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New Weighted Opial-Type Inequalities on Time Scales for Convex Functions

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Abstract: Our work is based on the multiple inequalities illustrated in 1967 by E. K. Godunova and V. I. Levin, in 1990 by Hwang and Yang and in 1993 by B. G. Pachpatte. With the help of the dynamic Jensen and Hölder inequality, we generalize a number of those inequalities to a general time scale. In addition to these generalizations, some integral and discrete inequalities will be obtained as special cases of our results.

Keywords: opial-type inequality; dynamic inequality; convexity; time scale

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1. Introduction

The following inequality [1] is well-known in the literature as Opial's inequality.

Theorem 1. *If δ is an absolutely continuous function on $[0,h]_{\mathbb{R}}$ with $\delta(0)=\delta(h)=0$, then*

$$\int_0^h |\delta(t)\delta'(t)|dt \le \frac{h}{4}\int_0^h |\delta'(t)|^2 dt. \tag{1}$$

In 1967 E. K. Godunova and V. I. Levin [2] proved the following two theorems which are a generalization of Opial's inequality (1).

Theorem 2. *Let δ be real-valued absolutely continuous function on $[a,b]_{\mathbb{R}}$ with $\delta(a)=0$. Let f be real-valued convex and increasing function on $[0,\infty)_{\mathbb{R}}$ with $f(0)=0$. Then, the following inequality holds*

$$\int_a^b f'(|\delta(t)|)|\delta'(t)|\mathrm{d}t \leqslant f\left(\int_a^b |\delta'(t)|\mathrm{d}t\right).$$

Theorem 3. *Let δ be real-valued absolutely continuous function on $[a,b]_{\mathbb{R}}$ with $\delta(a)=\delta(b)=0$. Assume f and g are real-valued convex and increasing functions on $[0,\infty)_{\mathbb{R}}$ with $f(0)=0$. Further let $p\ge 0$ on $[a,b]_{\mathbb{R}}$ and $\int_a^b p(t)\mathrm{d}t=1$. Then, the following inequality holds*

$$\int_a^b f'(|\delta(t)|)|\delta'(t)|\mathrm{d}t \le 2f\left(g^{-1}\left(\int_a^b p(t)g\left(\frac{|\delta'(t)|}{2p(t)}\right)\mathrm{d}t\right)\right). \tag{2}$$

In 1990, Hwang and Yang [3] established the following result:

Theorem 4. *Assuming $f \geq 0$ and $g \geq 0$ are continuous functions on $[0, \infty)_{\mathbb{R}}$ with $f(0) = 0$ such that $f' \geq 0$ and $g' \geq 0$ exist and non-decreasing continuous functions on $[0, \infty)_{\mathbb{R}}$. Suppose x and y are absolutely continuous functions on $[a, \tau]_{\mathbb{R}}$, and $x(a) = y(a) = 0$. Then, for all $m \geq 1$, we get*

$$\begin{aligned} &\int_a^\tau \left[f(|\delta(t)|^m)g'(|\gamma(t)|^m)|\gamma'(t)|^m + g(|\gamma(t)|^m)f'(|\delta(t)|^m)|\delta'(t)|^m\right]\mathrm{d}t \\ &\leq \frac{1}{\lambda} f\left(\lambda \int_a^\tau |\delta'(t)|^m \mathrm{d}t\right) g\left(\lambda \int_a^\tau |\gamma'(t)|^m \mathrm{d}t\right), \end{aligned} \tag{3}$$

where $\lambda = (\tau - a)^{m-1}$.

S. Hilger [4], suggested time scales theory to unify discrete and continuous analysis. Continuous calculus, discrete calculus, and quantum calculus can be said as the three most common examples of calculus on time scales i.e., for continuous calculus $\mathbb{T} = \mathbb{R}$, for discrete calculus $\mathbb{T} = h\mathbb{Z}$ and for quantum calculus $\mathbb{T} = \overline{q^{\mathbb{Z}}} = \{q^z : z \in \mathbb{Z}\} \cup \{0\}$ where $q > 1$. The book due to Bohner and Peterson [5] on the subject of time scales briefs and organizes much of time scales calculus. For some Opial-type integral, dynamic inequalities and other types of inequalities on time scales, see the papers [6–32]. More results on inequalities see, [33–53].

The following essential relations on some time scales such as $\mathbb{R}$, $\mathbb{Z}$, $h\mathbb{Z}$ and $\overline{q^{\mathbb{Z}}}$ will be used in the following section. Note that:

(i) If $\mathbb{T} = \mathbb{R}$, then

$$\sigma(t) = t, \quad \mu(t) = 0, \quad \psi^\Delta(t) = \psi'(t), \quad \int_\alpha^\beta \psi(t)\Delta t = \int_\alpha^\beta \psi(t)dt. \tag{4}$$

(ii) If $\mathbb{T} = \mathbb{Z}$, then

$$\sigma(t) = t+1, \quad \mu(t) = 1, \quad \psi^\Delta(t) = \psi(t+1) - \psi(t), \quad \int_\alpha^\beta \psi(t)\Delta t = \sum_{t=\alpha}^{\beta-1} \psi(t). \tag{5}$$

(iii) If $\mathbb{T} = h\mathbb{Z}$, then

$$\sigma(t) = t+h, \quad \mu(t) = h, \quad \psi^\Delta(t) = \frac{\psi(t+h) - \psi(t}{h}), \quad \int_\alpha^\beta \psi(t)\Delta t = \sum_{t=\frac{\alpha}{h}}^{\frac{\beta}{h}-1} \psi(th)h. \tag{6}$$

(iv) If $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, then

$$\sigma(t) = qt, \quad \mu(t) = (q-1)t, \quad \psi^\Delta(t) = \tfrac{\psi(qt)-\psi(t)}{(q-1)t}, \quad \textstyle\int_\alpha^\beta \psi(t)\Delta t = (q-1)\sum_{t=(\log_q \alpha)}^{(\log_q \beta)-1} \psi(q^t)q^t. \tag{7}$$

Next is Hölder's and Jensen's inequality:

Lemma 1 (Hölder's inequality [5])**.** *Let $a, b \in \mathbb{T}$. For $f, g \in C_{rd}(\mathbb{T}, \mathbb{R})$, we have*

$$\int_a^b f(t)g(t)\Delta t \leq \left[\int_a^b f^p(t)\Delta t\right]^{\frac{1}{p}} \left[\int_a^b g^q(t)\Delta t\right]^{\frac{1}{q}},$$

where $p > 1$ and ${}^1/_p + {}^1/_q = 1$.

Lemma 2 (Jensen's inequality [16]). *Let a, $b \in \mathbb{T}$ and c, $d \in \mathbb{R}$. Assume that $g \in C_{rd}\big([a,b]_{\mathbb{T}},[c,d]\big)$ and $r \in C_{rd}\big([a,b]_{\mathbb{T}},\mathbb{R}\big)$ are nonnegative with $\int_a^b r(t)\Delta t > 0$. If $\check{\Phi} \in C_{rd}\big((c,d),\mathbb{R}\big)$ is a convex function, then*

$$\check{\Phi}\left(\frac{\int_a^b r(t)g(t)\Delta t}{\int_a^b r(t)\Delta t}\right) \leqslant \frac{\int_a^b r(t)\check{\Phi}(g(t))\Delta t}{\int_a^b r(t)\Delta t}.$$

Lemma 3 (Chain rule [5]). *Assume $g : \mathbb{R} \to \mathbb{R}$, is continuous, $g : \mathbb{T} \to \mathbb{R}$ is delta-differentiable on $\mathbb{T}^\kappa$, and $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then there exists c in the real interval $[t, \sigma(t)]$ with*

$$(f \circ g)^\Delta(t) = f'(g(c))g^\Delta(t). \tag{8}$$

In the proofs of our results, the following inequality will be used:

$$a^k + b^k \le (a+b)^k \le 2^{k-1}\big(a^k + b^k\big), \quad \text{if } a \ge 0, b \ge 0 \text{ and } k \ge 1. \tag{9}$$

In this article, we prove some dynamic Opial-type inequalities involving convex functions on time scales. Our results generalize some of the mentioned results of Pachpatte [54,55], E. K. Godunova and V. I. Levin [2] and Hwang and Yang [3], in time scales. Furthermore, our results extend some existing dynamic Opial-type inequalities in the literature, and give some integral and discrete inequalities as special cases.

2. Main Results

Theorem 5. *Assuming $\mathbb{T}$ is a time scale with Γ, $\alpha \in \mathbb{T}$, $g \ge 0$ and $f \ge 0$ are continuous functions on $[0,\infty)_{\mathbb{R}}$ with $f(0) = 0$ such that $g' \ge 0$ and $f' \ge 0$ exist and non-decreasing continuous functions on $[0,\infty)_{\mathbb{R}}$. Suppose x and y are rd-continuous functions on $[\alpha,\Gamma]_{\mathbb{T}}$, and $x(\alpha) = y(\alpha) = 0$. Then, for all $m \ge 1$, we get*

$$\begin{aligned}&\int_\alpha^\Gamma \Big[f\big(|x(\theta)|^m\big)g'\big(|y(\theta)|^m\big)\big|y^\Delta(\theta)\big|^m + g\big(|y(\theta)|^m\big)f'\big(|x(\theta)|^m\big)\big|x^\Delta(\theta)\big|^m\Big]\Delta\theta \\ &\le \frac{1}{\lambda_1} f\left(\lambda_1 \int_\alpha^\Gamma \big|x^\Delta(\theta)\big|^m \Delta\theta\right) g\left(\lambda_1 \int_\alpha^\Gamma \big|y^\Delta(\theta)\big|^m \Delta\theta\right),\end{aligned} \tag{10}$$

where $\lambda_1 = (\Gamma - \alpha)^{m-1}$.

Proof. For $t \in [\alpha,\Gamma]_{\mathbb{T}}$. Define $\varphi(t) := \int_\alpha^t \big|x^\Delta(\theta)\big|^m \Delta\theta$ and $\psi(t) := \int_\alpha^t \big|y^\Delta(\theta)\big|^m \Delta\theta$ so that $\varphi^\Delta(t) = \big|x^\Delta(t)\big|^m$ and $\psi^\Delta(t) = \big|y^\Delta(t)\big|^m$. Thus,

$$\begin{aligned}|x(t)|^m &= \left|\int_\alpha^t x^\Delta(\theta)\Delta\theta\right|^m \le \left(\int_\alpha^t \big|x^\Delta(\theta)\big|\Delta\theta\right)^m, \\ |y(t)|^m &= \left|\int_\alpha^t y^\Delta(\theta)\Delta\theta\right|^m \le \left(\int_\alpha^t \big|y^\Delta(\theta)\big|\Delta\theta\right)^m.\end{aligned} \tag{11}$$

Next, applying the dynamic Hölder's inequality (1) on (11) with indices m and $\frac{m}{m-1}$, we get

$$\begin{aligned}|x(t)|^m &\le (t-\alpha)^{m-1}\left(\int_\alpha^t \big|x^\Delta(\theta)\big|^m \Delta\theta\right) \\ &\le (t-\alpha)^{m-1}\left(\int_\alpha^t \big|x^\Delta(\theta)\big|^m \Delta\theta\right) \\ &\le \lambda_1 \varphi(t).\end{aligned} \tag{12}$$

and similarly
$$|y(t)|^m \le \lambda_1 \psi(t). \tag{13}$$

Since $f \ge 0$, $f' \ge 0$, $g \ge 0$, and $g' \ge 0$ are non-decreasing continuous on $[0,\infty)_{\mathbb{R}}$. Since $\sigma(t) \ge t$, we get $g(\lambda_1\psi) \le g^{\sigma}(\lambda_1\psi)$, and we find that

$$\begin{aligned}
&\int_{\alpha}^{\Gamma}\left[f(|x(\theta)|^m)g'(|y(\theta)|^m)\left|y^{\Delta}(\theta)\right|^m + g(|y(\theta)|^m)f'(|x(\theta)|^m)\left|x^{\Delta}(\theta)\right|^m\right]\Delta\theta \\
&\le \int_{\alpha}^{\Gamma}\left[f(\lambda_1\varphi(\theta))g'(\lambda_1\psi(\theta))\psi^{\Delta}(\theta) + g(\lambda_1\psi(\theta))f'(\lambda_1\varphi(\theta))\varphi^{\Delta}(\theta)\right]\Delta\theta \\
&\le \int_{\alpha}^{\Gamma}\left[f(\lambda_1\varphi(\theta))g'(\lambda_1\psi(\theta))\psi^{\Delta}(\theta) + g^{\sigma}(\lambda_1\psi(\theta))f'(\lambda_1\varphi(\theta))\varphi^{\Delta}(\theta)\right]\Delta\theta.
\end{aligned} \tag{14}$$

From Lemma 3 for $c \in [t,\sigma(t)]_{\mathbb{R}}$, we get

$$\frac{1}{\lambda_1}f^{\Delta}(\lambda_1\varphi)(t) = f'(\lambda_1\varphi(c))\varphi^{\Delta}(t) \ge f'(\lambda_1\varphi(t))\varphi^{\Delta}(t), \tag{15}$$

and similarly
$$\frac{1}{\lambda_1}g^{\Delta}(\lambda_1\psi(t)) \ge g'(\lambda_1\psi(t))\psi^{\Delta}(t). \tag{16}$$

Then, by substituting (15) and (16) into (14), we get

$$\begin{aligned}
&\int_{\alpha}^{\Gamma}\left[f(|x(\theta)|^m)g'(|y(\theta)|^m)\left|y^{\Delta}(\theta)\right|^m + g(|y(\theta)|^m)f'(|x(\theta)|^m)\left|x^{\Delta}(\theta)\right|^m\right]\Delta\theta \\
&\le \int_{\alpha}^{\Gamma}\left[f(\lambda_1\varphi(\theta))g'(\lambda_1\psi(\theta))\psi^{\Delta}(\theta) + g^{\sigma}(\lambda_1\psi(\theta))f'(\lambda_1\varphi(\theta))\varphi^{\Delta}(\theta)\right]\Delta\theta \\
&= \frac{1}{\lambda_1}\int_{\alpha}^{\Gamma}\left[f(\lambda_1\varphi)g(\lambda_1\psi)\right]^{\Delta}(\theta)\Delta\theta \\
&= \frac{1}{\lambda_1}f\left(\lambda_1\int_{\alpha}^{\Gamma}\left|x^{\Delta}(\theta)\right|^m\Delta\theta\right)g\left(\lambda_1\int_{\alpha}^{\Gamma}\left|y^{\Delta}(\theta)\right|^m\Delta\theta\right),
\end{aligned}$$

which is the desired inequality (10). This proves our claim. □

Remark 1. *When $\mathbb{T} = \mathbb{R}$ in Theorem 5, then, by the relation (4), we obtain Hwang and Yang inequality (3).*

Corollary 1. *If we take $\mathbb{T} = h\mathbb{Z}$ in Theorem 5, then, by the relation (6), inequality (10) becomes*

$$\begin{aligned}
&h\sum_{n=\frac{\alpha}{h}}^{\frac{\Gamma}{h}-1}\left[f(|x(nh)|^m)g'(|y(nh)|^m)\left|\Delta y(nh)\right|^m + g(|y(nh)|^m)f'(|x(nh)|^m)\left|\Delta x(nh)\right|^m\right] \\
&\le \frac{1}{\lambda_1}f\left(h\lambda_1\sum_{n=\frac{\alpha}{h}}^{\frac{\Gamma}{h}-1}\left|\Delta x(nh)\right|^m\right)g\left(h\lambda_1\sum_{n=\frac{\alpha}{h}}^{\frac{\Gamma}{h}-1}\left|\Delta y(nh)\right|^m\right).
\end{aligned}$$

Remark 2. *In Corollary 2, if we take $h = 1$, then inequality (10) becomes*

$$\begin{aligned}
&\sum_{n=\alpha}^{\Gamma-1}\left[f(|x(n)|^m)g'(|y(n)|^m)\left|\Delta y(n)\right|^m + g(|y(n)|^m)f'(|x(n)|^m)\left|\Delta x(n)\right|^m\right] \\
&\le \frac{1}{\lambda_1}f\left(\lambda_1\sum_{n=\alpha}^{\Gamma-1}\left|\Delta x(n)\right|^m\right)g\left(\lambda_1\sum_{n=\alpha}^{\Gamma-1}\left|\Delta y(n)\right|^m\right).
\end{aligned}$$

Corollary 2. *If we take $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ in Theorem 5, then, by the relation (7), inequality (10) becomes*

$$(q-1)\sum_{n=(\log_q \alpha)}^{(\log_q \Gamma)-1}\Big[f\big(|x(q^n)|^m\big)g'\big(|y(q^n)|^m\big)\big|\Delta y(q^n)\big|^m + g\big(|y(q^n)|^m\big)f'\big(|x(q^n)|^m\big)\big|\Delta x(q^n)\big|^m\Big]q^n$$
$$\leq \frac{1}{\lambda_1}f\bigg((q-1)\lambda_1\sum_{n=(\log_q \alpha)}^{(\log_q \Gamma)-1}\big|\Delta x(q^n)\big|^m q^n\bigg)g\bigg((q-1)\lambda_1\sum_{n=(\log_q \alpha)}^{(\log_q \Gamma)-1}\big|\Delta y(q^n)\big|^m q^n\bigg).$$

Remark 3. *If $g = m = 1$, and $x = y$ in inequality (10), then, we obtain*

$$\int_\alpha^\Gamma f'\big(|x(\theta)|\big)\big|x^\Delta(\theta)\big|\Delta\theta \leq f\bigg(\int_\alpha^\Gamma \big|x^\Delta(\theta)\big|\Delta\theta\bigg) \tag{17}$$

with weak conditions for f.

Remark 4. *If $\mathbb{T} = \mathbb{R}$, then inequality (17) gives E. K. Godunova and V. I. Levin's inequality in [2].*

Corollary 3. *Assuming $f(t) = t^{\frac{\ell+m}{m}}$, $g = 1$, $x = y$ and $\ell \geq 0$ in Theorem 5. Then, from (10), and Lemma 1 with indices $\frac{\ell+m}{\ell}$ and $\frac{\ell+m}{m}$, we obtain*

$$\begin{aligned}\int_\alpha^\Gamma |x(\theta)|^\ell\big|x^\Delta(\theta)\big|^m\Delta\theta &\leq \frac{m}{\ell+m}\frac{1}{\lambda_1}\bigg(\lambda_1\int_\alpha^\Gamma \big|x^\Delta(\theta)\big|^m\Delta\theta\bigg)^{\frac{\ell+m}{m}} \\ &\leq \frac{m}{\ell+m}\lambda_1^{\frac{\ell}{m}}(\Gamma-\alpha)^{\frac{\ell}{m}}\int_\alpha^\Gamma\big|x^\Delta(\theta)\big|^{\ell+m}\Delta\theta \\ &= \frac{m}{\ell+m}(\Gamma-\alpha)^\ell\int_\alpha^\Gamma\big|x^\Delta(\theta)\big|^{\ell+m}\Delta\theta.\end{aligned} \tag{18}$$

Corollary 4. *When $m = 1$ and $\ell = 1$ in (18), we obtain the following inequality*

$$\int_\alpha^\Gamma |x(\theta)|\big|x^\Delta(\theta)\big|\Delta\theta \leq \frac{(\Gamma-\alpha)}{2}\int_\alpha^\Gamma\big|x^\Delta(\theta)\big|^2\Delta\theta. \tag{19}$$

Corollary 5. *When $\mathbb{T} = \mathbb{R}$, $m = 1$ and $\alpha = 0$ in (18), we obtain the inequality of Hua [56]*

$$\int_0^\Gamma |x(\theta)|^\ell\big|x'(\theta)\big|\mathrm{d}\theta \leq \frac{\Gamma^\ell}{\ell+1}\int_0^\Gamma\big|x'(\theta)\big|^{\ell+1}\mathrm{d}\theta. \tag{20}$$

Moreover, in (20), equality holds if and only if $x(t) = ct$.

Corollary 6. *If $\mathbb{T} = \mathbb{R}$, then (18), gives the inequality of Yang [57].*

$$\int_\alpha^\Gamma |x(\theta)|^\ell\big|x'(\theta)\big|^m\mathrm{d}\theta \leq \frac{m}{\ell+m}(\Gamma-\alpha)^\ell\int_\alpha^\Gamma\big|x'(\theta)\big|^{\ell+m}\mathrm{d}\theta. \tag{21}$$

Corollary 7. *Assuming $p \geq 0$ is rd-continuous on $[\alpha,\Gamma]_{\mathbb{T}}$ with $\int_\alpha^\Gamma \frac{\Delta\theta}{p(\theta)} < \infty$, and let $q > 0$ is non-increasing bounded on $[\alpha,\Gamma]_{\mathbb{T}}$. By using $f(t) = t^2$, $g(t) = 1$, $m = 1$ and $x(t) = y(t) = \int_\alpha^t \sqrt{q(\theta)}|z^\Delta(\theta)|\Delta\theta$ and $x^\Delta(t) = \sqrt{q(t)}|z^\Delta(t)|$. Then, we get from Theorem 5, the following inequality*

$$2\int_\alpha^\Gamma\bigg(\int_\alpha^\theta\sqrt{q(\xi)}\big|x^\Delta(\xi)\big|\Delta\xi\bigg)\sqrt{q(\theta)}\big|z^\Delta(\theta)\big|\Delta\theta \leq \bigg(\int_\alpha^\Gamma\sqrt{q(\theta)}\big|z^\Delta(\theta)\big|\Delta\theta\bigg)^2.$$

However, since

$$\int_{\alpha}^{t} \sqrt{q(\theta)}|z^{\Delta}(\theta)|\Delta\theta \geq \sqrt{q(t)} \int_{\alpha}^{t} |z^{\Delta}(\theta)|\Delta\theta \geq \sqrt{q(t)}z(t),$$

it follows that

$$2\int_{\alpha}^{\Gamma} q(\theta)|x(\theta)||x^{\Delta}(\theta)|\Delta\theta \leq \left(\int_{\alpha}^{\Gamma} \frac{1}{\sqrt{p(\theta)}}\sqrt{p(\theta)q(\theta)}|x^{\Delta}(\theta)|\Delta\theta\right)^{2}.$$

By applying the Cauchy–Schwarz inequality, we get that

$$\int_{\alpha}^{\Gamma} q(\theta)|x(\theta)||x^{\Delta}(\theta)|\Delta\theta \leq \frac{1}{2}\int_{\alpha}^{\Gamma} \frac{\Delta\theta}{p(\theta)} \int_{\alpha}^{\Gamma} p(\theta)q(\theta)|x^{\Delta}(\theta)|^{2}\Delta\theta. \tag{22}$$

Remark 5. *When $\mathbb{T} = \mathbb{R}$, the inequality (22), reduces to Yang inequality [57]*

$$\int_{\alpha}^{\Gamma} q(\theta)|x(\theta)||x'(\theta)|\mathrm{d}\theta \leq \frac{1}{2}\int_{\alpha}^{\Gamma} \frac{\mathrm{d}\theta}{p(\theta)} \int_{\alpha}^{\Gamma} p(\theta)q(\theta)|x'(\theta)|^{2}\mathrm{d}\theta.$$

Corollary 8. *Take $f(t) = t^2$, $g(t) = 1$, $m = 1$ and $x = y$ in Theorem 5, the inequality (10) becomes*

$$\begin{aligned}\int_{\alpha}^{\Gamma} |x(\theta)||x^{\Delta}(\theta)|\Delta\theta &\leq \frac{1}{2}\left(\int_{\alpha}^{\Gamma} |x^{\Delta}(\theta)|\Delta\theta\right)^{2} \\ &= \frac{1}{2}\left(\int_{\alpha}^{\Gamma} p^{\frac{1}{\hat{\nu}}}(\theta)p^{-\frac{1}{\hat{\nu}}}(\theta)|x^{\Delta}(\theta)|\Delta\theta\right)^{2}.\end{aligned}$$

From Lemma 1 with indices $\hat{\nu}$, $\hat{\mu}$ such that $\frac{1}{\hat{\nu}} + \frac{1}{\hat{\mu}} = 1$, we obtain

$$\int_{\alpha}^{\Gamma} |x(\theta)||x^{\Delta}(\theta)|\Delta\theta \leq \frac{1}{2}\left(\int_{\alpha}^{\Gamma} p^{1-\hat{\mu}}(\theta)\Delta\theta\right)^{\frac{2}{\hat{\mu}}} \left(\int_{\alpha}^{\Gamma} p(\theta)|x^{\Delta}(\theta)|^{\hat{\nu}}\Delta\theta\right)^{\frac{2}{\hat{\nu}}}. \tag{23}$$

Remark 6. *If $\mathbb{T} = \mathbb{R}$, then (23), gives the inequality of Maroni [58].*

$$\int_{\alpha}^{\Gamma} |x(\theta)||x^{\Delta}(\theta)|\mathrm{d}\theta \leq \frac{1}{2}\left(\int_{\alpha}^{\Gamma} p^{1-\hat{\mu}}(\theta)\mathrm{d}\theta\right)^{\frac{2}{\hat{\mu}}} \left(\int_{\alpha}^{\Gamma} p(\theta)|x'(\theta)|^{\nu}\mathrm{d}\theta\right)^{\frac{2}{\nu}}. \tag{24}$$

Theorem 6. *Under the hypotheses of Theorem 5. Let $p^* > 0$ and $p > 0$ be defined on $[\alpha, \Gamma]_{\mathbb{T}}$ with $\int_{\alpha}^{\Gamma} p(\theta)\Delta\theta = 1$ and $\int_{\alpha}^{\Gamma} p^*(\theta)\Delta\theta = 1$. Further, let $\eta > 0$ be convex and increasing on $[0, \infty)_{\mathbb{R}}$. Then, for all $m \geq 1$, the following inequality holds*

$$\begin{aligned}&\int_{\alpha}^{\Gamma} \left[f(|x(\theta)|^{m})g'(|y(\theta)|^{m})|y^{\Delta}(\theta)|^{m} + g(|y(\theta)|^{m})f'(|x(\theta)|^{m})|x^{\Delta}(\theta)|^{m}\right]\Delta\theta \\ &\leq \frac{1}{\lambda_1} f\left(2\lambda_1\eta^{-1}\left(\int_{\alpha}^{\Gamma} p(\theta)\eta\left(\frac{|x^{\Delta}(\theta)|^{m}}{2p(\theta)}\right)\Delta\theta\right)\right) \\ &\quad\times g\left(2\lambda_1\eta^{-1}\left(\int_{\alpha}^{\Gamma} p^*(\theta)\eta\left(\frac{|y^{\Delta}(\theta)|^{m}}{2p^*(\theta)}\right)\Delta\theta\right)\right),\end{aligned} \tag{25}$$

where λ_1 defined as in Theorem 5.

Proof. Dynamic Jensen inequality (2) provides

$$\eta\left(\frac{1}{2}\int_{\alpha}^{\Gamma}|x^{\Delta}(\theta)|^{m}\Delta\theta\right) \leq \int_{\alpha}^{\Gamma} p(\theta)\eta\left(\frac{|x^{\Delta}(\theta)|^{m}}{2p(\theta)}\right)\Delta\theta.$$

Further, since η is increasing, we have

$$\int_{\alpha}^{\Gamma}|x^{\Delta}(\theta)|^{m}\Delta\theta \leq 2\eta^{-1}\left(\int_{\alpha}^{\Gamma} p(\theta)\eta\left(\frac{|x^{\Delta}(\theta)|^{m}}{2p(\theta)}\right)\Delta\theta\right). \tag{26}$$

Similarly, we have

$$\int_{\alpha}^{\Gamma}|y^{\Delta}(\theta)|^{m}\Delta\theta \leq 2\eta^{-1}\left(\int_{\alpha}^{\Gamma} p^{*}(\theta)\eta\left(\frac{|y^{\Delta}(\theta)|^{m}}{2p^{*}(\theta)}\right)\Delta\theta\right). \tag{27}$$

From (10), (26) and (27) immediately gives

$$\begin{aligned}
&\int_{\alpha}^{\Gamma}\left[f(|x(\theta)|^{m})g'(|y(\theta)|^{m})\left|y^{\Delta}(\theta)\right|^{m}+g(|y(\theta)|^{m})f'(|x(\theta)|^{m})\left|x^{\Delta}(\theta)\right|^{m}\right]\Delta\theta\\
&\leq \frac{1}{\lambda_1}f\left(2\lambda_1\eta^{-1}\left(\int_{\alpha}^{\Gamma} p(\theta)\eta\left(\frac{|x^{\Delta}(\theta)|^{m}}{2p(\theta)}\right)\Delta\theta\right)\right)\\
&\quad\times g\left(2\lambda_1\eta^{-1}\left(\int_{\alpha}^{\Gamma} p^{*}(\theta)\eta\left(\frac{|y^{\Delta}(\theta)|^{m}}{2p^{*}(\theta)}\right)\Delta\theta\right)\right).
\end{aligned}$$

This gives our claim. □

Remark 7. *When $\mathbb{T}=\mathbb{R}$ in Theorem 6, then, by the relation (4), we get the inequality of Hwang and Yang [3].*

Corollary 9. *If we take $\mathbb{T}=h\mathbb{Z}$ in Theorem 6, then, by the relation (6), inequality (25) becomes*

$$\begin{aligned}
&h\sum_{n=\frac{\alpha}{h}}^{\frac{\Gamma}{h}-1}\left[f(|x(nh)|^{m})g'(|y(nh)|^{m})\left|\Delta y(nh)\right|^{m}+g(|y(nh)|^{m})f'(|x(nh)|^{m})\left|\Delta x(nh)\right|^{m}\right]\\
&\leq \frac{1}{\lambda_1}f\left(2h\lambda_1\eta^{-1}\left(\sum_{n=\frac{\alpha}{h}}^{\frac{\Gamma}{h}-1} p(nh)\eta\left(\frac{|\Delta x(nh)|^{m}}{2p(nh)}\right)\right)\right)\\
&\quad\times g\left(2h\lambda_1\eta^{-1}\left(\sum_{n=\frac{\alpha}{h}}^{\frac{\Gamma}{h}-1} p^{*}(nh)\eta\left(\frac{|\Delta y(nh)|^{m}}{2p^{*}(nh)}\right)\right)\right).
\end{aligned}$$

Remark 8. *In Corollary 10, if we take $h=1$, then inequality (25) becomes*

$$\begin{aligned}
&\sum_{n=\alpha}^{\Gamma-1}\left[f(|x(n)|^{m})g'(|y(n)|^{m})\left|\Delta y(n)\right|^{m}+g(|y(n)|^{m})f'(|x(n)|^{m})\left|\Delta x(n)\right|^{m}\right]\\
&\leq \frac{1}{\lambda_1}f\left(2\lambda_1\eta^{-1}\left(\sum_{n=\alpha}^{\Gamma-1} p(n)\eta\left(\frac{|\Delta x(n)|^{m}}{2p(n)}\right)\right)\right)\\
&\quad\times g\left(2\lambda_1\eta^{-1}\left(\sum_{n=\alpha}^{\Gamma-1} p^{*}(n)\eta\left(\frac{|\Delta y(n)|^{m}}{2p^{*}(n)}\right)\right)\right).
\end{aligned}$$

Corollary 10. *If we take $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ in Theorem 6, then, by the relation (7), inequality (25) becomes*

$$
\begin{aligned}
&(q-1)\sum_{n=(\log_q \alpha)}^{(\log_q \Gamma)-1} \left[f\left(|x(q^n)|^m\right)g'\left(|y(q^n)|^m\right)\left|\Delta y(q^n)\right|^m + g\left(|y(q^n)|^m\right)f'\left(|x(q^n)|^m\right)\left|\Delta x(q^n)\right|^m\right]q^n \\
&\quad\leq \frac{1}{\lambda_1} f\left(2(q-1)\lambda_1\eta^{-1}\left(\sum_{n=(\log_q \alpha)}^{(\log_q \Gamma)-1} q^n p(q^n)\eta\left(\frac{|\Delta x(q^n)|^m}{2p(q^n)}\right)\right)\right) \\
&\quad\times g\left(2(q-1)\lambda_1\eta^{-1}\left(\sum_{n=(\log_q \alpha)}^{(\log_q \Gamma)-1} q^n p^*(q^n)\eta\left(\frac{|\Delta y(q^n)|^m}{2p^*(q^n)}\right)\right)\right).
\end{aligned}
$$

Remark 9. *When $\mathbb{T} = \mathbb{R}$, if we take $m = 1$ and $p(t) = p^*(t)$, in the inequality (25), we get Pachpatte inequality [54]*

$$
\begin{aligned}
&\int_\alpha^\Gamma \left[f(|x(\theta)|)g'(|y(\theta)|)|y'(\theta)| + g(|y(\theta)|)f'(|x(\theta)|)|x'(\theta)|\right]\mathrm{d}\theta \\
&\leq f\left(\eta^{-1}\left(\int_\alpha^\Gamma p^*(\theta)\eta\left(\frac{|x'(\theta)|}{p^*(\theta)}\right)\mathrm{d}\theta\right)\right) \\
&\quad\times g\left(\eta^{-1}\left(\int_\alpha^\Gamma p^*(\theta)\eta\left(\frac{|y'(\theta)|}{p^*(\theta)}\right)\mathrm{d}\theta\right)\right).
\end{aligned} \tag{28}
$$

Remark 10. *Taking $f(\theta) = g(\theta)$ and $x(\theta) = y(\theta)$ in inequality (28), we get the Pachpatte inequality [54]*

$$
\int_\alpha^\Gamma f(|x(\theta)|)f'(|x(\theta)|)|x'(\theta)|\mathrm{d}\theta \leq \frac{1}{2}\left[f\left(\eta^{-1}\left(\int_\alpha^\Gamma p(\theta)\eta\left(\frac{|x'(\theta)|}{p(\theta)}\right)\mathrm{d}\theta\right)\right)\right]^2.
$$

Corollary 11. *Suppose $\mathbb{T}$ is a time scale, α, β, $(\alpha+\beta)/2 \in \mathbb{T}$, and f, g are defined as in Theorem 5. Furthermore, Assume x and y are rd-continuous functions on $[\alpha,\beta]_{\mathbb{T}}$ such that $x(\alpha) = y(\alpha) = 0$ and $x(\beta) = y(\beta) = 0$. Then, we get*

$$
\begin{aligned}
&\int_\alpha^\beta \left[f(|x(\theta)|^m)g'(|y(\theta)|^m)|y^\Delta(\theta)|^m + g(|y(\theta)|^m)f'(|x(\theta)|^m)|x^\Delta(\theta)|^m\right]\Delta\theta \\
&\leq \frac{2^{m-1}}{\lambda_2} g\left(2^{1-m}\lambda_2\int_\alpha^\beta |y^\Delta(\theta)|^m\Delta\theta\right) \\
&\quad\times\left[f\left(2^{1-m}\lambda_2\int_\alpha^{\frac{\alpha+\beta}{2}} |x^\Delta(\theta)|^m\Delta\theta\right) + f\left(2^{1-m}\lambda_2\int_{\frac{\alpha+\beta}{2}}^\beta |x^\Delta(\theta)|^m\Delta\theta\right)\right],
\end{aligned} \tag{29}
$$

where $\lambda_2 = (\beta-\alpha)^{m-1}$.

Proof. Let $\Gamma \in [\alpha,\beta]_{\mathbb{T}}$, the functions x and y satisfy the conditions of Theorem 5 on $[\alpha,\Gamma]_{\mathbb{T}}$. Thus, inequality (10) holds. Next, in the interval $[\Gamma,\beta]_{\mathbb{T}}$, the functions x and y are rd-continuous, and $x(\beta) = y(\beta) = 0$. Thus, by defining $\varphi(t) = \int_t^\beta |x^\Delta(\theta)|^m\Delta\theta$, $\phi(t) = \int_t^\beta |y^\Delta(\theta)|^m\Delta\theta$ $t \in [\Gamma,\beta]$, and following an argument similar to Theorem 5, we obtain

$$
\begin{aligned}
&\int_\Gamma^\beta \left[f(|x(\theta)|^m)g'(|y(\theta)|^m)|y^\Delta(\theta)|^m + g(|y(\theta)|^m)f'(|x(\theta)|^m)|x^\Delta(\theta)|^m\right]\Delta\theta \\
&\leq \frac{1}{\lambda_3} f\left(\lambda_3\int_\Gamma^\beta |x^\Delta(\theta)|^m\Delta\theta\right) g\left(\lambda_3\int_\Gamma^\beta |y^\Delta(\theta)|^m\Delta\theta\right),
\end{aligned} \tag{30}
$$

where $\lambda_3 = (\beta - \Gamma)^{m-1}$. A combination of the inequalities (10) and (30), we get

$$\begin{aligned}
&\int_{\alpha}^{\beta} \left[f(|x(\theta)|^m)g'(|y(\theta)|^m)|y^{\Delta}(\theta)|^m + g(|y(\theta)|^m)f'(|x(\theta)|^m)|x^{\Delta}(\theta)|^m\right]\Delta\theta \\
&= \int_{\alpha}^{\Gamma} \left[f(|x(\theta)|^m)g'(|y(\theta)|^m)|y^{\Delta}(\theta)|^m + g(|y(\theta)|^m)f'(|x(\theta)|^m)|x^{\Delta}(\theta)|^m\right]\Delta\theta \\
&\quad + \int_{\Gamma}^{\beta} \left[f(|x(\theta)|^m)g'(|y(t)|^m)|y^{\Delta}(\theta)|^m + g(|y(\theta)|^m)f'(|x(\theta)|^m)|x^{\Delta}(\theta)|^m\right]\Delta\theta \\
&\leq \frac{1}{\lambda_1} f\left(\lambda_1 \int_{\alpha}^{\Gamma} |x^{\Delta}(\theta)|^m \Delta\theta\right) g\left(\lambda_1 \int_{\alpha}^{\Gamma} y^{\Delta}(\theta)|^m \Delta\theta\right) \\
&\quad + \frac{1}{\lambda_3} f\left(\lambda_3 \int_{\Gamma}^{\beta} |x^{\Delta}(\theta)|^m \Delta\theta\right) g\left(\lambda_3 \int_{\Gamma}^{\beta} |y^{\Delta}(\theta)|^m \Delta\theta\right),
\end{aligned} \tag{31}$$

for $\Gamma = \frac{\alpha+\beta}{2}$, we find that $\lambda_1 = \lambda_3 = (\frac{\beta-\alpha}{2})^{m-1}$, where $\lambda_2 = (\beta - \alpha)^{m-1}$, then $\lambda_1 = \lambda_3 = \frac{\lambda_2}{2^{m-1}}$. Then, by substituting in (31), we get

$$\begin{aligned}
&\int_{\alpha}^{\beta} \left[f(|x(\theta)|^m)g'(|y(\theta)|^m)|y^{\Delta}(\theta)|^m + g(|y(\theta)|^m)f'(|x(\theta)|^m)|x^{\Delta}(\theta)|^m\right]\Delta\theta \\
&\leq \frac{2^{m-1}}{\lambda_2} f\left(2^{1-m}\lambda_2 \int_{\alpha}^{\frac{\alpha+\beta}{2}} |x^{\Delta}(\theta)|^m \Delta\theta\right) g\left(2^{1-m}\lambda_2 \int_{\alpha}^{\frac{\alpha+\beta}{2}} |y^{\Delta}(\theta)|^m \Delta\theta\right) \\
&\quad + \frac{2^{m-1}}{\lambda_2} f\left(2^{1-m}\lambda_2 \int_{\frac{\alpha+\beta}{2}}^{\beta} |x^{\Delta}(\theta)|^m \Delta\theta\right) g\left(2^{1-m}\lambda_2 \int_{\frac{\alpha+\beta}{2}}^{\beta} |y^{\Delta}(\theta)|^m \Delta\theta\right).
\end{aligned} \tag{32}$$

Since g is non-decreasing function, we have

$$\begin{aligned}
&\int_{\alpha}^{\beta} \left[f(|x(\theta)|^m)g'(|y(\theta)|^m)|y^{\Delta}(\theta)|^m + g(|y(\theta)|^m)f'(|x(t)|^m)|x^{\Delta}(\theta)|^m\right]\Delta\theta \\
&\leq \frac{2^{m-1}}{\lambda_2} g\left(2^{1-m}\lambda_2 \int_{\alpha}^{\beta} |y^{\Delta}(\theta)|^m \Delta\theta\right) \left[f\left(2^{1-m}\lambda_2 \int_{\alpha}^{\frac{\alpha+\beta}{2}} |x^{\Delta}(\theta)|^m \Delta\theta\right) \right. \\
&\quad \left. + f\left(2^{1-m}\lambda_2 \int_{\frac{\alpha+\beta}{2}}^{\beta} |x^{\Delta}(\theta)|^m \Delta\theta\right) \right].
\end{aligned}$$

This gives our claim (29). □

Corollary 12. *If we take $\mathbb{T} = h\mathbb{Z}$ in Corollary 11, then, by the relation (6), inequality (29) becomes*

$$\begin{aligned}
&h \sum_{n=\frac{\alpha}{h}}^{\frac{\beta}{h}-1} \left[f(|x(nh)|^m)g'(|y(nh)|^m)|\Delta y(nh)|^m + g(|y(nh)|^m)f'(|x(nh)|^m)|\Delta x(nh)|^m\right] \\
&\leq \frac{2^{m-1}}{\lambda_2} g\left(2^{1-m}\lambda_2 h \sum_{n=\frac{\alpha}{h}}^{\frac{\beta}{h}-1} |\Delta y(nh)|^m\right) \\
&\quad \times \left[f\left(2^{1-m}\lambda_2 h \sum_{n=\frac{\alpha}{h}}^{\frac{\alpha+\beta}{2h}-1} |\Delta x(nh)|^m\right) + f\left(2^{1-m}\lambda_2 h \sum_{n=\frac{\alpha+\beta}{2h}}^{\frac{\beta}{h}-1} |\Delta x(nh)|^m\right) \right].
\end{aligned}$$

Remark 11. *In Corollary 12, if we take $h = 1$, then inequality (29) becomes*

$$\begin{aligned}
&\sum_{n=\alpha}^{\beta-1} [f(|x(n)|^m)g'(|y(n)|^m)|\Delta y(n)|^m + g(|y(n)|^m)f'(|x(n)|^m)|\Delta x(n)|^m] \\
&\leq \frac{2^{m-1}}{\lambda_2} g\left(2^{1-m}\lambda_2 \sum_{n=\alpha}^{\beta-1} |\Delta y(n)|^m\right) \\
&\quad \times \left[f\left(2^{1-m}\lambda_2 \sum_{n=\alpha}^{\frac{\alpha+\beta}{2}-1} |\Delta x(n)|^m\right) + f\left(2^{1-m}\lambda_2 \sum_{n=\frac{\alpha+\beta}{2}}^{\beta-1} |\Delta x(n)|^m\right)\right].
\end{aligned}$$

Corollary 13. *If we take $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ in Corollary 11, then, by the relation (7), inequality (29) becomes*

$$\begin{aligned}
&(q-1) \sum_{n=(\log_q \alpha)}^{(\log_q \beta)-1} [f(|x(q^n)|^m)g'(|y(q^n)|^m)|\Delta y(q^n)|^m + g(|y(q^n)|^m)f'(|x(q^n)|^m)|\Delta x(q^n)|^m] q^n \\
&\leq \frac{2^{m-1}}{\lambda_2} g\left(2^{1-m}\lambda_2(q-1) \sum_{n=(\log_q \alpha)}^{(\log_q \beta)-1} |\Delta y(q^n)|^m q^n\right) \\
&\quad \times \left[f\left(2^{1-m}\lambda_2(q-1) \sum_{n=(\log_q \alpha)}^{(\log_q \frac{\alpha+\beta}{2})-1} |\Delta x(q^n)|^m q^n\right)\right. \\
&\quad \left. + f\left(2^{1-m}\lambda_2(q-1) \sum_{n=(\log_q \frac{\alpha+\beta}{2})}^{(\log_q \beta)-1} |\Delta x(q^n)|^m q^n\right)\right].
\end{aligned}$$

Corollary 14. *In Corollary 11. For $m = 1$, we can get the following inequality*

$$\begin{aligned}
&\int_\alpha^\beta [f(|x(\theta)|)g'(|y(\theta)|)|y^\Delta(\theta)| + g(|y(t)|)f'(|x(\theta)|)|x^\Delta \theta|]\Delta\theta \\
&\leq 2f\left(\frac{1}{2}\int_\alpha^\beta |x^\Delta(\theta)|\Delta\theta\right) g\left(\frac{1}{2}\int_\alpha^\beta |y^\Delta(\theta)|\Delta\theta\right).
\end{aligned} \tag{33}$$

Proof. By substituting in (32) by $m = 1$, we get

$$\begin{aligned}
&\int_\alpha^\beta [f(|x(\theta)|)g'(|y(\theta)|)|y^\Delta(\theta)| + g(|y(\theta)|)f'(|x(\theta)|)|x^\Delta(\theta)|]\Delta\theta \\
&\leq f\left(\int_\alpha^{\frac{\alpha+\beta}{2}} |x^\Delta(\theta)|\Delta\theta\right) g\left(\int_\alpha^{\frac{\alpha+\beta}{2}} |y^\Delta(\theta)|\Delta\theta\right) \\
&\quad + f\left(\int_{\frac{\alpha+\beta}{2}}^\beta |x^\Delta(\theta)|\Delta\theta\right) g\left(\int_{\frac{\alpha+\beta}{2}}^\beta |y^\Delta(\theta)|\Delta\theta\right).
\end{aligned} \tag{34}$$

If we choose Γ such that

$$\int_\alpha^\Gamma \left|x^\Delta(\theta)\right| \Delta\theta = \int_\Gamma^\beta \left|x^\Delta(\theta)\right| \Delta\theta = \frac{1}{2}\int_\alpha^\beta \left|x^\Delta(\theta)\right| \Delta\theta. \tag{35}$$

Then, by substituting from (35) in (34), we can get

$$\begin{aligned}&\int_\alpha^\beta \left[f(|x(\theta)|)g'(|y(\theta)|)|y^\Delta(\theta)| + g(|y(\theta)|)f'(|x(\theta)|)|x^\Delta(\theta)|\right]\Delta\theta\\&\le 2f\left(\frac{1}{2}\int_\alpha^\beta |x^\Delta(\theta)|\Delta\theta\right)g\left(\frac{1}{2}\int_\alpha^\beta |y^\Delta(\theta)|\Delta\theta\right).\end{aligned}$$

This gives our claim. □

Corollary 15. *Let $f(t) = t^{\frac{\ell+m}{m}}$, $\ell \ge 0$, $g(t) = 1$ and $x(t) = y(t)$ in (29), to obtain*

$$\begin{aligned}\int_\alpha^\beta \frac{\ell+m}{m}|x(\theta)|^\ell |x^\Delta(\theta)|^m \Delta\theta \le \left(\frac{\beta-\alpha}{2}\right)^{\frac{\ell(m-1)}{m}}\Bigg[\left(\int_\alpha^{\frac{\alpha+\beta}{2}} |x^\Delta(\theta)|^m\Delta\theta\right)^{\frac{\ell+m}{m}}&\\ + \left(\int_{\frac{\alpha+\beta}{2}}^\beta |x^\Delta(\theta)|^m\Delta\theta\right)^{\frac{\ell+m}{m}}\Bigg].&\end{aligned} \tag{36}$$

By using the inequality (9) in inequality (36), we get

$$\begin{aligned}\int_\alpha^\beta |x(\theta)|^\ell |x^\Delta(\theta)|^m\Delta\theta \le{}& \frac{m}{\ell+m}\left(\frac{\beta-\alpha}{2}\right)^{\frac{\ell(m-1)}{m}}\\&\times\left[\left(\int_\alpha^{\frac{\alpha+\beta}{2}} |x^\Delta(\theta)|^m\Delta\theta\right)^{\frac{\ell+m}{m}} + \left(\int_{\frac{\alpha+\beta}{2}}^\beta |x^\Delta(\theta)|^m\Delta\theta\right)^{\frac{\ell+m}{m}}\right]\\ \le{}& \frac{m}{\ell+m}\left(\frac{\beta-\alpha}{2}\right)^{\frac{\ell(m-1)}{m}}\left[\left(\int_\alpha^{\frac{\alpha+\beta}{2}} |x^\Delta(\theta)|^m\Delta\theta\right) + \left(\int_{\frac{\alpha+\beta}{2}}^\beta |x^\Delta(\theta)|^m\Delta\theta\right)\right]^{\frac{\ell+m}{m}}\\ ={}& \frac{m}{\ell+m}\left(\frac{\beta-\alpha}{2}\right)^{\frac{\ell(m-1)}{m}}\left(\int_\alpha^\beta |x^\Delta(\theta)|^m\Delta\theta\right)^{\frac{\ell+m}{m}}.\end{aligned} \tag{37}$$

From Lemma 1 with indices $\frac{\ell+m}{\ell}$ and $\frac{\ell+m}{m}$ on (37), we have

$$\int_\alpha^\beta |x(\theta)|^\ell |x^\Delta(\theta)|^m\Delta\theta \le \frac{m}{\ell+m}\left(\frac{\beta-\alpha}{2}\right)^\ell \int_\alpha^\beta |x^\Delta(\theta)|^{\ell+m}\Delta\theta. \tag{38}$$

Remark 12. *Taking $\mathbb{T} = \mathbb{R}$, then (38) gives Yang inequality [59]*

$$\int_\alpha^\beta |x(\theta)|^\ell |x'(\theta)|^m \mathrm{d}t \le \frac{m}{\ell+m}\left(\frac{\beta-\alpha}{2}\right)^\ell \int_\alpha^\beta |x'(\theta)|^{\ell+m}\mathrm{d}\theta.$$

Theorem 7. *Under the hypotheses of Theorem 5. Assuming that $s \ge 0$, $r \ge 0$ and $s^\Delta \ge 0$, $r^\Delta \ge 0$, $t \in [\alpha, \Gamma]_\mathbb{T}$ and $r(\alpha) = 0$, $s(\alpha) = 0$ and $\chi \ge 0$ and $\pi \ge 0$ are convex and increasing functions on $(0, \infty)_\mathbb{R}$. Then, we get*

$$\begin{aligned}&\int_\alpha^\Gamma \Bigg[s^\Delta(\theta)\pi\left(\frac{|y^\Delta(\theta)|}{s^\Delta(\theta)}\right)f\left(r(\theta)\chi\left(\frac{|x(\theta)|}{r(\theta)}\right)\right)g'\left(s(\theta)\pi\left(\frac{|y(\theta)|}{s(\theta)}\right)\right)\\&\quad + r^\Delta(\theta)\chi\left(\frac{|x^\Delta(\theta)|}{r^\Delta(\theta)}\right)f'\left(r(\theta)\chi\left(\frac{|x(\theta)|}{r(\theta)}\right)\right)g\left(s(\theta)\pi\left(\frac{|y(\theta)|}{s(\theta)}\right)\right)\Bigg]\Delta\theta\\&\le f\left(\int_\alpha^\Gamma r^\Delta(\theta)\chi\left(\frac{|x^\Delta(\theta)|}{r^\Delta(\theta)}\right)\Delta\theta\right)g\left(\int_\alpha^\Gamma s^\Delta(\theta)\pi\left(\frac{|y^\Delta(\theta)|}{s^\Delta(\theta)}\right)\Delta\theta\right).\end{aligned} \tag{39}$$

Proof. For $t \in [\alpha, \Gamma]_{\mathbb{T}}$, we define $\varphi(t) := \int_\alpha^t |x^\Delta(\theta)|\Delta\theta$ and $\psi(t) := \int_\alpha^t |y^\Delta(\theta)|\Delta\theta$ so that $\varphi^\Delta(t) = |x^\Delta(t)|$ and $\psi^\Delta(t) = |y^\Delta(t)|$.

Thus,

$$|x(t)| = \left|\int_\alpha^t x^\Delta(\theta)\Delta\theta\right| \leq \int_\alpha^t |x^\Delta(\theta)|\Delta\theta = \varphi(t),$$
$$|y(t)| = \left|\int_\alpha^t y^\Delta(\theta)\Delta\theta\right| \leq \int_\alpha^t |y^\Delta(\theta)|\Delta\theta = \psi(t).$$

Then, we obtain

$$\frac{|x(t)|}{|r(t)|} \leq \frac{\frac{\int_\alpha^t r^\Delta(\theta)\varphi^\Delta(\theta)\Delta\theta}{r^\Delta(\theta)}}{\int_\alpha^t r^\Delta(\theta)\Delta\theta},$$
$$\frac{|y(t)|}{|s(t)|} \leq \frac{\frac{\int_\alpha^t s^\Delta(\theta)\psi^\Delta(\theta)\Delta\theta}{s^\Delta(\theta)}}{\int_\alpha^t s^\Delta(\theta)\Delta\theta}.$$

Thus, from Lemma 2, we get

$$\chi\left(\frac{|x(t)|}{r(t)}\right) \leq \frac{1}{r(t)}\int_\alpha^t r^\Delta(\theta)\chi\left(\frac{\varphi^\Delta(\theta)}{r^\Delta(\theta)}\right)\Delta\theta.$$

$$\pi\left(\frac{|y(t)|}{s(t)}\right) \leq \frac{1}{s(t)}\int_\alpha^t s^\Delta(\theta)\pi\left(\frac{\psi^\Delta(\theta)}{s^\Delta(\theta)}\right)\Delta\theta.$$

Using the above inequalities, we get

$$\begin{aligned}
&\int_\alpha^\Gamma \left[s^\Delta(\theta)\pi\left(\frac{|y^\Delta(\theta)|}{s^\Delta(\theta)}\right)f\left(r(\theta)\chi\left(\frac{|x(\theta)|}{r(\theta)}\right)\right)g'\left(s(\theta)\pi\left(\frac{|y(\theta)|}{s(\theta)}\right)\right)\right.\\
&\qquad\left.+\, r^\Delta(\theta)\chi\left(\frac{|x^\Delta(\theta)|}{r^\Delta(\theta)}\right)f'\left(r(\theta)\chi\left(\frac{|x(\theta)|}{r(\theta)}\right)\right)g\left(s(\theta)\pi\left(\frac{|y(\theta)|}{s(\theta)}\right)\right)\right]\Delta\theta\\
&\leq \int_\alpha^\Gamma \left[s^\Delta(\theta)\pi\left(\frac{\psi^\Delta(\theta)}{s^\Delta(\theta)}\right)f\left(\int_\alpha^t r^\Delta(\theta)\chi\frac{\varphi^\Delta(\theta)}{r^\Delta(\theta)}\Delta\theta\right)g'\left(\int_\alpha^t s^\Delta(\theta)\pi\left(\frac{\psi^\Delta(\theta)}{s^\Delta(\theta)}\right)\Delta\theta\right)\right.\\
&\qquad\left.+\, r^\Delta(\theta)\chi\left(\frac{\varphi^\Delta(\theta)}{r^\Delta(\theta)}\right)f'\left(\int_\alpha^t r^\Delta(\theta)\chi\left(\frac{\varphi^\Delta(\theta)}{r^\Delta(\theta)}\right)\Delta\theta\right)g\left(\int_\alpha^t s^\Delta(\theta)\pi\left(\frac{\psi^\Delta(\theta)}{s^\Delta(\theta)}\right)\Delta\theta\right)\right]\Delta\theta.
\end{aligned} \tag{40}$$

Since f, r, χ, and φ are increasing and $t \leq \sigma(t)$, we get

$$f\left(\int_\alpha^t r^\Delta(\theta)\chi\Big(\frac{\varphi^\Delta(\theta)}{r^\Delta(\theta)}\Big)\Delta\theta\right) \leq f^\sigma\left(\int_\alpha^t r^\Delta(\theta)\chi\Big(\frac{\varphi^\Delta(\theta)}{r^{\Delta\cdot}(\theta)}\Big)\Delta\theta\right).$$

Then by substituting in (40), we get

$$\begin{aligned}
&\int_{\alpha}^{\Gamma}\Bigg[s^{\Delta}(\theta)\pi\left(\frac{|y^{\Delta}(\theta)|}{s^{\Delta}(\theta)}\right)f\left(r(\theta)\chi\left(\frac{|x(\theta)|}{r(\theta)}\right)\right)g'\left(s(\theta)\pi\left(\frac{|y(\theta)|}{s(\theta)}\right)\right)\\
&\quad+r^{\Delta}(\theta)\chi\left(\frac{|x^{\Delta}(\theta)|}{r^{\Delta}(\theta)}\right)f'\left(r(\theta)\chi\left(\frac{|x(\theta)|}{r(\theta)}\right)\right)g\left(s(\theta)\pi\left(\frac{|y(\theta)|}{s(\theta)}\right)\right)\Bigg]\Delta\theta\\
&\quad\leq\int_{\alpha}^{\Gamma}\Bigg[s^{\Delta}(\theta)\pi\left(\frac{\psi^{\Delta}(\theta)}{s^{\Delta}(\theta)}\right)f^{\sigma}\left(\int_{\alpha}^{t}r^{\Delta}(\theta)\chi\left(\frac{\varphi^{\Delta}(\theta)}{r^{\Delta}(\theta)}\right)\Delta\theta\right)g'\left(\int_{\alpha}^{t}s^{\Delta}(\theta)\pi\left(\frac{\psi^{\Delta}(\theta)}{s^{\Delta}(\theta)}\right)\Delta\theta\right)\\
&\quad+r^{\Delta}(\theta)\chi\left(\frac{\varphi^{\Delta}(\theta)}{r^{\Delta}(\theta)}\right)f'\left(\int_{\alpha}^{t}r^{\Delta}(\theta)\chi\left(\frac{\varphi^{\Delta}(\theta)}{r^{\Delta}(\theta)}\right)\Delta\theta\right)g\left(\int_{\alpha}^{t}s^{\Delta}(\theta)\pi\left(\frac{\psi^{\Delta}(\theta)}{s^{\Delta}(\theta)}\right)\Delta\theta\right)\Bigg]\Delta\theta.
\end{aligned}\tag{41}$$

From Lemma 3 for $c\in[t,\sigma(t)]$, we get

$$\begin{aligned}
f^{\Delta}\left(\int_{\alpha}^{t}r^{\Delta}(\theta)\chi\left(\frac{\varphi^{\Delta}(\theta)}{r^{\Delta}(\theta)}\Delta\theta\right)\right)&=r^{\Delta}(t)\chi\left(\frac{\varphi^{\Delta}(t)}{r^{\Delta}(t)}\right)f'\left(\int_{\alpha}^{c}r^{\Delta}(\theta)\chi\left(\frac{\varphi^{\Delta}(\theta)}{r^{\Delta}(\theta)}\right)\Delta\theta\right)\\
&\geq r^{\Delta}(t)\chi\left(\frac{\varphi^{\Delta}(t)}{r^{\Delta}(t)}\right)f'\left(\int_{\alpha}^{t}r^{\Delta}(\theta)\chi\left(\frac{\varphi^{\Delta}(\theta)}{r^{\Delta}(\theta)}\right)\Delta\theta\right).
\end{aligned}\tag{42}$$

$$\begin{aligned}
g^{\Delta}\left(\int_{\alpha}^{t}s^{\Delta}(\theta)\pi\left(\frac{\psi^{\Delta}(\theta)}{s^{\Delta}(\theta)}\Delta\theta\right)\right)&=s^{\Delta}(t)\chi\left(\frac{\psi^{\Delta}(t)}{s^{\Delta}(t)}\right)g'\left(\int_{\alpha}^{c}s^{\Delta}(\theta)\chi\left(\frac{\psi^{\Delta}(\theta)}{s^{\Delta}(\theta)}\right)\Delta\theta\right)\\
&\geq s^{\Delta}(t)\chi\left(\frac{\varphi^{\Delta}(t)}{s^{\Delta}(t)}\right)g'\left(\int_{\alpha}^{t}s^{\Delta}(\theta)\pi\left(\frac{\psi^{\Delta}(\theta)}{s^{\Delta}(\theta)}\right)\Delta\theta\right).
\end{aligned}\tag{43}$$

Then, by substituting from (42) and (42) in (41), we get

$$\begin{aligned}
&\int_{\alpha}^{\Gamma}\Bigg[s^{\Delta}(\theta)\pi\left(\frac{|y^{\Delta}(\theta)|}{s^{\Delta}(\theta)}\right)f\left(r(\theta)\chi\left(\frac{|x(\theta)|}{r(\theta)}\right)\right)g'\left(s(\theta)\pi\left(\frac{|y(\theta)|}{s(\theta)}\right)\right)\\
&\quad+r^{\Delta}(\theta)\chi\left(\frac{|x^{\Delta}(\theta)|}{r^{\Delta}(\theta)}\right)f'\left(r(\theta)\chi\left(\frac{|x(\theta)|}{r(\theta)}\right)\right)g\left(s(\theta)\pi\left(\frac{|y(\theta)|}{s(\theta)}\right)\right)\Bigg]\Delta\theta\\
&\leq\int_{\alpha}^{\Gamma}\Bigg[g^{\Delta}\left(\int_{\alpha}^{t}s^{\Delta}(\theta)\pi\left(\frac{\psi^{\Delta}(\theta)}{s^{\Delta}(\theta)}\right)\Delta\theta\right)f^{\sigma}\left(\int_{\alpha}^{t}r^{\Delta}(\theta)\chi\left(\frac{\varphi^{\Delta}(\theta)}{r^{\Delta}(\theta)}\right)\Delta\theta\right)\\
&\quad+f^{\Delta}\left(\int_{\alpha}^{t}r^{\Delta}(\theta)\chi\left(\frac{\varphi^{\Delta}(\theta)}{r^{\Delta}(\theta)}\right)\Delta\theta\right)g\left(\int_{\alpha}^{t}s^{\Delta}(\theta)\pi\left(\frac{\psi^{\Delta}(\theta)}{s^{\Delta}(\theta)}\right)\Delta\theta\right)\Bigg]\Delta\theta\\
&=\int_{\alpha}^{\Gamma}\left[f\left(\int_{\alpha}^{t}r^{\Delta}(\theta)\chi\left(\frac{\varphi^{\Delta}(\theta)}{r^{\Delta}(\theta)}\right)\Delta\theta\right)g\left(\int_{\alpha}^{t}s^{\Delta}(\theta)\pi\left(\frac{\psi^{\Delta}(\theta)}{s^{\Delta}(\theta)}\right)\Delta\theta\right)\right]^{\Delta}\\
&=f\left(\int_{\alpha}^{\Gamma}r^{\Delta}(\theta)\chi\left(\frac{|x^{\Delta}(\theta)|}{r^{\Delta}(\theta)}\right)\Delta\theta\right)g\left(\int_{\alpha}^{\Gamma}s^{\Delta}(\theta)\pi\left(\frac{|y^{\Delta}(\theta)|}{s^{\Delta}(\theta)}\right)\Delta\theta\right).
\end{aligned}$$

This gives our claim. □

Remark 13. *Taking $\mathbb{T}=\mathbb{R}$, then, by the relation (6), inequality (39) gives Pachpatte inequality in [55]*

$$
\begin{aligned}
&\int_{\alpha}^{\Gamma}\left[s'(\theta)\pi\left(\frac{|y'(\theta)|}{s'(\theta)}\right)f\left(r(\theta)\chi\left(\frac{|x(\theta)|}{r(\theta)}\right)\right)g'\left(s(\theta)\pi\left(\frac{|y(\theta)|}{s(\theta)}\right)\right)\right.\\
&\qquad\left.+r'(\theta)\chi\Big(\frac{|x'(\theta)|}{r'(\theta)}\Big)f'\left(r(\theta)\chi\Big(\frac{|x(\theta)|}{r(\theta)}\Big)\right)g\left(s(\theta)\pi\left(\frac{|y(\theta)|}{s(\theta)}\right)\right)\right]\mathrm{d}\theta\\
&\leq f\left(\int_{\alpha}^{\Gamma}r'(\theta)\chi\left(\frac{|x'(\theta)|}{r'(\theta)}\right)\mathrm{d}\theta\right)g\Big(\int_{\alpha}^{\Gamma}s'(\theta)\pi\left(\frac{|y'(\theta)|}{s'(\theta)}\right)\mathrm{d}\theta\Big).
\end{aligned}
$$

Corollary 16. *If we take $\mathbb{T}=h\mathbb{Z}$ in Theorem 7, then, by the relation (6), inequality (39) becomes*

$$
\begin{aligned}
&h\sum_{n=\frac{\alpha}{h}}^{\frac{\Gamma}{h}-1}\left[\Delta s(nh)\pi\left(\frac{|\Delta y(nh)|}{\Delta s(nh)}\right)f\left(r(nh)\chi\left(\frac{|x(nh)|}{r(nh)}\right)\right)g'\left(s(nh)\pi\left(\frac{|y(nh)|}{s(nh)}\right)\right)\right.\\
&\qquad\left.+\Delta r(nh)\chi\Big(\frac{|\Delta x(nh)|}{\Delta r(nh)}\Big)f'\left(r(nh)\chi\Big(\frac{|x(nh)|}{r(nh)}\Big)\right)g\left(s(nh)\pi\left(\frac{|y(nh)|}{s(nh)}\right)\right)\right]\\
&\leq f\left(h\sum_{n=\frac{\alpha}{h}}^{\frac{\Gamma}{h}-1}\Delta r(nh)\chi\left(\frac{|\Delta x(nh)|}{\Delta r(nh)}\right)\right)g\left(h\sum_{n=\frac{\alpha}{h}}^{\frac{\Gamma}{h}-1}\Delta s(nh)\pi\left(\frac{|\Delta y(nh)|}{\Delta s(nh)}\right)\right).
\end{aligned}
$$

Remark 14. *In Corollary 16, if we take $h=1$, then inequality (39) becomes*

$$
\begin{aligned}
&\sum_{n=\alpha}^{\Gamma-1}\left[\Delta s(n)\pi\left(\frac{|\Delta y(n)|}{\Delta s(n)}\right)f\left(r(n)\chi\left(\frac{|x(n)|}{r(n)}\right)\right)g'\left(s(n)\pi\left(\frac{|y(n)|}{s(n)}\right)\right)\right.\\
&\qquad\left.+\Delta r(n)\chi\Big(\frac{|\Delta x(n)|}{\Delta r(n)}\Big)f'\left(r(n)\chi\Big(\frac{|x(n)|}{r(n)}\Big)\right)g\left(s(n)\pi\left(\frac{|y(n)|}{s(n)}\right)\right)\right]\\
&\leq f\left(\sum_{n=\alpha}^{\Gamma-1}\Delta r(n)\chi\left(\frac{|\Delta x(n)|}{\Delta r(n)}\right)\right)g\left(\sum_{n=\alpha}^{\Gamma-1}\Delta s(n)\pi\left(\frac{|\Delta y(n)|}{\Delta s(n)}\right)\right).
\end{aligned}
$$

Corollary 17. *If we take $\mathbb{T}=\overline{q^{\mathbb{Z}}}$ in Theorem 7, then, by the relation (7), inequality (39) becomes*

$$
\begin{aligned}
&(q-1)\sum_{n=(\log_q\alpha)}^{(\log_q\Gamma)-1}\left[\Delta s(q^n)\pi\left(\frac{|\Delta y(q^n)|}{\Delta s(q^n)}\right)f\left(r(q^n)\chi\left(\frac{|x(q^n)|}{r(q^n)}\right)\right)g'\left(s(q^n)\pi\left(\frac{|y(q^n)|}{s(q^n)}\right)\right)\right.\\
&\qquad\left.+\Delta r(q^n)\chi\Big(\frac{|\Delta x(q^n)|}{\Delta r(q^n)}\Big)f'\left(r(q^n)\chi\Big(\frac{|x(q^n)|}{r(q^n)}\Big)\right)g\left(s(q^n)\pi\left(\frac{|y(q^n)|}{s(q^n)}\right)\right)\right]q^n\\
&\leq f\left((q-1)\sum_{n=(\log_q\alpha)}^{(\log_q\Gamma)-1}q^n\Delta r(q^n)\chi\left(\frac{|\Delta x(q^n)|}{\Delta r(q^n)}\right)\right)\\
&\qquad\times g\left((q-1)\sum_{n=(\log_q\alpha)}^{(\log_q\Gamma)-1}q^n\Delta s(q^n)\pi\left(\frac{|\Delta y(q^n)|}{\Delta s(q^n)}\right)\right).
\end{aligned}
$$

Remark 15. *For $s(t) = r(t)$, $\chi(t) = \pi(t)$, $y(t) = x(t)$ and $g(t) = f(t)$, the inequality (39) becomes*

$$\begin{aligned}
&\int_\alpha^\Gamma r^\Delta(\theta)\chi\left(\frac{|x^\Delta(\theta)|}{r^\Delta(\theta)}\right) f\left(r(\theta)\chi\left(\frac{|x(\theta)|}{r(\theta)}\right)\right) f'\left(r(\theta)\chi\left(\frac{|x(\theta)|}{r(\theta)}\right)\right)\Delta\theta \\
&\le \frac{1}{2}\left[f\left(\int_\alpha^\Gamma r^\Delta(\theta)\chi\left(\frac{|x^\Delta(\theta)|}{r^\Delta(\theta)}\right)\Delta\theta\right)\right]^2.
\end{aligned} \tag{44}$$

Corollary 18. *With the assumptions of Theorem 7. Suppose $\omega \ge 0$, $t \in [\alpha,\Gamma]_{\mathbb{T}}$ and $\int_\alpha^\Gamma \omega(\theta)\Delta\theta = 1$. Assuming $\Psi \ge 0$ is increasing and convex on $[0,\infty)_{\mathbb{R}}$. Then, we get*

$$\begin{aligned}
&\int_\alpha^\Gamma \Bigg[s^\Delta(\theta)\pi\Big(\frac{|y^\Delta(\theta)|}{s^\Delta(\theta)}\Big) f\left(r(\theta)\chi\Big(\frac{|x(\theta)|}{r(\theta)}\Big)\right) g'\left(s(\theta)\pi\Big(\frac{|y(\theta)|}{s(\theta)}\Big)\right) \\
&\quad + r^\Delta(\theta)\chi\Big(\frac{|x^\Delta(\theta)|}{r^\Delta(\theta)}\Big) f'\left(r(\theta)\chi\Big(\frac{|x(\theta)|}{r(\theta)}\Big)\right) g\left(s(\theta)\pi\Big(\frac{|y(\theta)|}{s(\theta)}\Big)\right)\Bigg]\Delta\theta \\
&\le f\left(\Psi^{-1}\left(\int_\alpha^\Gamma \omega(\theta)\Psi\left(\frac{r^\Delta(\theta)\chi\big(\frac{|x^\Delta(\theta)|}{r^\Delta(\theta)}\big)}{\omega(\theta)}\right)\Delta\theta\right)\right) g\left(\Psi^{-1}\left(\int_\alpha^\Gamma \omega(\theta)\Psi\left(\frac{s^\Delta(\theta)\pi\big(\frac{|y^\Delta(\theta)|}{s^\Delta(\theta)}\big)}{\omega(\theta)}\right)\Delta\theta\right)\right).
\end{aligned} \tag{45}$$

Proof. Since

$$\int_\alpha^\Gamma r^\Delta(\theta)\chi\left(\frac{|x^\Delta(\theta)|}{r^\Delta(\theta)}\right) \le \frac{\int_\alpha^\Gamma \frac{r^\Delta(\theta)\chi\big(\frac{|x^\Delta(\theta)|}{r^\Delta(\theta)}\big)}{\omega(\theta)}\omega(\theta)\Delta\theta}{\int_\alpha^\Gamma \omega(\theta)\Delta\theta}$$

and

$$\int_\alpha^\Gamma s^\Delta(\theta)\pi\left(\frac{|y^\Delta(\theta)|}{s^\Delta(\theta)}\right) \le \frac{\int_\alpha^\Gamma \frac{s^\Delta(\theta)\pi\big(\frac{|y^\Delta(\theta)|}{s^\Delta(\theta)}\big)}{\omega(\theta)}\omega(\theta)\Delta\theta}{\int_\alpha^\Gamma \omega(\theta)\Delta\theta}.$$

From Lemma 2, we have

$$\Psi\left(\int_\alpha^\Gamma r^\Delta(\theta)\chi\Big(\frac{|x^\Delta(\theta)|}{r^\Delta(\theta)}\Big)\right) \le \int_\alpha^\Gamma \omega(\theta)\Psi\left(\frac{r^\Delta(\theta)\pi\big(\frac{|x^\Delta(\theta)|}{r^\Delta(\theta)}\big)}{\omega(\theta)}\right)\Delta\theta$$

and

$$\Psi\left(\int_\alpha^\Gamma s^\Delta(\theta)\pi\Big(\frac{|y^\Delta(\theta)|}{s^\Delta(\theta)}\Big)\right) \le \int_\alpha^\Gamma \omega(\theta)\Psi\left(\frac{s^\Delta(\theta)\pi\big(\frac{|y^\Delta(\theta)|}{s^\Delta(\theta)}\big)}{\omega(\theta)}\right)\Delta\theta$$

and hence

$$\int_\alpha^\Gamma r^\Delta(\theta)\chi\left(\frac{|x^\Delta(\theta)|}{r^\Delta(\theta)}\right) \le \Psi^{-1}\left(\int_\alpha^\Gamma \omega(\theta)\Psi\left(\frac{r^\Delta(\theta)\pi\big(\frac{|x^\Delta(\theta)|}{r^\Delta(\theta)}\big)}{\omega(\theta)}\right)\Delta\theta\right) \tag{46}$$

and

$$\int_\alpha^\Gamma s^\Delta(\theta)\pi\left(\frac{|y^\Delta(\theta)|}{s^\Delta(\theta)}\right) \le \Psi^{-1}\left(\int_\alpha^\Gamma \omega(\theta)\Psi\left(\frac{s^\Delta(\theta)\pi\big(\frac{|y^\Delta(\theta)|}{s^\Delta(\theta)}\big)}{\omega(\theta)}\right)\Delta\theta\right) \tag{47}$$

Using (46) and (47) in (39), we get (45). This completes the proof. □

3. Conclusions

In this paper, with the help of the dynamic Jensen inequality, dynamic Hölder inequality and a simple consequence of Keller's chain rule on time scales, we generalized a number of Opial-type inequalities to a general time scale. Besides that, in order to illustrate the theorems for each type of inequality applied to various time scales such as $\mathbb{R}$, $h\mathbb{Z}$, $\overline{q^{\mathbb{Z}}}$ and $\mathbb{Z}$ as a sub case of $h\mathbb{Z}$. For future studies, researchers may obtain some different generalizations for dynamic Opial inequality and its companion inequalities by using the results presented in this paper.

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References

1. Opial, Z. Sur une inégalité. *Ann. Polon. Math.* **1960**, *8*, 29–32. [CrossRef]
2. Godunova, E.K.; Levin, V.I. An inequality of Maroni. *Mat. Zametki* **1967**, *2*, 221–224. [CrossRef]
3. Hwang, T.S.; Yang, G.S. On integral inequalities related to Opial's inequality. *Tamkang J. Math.* **1990**, *21*, 177–183.
4. Hilger, S. Ein maßkettenkalkül mit anwendung auf zentrumsmannigfaltigkeiten. Ph.D. Thesis, Universität Würzburg, Würzburg, Germany, 1988.
5. Bohner, M.; Peterson, A. *Dynamic Equations on Time Scales: An Introduction with Applications*; Birkhäuser Boston, Inc.: Boston, MA, USA, 2001. [CrossRef]
6. Karpuz, B.; Kaymakçalan, B.; Özkan, U.M. Some multi-dimenstonal Opial-type inequalities on time scales. *J. Math. Ineq.* **2010**, *4*, 207–216. [CrossRef]
7. Zhao, C.J.; Cheung, W.S. On Opial-type integral inequalities and applications. *Math. Inequal. Appl.* **2014**, *17*, 223–232. [CrossRef]
8. Cheung, W.S. Opial-type inequalities with m functions in n variables. *Mathematika* **1992**, *39*, 319–326. [CrossRef]
9. Cheung, W.S. Some generalized Opial-type inequalities. *J. Math. Anal. Appl.* **1991**, *162*, 317–321. [CrossRef]
10. Zhao, C.J.; Cheung, W.S. On Opial's type inequalities for an integral operator with homogeneous kernel. *J. Inequalities Appl.* **2012**, *2012*, 123. [CrossRef]
11. Abdeldaim, A.; El-Deeb, A.; Ahmed, R.G. On Retarded Nonlinear Integral Inequalities Of Gronwall and Applications. *J. Math. Inequal.* **2019**, *13*, 1023–1038. [CrossRef]
12. El-Deeb, A.; El-Sennary, H.; Agarwal, P. Some opial-type inequalities with higher order delta derivatives on time scales. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales Serie A Matemáticas* **2020**, *114*, 29. [CrossRef]
13. El-Deeb, A.A.; Makharesh, S.D.; Baleanu, D. Dynamic Hilbert-Type Inequalities with Fenchel-Legendre Transform. *Symmetry* **2020**, *12*, 582. [CrossRef]
14. El-Deeb, A.A.; Khan, Z.A. Certain new dynamic nonlinear inequalities in two independent variables and applications. *Bound. Value Probl.* **2020**, *2020*, 31. [CrossRef]
15. Abdeldaim, A.; El-Deeb, A.A.; Agarwal, P.; El-Sennary, H.A. On some dynamic inequalities of Steffensen type on time scales. *Math. Methods Appl. Sci.* **2018**, *41*, 4737–4753. doi:10.1002/mma.4927. [CrossRef]
16. Agarwal, R.; O'Regan, D.; Saker, S. *Dynamic Inequalities on Time Scales*; Springer: Cham, Switzerland, 2014, Volume 2014.
17. Akin-Bohner, E.; Bohner, M.; Akin, F. Pachpatte inequalities on time scales. *JIPAM. J. Inequal. Pure Appl. Math.* **2005**, *6*, 23.
18. Bohner, M.; Matthews, T. The Grüss inequality on time scales. *Commun. Math. Anal.* **2007**, *3*, 1–8.
19. Bohner, M.; Matthews, T. Ostrowski inequalities on time scales. *JIPAM J. Inequal. Pure Appl. Math.* **2008**, *9*, 8.
20. Dinu, C. Hermite-Hadamard inequality on time scales. *J. Inequal. Appl.* **2008**, 24. [CrossRef]
21. El-Deeb, A.A. On some generalizations of nonlinear dynamic inequalities on time scales and their applications. *Appl. Anal. Discret. Math.* **2019**, 440–462. [CrossRef]
22. El-Deeb, A.A.; Cheung, W.S. A variety of dynamic inequalities on time scales with retardation. *J. Nonlinear Sci. Appl.* **2018**, *11*, 1185–1206. [CrossRef]

23. El-Deeb, A.A.; El-Sennary, H.A.; Khan, Z.A. Some Steffensen-type dynamic inequalities on time scales. *Adv. Differ. Equ.* **2019**, 246. [CrossRef]
24. El-Deeb, A.A.; Elsennary, H.A.; Cheung, W.S. Some reverse Hölder inequalities with Specht's ratio on time scales. *J. Nonlinear Sci. Appl.* **2018**, *11*, 444–455. [CrossRef]
25. El-Deeb, A.A.; Elsennary, H.A.; Nwaeze, E.R. Generalized weighted Ostrowski, trapezoid and Grüss type inequalities on time scales. *Fasc. Math.* **2018**, 123–144. [CrossRef]
26. El-Deeb, A.A.; Xu, H.; Abdeldaim, A.; Wang, G. Some dynamic inequalities on time scales and their applications. *Adv. Differ. Equ.* **2019**, 19. [CrossRef]
27. El-Deeb, A.A. Some Gronwall-Bellman type inequalities on time scales for Volterra-Fredholm dynamic integral equations. *J. Egypt. Math. Soc* **2018**, *26*, 1–17. [CrossRef]
28. Hilscher, R. A time scales version of a Wirtinger-type inequality and applications. *J. Comput. Appl. Math.* **2002**, *141*, 219–226. [CrossRef]
29. Li, W.N. Some delay integral inequalities on time scales. *Comput. Math. Appl.* **2010**, *59*, 1929–1936. [CrossRef]
30. Řehák, P. Hardy inequality on time scales and its application to half-linear dynamic equations. *J. Inequal. Appl.* **2005**, 495–507. [CrossRef]
31. Saker, S.H.; El-Deeb, A.A.; Rezk, H.M.; Agarwal, R.P. On Hilbert's inequality on time scales. *Appl. Anal. Discrete Math.* **2017**, *11*, 399–423. [CrossRef]
32. Tian, Y.; El-Deeb, A.A.; Meng, F. Some nonlinear delay Volterra-Fredholm type dynamic integral inequalities on time scales. *Discret. Dyn. Nat. Soc.* **2018**, 8. [CrossRef]
33. Saker, S.H. Opial's type inequalities on time scales and some applications. *Ann. Polon. Math.* **2012**, *104*, 243–260. [CrossRef]
34. Saker, S.H. Some Opial-type inequalities on time scales. *Abstr. Appl. Anal.* **2011**. [CrossRef]
35. Saker, S.H. New inequalities of Opial's type on time scales and some of their applications. *Discrete Dyn. Nat. Soc.* **2012**, 23. [CrossRef]
36. Bohner, M.J.; Mahmoud, R.R.; Saker, S.H. Discrete, continuous, delta, nabla, and diamond-alpha Opial inequalities. *Math. Inequal. Appl.* **2015**, *18*, 923–940. [CrossRef]
37. Saker, S.H. Some new inequalities of Opial's type on time scales. *Abstr. Appl. Anal.* **2012**, 14. [CrossRef]
38. Karpuz, B.; Özkan, U.M. Some generalizations for Opial's inequality involving several functions and their derivatives of arbitrary order on arbitrary time scales. *Math. Inequal. Appl.* **2011**, *14*, 79–92. [CrossRef]
39. Li, J.D. Opial-type integral inequalities involving several higher order derivatives. *J. Math. Anal. Appl.* **1992**, *167*, 98–110.
40. Lee, C.S. On Some Generalization of Inequalities of Opial, Yang and Shum. *Canad. Math. Bull.* **1980**, *23*. [CrossRef]
41. Abdeldaim, A.; El-Deeb, A.A. Some new retarded nonlinear integral inequalities with iterated integrals and their applications in retarded differential equations and integral equations. *J. Fract. Calc. Appl.* **2014**, *5*, 9. [CrossRef]
42. Abdeldaim, A.; El-Deeb, A.A. On generalized of certain retarded nonlinear integral inequalities and its applications in retarded integro-differential equations. *Appl. Math. Comput.* **2015**, *256*, 375–380. [CrossRef]
43. Abdeldaim, A.; El-Deeb, A.A. On some generalizations of certain retarded nonlinear integral inequalities with iterated integrals and an application in retarded differential equation. *J. Egypt. Math. Soc.* **2015**, *23*, 470–475. [CrossRef]
44. Abdeldaim, A.; El-Deeb, A.A. On some new nonlinear retarded integral inequalities with iterated integrals and their applications in integro-differential equations. *Br. J. Math. Comput. Sci.* **2015**, *5*, 479–491. [CrossRef]
45. Agarwal, R.P.; Lakshmikantham, V. *Uniqueness and Nonuniqueness Criteria for Ordinary Differential Equations*; Volume 6 of Series in Real Analysis; World Scientific Publishing: Singapore, 1993.
46. El-Deeb, A.A. *On Integral Inequalities and Their Applications Lambert*; LAP Lambert Academic Publishing: Saarbrücken, Germany, 2017.
47. El-Deeb, A.A. A Variety of Nonlinear Retarded Integral Inequalities of Gronwall Type and Their Applications. In *Advances in Mathematical Inequalities and Applications*; Springer: Berlin, Germany, 2018; pp. 143–164.
48. El-Deeb, A.A.; Ahmed, R.G. On some explicit bounds on certain retarded nonlinear integral inequalities with applications. *Adv. Inequalities Appl.* **2016**, *2016*, 19.

49. El-Deeb, A.A.; Ahmed, R.G. On some generalizations of certain nonlinear retarded integral inequalities for Volterra-Fredholm integral equations and their applications in delay differential equations. *J. Egypt. Math. Soc.* **2017**, *25*, 279–285. [CrossRef]
50. El-Owaidy, H.; Abdeldaim, A.; El-Deeb, A.A. On some new retarded nonlinear integral inequalities and their applications. *Math. Sci. Lett.* **2014**, *3*, 157. [CrossRef]
51. El-Owaidy, H.M.; Ragab, A.A.; Eldeeb, A.A.; Abuelela, W.M.K. On some new nonlinear integral inequalities of Gronwall-Bellman type. *Kyungpook Math. J.* **2014**, *54*, 555–575. [CrossRef]
52. El-Deeb, A.A.; Kh., F.M.; Ismail, G.A.F.; Khan, Z.A. Weighted dynamic inequalities of Opial-type on time scales. *Adv. Differ. Eq.* **2019**, *2019*, 393. [CrossRef]
53. Kh., F.M.; El-Deeb, A.A.; Abdeldaim, A.; Khan, Z.A. On some generalizations of dynamic Opial-type inequalities on time scales. *Adv. Differ. Eq.* **2019**, *2019*, 1–14. [CrossRef]
54. Pachpatte, B.G. On integral inequalities similar to Opial's inequality. *Demonstr. Math.* **1989**, *22*, 21–27. [CrossRef]
55. Pachpatte, B.G. A note on generalized Opial type inequalities. *Tamkang J. Math.* **1993**, *24*, 229–235.
56. Hua, L.G. On an inequality of Opial. *Sci. Sin.* **1965**, *14*, 789–790.
57. Yang, G.S. On a certain result of Z. Opial. *Proc. Jpn. Acad.* **1966**, *42*, 78–83. [CrossRef]
58. Maroni, P. Sur l'inégalité d'Opial-Beesack. *C. R. Acad. Sci. Paris Sér. A-B* **1967**, *264*, A62–A64.
59. Yang, G.S. A note on some integro-differential inequalities. *Soochow J. Math.* **1983**, *9*, 231–236.