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}}. \end{cases} \tag{3}$$

3. Main Results and Proofs

Let $X^0 = (X_t^0, t \geq 0)$ be the solution to the underlying ordinary differential equation under the true value of the parameters:

$$dX_t^0 = (\theta_1 - \theta_2 X_t^0)dt, X_0^0 = x_0. \tag{4}$$

Note that

$$X_{t_i} - X_{t_{i-1}} = \frac{1}{n}\theta_1 - \theta_2 \int_{t_{i-1}}^{t_i} X_s ds + \varepsilon \int_{t_{i-1}}^{t_i} \sqrt{X_s} dZ_s. \tag{5}$$

Then, we can give a more explicit decomposition for $\hat{\theta}_{1n,\varepsilon}$ and $\hat{\theta}_{2n,\varepsilon}$ as follows:

$$\begin{aligned} \hat{\theta}_{1n,\varepsilon} &= \theta_1 + \frac{n\theta_2 \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_s}{X_{t_{i-1}}} ds \sum_{i=1}^n X_{t_{i-1}}}{n^2 - \sum_{i=1}^n X_{t_{i-1}} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}} - \frac{n^2 \theta_2 \sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_s ds}{n^2 - \sum_{i=1}^n X_{t_{i-1}} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}} \\ &+ \frac{n^2 \varepsilon \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sqrt{X_s} dZ_s}{n^2 - \sum_{i=1}^n X_{t_{i-1}} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}} - \frac{n \varepsilon \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{\sqrt{X_s}}{X_{t_{i-1}}} dZ_s \sum_{i=1}^n X_{t_{i-1}}}{n^2 - \sum_{i=1}^n X_{t_{i-1}} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}} \\ &= \theta_1 + \frac{\theta_2 \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_s}{X_{t_{i-1}}} ds \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}}}{1 - \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} \frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}} - \frac{\theta_2 \sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_s ds}{1 - \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} \frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}} n \\ &+ \frac{\varepsilon \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sqrt{X_s} dZ_s}{1 - \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} \frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}} - \frac{\varepsilon \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{\sqrt{X_s}}{X_{t_{i-1}}} dZ_s \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}}}{1 - \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} \frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}}. \\ \hat{\theta}_{2n,\varepsilon} &= \frac{n^2 \theta_2 \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_s}{X_{t_{i-1}}} ds}{n^2 - \sum_{i=1}^n X_{t_{i-1}} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}} - \frac{n \theta_2 \sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_s ds \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}}{n^2 - \sum_{i=1}^n X_{t_{i-1}} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}} \\ &+ \frac{n \varepsilon \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sqrt{X_s} dZ_s \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}}{n^2 - \sum_{i=1}^n X_{t_{i-1}} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}} - \frac{n^2 \varepsilon \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{\sqrt{X_s}}{X_{t_{i-1}}} dZ_s}{n^2 - \sum_{i=1}^n X_{t_{i-1}} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}} \\ &= \frac{\theta_2 \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_s}{X_{t_{i-1}}} ds}{1 - \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} \frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}} - \frac{\theta_2 \sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_s ds \frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}}{1 - \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} \frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}} \\ &+ \frac{\varepsilon \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sqrt{X_s} dZ_s \frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}}{1 - \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} \frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}} - \frac{\varepsilon \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{\sqrt{X_s}}{X_{t_{i-1}}} dZ_s}{1 - \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} \frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}}. \end{aligned}$$

Before giving the theorems, we need to establish some preliminary results.

Lemma 1. [20] Let Z be a strictly α -stable Levy process and $\phi \in L_{a.s.}^\alpha$, where $L_{a.s.}^\alpha$ is the family of all real-valued (F_t) -predictable processes ϕ on $\Omega \times [0, \infty)$ such that for every $T > 0$, $\int_0^T |\phi(s, \omega)|^\alpha dt < \infty$ a.s. Then,

$$\int_0^t \phi(s) dZ_s = Z' \circ \int_0^t \phi_+^\alpha(s) ds - Z'' \circ \int_0^t \phi_-^\alpha(s) ds, \text{ a.s.}$$

If Z is symmetric, that is, $\beta = 0$, then there exists some α -stable Levy process $Z' \stackrel{d}{=} Z$, such that

$$\int_0^t \phi(s) dZ_s = Z' \circ \int_0^t |\phi(s)|^\alpha ds, \text{ a.s.}$$

Lemma 2. When $\varepsilon \rightarrow 0, n \rightarrow \infty$, we have

$$\sup_{0 \leq t \leq 1} |X_t - X_t^0| \xrightarrow{P} 0.$$

Proof. Observe that

$$X_t - X_t^0 = -\theta_2 \int_0^t (X_s - X_s^0) ds + \varepsilon \int_0^t \sqrt{X_s} dZ_s. \quad (6)$$

By using the Cauchy–Schwarz inequality, we find

$$\begin{aligned} |X_t - X_t^0|^2 &\leq 2\theta_2^2 \left| \int_0^t (X_s - X_s^0) ds \right|^2 + 2\varepsilon^2 \left| \int_0^t \sqrt{X_s} dZ_s \right|^2 \\ &\leq 2\theta_2^2 t \int_0^t |X_s - X_s^0|^2 ds + 2\varepsilon^2 \sup_{0 \leq t \leq 1} \left| \int_0^t \sqrt{X_s} dZ_s \right|^2. \end{aligned}$$

According to Gronwall's inequality, we obtain

$$|X_t - X_t^0|^2 \leq 2\varepsilon^2 e^{2\theta_2^2 t^2} \sup_{0 \leq t \leq 1} \left| \int_0^t \sqrt{X_s} dZ_s \right|^2. \quad (7)$$

Then, it follows that

$$\sup_{0 \leq t \leq 1} |X_t - X_t^0| \leq \sqrt{2}\varepsilon e^{\theta_2^2} \sup_{0 \leq t \leq 1} \left| \int_0^t \sqrt{X_s} dZ_s \right|. \quad (8)$$

Assume that $\inf_{0 \leq t \leq 1} X_t = X_N > 0$, $\sup_{0 \leq t \leq 1} X_t = X_M < \infty$ By the Markov inequality, for any given $\delta > 0$, when $\varepsilon \rightarrow 0$, we have

$$\begin{aligned} &\mathbb{P}(\sqrt{2}\varepsilon e^{\theta_2^2} \sup_{0 \leq t \leq 1} \left| \int_0^t \sqrt{X_s} dZ_s \right| > \delta) \\ &\leq \delta^{-1} \sqrt{2}\varepsilon e^{\theta_2^2} \mathbb{E}[\sup_{0 \leq t \leq 1} \left| \int_0^t \sqrt{X_s} dZ_s \right|] \\ &\leq C\delta^{-1} \sqrt{2}\varepsilon e^{\theta_2^2} \mathbb{E}[(\int_0^1 X_s^{\frac{\alpha}{2}} ds)^{\frac{1}{\alpha}}] \\ &\leq C\delta^{-1} \sqrt{2}\varepsilon e^{\theta_2^2} \mathbb{E}[X_M^{\frac{1}{2}}] \rightarrow 0, \end{aligned}$$

where C is a constant.

Therefore, it is easy to check that

$$\sup_{0 \leq t \leq 1} |X_t - X_t^0| \xrightarrow{P} 0. \quad (9)$$

The proof is complete. \square

Remark 1. In Lemma 2, the following moment inequalities for stable stochastic integrals has been used to obtain the results:

$$\mathbb{E}[\sup_{t \leq T} \left| \int_0^t |\phi(s)| dZ_s \right|] \leq C\mathbb{E}[(\int_0^T |\phi(t)|^\alpha dt)^{\frac{1}{\alpha}}].$$

The above moment inequalities for stable stochastic integrals were established in Theorems 3.1 and 3.2 of [21].

Proposition 1. When $\varepsilon \rightarrow 0, n \rightarrow \infty$, we have

$$\frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} \xrightarrow{P} \int_0^1 X_t^0 dt.$$

Proof. Since

$$\frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} = \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}}^0 + \frac{1}{n} \sum_{i=1}^n (X_{t_{i-1}} - X_{t_{i-1}}^0). \quad (10)$$

it is clear that

$$\frac{1}{n} \sum_{i=1}^n X_{t_{i-1}}^0 \xrightarrow{P} \int_0^1 X_t^0 dt. \quad (11)$$

According to Lemma 2, when $\varepsilon \rightarrow 0, n \rightarrow \infty$, we have

$$\left| \frac{1}{n} \sum_{i=1}^n (X_{t_{i-1}} - X_{t_{i-1}}^0) \right| \leq \frac{1}{n} \sum_{i=1}^n |X_{t_{i-1}} - X_{t_{i-1}}^0| \leq \sup_{0 \leq t \leq 1} |X_t - X_t^0| \xrightarrow{P} 0$$

Therefore, we obtain

$$\frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} \xrightarrow{P} \int_0^1 X_t^0 dt. \quad (12)$$

The proof is complete. \square

Proposition 2. When $\varepsilon \rightarrow 0, n \rightarrow \infty$, we have

$$\frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}} \xrightarrow{P} \int_0^1 \frac{1}{X_t^0} dt.$$

Proof. Since

$$\frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}} = \frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}^0} + \frac{1}{n} \sum_{i=1}^n \left(\frac{1}{X_{t_{i-1}}} - \frac{1}{X_{t_{i-1}}^0} \right). \quad (13)$$

it is clear that

$$\frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}^0} \xrightarrow{P} \int_0^1 \frac{1}{X_t^0} dt. \quad (14)$$

According to Lemma 2, when $\varepsilon \rightarrow 0, n \rightarrow \infty$, we have

$$\begin{aligned} \left| \frac{1}{n} \sum_{i=1}^n \left(\frac{1}{X_{t_{i-1}}} - \frac{1}{X_{t_{i-1}}^0} \right) \right| &= \left| \frac{1}{n} \sum_{i=1}^n \frac{X_{t_{i-1}}^0 - X_{t_{i-1}}}{X_{t_{i-1}} X_{t_{i-1}}^0} \right| \leq \frac{1}{n} \sum_{i=1}^n \frac{|X_{t_{i-1}}^0 - X_{t_{i-1}}|}{|X_{t_{i-1}} X_{t_{i-1}}^0|} \\ &\leq \sup_{0 \leq t \leq 1} \frac{|X_t - X_t^0|}{|X_t X_t^0|} \leq \sup_{0 \leq t \leq 1} \frac{|X_t - X_t^0|}{X_N^2} \xrightarrow{P} 0. \end{aligned}$$

Therefore, we obtain

$$\frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}} \xrightarrow{P} \int_0^1 \frac{1}{X_t^0} dt. \quad (15)$$

In the following theorem, the consistency of the least squares estimators is proved.

The proof is complete.

Theorem 1. When $\varepsilon \rightarrow 0, n \rightarrow \infty$ and $\varepsilon n^{1-\frac{1}{\alpha}} \rightarrow 0$, the least squares estimators $\hat{\theta}_{1n,\varepsilon}$ and $\hat{\theta}_{2n,\varepsilon}$ are consistent, namely

$$\hat{\theta}_{1n,\varepsilon} \xrightarrow{P} \theta_1, \quad \hat{\theta}_{2n,\varepsilon} \xrightarrow{P} \theta_2.$$

Proof. According to Propositions 1 and 2, it is clear that

$$1 - \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} \frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}} \xrightarrow{P} 1 - \int_0^1 X_t^0 dt \int_0^1 \frac{1}{X_t^0} dt. \tag{16}$$

When $\varepsilon \rightarrow 0, n \rightarrow \infty$, it can be checked that

$$\theta_2 \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_s}{X_{t_{i-1}}} ds \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} \xrightarrow{P} \theta_2 \int_0^1 \frac{X_t}{X_t^0} dt \int_0^1 X_t^0 dt, \tag{17}$$

and

$$\theta_2 \sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_s ds \xrightarrow{P} \theta_2 \int_0^1 X_t dt. \tag{18}$$

According to Lemma 2, we have

$$\theta_2 \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_s}{X_{t_{i-1}}} ds \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} - \theta_2 \sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_s ds \xrightarrow{P} 0. \tag{19}$$

By the Markov inequality, we have

$$\begin{aligned} P(|\varepsilon \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sqrt{X_s} dZ_s| > \delta) &\leq \delta^{-1} \varepsilon \sum_{i=1}^n \mathbb{E} |\int_{t_{i-1}}^{t_i} \sqrt{X_s} dZ_s| \\ &\leq 2c_2 \delta^{-1} \varepsilon \sum_{i=1}^n \mathbb{E} [\int_{t_{i-1}}^{t_i} X_s^{\frac{\alpha}{2}} ds]^{\frac{1}{\alpha}} \leq 2c_2 \delta^{-1} \varepsilon n^{1-\frac{1}{\alpha}} \mathbb{E}[X_M^{\frac{1}{2}}] \rightarrow 0, \end{aligned}$$

where c_2 is constant and implies that $\varepsilon \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sqrt{X_s} dZ_s \xrightarrow{P} 0$ as $\varepsilon \rightarrow 0, n \rightarrow \infty$ and $\varepsilon n^{1-\frac{1}{\alpha}} \rightarrow 0$.

With the results of Proposition 1 and (16), we have

$$\frac{\varepsilon \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sqrt{X_s} dZ_s}{1 - \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} \frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}} \xrightarrow{P} 0. \tag{20}$$

since

$$\begin{aligned} |\varepsilon \sum_{i=1}^n \frac{1}{X_{t_{i-1}}} \int_{t_{i-1}}^{t_i} \sqrt{X_s} dZ_s| &\leq \varepsilon \sum_{i=1}^n |\frac{1}{X_{t_{i-1}}}| \|\int_{t_{i-1}}^{t_i} \sqrt{X_s} dZ_s\| \\ &\leq \varepsilon \sum_{i=1}^n (|\frac{1}{X_{t_{i-1}}^0}| + |\frac{1}{X_{t_{i-1}}} - \frac{1}{X_{t_{i-1}}^0}|) \|\int_{t_{i-1}}^{t_i} \sqrt{X_s} dZ_s\| \\ &\leq \varepsilon \sum_{i=1}^n |\frac{1}{X_{t_{i-1}}^0}| \|\int_{t_{i-1}}^{t_i} \sqrt{X_s} dZ_s\| + \varepsilon \sup_{0 \leq t \leq 1} |\frac{1}{X_t} - \frac{1}{X_t^0}| \|\int_{t_{i-1}}^{t_i} \sqrt{X_s} dZ_s\|. \end{aligned}$$

By the Markov inequality, we have

$$\begin{aligned} P(|\varepsilon \sum_{i=1}^n |\frac{1}{X_{t_{i-1}}^0}| \|\int_{t_{i-1}}^{t_i} \sqrt{X_s} dZ_s\| > \delta) &\leq \delta^{-1} \varepsilon \sum_{i=1}^n |\frac{1}{X_{t_{i-1}}^0}| \mathbb{E} \|\int_{t_{i-1}}^{t_i} \sqrt{X_s} dZ_s\| \\ &\leq 2c_2 \delta^{-1} \varepsilon \sum_{i=1}^n |\frac{1}{X_{t_{i-1}}^0}| \mathbb{E} [\int_{t_{i-1}}^{t_i} X_s^{\frac{\alpha}{2}} ds]^{\frac{1}{\alpha}} \\ &\leq 2c_2 \delta^{-1} \varepsilon n^{1-\frac{1}{\alpha}} \frac{1}{n} \sum_{i=1}^n |\frac{1}{X_{t_{i-1}}^0}| \mathbb{E}[X_M^{\frac{1}{2}}], \end{aligned}$$

which implies that $\varepsilon \sum_{i=1}^n \left| \frac{1}{X_t^0} \right\| \int_{t_{i-1}}^{t_i} \sqrt{X_s} dZ_s \xrightarrow{P} 0$ as $\varepsilon \rightarrow 0, n \rightarrow \infty$ and $\varepsilon n^{1-\frac{1}{\alpha}} \rightarrow 0$. According to Lemma 2, when $\varepsilon \rightarrow 0, n \rightarrow \infty$, it is obvious that

$$\varepsilon \sup_{0 \leq t \leq 1} \left| \frac{1}{X_t} - \frac{1}{X_t^0} \right\| \int_{t_{i-1}}^{t_i} \sqrt{X_s} dZ_s \xrightarrow{P} 0. \tag{21}$$

Then, we have

$$\varepsilon \sum_{i=1}^n \frac{1}{X_{t_{i-1}}} \int_{t_{i-1}}^{t_i} \sqrt{X_s} dZ_s \xrightarrow{P} 0. \tag{22}$$

Therefore, by (16), (19), and (22), when $\varepsilon \rightarrow 0, n \rightarrow \infty$ and $\varepsilon n^{1-\frac{1}{\alpha}} \rightarrow 0$, we have

$$\hat{\theta}_{1n,\varepsilon} \xrightarrow{P} \theta_1.$$

Using the same methods in Theorem 1, it can be easily checked that

$$\theta_2 \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_s}{X_{t_{i-1}}} ds \xrightarrow{P} \theta_2, \tag{23}$$

$$\theta_2 \sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_s ds \frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}} \xrightarrow{P} \theta_2 \int_0^1 X_t^0 dt \int_0^1 \frac{1}{X_t^0} dt. \tag{24}$$

Then, according to (16), we have

$$\frac{\theta_2 \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_s}{X_{t_{i-1}}} ds}{1 - \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} \frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}} - \frac{\theta_2 \sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_s ds \frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}}{1 - \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} \frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}} \xrightarrow{P} \theta_2. \tag{25}$$

Together with the results that

$$\frac{\varepsilon \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sqrt{X_s} dZ_s \frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}}{1 - \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} \frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}} \xrightarrow{P} 0, \tag{26}$$

$$\frac{\varepsilon \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{\sqrt{X_s}}{X_{t_{i-1}}} dZ_s}{1 - \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} \frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}} \xrightarrow{P} 0, \tag{27}$$

when $\varepsilon \rightarrow 0, n \rightarrow \infty$ and $\varepsilon n^{1-\frac{1}{\alpha}} \rightarrow 0$, we have

$$\hat{\theta}_{2n,\varepsilon} \xrightarrow{P} \theta_2.$$

The proof is complete. \square

Theorem 2. When $\varepsilon \rightarrow 0, n \rightarrow \infty$ and $n\varepsilon \rightarrow \infty$,

$$\varepsilon^{-1}(\hat{\theta}_{1n,\varepsilon} - \theta_1) \xrightarrow{d} \frac{(\int_0^1 (X_t^0)^{\frac{\alpha}{2}} dt)^{\frac{1}{\alpha}} - \int_0^1 X_t^0 dt (\int_0^1 (X_t^0)^{-\frac{\alpha}{4}} dt)^{\frac{1}{\alpha}}}{1 - \int_0^1 X_t^0 dt \int_0^1 \frac{1}{X_t^0} dt} S_\alpha(1, 0, 0),$$

$$\varepsilon^{-1}(\hat{\theta}_{2n,\varepsilon} - \theta_2) \xrightarrow{d} \frac{(\int_0^1 (X_t^0)^\alpha dt)^{\frac{1}{\alpha}} \int_0^1 X_t^0 dt - (\int_0^1 (X_t^0)^{-\frac{\alpha}{4}} dt)^{\frac{1}{\alpha}}}{1 - \int_0^1 X_t^0 dt \int_0^1 \frac{1}{X_t^0} dt} S_\alpha(1, 0, 0).$$

Proof. According to the explicit decomposition for $\hat{\theta}_{1n,\varepsilon}$, it is obvious that

$$\begin{aligned} \varepsilon^{-1}(\hat{\theta}_{1n,\varepsilon} - \theta_1) &= \frac{\varepsilon^{-1}\theta_2 \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_s}{\bar{X}_{t_{i-1}}} ds \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}}}{1 - \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} \frac{1}{n} \sum_{i=1}^n \frac{1}{\bar{X}_{t_{i-1}}}} - \frac{\varepsilon^{-1}\theta_2 \sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_s ds}{1 - \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} \frac{1}{n} \sum_{i=1}^n \frac{1}{\bar{X}_{t_{i-1}}}} \\ &+ \frac{\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sqrt{X_s} dZ_s}{1 - \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} \frac{1}{n} \sum_{i=1}^n \frac{1}{\bar{X}_{t_{i-1}}}} - \frac{\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{\sqrt{X_s}}{\bar{X}_{t_{i-1}}} dZ_s \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}}}{1 - \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} \frac{1}{n} \sum_{i=1}^n \frac{1}{\bar{X}_{t_{i-1}}}}. \end{aligned}$$

From Lemma 2, when $\varepsilon \rightarrow 0, n \rightarrow \infty$ and $n\varepsilon \rightarrow \infty$,

$$\begin{aligned} |\varepsilon^{-1}\theta_2 \sum_{i=1}^n \frac{1}{\bar{X}_{t_{i-1}}} \int_{t_{i-1}}^{t_i} X_s ds| &\leq \varepsilon^{-1}\theta_2 \sum_{i=1}^n \frac{1}{\bar{X}_{t_{i-1}}} \|\int_{t_{i-1}}^{t_i} X_s ds\| \\ &\leq \varepsilon^{-1}n^{-1}\theta_2 \sum_{i=1}^n (|\frac{1}{\bar{X}_{t_{i-1}}} - \frac{1}{\bar{X}_{t_{i-1}}^0}| + |\frac{1}{\bar{X}_{t_{i-1}}^0}|) \sup_{t_{i-1} \leq t \leq t_i} |X_t| \xrightarrow{P} 0. \end{aligned}$$

then, it is easy to check that

$$\varepsilon^{-1}\theta_2 \sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_s ds \xrightarrow{P} 0.$$

Together with (12) and (16), we have

$$\frac{\varepsilon^{-1}\theta_2 \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_s}{\bar{X}_{t_{i-1}}} ds \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}}}{1 - \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} \frac{1}{n} \sum_{i=1}^n \frac{1}{\bar{X}_{t_{i-1}}}} \xrightarrow{P} 0, \quad (28)$$

and

$$\frac{\varepsilon^{-1}\theta_2 \sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_s ds}{1 - \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} \frac{1}{n} \sum_{i=1}^n \frac{1}{\bar{X}_{t_{i-1}}}} \xrightarrow{P} 0. \quad (29)$$

since

$$\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sqrt{X_s} dZ_s = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sqrt{X_s^0} dZ_s + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (\sqrt{X_s} - \sqrt{X_s^0}) dZ_s. \quad (30)$$

Using the Markov inequality and Holder’s inequality, for any given $\delta > 0$, we have

$$\begin{aligned}
 & P(|\sum_{i=1}^n \int_{t_{i-1}}^{t_i} (\sqrt{X_s} - \sqrt{X_s^0})dZ_s| > \delta) \\
 & \leq \delta^{-1} \sum_{i=1}^n \mathbb{E}[|\int_{t_{i-1}}^{t_i} (\sqrt{X_s} - \sqrt{X_s^0})dZ_s|] \\
 & \leq 2c_2\delta^{-1} \sum_{i=1}^n \mathbb{E}[(\int_{t_{i-1}}^{t_i} |\sqrt{X_s} - \sqrt{X_s^0}|^\alpha dZ_s)^\frac{1}{\alpha}] \\
 & \leq 2c_2\delta^{-1} \sum_{i=1}^n \mathbb{E}[(\int_{t_{i-1}}^{t_i} \frac{|X_s - X_s^0|^\alpha}{|\sqrt{X_s} + \sqrt{X_s^0}|^\alpha} dZ_s)^\frac{1}{\alpha}] \\
 & \leq 2c_2\delta^{-1} \sum_{i=1}^n \mathbb{E}[\frac{1}{2\sqrt{X_N}} (\int_{t_{i-1}}^{t_i} |X_s - X_s^0|^\alpha ds)^\frac{1}{\alpha}] \\
 & \leq 2c_2\delta^{-1} \sum_{i=1}^n (\mathbb{E}[\frac{1}{2\sqrt{X_N}}]^2)^\frac{1}{2} (\mathbb{E}[\int_{t_{i-1}}^{t_i} |X_s - X_s^0|^\alpha ds]^\frac{2}{\alpha})^\frac{1}{2} \\
 & \leq 2c_2\delta^{-1} (\mathbb{E}[\frac{1}{4X_N}])^\frac{1}{2} \sum_{i=1}^n (\mathbb{E}[\sup_{t_{i-1} \leq t \leq t_i} |X_t - X_t^0| n^{-\frac{2}{\alpha}}])^\frac{1}{2} \\
 & \leq 2c_2\delta^{-1} n^{-\frac{1}{\alpha}} (\mathbb{E}[\frac{1}{4X_N}])^\frac{1}{2} \sum_{i=1}^n (\mathbb{E}[\sqrt{2}\varepsilon e^{\theta_2} \sup_{t_{i-1} \leq t \leq t_i} |\int_0^t \sqrt{X_s} dZ_s|])^\frac{1}{2} \\
 & \leq 2^{\frac{5}{4}} c_2 \delta^{-1} n^{1-\frac{1}{\alpha}} \varepsilon^{\frac{1}{2}} e^{\theta_2} (\mathbb{E}[\frac{1}{4X_N}])^\frac{1}{2} (\mathbb{E}[\int_0^1 X_s^{\frac{\alpha}{2}} ds]^\frac{1}{\alpha})^\frac{1}{2} \\
 & \leq 2^{\frac{5}{4}} c_2 \delta^{-1} n^{1-\frac{1}{\alpha}} \varepsilon^{\frac{1}{2}} e^{\theta_2} (\mathbb{E}[\frac{1}{4X_N}])^\frac{1}{2} (\mathbb{E}[X_M]^\frac{1}{2})^\frac{1}{2} \\
 & \rightarrow 0,
 \end{aligned}$$

as $\varepsilon \rightarrow 0, n \rightarrow \infty$ and $n^{1-\frac{1}{\alpha}} \varepsilon^{\frac{1}{2}} \rightarrow 0$. Moreover,

$$\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sqrt{X_s^0} dZ_s = \int_0^1 \sum_{i=1}^n \sqrt{X_s^0} 1_{(t_{i-1}, t_i]}(s) dZ_s = Z' \circ \int_0^1 \sum_{i=1}^n (\sqrt{X_s^0} 1_{(t_{i-1}, t_i]}(s))^\alpha ds,$$

where $Z' \stackrel{d}{=} Z$. Since

$$\int_0^1 \sum_{i=1}^n (\sqrt{X_s^0} 1_{(t_{i-1}, t_i]}(s))^\alpha ds \rightarrow \int_0^1 (X_s^0)^\frac{\alpha}{2} ds, \tag{31}$$

it is clear that

$$Z' \circ \int_0^1 \sum_{i=1}^n (\sqrt{X_s^0} 1_{(t_{i-1}, t_i]}(s))^\alpha ds \xrightarrow{a.s.} Z' \circ \int_0^1 (X_s^0)^\frac{\alpha}{2} ds. \tag{32}$$

It immediately follows that

$$\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sqrt{X_s^0} dZ_s \xrightarrow{d} (\int_0^1 (X_t^0)^\frac{\alpha}{2} dt)^\frac{1}{\alpha} S_\alpha(1, 0, 0). \tag{33}$$

Then, from (12), (16), and (33), we have

$$\varepsilon^{-1}(\hat{\theta}_{1n,\varepsilon} - \theta_1) \xrightarrow{d} \frac{(\int_0^1 (X_t^0)^\frac{\alpha}{2} dt)^\frac{1}{\alpha} - \int_0^1 X_t^0 dt (\int_0^1 (X_t^0)^{-\frac{\alpha}{4}} dt)^\frac{1}{\alpha}}{1 - \int_0^1 X_t^0 dt \int_0^1 \frac{1}{X_t^0} dt} S_\alpha(1, 0, 0). \tag{34}$$

as

$$\begin{aligned} \varepsilon^{-1}(\hat{\theta}_{2n,\varepsilon} - \theta_2) &= \frac{\varepsilon^{-1}\theta_2 \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_s}{X_{t_{i-1}}} ds}{1 - \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} \frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}} - \frac{\varepsilon^{-1}\theta_2 \sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_s ds \frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}}{1 - \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} \frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}} - \varepsilon^{-1}\theta_2 \\ &+ \frac{\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sqrt{X_s} dZ_s \frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}}{1 - \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} \frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}} - \frac{\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{\sqrt{X_s}}{X_{t_{i-1}}} dZ_s}{1 - \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} \frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}}. \end{aligned}$$

According to above results, it is obvious that

$$\frac{\varepsilon^{-1}\theta_2 \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_s}{X_{t_{i-1}}} ds}{1 - \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} \frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}} - \frac{\varepsilon^{-1}\theta_2 \sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_s ds \frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}}{1 - \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} \frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}} - \varepsilon^{-1}\theta_2 \xrightarrow{P} 0, \tag{35}$$

and

$$\begin{aligned} &\frac{\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sqrt{X_s} dZ_s \frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}}{1 - \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} \frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}} - \frac{\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{\sqrt{X_s}}{X_{t_{i-1}}} dZ_s}{1 - \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} \frac{1}{n} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}} \\ &\xrightarrow{d} \frac{(\int_0^1 (X_t^\alpha) dt)^{\frac{1}{\alpha}} \int_0^1 X_t^0 dt - (\int_0^1 (X_t^\alpha)^{-\frac{\alpha}{4}} dt)^{\frac{1}{\alpha}}}{1 - \int_0^1 X_t^0 dt \int_0^1 \frac{1}{X_t^0} dt} S_\alpha(1, 0, 0). \end{aligned}$$

Then, we have

$$\varepsilon^{-1}(\hat{\theta}_{2n,\varepsilon} - \theta_2) \xrightarrow{d} \frac{(\int_0^1 (X_t^\alpha) dt)^{\frac{1}{\alpha}} \int_0^1 X_t^0 dt - (\int_0^1 (X_t^\alpha)^{-\frac{\alpha}{4}} dt)^{\frac{1}{\alpha}}}{1 - \int_0^1 X_t^0 dt \int_0^1 \frac{1}{X_t^0} dt} S_\alpha(1, 0, 0). \tag{36}$$

The proof is complete. □

4. Simulation

In this experiment, we generate a discrete sample $(X_{t_{i-1}})_{i=1,\dots,n}$ and compute $\hat{\theta}_{1n,\varepsilon}$ and $\hat{\theta}_{2n,\varepsilon}$ from the sample. We let $x_0 = 0.05$ and $\alpha = 1.8$. For every given true value of the parameters (θ_1, θ_2) , the size of the sample is represented as “Size n” and given in the first column of the table. In Table 1, $\varepsilon = 0.1$, the size is increasing from 1000 to 5000. In Table 2, $\varepsilon = 0.01$, the size is increasing from 10,000 to 50,000. Based on the ten-time average of the least squares estimation of the random number in the calculation model, the tables list the values of the least squares estimator (LSE) of θ_1 (“ $\theta_1 - LSE$ ”) and θ_2 (“ $\theta_2 - LSE$ ”), the absolute error (AE), and the relative error (RE) of the least squares estimator.

Table 1. Least squares estimator simulation results of θ_1 and θ_2 .

True		Average		AE		RE	
(θ_1, θ_2)	Size n	$\theta_1 - LSE$	$\theta_2 - LSE$	θ_1	θ_2	θ_1	θ_2
(1,1)	1000	1.2632	0.7568	0.2632	0.2432	26.32%	24.32%
	2000	1.1425	0.8673	0.1425	0.1327	14.25%	13.27%
	5000	1.0651	0.9586	0.0651	0.0414	6.51%	4.14%
(2,3)	1000	1.6573	3.2538	0.3427	0.2538	17.14%	8.46%
	2000	2.1836	3.1209	0.1836	0.1209	9.18%	4.03%
	5000	2.0528	3.0614	0.0528	0.0614	2.64%	2.05%

Table 2. Least squares estimator simulation results of θ_1 and θ_2 .

True (θ_1, θ_2)	Average Size n	Average		AE		RE	
		$\theta_1 - LSE$	$\theta_2 - LSE$	θ_1	θ_2	θ_1	θ_2
(1,1)	10,000	1.1346	0.8735	0.1346	0.1265	13.46%	12.65%
	20,000	1.0538	0.9359	0.0538	0.0641	5.38%	6.41%
	50,000	1.0010	0.9987	0.0010	0.0013	0.1%	0.13%
(2,3)	10,000	1.8645	3.1452	0.1355	0.1452	6.78%	4.84%
	20,000	2.0649	3.0722	0.0649	0.0722	3.25%	2.41%
	50,000	2.0028	3.0017	0.0028	0.0017	0.14%	0.06%

The two tables indicate that the absolute error between the estimator and the true value depends on the size of the true value samples for any given parameter. In Table 1, when $n = 5000$, the relative error of the estimators does not exceed 7%. In Table 2, when $n = 50,000$, the relative error of the estimators does not exceed 0.2%. The estimators are good.

In Figure 1, we let $\theta_1 = 1$ under $T = 500$, $\varepsilon = 0.1$ and $\varepsilon = 0.01$, respectively. In Figure 2, we let $\theta_2 = 2$ under $T = 500$, $\varepsilon = 0.1$ and $\varepsilon = 0.01$, respectively. The two figures indicate that when T is fixed and ε is small, the obtained estimators are closer to the true parameter value compared to that of the large ε . When T is large enough and ε is small enough, the obtained estimators are very close to the true parameter value. If we let T convergeto infinity and ε convergeto zero, the two estimators will converge to the true value.

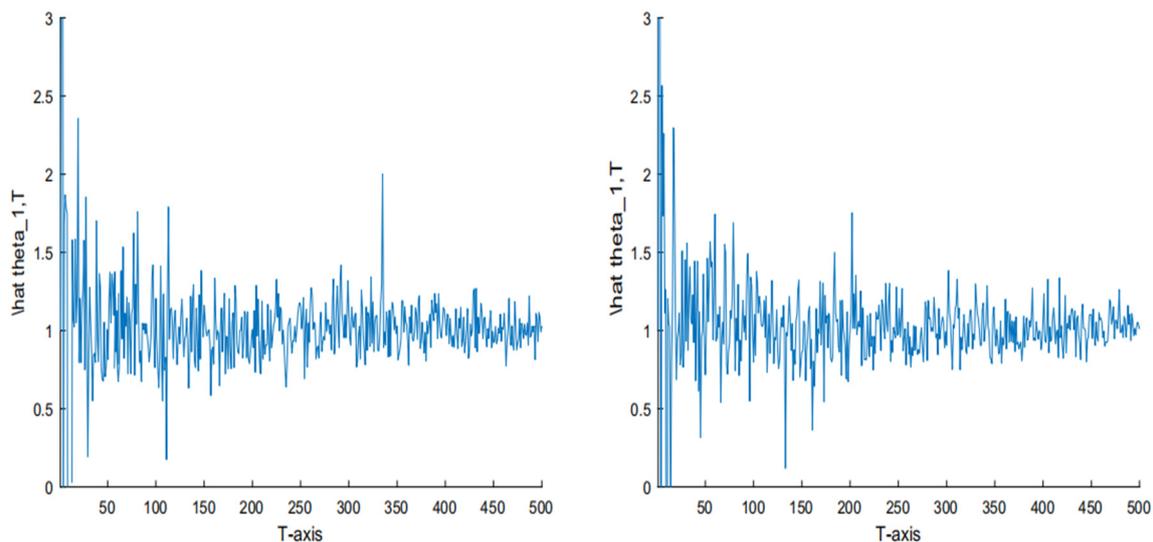


Figure 1. The simulation of the estimator $\hat{\theta}_{1,T}$ with $T = 500$ and $\theta_1 = 1$ under $\varepsilon = 0.1$ and $\varepsilon = 0.01$, respectively.

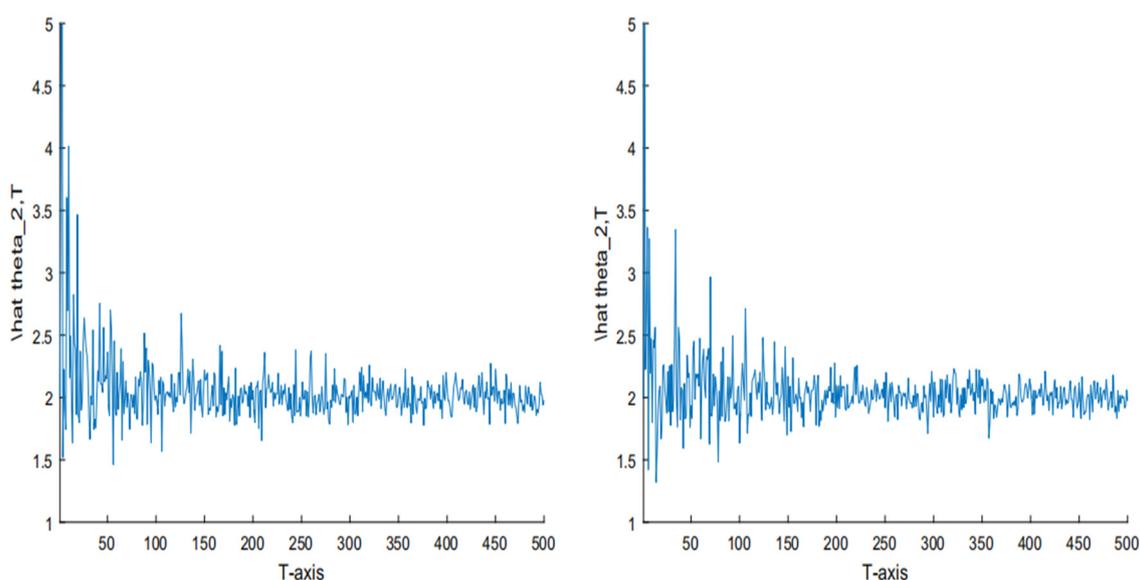


Figure 2. The simulation of the estimator $\hat{\theta}_{2,T}$ and θ_2 with $T = 500$ and $\theta_2 = 2$ under $\varepsilon = 0.1$ and $\varepsilon = 0.01$, respectively.

5. Conclusions

The aim of this paper was to study the parameter estimation problem for the Cox–Ingersoll–Ross model driven by small symmetrical α -stable noises from discrete observations. The contrast function was introduced to obtain the explicit formula of the least squares estimators and the error of estimation was given. The consistency and the rate of convergence of the least squares estimators were proved by Markov inequality, Cauchy–Schwarz inequality, and Gronwall’s inequality. The asymptotic distribution of the estimators were discussed as well.

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