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Ordered Gyrovector Spaces

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Abstract: The well-known construction scheme to define a partial order on a vector space is to use a proper convex cone. Applying this idea to the gyrovector space we construct the partial order, called a gyro-order. We also give several inequalities of gyrolines and cogyrolines in terms of the gyro-order.

Keywords: gyrogroup; gyrovector space; gyroline; cogyroline; partial order

1. Introduction

Since finding appropriate algebraic coordinatizations in geometric settings has been attempted first by R. Descartes and P. Fermat, the study of more general algebraic structures have been actively researched. Gyrogroups and gyrovector spaces with the non-associative binary operation $\oplus$, introduced by A. Ungar [1], are suitable generalization of groups and vector spaces. In particular, gyrovector spaces algebraically regulate many typical examples of analytic hyperbolic geometry. For instance, the Einstein gyrovector space, Möbius gyrovector space, and Proper Velocity (PV, in short) gyrovector space provide algebraic tools to study the Beltrami–Klein, Poincaré ball models, and PV space model of hyperbolic geometry, respectively. As many recent results shows that the theory of gyrogroups and gyrovector spaces can be applied to various areas such as the loop theory, the theory of special relativity, and quantum information, it has been widely studied [2–6].

To capture fundamental analogies between gyrogroups and groups, there is another binary operation, called a coaddition $\boxplus$ in a gyrogroup: see [1] for more information. The gyrovector space G has a gyrometric and cogyrometric satisfying the gyrotriangle and cogyrotriangle inequalities, respectively, such that

$$d_{\oplus}(x,y) = \| \ominus x \oplus y\|, \; d_{\boxplus}(x,y) = \| \ominus x \boxplus y\|$$

for any $x, y \in G$, where $\ominus x$ denotes the inverse of x. Curves on which the gyrotriangle and cogyrotriangle inequalities reduce to equalities are called gyrolines and cogyrolines. They are uniquely determined by given two points in the gyrovector space and play important roles in the concepts of gyrocentroid and gyroparallelogram law. Under settings of gyrovector spaces we have many different types of gyrolines. Especially, note that it coincides with a Riemannian geodesic, called the weighted geometric mean, on the Cartan–Hadamard manifold of positive definite Hermitian matrices with Riemannian trace metric.

The well-known construction scheme to define a partial order on the vector space V is to use a proper convex cone in V. A subset $C \subset V$ is called a proper convex cone if and only if

$$\alpha \mathbf{x} + \beta \mathbf{y} \in C, \; C \cap -C = \{\mathbf{0}\}$$

for any scalars $\alpha, \beta > 0$ and any $\mathbf{x}, \mathbf{y} \in C$. Using the proper convex cone C the relation defined as

$$\mathbf{x} \leq \mathbf{y} \text{ if and only if } \mathbf{y} - \mathbf{x} \in C,$$

gives us a partial order. The partial order can be applied to many research fields such as category theory, graph theory, and computer science. Applying this construction scheme to the gyrovector space, we define the partial order, what we call a gyro-order and an ordered gyrovector space for the gyrovector space equipped with the gyro-order. Furthermore, we show interesting inequalities about gyrogeodesics, that is, gyrolines and cogyrolines, in terms of the gyro-order.

2. Gyrovector Spaces

Let $\mathbf{B} := \{\mathbf{v} \in \mathbb{R}^3 : \|\mathbf{v}\| = \mathbf{v}^T\mathbf{v} < 1\}$ be the open unit ball in the 3-dimensional Euclidean space $\mathbb{R}^3$. In 1905 A. Einstein has introduced a relativistic sum of vectors in $\mathbf{B}$, which founded the theory of special relativity:

$$\mathbf{u} \oplus_E \mathbf{v} = \frac{1}{1+\mathbf{u}^T\mathbf{v}} \left\{ \mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}^T\mathbf{v})\mathbf{u} \right\}, \tag{1}$$

where $\gamma_{\mathbf{u}} := \dfrac{1}{\sqrt{1-\|\mathbf{u}\|^2}}$ is the well-known Lorentz gamma factor. We denote as $\mathbf{u}^T\mathbf{v}$ the Euclidean inner product of $\mathbf{u}$ and $\mathbf{v}$ written in matrix form. The formula (1) is a binary operation on the open unit ball $\mathbf{B}$, called the Einstein velocity addition.

To analyze the Einstein's relativistic sum abstractly, A. Ungar has introduced a group-like structure, called a gyrogroup, in several papers and books; see [1] and its bibliographies. His algebraic axioms are similar to those of a group, but a binary operation in the gyrogroup is neither associative nor commutative in general.

Definition 1. *A triple $(G, \oplus, e)$ is a gyrogroup if G is a nonempty set, $\oplus$ is a binary operation on G, and the following are satisfied for all $x, y, z \in G$.*

(G1) $e \oplus x = x \oplus e = x$ *(existence of identity);*
(G2) $x \oplus (\ominus x) = (\ominus x) \oplus x = e$ *(existence of inverses);*
(G3) There is an automorphism $\mathrm{gyr}[x, y] : G \to G$ *for each* $x, y \in G$ *such that*

$$x \oplus (y \oplus z) = (x \oplus y) \oplus \mathrm{gyr}[x, y]z \quad (\textit{gyroassociativity});$$

(G4) $\mathrm{gyr}[e, x] = id_G$*, where* id_G *represents the identity map on* G*;*
(G5) $\mathrm{gyr}[x \oplus y, y] = \mathrm{gyr}[x, y]$ *(loop property).*

A gyrogroup $(G, \oplus)$ is gyrocommutative if it holds

$$x \oplus y = \mathrm{gyr}[x, y](y \oplus x) \quad (\textit{gyrocommutativity}).$$

A gyrogroup is uniquely 2-divisible if for every $a \in G$, there exists a unique $x \in G$ such that $x \oplus x = a$.

We sometimes write as $x \ominus y := x \oplus (\ominus y)$ for any $x, y \in G$. Using (G4) and (G5) we obtain

$$\mathrm{gyr}[x, x] = \mathrm{gyr}[\ominus x, x] = id_G \tag{2}$$

for any $x \in G$. Note that (G3) and (G5) are also called the left gyroassociativity and left loop property. Furthermore, the following hold on a gyrogroup $(G, \oplus)$ [1] (Theorem 2.35):

$$\begin{aligned} (x \oplus y) \oplus z &= x \oplus (y \oplus \mathrm{gyr}[y, x]z) \\ \mathrm{gyr}[x, y] &= \mathrm{gyr}[x, y \oplus x] \end{aligned} \tag{3}$$

for any $x, y, z \in G$. These are called the right gyroassociativity and right loop property, respectively.

We call the map $\mathrm{gyr}[x, y]$ in (G3) the gyroautomorphism or Thomas gyration generated by x and y in a gyrogroup G, which is analogous to the precession map in a loop theory. Moreover, we have from (G2) and (G3)

$$\mathrm{gyr}[x, y]z = \ominus(x \oplus y) \oplus [x \oplus (y \oplus z)]$$

for any $x, y, z \in G$. It can be rewritten as

$$\mathrm{gyr}[x, y] = L_{\ominus(x \oplus y)} L_x L_y = L_{x \oplus y}^{-1} L_x L_y, \tag{4}$$

where L_x is the left translation by $x \in G$. The last equality follows from $L_x^{-1} = L_{\ominus x}$ due to the left cancellation law [1] (Theorem 2.10): for any $x, y \in G$

$$\ominus x \oplus (x \oplus y) = y.$$

In Euclidean space it plays a role of rotation in the plane generated by x and y, leaving the orthogonal complement fixed.

Definition 2. *For a gyrogroup $(G, \oplus)$, the gyrogroup cooperation is a binary operation in G defined by*

$$x \boxplus y = x \oplus \mathrm{gyr}[x, \ominus y] y$$

for any $x, y \in G$.

We simply write as $x \boxminus y := x \boxplus (\ominus y)$ for any $x, y \in G$. Note that $x \boxminus y = x \ominus \mathrm{gyr}[x, y] y$, since $\mathrm{gyr}[x, y](\ominus y) = \ominus \mathrm{gyr}[x, y] y$ by [1] (Theorem 2.10).

We give the equivalent conditions for the gyrogroup to be gyrocommutative.

Theorem 1. *Let $(G, \oplus)$ be a gyrogroup. The following are equivalent: for all $x, y \in G$*

(i) *G is gyrocommutative,*
(ii) *G satisfies the automorphic inverse property: $\ominus(x \oplus y) = \ominus x \ominus y$,*
(iii) *G satisfies the Bruck identity: $(x \oplus y) \oplus (x \oplus y) = x \oplus (y \oplus (y \oplus x))$,*
(iv) *The cooperation $\boxplus$ is commutative: $x \boxplus y = y \boxplus x$.*

Proof. The equivalence between (i) and (ii), between (i) and (iv) have been shown from [1] (Theorem 3.2), and [1] (Theorem 3.4), respectively.

(i) $\Rightarrow$ (iii): Let $(G, \oplus)$ be a gyrocommutative gyrogroup. Then for any $x, y \in G$, $x \oplus y = \mathrm{gyr}[x, y](y \oplus x)$. By (4) we have

$$\begin{aligned}(x \oplus y) \oplus (x \oplus y) &= L_{x \oplus y}(x \oplus y) = L_{x \oplus y} \mathrm{gyr}[x, y](y \oplus x) \\ &= L_{x \oplus y} L_{x \oplus y}^{-1} L_x L_y (y \oplus x) = L_x L_y (y \oplus x) = x \oplus (y \oplus (y \oplus x)).\end{aligned}$$

(iii) $\Rightarrow$ (i): Let a gyrogroup $(G, \oplus)$ satisfy the Bruck identity (iii). The item (iii) can be written as

$$L_{x \oplus y}(x \oplus y) = L_x L_y (y \oplus x).$$

Since the left translation is bijective from [1] (Theorem 2.22), we have by (4)

$$x \oplus y = L_{x \oplus y}^{-1} L_x L_y (y \oplus x) = \mathrm{gyr}[x, y](y \oplus x).$$

□

Analogous to construct a vector space from an additive group with scalar multiplication, we can define a gyrovector space from a gyrocommutative gyrogroup with certain scalar multiplication. The following definition of the gyrovector space is slightly different from Definition 6.2 in [1] introduced by A. Ungar.

Definition 3. *A gyrovector space is a triple $(G, \oplus, \odot)$, where $(G, \oplus)$ is a gyrocommutative gyrogroup and $\odot$ is a scalar multiplication defined by*

$$(t, x) \mapsto t \odot x : \mathbb{R} \times G \to G$$

that satisfies the following: for any $s, t \in \mathbb{R}$ and $a, b, x \in G$

(S1) $1 \odot x = x,\ 0 \odot x = 0 = t \odot 0$, *and* $(-1) \odot x = \ominus x$;
(S2) $(s + t) \odot x = s \odot x \oplus t \odot x$;
(S3) $s \odot (t \odot x) = (st) \odot x$;
(S4) $\mathrm{gyr}[a, b](t \odot x) = t \odot \mathrm{gyr}[a, b]x$.

Definition 4. *A gyrovector space $(G, \oplus, \odot)$ equipped with Hausdorff topology such that both maps $\oplus : G \times G \to G$ and $\odot : \mathbb{R} \times G \to G$ are continuous is called a topological gyrovector space.*

The gyroaddition $\oplus$ does not, in general, satisfy the distributivity with scalar multiplication, i.e.,

$$t \odot (x \oplus y) \neq t \odot x \oplus t \odot y$$

for $t \in \mathbb{R}$ and $x, y \in G$. On the other hand, it has been shown in [7] that the following are equivalent on the topological gyrovector space $(G, \oplus, \odot)$: for any $s, t \in \mathbb{R}$

(i) $\mathrm{gyr}[x, y] = id_G$;
(ii) $\mathrm{gyr}[s \odot x, t \odot y] = id_G$;
(iii) $t \odot (x \oplus y) = t \odot x \oplus t \odot y$.

Setting $x = y$ in the above equivalences, we obtain from (2) that

$$\mathrm{gyr}[s \odot x, t \odot x] = id_G. \tag{5}$$

At the beginning of this section the Einstein's relativistic sum of vectors in the open unit ball of $\mathbb{R}^3$ is introduced. On the other hand, it can be extended to the binary operation in the open unit ball of $\mathbb{R}^n$ and gives us a typical example of the topological gyrovector spaces.

Example 1. *For any $\mathbf{u}, \mathbf{v} \in \mathbf{B}$, where $\mathbf{B}$ is the open unit ball in $\mathbb{R}^n$, we define:*

$$\mathbf{u} \oplus_E \mathbf{v} = \frac{1}{1 + \mathbf{u}^T \mathbf{v}} \left\{ \mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}} \mathbf{v} + \frac{\gamma_{\mathbf{u}}}{1 + \gamma_{\mathbf{u}}} (\mathbf{u}^T \mathbf{v}) \mathbf{u} \right\}$$
$$\mathbf{u} \oplus_M \mathbf{v} = \frac{1}{1 + \mathbf{u}^T \mathbf{v} + \|\mathbf{u}\|^2 \|\mathbf{v}\|^2} \left\{ \left(1 + 2\mathbf{u}^T \mathbf{v} + \|\mathbf{v}\|^2\right) \mathbf{u} + \left(1 - \|\mathbf{u}\|^2\right) \mathbf{v} \right\}.$$

The sum $\mathbf{u} \oplus_M \mathbf{v}$ is called the Möbius addition of $\mathbf{u}$ and $\mathbf{v}$, known as Möbius translation: see formula (4.5.5) of [8].

We define a map $\odot : \mathbb{R} \times \mathbf{B} \to \mathbf{B}$ by

$$t \odot \mathbf{v} = \frac{(1 + \|\mathbf{v}\|)^t - (1 - \|\mathbf{v}\|)^t}{(1 + \|\mathbf{v}\|)^t + (1 - \|\mathbf{v}\|)^t} \frac{\mathbf{v}}{\|\mathbf{v}\|} = \tanh\left(t \tanh^{-1} \|\mathbf{v}\|\right) \frac{\mathbf{v}}{\|\mathbf{v}\|} \tag{6}$$

for $t \in \mathbb{R}$ and $\mathbf{v}(\neq \mathbf{0}) \in \mathbf{B}$, and define $t \odot \mathbf{0} := \mathbf{0}$. We call $(\mathbf{B}, \oplus_E, \odot)$ and $(\mathbf{B}, \oplus_M, \odot)$ the Einstein gyrovector space and the Möbius gyrovector space, respectively.

Example 2 ([4,7])**.** *We define two different binary operations $\oplus$ and $*$ on $\mathbb{P}_n$, the open convex cone of all $n \times n$ positive definite Hermitian matrices, such as*

$$A \oplus B = A^{1/2} B A^{1/2}, \ A * B = (AB^2 A)^{1/2}$$

for any $A, B \in \mathbb{P}_n$. Then $(\mathbb{P}_n, \oplus)$ and $(\mathbb{P}_n, *)$ are gyrocommutative gyrogroups, which are isomorphic via the squaring map. Moreover, we define a scalar multiplication $\circ$ by

$$\circ : \mathbb{R} \times \mathbb{P}_n \to \mathbb{P}_n, \; t \circ A = A^t$$

for any $t \in \mathbb{R}$. Then the systems $(\mathbb{P}_n, \oplus, \circ)$ and $(\mathbb{P}_n, *, \circ)$ form topological gyrovector spaces.

3. Gyrolines and Cogyrolines

The geodesic in the Euclidean space $\mathbb{R}^n$ is the straight line, which can be expressed as

$$x + t(-x + y) \text{ and } t(y - x) + x.$$

It is uniquely determined by given two distinct points x and y, and we call each of them the line representation by two points x and y. In the Euclidean geometry the above representations coincide via the associative algebra of vector spaces.

On the gyrovector space $(G, \oplus, \odot)$ we write the above expressions as

$$\begin{aligned} &x \oplus t \odot (\ominus x \oplus y), \\ &t \odot (y \boxminus x) \oplus x. \end{aligned}$$

We call them, respectively, the gyroline and the cogyroline determined by two distinct points x and y. They are totally different in the setting of gyrovector spaces, because of the non-associativity of a gyrovector space.

The midpoints of gyrolines and cogyrolines at $t = 1/2$ are called the gyromidpoint and the cogyromidpoint, respectively. They have an interesting connection as the unique solution of the simultaneous equations in [1] (Theorem 6.15).

Proposition 1 ([1] (Theorem 6.15))**.** *Let $x, y \in G$, where $(G, \oplus, \odot)$ is a gyrovector space. Then the following system of two equations*

$$\begin{cases} a \oplus b = x, \\ \ominus a \oplus b = y \end{cases}$$

for the unknowns a and b has the unique solution such that

$$a = \frac{1}{2} \odot (x \boxminus y), \qquad b = \frac{1}{2} \odot (x \boxminus y) \oplus y.$$

Note that the solution a is the gyromidpoint of x and $\ominus y$, and the solution b is the cogyromidpoint of x and y.

We study more gyrolines and cogyrolines with these midpoints in this section.

3.1. Gyrolines and Gyromidpoints

The gyroline passing through the points x and y in the gyrovector space $(G, \oplus, \odot)$ [1] (Definition 6.19) is given by

$$L : \mathbb{R} \times G \times G \to G, \; L(t; x, y) = x \oplus t \odot (\ominus x \oplus y). \tag{7}$$

It is uniquely determined by given two distinct points, and a left gyrotranslation preserves the notion of a gyroline by [1] (Theorem 6.21). In other words,

$$a \oplus L(t; x, y) = L(t; a \oplus x, a \;\oplus y) \tag{8}$$

for any $a \in G$. The gyromidpoint of x and y in G is given by

$$L\left(\frac{1}{2}; x, y\right) = x \oplus \frac{1}{2} \odot (\ominus x \oplus y) = \frac{1}{2} \odot (x \boxplus y).$$

Lemma 1. *Let $x, y \in G$, where $(G, \oplus, \odot)$ is a gyrovector space. Let $s, t, u \in \mathbb{R}$.*

(i) $L(t; \ominus x, \ominus y) = \ominus L(t; x, y)$.
(ii) $L(t; x, y) = L(1 - t; y, x)$.
(iii) $L(u; L(s; x, y), L(t; x, y)) = L((1 - u)s + ut; x, y)$.

Proof. Note that (i) can be proved by the automorphic inverse property in Theorem 1, and (ii) and (iii) have been shown from Lemma 6.27 and Theorem 6.20 in [1]. □

Corollary 1. *Let $x, y \in G$, where $(G, \oplus, \odot)$ is a gyrovector space. Then the equation $L(t; x, a) = y$ for the unknown a and nonzero t has a unique solution*

$$a = L\left(\frac{1}{t}; x, y\right).$$

Proof. By Lemma 1 (3), we have that $a = L\left(\frac{1}{t}; x, y\right)$ satisfies the equation $L(t; x, a) = y$. Suppose that $b \in G$ is another solution of the equation $L(t; x, b) = y$. Then $L(t; x, a) = L(t; x, b)$, that is,

$$x \oplus t \odot (\ominus x \oplus a) = x \oplus t \odot (\ominus x \oplus b).$$

By using the Left Cancellation Law, taking a scalar multiplication $\odot$ by $\dfrac{1}{t}$ on both sides, and using the Left Cancellation Law again, we obtain $a = b$. □

3.2. Cogyrolines and Cogyromidpoints

The cogyroline passing through the points x and y in the gyrovector space $(G, \oplus, \odot)$ is defined by

$$L^c : \mathbb{R} \times G \times G \to G,\ L^c(t; x, y) = t \odot (\ominus x \boxplus y) \oplus x. \tag{9}$$

It is uniquely determined by given two distinct points, and one can write from the commutativity of cooperation $\boxplus$ in Theorem 1 (iv) as

$$L^c(t; x, y) = t \odot (y \boxminus x) \oplus x.$$

Moreover, the point

$$L^c(1/2; x, y) = \frac{1}{2} \odot (\ominus x \boxplus y) \oplus x = \frac{1}{2} \odot (y \boxminus x) \oplus x$$

is called the cogyromidpoint of two points x and y. The last equality in the above follows from the commutativity of gyrogroup cooperation $\boxplus$ in Theorem 1 (iv).

Remark 1. *By Definition 3 (S3) and Lemma 1 (2) with $t = 1/2$, we have alternative expressions of the cogyroline.*

$$L^c(t; x, y) = 2t \odot \left(\frac{1}{2} \odot (y \boxminus x)\right) \oplus x = 2t \odot \left(\frac{1}{2} \odot (\ominus x \boxplus y)\right) \oplus x = 2t \odot L\left(\frac{1}{2}; \ominus x, y\right) \oplus x.$$

Lemma 2. *Let $x, y \in G$, where $(G, \oplus, \odot)$ is a gyrovector space. Then*

$$L^c\left(\frac{1}{2}; x, y\right) = \frac{1}{2} \odot \mathrm{gyr}\left[L^c\left(\frac{1}{2}; x, y\right), x\right](x \oplus y).$$

Proof. By the gyrocommutativity, Definition 3 (S4), the Bruck identity (3) of Theorem 1, Definition 3 (S3), the loop property (G5), and the Right Cancellation Law in [1] (Theorem 2.22), we have

$$\begin{aligned} 2 \odot L^c\left(\frac{1}{2}; x, y\right) &= 2 \odot \left\{\frac{1}{2} \odot (y \boxminus x) \oplus x\right\} \\ &= 2 \odot \mathrm{gyr}\left[\frac{1}{2} \odot (y \boxminus x), x\right] \left\{x \oplus \frac{1}{2} \odot (y \boxminus x)\right\} \\ &= \mathrm{gyr}\left[\frac{1}{2} \odot (y \boxminus x), x\right] 2 \odot \left\{x \oplus \frac{1}{2} \odot (y \boxminus x)\right\} \\ &= \mathrm{gyr}\left[\frac{1}{2} \odot (y \boxminus x) \oplus x, x\right] \{x \oplus ((y \boxminus x) \oplus x)\} \\ &= \mathrm{gyr}\left[L^c\left(\frac{1}{2}; x, y\right), x\right] (x \oplus y). \end{aligned}$$

□

The following are the basic properties of cogyrolines analogous to those of gyrolines.

Lemma 3. *Let $x, y \in G$, where $(G, \oplus, \odot)$ is a gyrovector space. Let $s, t, u \in \mathbb{R}$.*

(i) $L^c(t; \ominus x, \ominus y) = \ominus L^c(t; x, y)$.
(ii) $L^c(t; x, y) = L^c(1 - t; y, x)$.
(iii) $L^c(u; L^c(s; x, y), L^c(t; x, y)) = L^c((1 - u)s + ut; x, y)$.

Proof. Note that (i) can be proved by the automorphic inverse property and the commutativity of cooperation $\boxplus$ in Theorem 1, and (ii) and (iii) have been shown from Lemma 6.59 and Theorem 6.54 in [1]. □

Similar to Corollary 1 we obtain the following for cogyrolines.

Corollary 2. *Let $x, y \in G$, where $(G, \oplus, \odot)$ is a gyrovector space. Then the equation $L^c(t; x, a) = y$ for the unknown a and nonzero t has a unique solution*

$$a = L^c\left(\frac{1}{t}; x, y\right).$$

We obtain the connection between gyrolines and cogyrolines under certain condition.

Proposition 2. *Let $x, y \in G$ satisfying that $\mathrm{gyr}[x, y] = id_G$, where $(G, \oplus, \odot)$ is a gyrovector space. Then for any $t \in \mathbb{R}$,*

$$L(t; x, y) = \mathrm{gyr}[x, t \odot (\ominus x \oplus y)] L^c(t; x, y).$$

Proof. In [1] (Theorem 2.34) it was proven that $\mathrm{gyr}[\ominus x, \ominus y] = \mathrm{gyr}[x, y] = id_G$ for any $x, y \in G$. So

$$y \boxminus x = (\ominus x) \boxplus y = (\ominus x) \oplus \mathrm{gyr}[\ominus x, \ominus y] y = \ominus x \oplus y.$$

By applying the gyrocommutativity, we have

$$L^c(t; x, y) = t \odot (\ominus x \oplus y) \oplus x = \mathrm{gyr}[t \odot (\ominus x \oplus y), x](x \oplus t \odot (\ominus x \oplus y)).$$

By using the inversive symmetry of gyroautomorphism in [1] (Theorem 2.34), we obtain the desired identity. □

4. Gyro-Order

For a gyrovector space $(G, \oplus, \otimes)$ with the identity element e, assume that $\mathcal{C}$ is a subset of G satisfying for any $x, y \in \mathcal{C}$

(C1) $t \otimes x \in \mathcal{C}$ for any $t \geq 0$,
(C2) $x \oplus y \in \mathcal{C}$,
(C3) $\mathrm{gyr}[a,b](\mathcal{C}) \subseteq \mathcal{C}$ for any $a, b \in G$,
(C4) $(\ominus\mathcal{C}) \cap \mathcal{C} = \{e\}$,

where $\mathrm{gyr}[a,b](\mathcal{C}) := \{\mathrm{gyr}[a,b]x : x \in \mathcal{C}\}$ and $\ominus\mathcal{C} = \{\ominus x : x \in \mathcal{C}\}$. We define a relation $\leq$ such as for any $x, y \in G$

$$x \leq y \text{ if and only if } \ominus x \oplus y \in \mathcal{C}. \tag{10}$$

Alternatively, $x \ominus y \in \ominus\mathcal{C}$ by the automorphic inverse property in Theorem 1 (ii).

Proposition 3. *The relation $\leq$ defined in (10) is a partial order on a gyrovector space $(G, \oplus, \otimes)$.*

Proof. Let $x, y, z \in G$.
(Reflexive) Since $\ominus x \oplus x = e \in \mathcal{C}$, we can easily have $x \leq x$.
(Anti-symmetric) Assume $x \leq y$ and $y \leq x$, that is, $\ominus x \oplus y \in \mathcal{C}$ and $\ominus y \oplus x \in \mathcal{C}$. Then by the automorphic inverse property in Theorem 1 (ii), we have $\ominus x \oplus y = \ominus(x \oplus (\ominus y)) \in \ominus\mathcal{C}$. Moreover,

$$x \oplus (\ominus y) = \mathrm{gyr}[x, \ominus y](\ominus y \oplus x) \in \mathrm{gyr}[x, \ominus y](\mathcal{C}) \subseteq \mathcal{C}$$

from the gyrocommutativity and (C3). Thus, $\ominus x \oplus y = e$ by (C4), and conclude $x = y$.
(Transitive) Assume $x \leq y$ and $y \leq z$, that is, $\ominus x \oplus y \in \mathcal{C}$ and $\ominus y \oplus z \in \mathcal{C}$. Then we have $\mathrm{gyr}[\ominus x, y](\ominus y \oplus z) \in \mathcal{C}$ from (C3). By [1] (Theorem 2.15) and (C2),

$$\ominus x \oplus z = (\ominus x \oplus y) \oplus \mathrm{gyr}[\ominus x, y](\ominus y \oplus z) \in \mathcal{C}.$$

Thus, $x \leq z$. □

Definition 5. *The partial order $\leq$ on the gyrovector space G defined in (10) is called a gyro-order. Moreover, we call $(G, \leq)$ an ordered gyrovector space.*

Example 3. *Let us consider the gyrovector space $(\mathbb{P}_n, *, \circ)$ in Example 2. Let $\mathcal{C} = \{X \in \mathbb{P}_n : X \geq I\}$, where I is the identity matrix. Assume that $X, Y \in \mathcal{C}$ and $A, B \in \mathbb{P}_n$.*

(C1) Since $t \circ X = X^t \geq I$ for any $t \geq 0$, we have $t \circ X \in \mathcal{C}$.
(C2) Since $X * Y = (XY^2X)^{\frac{1}{2}} \geq X \geq I$ by order preserving of the congruence transformation and the square root map, we have $X * Y \in \mathcal{C}$.
(C3) Note from the gyroassociativity in (G3) that the gyroautomorphism on $(\mathbb{P}_n, *)$ generated by A and B is given by

$$\mathrm{gyr}[A,B]C = \left[(AB^2A)^{-1/2}ABC^2BA(AB^2A)^{-1/2}\right]^{1/2}$$

for any $C \in \mathbb{P}_n$. Since $X \in \mathcal{C}$,

$$\mathrm{gyr}[A,B]X \geq \left[(AB^2A)^{-1/2}ABIBA(AB^2A)^{-1/2}\right]^{1/2} = I.$$

Thus, $\mathrm{gyr}[A,B]X \in \mathcal{C}$.
(C4) Assume that $X \in \mathcal{C}$ and $X \in \ominus\mathcal{C}$. Then $X \geq I$ and $X^{-1} \geq I$. Thus, $X = I$.

Via Proposition 3 we obtain the partial order $\leq$ on the gyrovector space $(\mathbb{P}_n, *, \circ)$ such as $X \leq Y$ if and only if

$$(X^{-1}Y^2X^{-1})^{\frac{1}{2}} \geq I.$$

By using order-preserving of the square root map, one can see that it coincides with the well-known Loewner partial order.

Several fundamental properties of the gyro-order on $(G, \leq)$ are following.

Proposition 4. *If $x \leq y$ for any $x, y \in G$, then $\ominus y \leq \ominus x$.*

Proof. Assume that $x \leq y$, that is, $\ominus x \oplus y \in \mathcal{C}$. Then we have

$$\ominus(\ominus y) \oplus (\ominus x) = y \oplus (\ominus x) = \mathrm{gyr}[y, \ominus x](\ominus x \oplus y) \in \mathrm{gyr}[y, \ominus x](\mathcal{C}) \subseteq \mathcal{C}$$

The first identity holds from Theorem 1 (ii), the second from the gyrocommutativity, and the last inclusion holds from (C3). Thus, $\ominus y \leq \ominus x$ whenever $x \leq y$. □

Lemma 4. *Let $p, q \in \mathbb{R}$. Then the following hold:*

(i) if $p \leq q$, then $p \otimes x \leq q \otimes x$ for $x \in \mathcal{C}$;
(ii) if $x \leq y$ for $x, y \in G$, then $a \oplus x \leq a \oplus y$ for any $a \in G$.

Proof. By using (S2) and (C1), we can easily prove (i). For (ii), let $x \leq y$ for $x, y \in G$. Then $\ominus x \oplus y \in \mathcal{C}$. so by [1] (Theorem 2.16) and (C3), we have

$$\ominus(a \oplus x) \oplus (a \oplus y) = \mathrm{gyr}[a, x](\ominus x \oplus y) \in \mathrm{gyr}[a, x](\mathcal{C}) \subseteq \mathcal{C}$$

for any $a \in G$. Hence, $a \oplus x \leq a \oplus y$. □

Remark 2. *Lemma 4 (2) says that the left translation preserves the gyro-order.*

5. Inequalities

Throughout this section, we consider that $(G, \oplus, \odot)$ is the topological gyrovector space equipped with the gyro-order $\leq$ satisfying

$$\frac{1}{2} \odot x \leq \frac{1}{2} \odot y \tag{11}$$

whenever $x \leq y$.

We first show that the midpoint map on the ordered gyrovector space $(G, \leq)$ satisfying (11) is monotone.

Proposition 5. *Let $x, y \in G$ satisfying $x \leq y$. Then*

$$L\left(\frac{1}{2}; a, x\right) \leq L\left(\frac{1}{2}; a, y\right)$$

for any $a \in G$.

Proof. If $x \leq y$ for $x, y \in G$, then $\ominus a \oplus x \leq \ominus a \oplus y$ by Lemma 4 (ii). By (11)

$$\frac{1}{2} \odot (\ominus a \oplus x) \leq \frac{1}{2} \odot (\ominus a \oplus y).$$

Again by Lemma 4 (ii), we proved. □

Theorem 2. *Let $x, y \in G$ satisfying $x \leq y$. Then $t \odot x \leq t \odot y$ for all $t \in [0, 1]$.*

Proof. Let $x \le y$ for $x, y \in G$. Put $T := \{t \in [0,1] : t \odot x \le t \odot y\}$. Then clearly $0, 1 \in T$, and $\frac{1}{2} \in T$ due to (11).

Let $s, t \in T$. By Proposition 5 and Lemma 1 (ii) with $t = 1/2$, we have

$$\begin{aligned}\left(\frac{s+t}{2}\right) \odot x &= L\left(\frac{1}{2}; s \odot x, t \odot x\right) \\ &\le L\left(\frac{1}{2}; s \odot x, t \odot y\right) = L\left(\frac{1}{2}; t \odot y, s \odot x\right) \\ &\le L\left(\frac{1}{2}; t \odot y, s \odot y\right) = \left(\frac{s+t}{2}\right) \odot y.\end{aligned}$$

Thus, T contains all dyadic rational numbers in $[0,1]$. Since the dyadic rational numbers are dense in $[0,1]$ and the scalar multiplication is continuous, $T = [0,1]$. □

Proposition 6. *Assume that $x_1 \le y_1$ and $x_2 \le y_2$ for any $x_1, x_2, y_1, y_2 \in G$. Then*

$$L(t; x_1, x_2) \le L(t; y_1, y_2),$$

where $t \in [0,1]$.

Proof. From Lemma 4 (ii), Theorem 2, and Lemma 1 (ii), we have

$$\begin{aligned}x_1 \oplus t \odot (\ominus x_1 \oplus x_2) &\le x_1 \oplus t \odot (\ominus x_1 \oplus y_2) = y_2 \oplus (1-t) \odot (\ominus y_2 \oplus x_1) \\ &\le y_2 \oplus (1-t) \odot (\ominus y_2 \oplus y_1) = y_1 \oplus t \odot (\ominus y_1 \oplus y_2).\end{aligned}$$

□

Proposition 7. *Let $t \in [0,1]$. Then*

$$L(t; p \odot x, p \odot y) \le e \text{ for all } p \ge 1,$$

whenever $L(t; x, y) \le e$, where e is the identity in the gyrovector space G.

Proof. Suppose that $L(t; x, y) \le e$ for any $t \in [0,1]$. Let $z := \ominus x \oplus y$. Then $y = x \oplus z$ and $x \oplus t \odot z = L(t; x, y) \le e$, so $t \odot z \le \ominus x \oplus e = \ominus x$ by Lemma 4 (ii) and (G1). Applying Proposition 4 and (S3), we have

$$x \le (-t) \odot z. \tag{12}$$

For $r \in [1,2]$, choose $\lambda \in [0,1]$ such that $r = 2 - \lambda$. By Theorem 2 and (S3),

$$(1-\lambda) \odot x \le -t(1-\lambda) \odot z. \tag{13}$$

Moreover,

$$\begin{aligned}(2-\lambda) \odot y &= 2 \odot y \oplus (-\lambda) \odot y = 2 \odot (x \oplus z) \oplus (-\lambda) \odot (x \oplus z) \\ &= \{x \oplus (2 \odot z \oplus x)\} \oplus (-\lambda) \odot (x \oplus z) \\ &= x \oplus \{(2 \odot z \oplus x) \oplus (-\lambda) \odot \mathrm{gyr}[2 \odot z \oplus x, x](x \oplus z)\} \\ &= x \oplus \{(2 \odot z \oplus x) \oplus (-\lambda) \odot \mathrm{gyr}[2 \odot z, x](x \oplus z)\} \\ &= x \oplus \{2 \odot z \oplus (x \oplus (-\lambda) \odot (x \oplus z))\} \\ &= x \oplus \{2 \odot z \oplus L(\lambda; x, \ominus z)\} \\ &= x \oplus L(\lambda; 2 \odot z \oplus x, z).\end{aligned}$$

The third identity holds from Theorem 1 (iii) and (2), the fourth from (3), the fifth from (G5), the sixth from (G3), the seventh from Theorem 1 (ii), and the last from (8). Then

$$\begin{aligned}
&L(t;(2-\lambda)\odot x,(2-\lambda)\odot y)\\
&=(2-\lambda)\odot x\oplus t\odot\{-(2-\lambda)\odot x\oplus(x\oplus L(\lambda;2\odot z\oplus x,z))\}\\
&=(2-\lambda)\odot x\oplus t\odot\{-(1-\lambda)\odot x\oplus L(\lambda;2\odot z\oplus x,z)\}\\
&=x\oplus[(1-\lambda)\odot x\oplus t\odot\{-(1-\lambda)\odot x\oplus L(\lambda;2\odot z\oplus x,z)\}]\\
&=x\oplus L(t;(1-\lambda)\odot x,L(\lambda;2\odot z\oplus x,z))\\
&\leq x\oplus L(t;-t(1-\lambda)\odot z,L(\lambda;(2-t)\odot z,z))=x\oplus t\odot z\leq e.
\end{aligned}$$

The second and the third hold from (G3) and (5), and the inequality holds from Proposition 6 with (12) and (13).

The preceding result yields that

$$L(t;r\odot x,r\odot y)\leq e\implies L(t;2r\odot x,2r\odot y)\leq e.$$

Using mathematical induction the assertion is true for $r=2^n(2-\lambda)$, where $n\in\mathbb{N}$ and $0\leq\lambda\leq 1$. Thus, it holds for all real numbers. □

Proposition 8. *Let $x,y\in G$ such that $x\leq y$. Then*

$$L\left(\frac{q}{q+r};q\odot x,(-r)\odot y\right)\leq e \tag{14}$$

for all $q,r\geq 0$.

Proof. When $q=0$ or $r=0$, it is trivial. We first prove the inequality (14) by mathematical induction for a natural number r. Since

$$L\left(\frac{q}{q+1};q\odot x,(-1)\odot y\right)\leq L\left(\frac{q}{q+1};q\odot x,(-1)\odot x\right)=e,$$

the inequality (14) is true for $r=1$. Assume that

$$L\left(\frac{q}{q+k};q\odot x,(-k)\odot y\right)\leq e$$

for some natural number $r=k$. From Lemma 1 (ii), $L\left(\frac{k}{q+k};(-k)\odot y,q\odot x\right)\leq e$. Using Lemma 4 (ii),

$$\left(\frac{k}{q+k}\right)\odot(k\odot y\oplus q\odot x)=k\odot y\oplus L\left(\frac{k}{q+k};(-k)\odot y,q\odot x\right)\leq k\odot y\oplus e=k\odot y.$$

By Theorem 2, we have

$$\left(\frac{1}{q+k}\right)\odot(k\odot y\oplus q\odot x)\leq y. \tag{15}$$

Then

$$\begin{aligned}
&L\left(\frac{q}{q+k+1}; q\odot x, -(k+1)\odot y\right)\\
&= (-k)\odot y \oplus L\left(\frac{q}{q+k+1}; k\odot y\oplus q\odot x, (-1)\odot y\right)\\
&\leq (-k)\odot y \oplus L\left(\frac{q}{q+k+1}; k\odot y\oplus q\odot x, \left(\frac{-1}{q+k}\right)\odot (k\odot y\oplus q\odot x)\right)\\
&= (-k)\odot y \oplus \left(\frac{k}{q+k}\right)\odot (k\odot y\oplus q\odot x)\\
&= L\left(\frac{k}{q+k}; (-k)\odot y, q\odot x\right) \leq e.
\end{aligned}$$

The first identity holds from (8), and the inequality from Proposition 6 and Proposition 4 with (15). So (14) holds for all natural numbers r.

Consider $0 \leq r \leq 1$. By Theorem 2 and Proposition 4, $(-r)\odot y \leq (-r)\odot x$, so

$$L\left(\frac{q}{q+r}; q\odot x, (-r)\odot y\right) \leq L\left(\frac{q}{q+r}; q\odot x, (-r)\odot x\right) = e.$$

Arbitrary real number r can be written as $r = n+\lambda$ for a natural number n and $0 \leq \lambda \leq 1$. Then $\left(\frac{n}{q+n}\right)\odot (n\odot y\oplus q\odot x) \leq n\odot y$ since (14) holds for a natural number n. By Theorem 2 for $t = \frac{\lambda}{n}$, we have

$$\left(\frac{\lambda}{q+n}\right)\odot (n\odot y\oplus q\odot x) \leq \lambda\odot y.$$

Therefore,

$$\begin{aligned}
&L\left(\frac{q}{q+r}; q\odot x, (-r)\odot y\right) = L\left(\frac{r}{q+r}; (-r)\odot y, q\odot x\right)\\
&= L\left(\frac{n+\lambda}{q+n+\lambda}; (-n-\lambda)\odot y, q\odot x\right)\\
&= (-n)\odot y\oplus L\left(\frac{n+\lambda}{q+n+\lambda}; (-\lambda)\odot y, n\odot y\oplus q\odot x\right)\\
&\leq (-n)\odot y\oplus L\left(\frac{n+\lambda}{q+n+\lambda}; \left(\frac{-\lambda}{q+n}\right)\odot (n\odot y\oplus q\odot x), n\odot y\oplus q\odot x\right)\\
&= (-n)\odot y\oplus \left(\frac{n}{q+n}\right)\odot (n\odot y\oplus q\odot x)\\
&= L\left(\frac{n}{q+n}; (-n)\odot y, q\odot x\right) = L\left(\frac{q}{q+n}; q\odot x, (-n)\odot y\right) \leq e.
\end{aligned}$$

□

By the monotonicity of gyrolines in Proposition 6, we can see that for any $x, y \in G$ and any $t \in [0,1]$,

$$x \leq L(t; x, y) \leq y$$

whenever $x \leq y$. Meanwhile, the cogyroline $L^c(t; x, y)$ does not satisfy it, but has the following property.

Proposition 9. *Let $x, y \in G$ and $t \in [0,1]$. Then the following are equivalent:*

$$x \leq L^c(t; x, y) \;\text{ and }\; L^c(t; y, x) \leq y.$$

Proof. Suppose that $x \leq L^c(t;x,y)$. Then we have from Lemma 4 (ii), Lemma 3 (ii) and the left cancellation that

$$\begin{aligned}\ominus(1-t)\odot(x\boxminus y)\oplus x &\leq \ominus(1-t)\odot(x\boxminus y)\oplus L^c(t;x,y)\\ &= \ominus(1-t)\odot(x\boxminus y)\oplus L^c(1-t;y,x) = y.\end{aligned}$$

Note from the Cogyroautomorphic Inverse Theorem in [1] (Theorem 2.38) that

$$\ominus(1-t)\odot(x\boxminus y)\oplus x = (1-t)\odot(y\boxminus x)\oplus x = L^c(1-t;x,y) = L^c(t;y,x).$$

Thus, $L^c(t;y,x) \leq y$. We can prove the reverse implication via a similar process. □

6. Closing Remarks and Acknowledgement

Since A. Ungar has first introduced the notion of gyrogroup and gyrovector space, many papers and consequences in algebra, hyperbolic geometry, quantum information, and the theory of special relativity have been appeared. Especially, (uniquely 2-divisible) gyrocommutative gyrogroups are equivalent to Bruck loop (B-loop or dyadic symmetric set) with the same operation [5,6]. In this paper we constructed a partial order on a gyrovector space, called a gyro-order, and showed several inequalities about gyrolines and cogyrolines. The notion of gyro-order is new, and our scheme when applying to the gyrovector space of positive definite Hermitian matrices with certain gyrogroup operation coincides with the well-known Loewner order. On the other hand, there are a lot of different structures of gyrovector spaces, so we expect that by applying our construction scheme of the partial order to a various examples of gyrovector spaces, numerous interesting inequalities can be derived.

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References

1. Ungar, A.A. *Analytic Hyperbolic Geometry and Albert Einstein's Special Theory of Relativity*; World Scientific Press: Singapore, 2008.
2. Abe, T.; Watanabe, K. Finitely generated gyrovector subspaces and orthogonal gyrodecomposition in the Möbius gyrovector space. *J. Math. Anal. Appl.* **2017**, *449*, 77–90. [CrossRef]
3. Kim, S. Distances of qubit density matrices on Bloch sphere. *J. Math. Phys.* **2011**, *52*, 102303. [CrossRef]
4. Kim, S.; Lawson, J. Unit balls, Lorentz boosts, and hyperbolic geometry. *Results Math.* **2013**, *63*, 1225–1242. [CrossRef]
5. Lawson, J.; Lim, Y. Symmetric sets with midpoints and algebraically equivalent theories. *Results Math.* **2004**, *46*, 37–56. [CrossRef]
6. Sabinin, L.V.; Sabinina, L.L.; Sbitneva, L.V. On the notion of a gyrogroup. *Aequ. Math.* **1998**, *56*, 11–17. [CrossRef]
7. Kim, S. Distributivity on the gyrovector spaces. *Kyungpook Math. J.* **2015**, *55*, 13–20. [CrossRef]
8. Ratcliffe, J. *Foundations of Hyperbolic Manifolds*, 2nd ed.; Springer: Berlin, Germany, 2005.