symmetry

MDPI

Article

On the Chebyshev Polynomials and Some of Their Reciprocal Sums

Wenpeng Zhang [1,2] and Di Han [2,*]

1 School of Science, Xi'an Technological University, Xi'an 710021, China; wpzhang@nwu.edu.cn
2 School of Mathematics, Northwest University, Xi'an 710127, China
* Correspondence: dhan@nwu.edu.cn

Received: 27 February 2020; Accepted: 8 April 2020; Published: 2 May 2020

Abstract: In this paper, we utilize the mathematical induction, the properties of symmetric polynomial sequences and Chebyshev polynomials to study the calculating problems of a certain reciprocal sums of Chebyshev polynomials, and give two interesting identities for them. These formulae not only reveal the close relationship between the trigonometric function and the Riemann ζ-function, but also generalized some existing results. At the same time, an error in an existing reference is corrected.

Keywords: chebyshev polynomials; symmetric polynomial sequence; computational formula; mathematical induction; identity

MSC: 11B37; 11B83

1. Introduction

For any non-negative integer $n \geq 0$, the famous Chebyshev polynomials of the first kind $T_n(x)$ and the second kind $U_n(x)$ (see [1,2]) are defined by the second order linear recurrence formulae $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ for all integers $n \geq 1$ with $T_0(x) = 1$ and $T_1(x) = x$; $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$ for all integers $n \geq 1$ with $U_0(x) = 1$ and $U_1(x) = 2x$.

The general terms that are easy to deduce from the recursive relationships are

$$T_n(x) = \frac{1}{2}\left(\alpha^n + \beta^n\right) \quad \text{and} \quad U_n(x) = \frac{1}{2\sqrt{x^2-1}}\left(\alpha^{n+1} - \beta^{n+1}\right),$$

where $\alpha = x + \sqrt{x^2-1}$ and $\beta = x - \sqrt{x^2-1}$.

The generation functions of the Chebyshev polynomials $T_n(x)$ and $U_n(x)$ are

$$\frac{1-xz}{1-2xz+z^2} = \sum_{n=1}^{\infty} T_n(x)\cdot z^n, \quad (|x|<1, |z|<1)$$

and

$$\frac{1}{1-2xz+z^2} = \sum_{n=1}^{\infty} U_n(x)\cdot z^n, \quad (|x|<1, |z|<1).$$

Taking $x = \cos\theta$ in $T_n(x)$ and $U_n(x)$, then we also have the following identities

$$T_n(\cos\theta) = \cos(n\theta), \ \ U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}. \tag{1}$$

Since these polynomials have an important position in the theory and application of mathematics, many specialists and scholars have studied their various properties, and obtained a series of interesting conclusions. It is worth mentioning that T. Kim and their team to do a lot of important research work (see [3–8]). Other papers related to Chebyshev polynomials can also be found in [9–22]. For example, T. T. Wang and H. Zhang [9] and W. P. Zhang and T. T. Wang [10] obtained some exact expressions for the derivative and integral of the Chebyshev polynomials of the first kind in terms of the Chebyshev polynomials of the first kind. Y. Ma and X. X. Lv [12] considered the calculating problem of a certain reciprocal sums of Chebyshev polynomials, and obtained some identities. That is, for $k = 1,\ 2$ and 3, Y. Ma and X. X. Lv [12] gave some identities for the summations

$$\sum_{a=1}^{q-1} T_a^{-2k}\left(\cos\left(\frac{\pi h}{q}\right)\right) \text{ and } \sum_{a=1}^{q-1} U_{a-1}^{-2k}\left(\cos\left(\frac{\pi h}{q}\right)\right), \tag{2}$$

where, as usual, q is an odd number and h is an integer co-prime to q, i.e., $(h, q) = 1$.

Unfortunately, it is very difficult to obtain an identity for Equation (2) with $k \geq 4$ by the methods in [12]. Inspired by Y. Ma and X. X. Lv [12], in this paper, we utilize the mathematical induction, the properties of symmetric polynomial sequences, and Chebyshev polynomials to study these problems, and prove two generalized conclusions. In other words, we prove the following two results:

Theorem 1. *Let q be an odd number and $q \geq 3$. For any positive integer k and integer h with $(h, q) = 1$, we have the identity*

$$\begin{aligned}\sum_{a=1}^{q-1} U_{a-1}^{-2k}\left(\cos\left(\frac{\pi h}{q}\right)\right) &= \sin^{2k}\left(\frac{\pi h}{q}\right)\cdot \sum_{a=1}^{q-1}\sin^{-2k}\left(\frac{\pi a}{q}\right)\\ &= \frac{2\cdot \sin^{2k}\left(\frac{\pi h}{q}\right)}{(2k-1)!}\sum_{i=1}^{k}\frac{S(k-1,k-i)\cdot(2i-1)!\cdot\left(q^{2i}-1\right)}{\pi^{2i}}\zeta(2i)\\ &= \frac{\sin^{2k}\left(\frac{\pi h}{q}\right)}{(2k-1)!}\sum_{i=1}^{k}\frac{S(k-1,k-i)\cdot\left(q^{2i}-1\right)}{2i}\cdot(-1)^{i+1}\cdot B_{2i},\end{aligned}$$

where $\zeta(s)$ denotes the Riemann ζ-function, B_{2k} denotes the Bernoulli numbers, and $S(k-1, i)$ are defined by $\prod_{i=0}^{k-1}\left(x + (2i)^2\right) = \sum_{i=0}^{k-1} S(k-1, i)\cdot x^{k-i}$, and $S(0, 0) = 1$.

Theorem 2. *Let q be an odd number and $q \geq 3$. For any positive integer k and integer h with $(h, q) = 1$, we have the identity*

$$\begin{aligned}\sum_{a=1}^{q-1} T_a^{-2k}\left(\cos\left(\frac{\pi h}{q}\right)\right) &= \sum_{a=1}^{q-1}\cos^{-2k}\left(\frac{\pi a h}{q}\right) = \sum_{a=1}^{q-1}\cos^{-2k}\left(\frac{\pi a}{q}\right)\\ &= \frac{2}{(2k-1)!}\cdot\sum_{i=1}^{k}\frac{S(k-1,k-i)\cdot\left(2^{2i}-1\right)\cdot(2i-1)!\cdot\left(q^{2i}-1\right)}{\pi^{2i}}\cdot\zeta(2i)\\ &= \frac{1}{(2k-1)!}\cdot\sum_{i=1}^{k}\frac{S(k-1,k-i)\cdot\left(2^{2i}-1\right)\cdot\left(q^{2i}-1\right)}{2i}\cdot(-1)^{i+1}\cdot B_{2i},\end{aligned}$$

where we use the identity (see [1], Theorem 12.17)

$$\zeta(2k) = (-1)^{k+1}\cdot\frac{(2\pi)^{2k}\cdot B_{2k}}{2\cdot(2k)!} \text{ for all positive integers } k.$$

Note that $S(0,0) = S(1,0) = S(2,0) = 1$, $S(1,1) = 4$, $S(2,1) = 20$, $S(2,2) = 64$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, and $B_6 = \frac{1}{42}$; from Theorems 1 and 2, we can immediately deduce the following two corollaries:

Corollary 1. *[12] Let $q > 1$ be an odd number. For any integer h and $(h, q) = 1$, we have the identity*

$$\sum_{a=1}^{q-1} U_{a-1}^{-2}\left(\cos\left(\frac{\pi h}{q}\right)\right) = \sin^2\left(\frac{\pi h}{q}\right)\cdot\frac{(q^2-1)}{3};$$

$$\sum_{a=1}^{q-1} U_{a-1}^{-4}\left(\cos\left(\frac{\pi h}{q}\right)\right) = \sin^4\left(\frac{\pi h}{q}\right)\cdot\frac{(q^2+11)(q^2-1)}{45}$$

and

$$\sum_{a=1}^{q-1} U_{a-1}^{-6}\left(\cos\left(\frac{\pi h}{q}\right)\right) = \sin^6\left(\frac{\pi h}{q}\right)\cdot\frac{(q^2-1)(2q^4+23q^2+191)}{945}.$$

Corollary 2. *[12] Let $q > 1$ be an odd number. For any integer h and $(h, q) = 1$, we have the identity*

$$\sum_{a=1}^{q-1} T_a^{-2}\left(\cos\left(\frac{\pi h}{q}\right)\right) = \sum_{a=1}^{q-1}\cos^{-2}\left(\frac{\pi ah}{q}\right) = q^2-1;$$

$$\sum_{a=1}^{q-1} T_a^{-4}\left(\cos\left(\frac{\pi h}{q}\right)\right) = \sum_{a=1}^{q-1}\cos^{-4}\left(\frac{\pi ah}{q}\right) = \frac{(q^2-1)(q^2+3)}{3}$$

and

$$\sum_{a=1}^{q-1} T_a^{-6}\left(\cos\left(\frac{\pi h}{q}\right)\right) = \sum_{a=1}^{q-1}\cos^{-6}\left(\frac{\pi ah}{q}\right) = \frac{(q^2-1)(2q^4+7q^2+15)}{15}.$$

Some notes: It is clear that there are some calculation mistakes in [12]. In fact, for $k = 3$, the corresponding results in [12] are

$$\sum_{a=1}^{q-1} U_{a-1}^{-6}\left(\cos\left(\frac{\pi h}{q}\right)\right) = \sin^6\left(\frac{\pi h}{q}\right)\cdot\frac{(q^2-1)(2q^2-11)(q^2+17)}{945}$$

and

$$\sum_{a=1}^{q-1} T_a^{-6}\left(\cos\left(\frac{\pi h}{q}\right)\right) = \sum_{a=1}^{q-1}\cos^{-6}\left(\frac{\pi ah}{q}\right) = \frac{(q^2-1)(2q^4+7q^2-363)}{15}.$$

That is to say, Theorems 1 and 2 in [12] are not correct for $k = 3$. Our theorems obtain a generalized conclusion for all integers $k \geq 1$. Thus, our results not only reveal the close connection between a certain trigonometric functions and the Riemann ζ-function, but also generalize some existing results. At the same time, an error in the existing [12] is corrected.

It is clear that $\{S(h,i)\}$ $(0 \leq i \leq h)$ is a symmetric polynomial sequence; it can be calculated by the recursive formula $S(h, i+1) = (2h)^2 \cdot S(h-1, i) + S(h-1, i+1)$ for all integers $0 \leq i \leq h-2$, $S(h, 0) = 1$ and $S(h, h) = 4^h \cdot (h!)^2$. This also reflects the advantages of our theorems. Here, we give partial values of $S(k, i)$, as shown in Table 1.

Table 1. Values of $S(h,i)$.

$S(h,i)$	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
$h=0$	1						
$h=1$	1	4					
$h=2$	1	20	64				
$h=3$	1	56	784	2304			
$h=4$	1	120	4368	52,480	147,456		
$h=5$	1	220	16,368	489,280	5,395,456	14,745,600	
$h=6$	1	364	48,048	2,846,272	75,851,776	791,691,264	2,123,366,400

In Table 1, the first three lines are the values of $S(h,i)$ corresponding to Corollaries 1 and 2, which are no longer listed separately.

2. Several Lemmas

To facilitate the proofs of our theorems, we need following four basic lemmas.

Lemma 1. *Let $f(s) = \frac{\pi^2}{\cos^2(\pi s)}$. For any positive integer k, we have*

$$\frac{(2k-1)!\cdot\pi^{2k}}{\cos^{2k}(s)} = \sum_{i=0}^{k-1} S(k-1,i)\cdot\pi^{2i}\cdot f^{(2k-2i-2)}(s),$$

where, as usual, $f^{(n)}(s)$ denotes the n-order derivative of $f(s)$, the constants $S(k-1,i)$ are defined as $\prod_{i=0}^{k-1}\left(x+(2i)^2\right) = \sum_{i=0}^{k-1} S(k-1,i)\cdot x^{k-i}$, and $S(0,0)=1$.

Proof. We prove this main lemma by mathematical induction. Note that $S(0,0)=1$; thus, from the definition of $f(s)$, we know that Lemma 1 is correct for $k=1$. From the definition and properties of the derivative, we have

$$f'(s) = \frac{2\pi^3\cdot\sin(\pi s)}{\cos^3(\pi s)} \quad\text{and}\quad f''(s) = \frac{3!\cdot\pi^4}{\cos^4(\pi s)} - \frac{2^2\cdot\pi^4}{\cos^2(\pi s)}. \tag{3}$$

It is clear that Equation (3) implies

$$\frac{3!\cdot\pi^4}{\cos^4(\pi s)} = f''(s) + 2^2\cdot\pi^2\cdot f(s) = \sum_{i=0}^{1} S(1,i)\cdot\pi^{2i}\cdot f^{(2-2i)}(s). \tag{4}$$

Thus, Equation (4) implies that Lemma 1 is correct for $k=2$.

Assuming that Lemma 1 is correct for all integers $1\le k\le h$, that is,

$$\frac{(2h-1)!\cdot\pi^{2h}}{\cos^{2h}(\pi s)} = \sum_{i=0}^{h-1} S(h-1,i)\cdot\pi^{2i}\cdot f^{(2h-2-2i)}(s), \tag{5}$$

then, from Equation (5) and the properties of the derivative, we have

$$\frac{(2h)!\cdot\pi^{2h+1}\cdot\sin(\pi s)}{\cos^{2h+1}(\pi s)} = \sum_{i=0}^{h-1} S(h-1,i)\cdot\pi^{2i}\cdot f^{(2h-2i-1)}(s)$$

and

$$\frac{(2h)!\cdot\pi^{2h+2}}{\cos^{2h}(\pi s)} + \frac{(2h+1)!\cdot\pi^{2h+2}\sin^2(\pi s)}{\cos^{2h+2}(\pi s)} = \sum_{i=0}^{h-1} S(h-1,i)\cdot\pi^{2i}\cdot f^{(2h-2i)}(s). \tag{6}$$

Note that $\sin^2(s) + \cos^2(s) = 1$ and $(2h)^2 \cdot S(h-1,i) + S(h-1,i+1) = S(h,i+1)$ for all integers $0 \le i \le h-2$. From Equations (5) and (6), we have

$$
\begin{aligned}
&\frac{(2h+1)! \cdot \pi^{2h+2}}{\cos^{2h+2}(\pi s)} = \frac{(2h)^2 \cdot (2h-1)! \cdot \pi^{2k+2}}{\cos^{2h}(\pi s)} + \sum_{i=0}^{h} S(h-1,i) \cdot \pi^{2i} \cdot f^{(2h-2i)}(s) \\
&= (2h)^2 \cdot \pi^2 \cdot \sum_{i=0}^{h-1} S(h-1,i) \cdot \pi^{2i} \cdot f^{(2h-2i-2)}(s) + \sum_{i=0}^{h-1} S(h-1,i) \cdot \pi^{2i} \cdot f^{(2h-2i)}(s) \\
&= (2h)^2 \cdot S(h-1,h-1) \cdot \pi^{2h} \cdot f(s) + S(h-1,0) \cdot f^{(2h)}(s) \\
&\quad + (2h)^2 \cdot \sum_{i=0}^{h-2} S(h-1,i) \cdot \pi^{2i+2} \cdot f^{(2h-2i-2)}(s) \\
&\quad + \sum_{i=1}^{h-1} S(h-1,i) \cdot \pi^{2i} \cdot f^{(2h-2i)}(s) \\
&= S(h,h) \cdot \pi^{2h} \cdot f(s) + \sum_{i=0}^{h-2} S(h-1,i+1) \cdot \pi^{2i+2} \cdot f^{(2h-2i-2)}(s) \\
&\quad + S(h,0) \cdot f^{(2h)}(s) + (2h)^2 \cdot \sum_{i=0}^{h-2} S(h-1,i) \cdot \pi^{2i+2} \cdot f^{(2h-2i-2)}(s) \\
&= \sum_{i=0}^{h-2} \Big((2h)^2 \cdot S(h-1,i) + S(h-1,i+1) \Big) \cdot \pi^{2i+2} \cdot f^{(2h-2i)}(s) \\
&+ S(h,h) \cdot \pi^{2h} \cdot f(s) + S(h,0) \cdot f^{(2h)}(s) \\
&= S(h,0) \cdot f^{(2h)}(s) + \sum_{i=0}^{h-2} S(h,i+1) \cdot \pi^{2i+2} \cdot f^{(2h-2i-2)}(s) + S(h,h) \cdot \pi^{2h} \cdot f(s) \\
&= S(h,0) \cdot f^{(2h)}(s) + \sum_{i=1}^{h-1} S(h,i) \cdot \pi^{2i} \cdot f^{(2h-2i)}(s) + S(h,h) \cdot \pi^{2h} \cdot f(s) \\
&= \sum_{i=0}^{h} S(h,i) \cdot \pi^{2i} \cdot f^{(2h-2i)}(s).
\end{aligned} \tag{7}
$$

Equation (7) implies that Lemma 1 is correct for $k = h+1$.

This proves Lemma 1 by mathematical induction. □

Lemma 2. *Let $g(s) = \frac{\pi^2}{\sin^2(\pi s)}$ and $h(s) = \frac{(2\pi)^2}{\sin^2(2\pi s)}$. For any positive integer k, we have the identities*

$$
\frac{(2k-1)! \cdot \pi^{2k}}{\sin^{2k}(\pi s)} = \sum_{i=0}^{k-1} S(k-1,i) \cdot \pi^{2i} \cdot g^{(2k-2i-2)}(s)
$$

and

$$
\frac{(2k-1)! \cdot \pi^{2k}}{\sin^{2k}(2\pi s)} = \sum_{i=0}^{k-1} S(k-1,i) \cdot (2\pi)^{2i} \cdot g^{(2k-2i-2)}(s),
$$

where the constants $S(k-1,i)$ are defined as in Lemma 1.

Proof. Noting that $\sin^2(\pi s) + \cos^2(\pi s) = 1$, we have

$$
\left(\frac{(2k-1)! \cdot \pi^{2k}}{\sin^{2k}(\pi s)} \right)' = -\frac{(2k)! \cdot \pi^{2k+1} \cdot \cos(\pi s)}{\sin^{2k+1}(\pi s)}
$$

and

$$\left(\frac{(2k-1)!\cdot\pi^{2k}}{\sin^{2k}(\pi s)}\right)'' = \frac{(2k)!\cdot\pi^{2k+2}}{\sin^{2k}(\pi s)} + \frac{(2k+1)!\cdot\pi^{2k+2}\cdot\cos^2(\pi s)}{\sin^{2k+2}(\pi s)}$$
$$= \frac{(2k+1)!\cdot\pi^{2k+2}}{\sin^{2k+2}(\pi s)} - \frac{(2k)^2\cdot(2k-1)!\cdot\pi^{2k+2}}{\sin^{2k}(\pi s)}.$$

Thus, it is easy to deduce the identities

$$\frac{3!\cdot\pi^4}{\sin^4(\pi s)} = \sum_{i=0}^{1} S(1,i)\cdot\pi^{2i}\cdot g^{(2-2i)}(s) \tag{8}$$

and

$$\frac{3!\cdot(2\pi)^4}{\sin^4(2\pi s)} = \sum_{i=0}^{1} S(1,i)\cdot(2\pi)^{2i}\cdot h^{(2-2i)}(s). \tag{9}$$

Then, from Equations (8) and (9) and mathematical induction, we can deduce Lemma 2. □

Lemma 3. *Let $q > 1$ be an odd number, $g(s) = \frac{\pi^2}{\sin^2(\pi s)}$. For any positive integer k, we have*

$$\sum_{a=1}^{q-1} g^{(2k-2)}\left(\frac{a}{q}\right) = 2\cdot(2k-1)!\cdot\left(q^{2k}-1\right)\cdot\zeta(2k),$$

where $\zeta(s)$ denotes the Riemann ζ-function.

Proof. From [23] (see Corollary 6, Section 3, Chapter 5), we have the identity

$$\sin(\pi s) = \pi s\cdot\prod_{n=1}^{\infty}\left(1-\frac{s^2}{n^2}\right). \tag{10}$$

Then, from Equation (10) and the properties of the derivative, we also have

$$g(s) = \frac{\pi^2}{\sin^2(\pi s)} = \frac{1}{s^2} + \sum_{n=1}^{\infty}\left(\frac{1}{(n+s)^2}+\frac{1}{(n-s)^2}\right). \tag{11}$$

In general, for any positive integer k, we have

$$g^{(2k-2)}(s) = (2k-1)!\cdot\left(\frac{1}{s^{2k}} + \sum_{n=1}^{\infty}\left(\frac{1}{(n+s)^{2k}}+\frac{1}{(n-s)^{2k}}\right)\right). \tag{12}$$

Taking $s = \frac{a}{q}$ in Equation (12), and then sum over all $1 \le a \le q-1$. From the definition of the Riemann zeta-function, we have

$$\sum_{a=1}^{q-1} g^{(2k-2)}\left(\frac{a}{q}\right) = (2k-1)!\cdot\sum_{a=1}^{q-1}\left(\frac{q^{2k}}{a^{2k}} + \sum_{n=1}^{\infty}\left(\frac{1}{\left(n+\frac{a}{q}\right)^{2k}}+\frac{1}{\left(n-\frac{a}{q}\right)^{2k}}\right)\right)$$
$$= (2k-1)!\cdot\left(\sum_{n=0}^{\infty}\sum_{a=1}^{q-1}\frac{q^{2k}}{(nq+a)^{2k}} + \sum_{n=1}^{\infty}\sum_{a=1}^{q-1}\frac{q^{2k}}{(nq-a)^{2k}}\right)$$
$$= 2\cdot(2k-1)!\cdot\left(q^{2k}\cdot\sum_{n=0}^{\infty}\sum_{a=1}^{q}\frac{1}{(nq+a)^{2k}} - \sum_{n=1}^{\infty}\frac{1}{n^{2k}}\right)$$

$$= 2 \cdot (2k-1)! \cdot \left(q^{2k}-1\right) \cdot \zeta(2k).$$

This proves Lemma 3. □

Lemma 4. *Let $q > 1$ be an odd number, $g(s) = \frac{\pi^2}{\sin^2(\pi s)}$ and $h(s) = \frac{(2\pi)^2}{\sin^2(2\pi s)}$. For any integer $k \geq 1$, we have the identity*

$$\sum_{a=1}^{q-1} h^{(2k-2)}\left(\frac{a}{q}\right) = 2^{2k} \cdot \sum_{a=1}^{q-1} g^{(2k-2)}\left(\frac{a}{q}\right).$$

Proof. If $k = 1$, then note that $(2, q) = 1$; from the properties of the complete residue system mod q and the definitions of $g(s)$ and $h(s)$, we have

$$\sum_{a=1}^{q-1} h\left(\frac{a}{q}\right) = 2^2 \cdot \sum_{a=1}^{q-1} \frac{\pi^2}{\sin^2\left(\frac{2\pi a}{q}\right)} = 2^2 \cdot \sum_{a=1}^{q-1} \frac{\pi^2}{\sin^2\left(\frac{\pi a}{q}\right)} = 2^2 \cdot \sum_{a=1}^{q-1} g\left(\frac{a}{q}\right). \tag{13}$$

Thus, Lemma 4 is correct for $k = 1$. Then, note that the identity

$$\sum_{a=1}^{q-1} \frac{(2\pi)^{2k}}{\sin^{2k}\left(\frac{2\pi a}{q}\right)} = 2^{2k} \cdot \sum_{a=1}^{q-1} \frac{\pi^{2k}}{\sin^{2k}\left(\frac{\pi a}{q}\right)} \tag{14}$$

holds for all positive integers k. Thus, Lemma 4 follows from Equations (13) and (14) and mathematical induction. □

3. Proofs of the Theorems

In this section, we use the lemmas in Section 2 to complete the proofs of our results. First, we prove Theorem 1. For any odd number $q \geq 3$ and integer h with $(h, q) = 1$, taking $s = \cos\left(\frac{\pi h}{q}\right)$, from Lemmas 2 and 3 and the properties of the complete residue system mod q, we have the identity

$$\begin{aligned}
&\sum_{a=1}^{q-1} \frac{1}{U_{a-1}^{2k}\left(\cos\left(\frac{\pi h}{q}\right)\right)} = \sin^{2k}\left(\frac{\pi h}{q}\right) \sum_{a=1}^{q-1} \frac{1}{\sin^{2k}\left(\frac{\pi h a}{q}\right)} \\
&= \sin^{2k}\left(\frac{\pi h}{q}\right) \sum_{a=1}^{q-1} \frac{1}{\sin^{2k}\left(\frac{\pi a}{q}\right)} = \frac{\sin^{2k}\left(\frac{\pi h}{q}\right)}{(2k-1)! \cdot \pi^{2k}} \cdot \sum_{a=1}^{q-1} \frac{(2k-1)! \cdot \pi^{2k}}{\sin^{2k}\left(\frac{\pi a}{q}\right)} \\
&= \frac{\sin^{2k}\left(\frac{\pi h}{q}\right)}{(2k-1)! \cdot \pi^{2k}} \cdot \sum_{i=0}^{k-1} S(k-1, i) \cdot \pi^{2i} \cdot \sum_{a=1}^{q-1} g^{(2k-2i-2)}\left(\frac{a}{q}\right) \\
&= \frac{\sin^{2k}\left(\frac{\pi h}{q}\right)}{(2k-1)! \cdot \pi^{2k}} \cdot \sum_{i=1}^{k} S(k-1, k-i) \cdot \pi^{2k-2i} \cdot \sum_{a=1}^{q-1} g^{(2i-2)}\left(\frac{a}{q}\right) \\
&= \frac{2 \cdot \sin^{2k}\left(\frac{\pi h}{q}\right)}{(2k-1)! \cdot \pi^{2k}} \cdot \sum_{i=1}^{k} S(k-1, k-i) \cdot \pi^{2k-2i} \cdot (2i-1)! \cdot \left(q^{2i}-1\right) \cdot \zeta(2i) \\
&= \frac{2 \cdot \sin^{2k}\left(\frac{\pi h}{q}\right)}{(2k-1)!} \cdot \sum_{i=1}^{k} \frac{S(k-1, k-i) \cdot (2i-1)! \cdot \left(q^{2i}-1\right)}{\pi^{2i}} \cdot \zeta(2i).
\end{aligned}$$

This proves Theorem 1.

Now, we prove Theorem 2. Note that the identity

$$f(s) = \frac{\pi^2}{\cos^2(\pi s)} = \frac{(2\pi)^2}{\sin^2(2\pi s)} - \frac{\pi^2}{\sin^2(\pi s)} = h(s) - g(s).$$

Thus, from this identity and Lemma 1, we have

$$\begin{aligned} \frac{(2k-1)! \cdot \pi^{2k}}{\cos^{2k}(s)} &= \sum_{i=0}^{k-1} S(k-1,i) \cdot \pi^{2i} \cdot f^{(2k-2i-2)}(s) \\ &= \sum_{i=0}^{k-1} S(k-1,i) \cdot \pi^{2i} \cdot \left(h^{(2k-2i-2)}(s) - g^{(2k-2i-2)}(s)\right). \end{aligned} \tag{15}$$

From Equation (15), Lemmas 3 and 4, and the properties of the complete residue system mod q, we have

$$\begin{aligned} &\sum_{a=1}^{q-1} \frac{1}{T_a^{2k}\left(\cos\left(\frac{\pi h}{q}\right)\right)} = \sum_{a=1}^{q-1} \frac{1}{\cos^{2k}\left(\frac{\pi h a}{q}\right)} = \frac{1}{(2k-1)! \cdot \pi^{2k}} \sum_{a=1}^{q-1} \frac{(2k-1)! \cdot \pi^{2k}}{\cos^{2k}\left(\frac{\pi a}{q}\right)} \\ = \; & \frac{1}{(2k-1)! \cdot \pi^{2k}} \cdot \sum_{i=0}^{k-1} S(k-1,i) \cdot \pi^{2i} \cdot \sum_{a=1}^{q-1} \left(h^{(2k-2i-2)}\left(\frac{a}{q}\right) - g^{(2k-2i-2)}\left(\frac{a}{q}\right)\right) \\ = \; & \frac{1}{(2k-1)! \cdot \pi^{2k}} \cdot \sum_{i=0}^{k-1} S(k-1,i) \cdot \pi^{2i} \cdot \sum_{a=1}^{q-1} 2^{2k-2i} \cdot g^{(2k-2i-2)}\left(\frac{a}{q}\right) \\ & - \frac{1}{(2k-1)! \cdot \pi^{2k}} \cdot \sum_{i=0}^{k-1} S(k-1,i) \cdot \pi^{2i} \cdot \sum_{a=1}^{q-1} g^{(2k-2i-2)}\left(\frac{a}{q}\right) \\ = \; & \frac{1}{(2k-1)! \cdot \pi^{2k}} \cdot \sum_{i=0}^{k-1} S(k-1,i) \cdot \pi^{2i} \cdot \left(2^{2k-2i} - 1\right) \cdot \sum_{a=1}^{q-1} g^{(2k-2i-2)}\left(\frac{a}{q}\right) \\ = \; & \frac{1}{(2k-1)!} \cdot \sum_{i=1}^{k} S(k-1,k-i) \cdot \pi^{-2i} \cdot \left(2^{2i} - 1\right) \cdot \sum_{a=1}^{q-1} g^{(2i-2)}\left(\frac{a}{q}\right) \\ = \; & \frac{2}{(2k-1)!} \cdot \sum_{i=1}^{k} \frac{S(k-1,k-i) \cdot \left(2^{2i}-1\right) \cdot (2i-1)! \cdot \left(q^{2i}-1\right)}{\pi^{2i}} \cdot \zeta(2i). \end{aligned}$$

Then, we complete the proof of Theorem 2.

4. Conclusions

In this paper, we obtain two main results. Theorem 1 establishes a generalized calculation formula for a certain reciprocal sums of Chebyshev polynomials of the first kind. Theorem 2 establishes a generalized calculation formula for a certain reciprocal sums of Chebyshev polynomials of the second kind. As two special cases or two corollaries of our theorems, we give a new proof of the results in [12], and we also point out two computational errors in [12].

Author Contributions: All authors have equally contributed to this work. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the B.R.P.N.S.(2017JK1002) of Shaanxi Province and N. S. F. (11701447) of China.

Acknowledgments: The authors would like to thank the reviewers for their very helpful and detailed comments.

Conflicts of Interest: The authors declare that there are no conflict of interest regarding the publication of this paper.

References

1. Borwein, P.; Erdélyi, T. *Polynomials and Polynomial Inequalities*; Springer: New York, NY, USA, 1995.
2. Li, X.X. Some identities involving Chebyshev polynomials. *Math. Probl. Eng.* **2015**, *2015*. [CrossRef]
3. Kim, T.; Dolgy, D.V.; Kim, D.S. Representing sums of finite products of Chebyshev polynomials of the second kind and Fibonacci polynomials in terms of Chebyshev polynomials. *Adv. Stud. Contemp. Math.* **2018**, *28*, 321–336.
4. Kim, T.; Kim, D.S.; Dolgy, D.V.; Kim, D. Representation by several orthogonal polynomials for sums of finite products of Chebyshev polynomials of the first, third and fourth kinds. *Adv. Differ. Equ.* **2019**, *2019*, 110. [CrossRef]
5. Kim, T.; Kim, D.S.; Dolgy, D.V.; Kwon, J. Representing Sums of Finite Products of Chebyshev Polynomials of the First Kind and Lucas Polynomials by Chebyshev Polynomials. *Mathematics* **2019**, *2019*, 26. [CrossRef]
6. Kim, T.; Kim, D.S.; Jang, L.-C.; Jang, G.-W. Fourier Series for Functions Related to Chebyshev Polynomials of the First Kind and Lucas Polynomials. *Mathematics* **2018**, *2018*, 276. [CrossRef]
7. Dolgy, D.V.; Kim, D.S.; Kim, T.; Kwon, J. Connection Problem for Sums of Finite Products of Chebyshev Polynomials of the Third and Fourth Kinds. *Symmetry* **2018**, *2018*, 617. [CrossRef]
8. Kim, T.; Kim, D.S.; Jang, L.-C.; Dolgy, D.V. Representing by several orthogonal polynomials for sums of finite products of Chebyshev polynomials of the first kind and Lucas polynomials. *Adv. Differ. Equ.* **2019**, *2019*, 162. [CrossRef]
9. Wang, T.T.; Zhang, H. Some identities involving the derivative of the first kind Chebyshev polynomials. *Math. Probl. Eng.* **2015**, *2015*. [CrossRef]
10. Zhang, W.P.; Wang, T.T. Two identities involving the integral of the first kind Chebyshev polynomials. *Bull. Math. Soc. Sci. Math. Roum.* **2017**, *60*, 91–98.
11. Lv, X.X.; Shen, S.M. On Chebyshev polynomials and their applications. *Adv. Differ. Equ.* **2017**, *2017*, 343. [CrossRef]
12. Ma, Y.; Lv, X.X. Some identities involving the reciprocal sums of one kind Chebyshev polynomials. *Math. Probl. Eng.* **2017**, *2017*. [CrossRef]
13. Chen, L.; Zhang, W.P. Chebyshev polynomials and their some interesting applications. *Adv. Differ. Equ.* **2017**, *2017*, 303.
14. Clemente, C. Identities and generating functions on Chebyshev polynomials. *Georgian Math. J.* **2012**, *19*, 427–440.
15. Lee, C.; Wong, K.B. On Chebyshev's Polynomials and Certain Combinatorial Identities. *Bull. Malays. Math. Sci.* **2011**, *34*, 279–286.
16. Bircan, N.; Pommerenke, C. On Chebyshev polynomials and $\mathrm{GL}(2, Z/pZ)$. *Bull. Math. Soc. Sci. Math. Roum.* **2012**, *103*, 353–364.
17. Altinkaya, S.; Yalcin, S. On the Chebyshev polynomial coefficient problem of Bi-Bazilevic functions. *TWMS J. Appl. Eng. Math.* **2020**, *10*, 251–258.
18. Foucart, S.; Jean, B. Computation of Chebyshev Polynomials for Union of Intervals. *Comput. Methods Funct. Theory* **2019**, *19*, 625–641. [CrossRef]
19. Heydari, M.H. A direct method based on the Chebyshev polynomials for a new class of nonlinear variable-order fractional 2D optimal control problems. *J. Frankl. Inst.-Eng. Appl. Math.* **2019**, *356*, 8216–8236. [CrossRef]
20. Zhang, Y.X.; Chen, Z.Y. A new identity involving the Chebyshev polynomials. *Mathematics* **2018**, *6*, 244. [CrossRef]
21. Wang, S.Y. Some new identities of Chebyshev polynomials and their applications. *Adv. Differ. Equ.* **2015**, *2015*, 355.
22. Zhang, W.P. On Chebyshev polynomials and Fibonacci numbers. *Fibonacci Q.* **2002**, *40*, 424–428.
23. Pan, C.D.; Pan, C.B. *Basic Analytic Number Theory*; Harbin Institute of Technology Press: Harbin, China, 2016.