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Hyers–Ulam Stability and Existence of Solutions to the Generalized Liouville–Caputo Fractional Differential Equations

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Abstract: The aim of this paper is to study the stability of generalized Liouville–Caputo fractional differential equations in Hyers–Ulam sense. We show that three types of the generalized linear Liouville–Caputo fractional differential equations are Hyers–Ulam stable by a ρ-Laplace transform method. We establish existence and uniqueness of solutions to the Cauchy problem for the corresponding nonlinear equations with the help of fixed point theorems.

Keywords: generalized Liouville–Caputo fractional differential equations; Laplace transform; Hyers–Ulam stability

1. Introduction

Because fractional calculus has a good global correlation performance to reflect the historical dependence process of the development of system functions, and can also describe the attributes of the dynamic system itself, it becomes a powerful mathematical tool to describe some complex movements, irregular phenomena, memory features, and other aspects. Fractional calculus theory was widely used by mathematicians as well as chemists, engineers, economists, biologists, and physicists (see [1–5]). In 1876, Riemann proposed the definition of the Riemann–Liouville derivative. Caputo first proposed another definition of fractional derivative via a modified Riemann–Liouville fractional integral at the beginning of the 20th century, namely a Caputo fractional derivative. Caputo and Fabrizio [6] introduced a new nonlocal derivative without a singular kernel and obtained the new Caputo–Fabrizio fractional derivative of order $0 < \alpha < 1$. Theoretical research and application of Caputo–Fabrizio fractional can be referred to [7–13]. Butzer et al. [14–18] study properties of the Hadamard fractional integral and the derivative. In [19,20], Katugampola introduced a new fractional integral and fractional derivative, which generalizes the Riemann–Liouville and the Hadamard integrals and derivative into a single form, respectively.

Hyers–Ulam stability has been one of the most active research topics in differential equations, and obtained a series of results (see [21–30]). Recently, Alqifiary et al. [22] obtained generalized Hyers–Ulam stability of linear differential equations. Razaei et al. [31] proved that the Hyers–Ulam stability of linear differential equations. Wang et al. [32] proved that two types of fractional linear differential equations are Hyers–Ulam stable. Shen et al. [33] deal with the Ulam stability of linear fractional differential equations with constant coefficients. Liu et al. [34] proved the Hyers–Ulam stability of linear Caputo–Fabrizio fractional differential equations. Liu et al. [35] studied the Hyers–Ulam stability of linear Caputo–Fabrizio fractional differential equations with the Mittag–Leffler kernel. Laplace transform method is used to deal with linear equations and fixed point approach and Gronwall inequality are used to deal with nonlinear equations.

For some differential equations describing physical models and practical problems, it is very difficult to find their exact solutions and the method of finding its exact solution (if they exist) is also very complicated. In order to construct explicit solutions to differential equations with constant coefficients and in the frame of Riemann–Liouville, Caputo and Riez fractional derivatives, integral transforms including Laplace, Mellin, and Fourier were found to be strong tools. One of the main difficulty is to find some appropriate transformations in order to find analytic solutions to some classes of fractional differential equations. In order to extend the possibility of working in a large class of functions, Jarad et al. [36] present a modified Laplace transform that it call ρ-Laplace transform, study its properties, and prove its own convolution theorem.

Motivated by [36], we apply the ρ-Laplace transform method to study the Hyers–Ulam stability of the following linear differential equations:

$$(\mathbb{D}_c^{\alpha,\rho} f)(t) = g(t),\ 0 < \alpha < 1,\ \rho > 0, \tag{1}$$

and

$$(\mathbb{D}_c^{\alpha,\rho} f)(t) - \lambda f(t) = g(t),\ 0 < \alpha < 1,\ \rho > 0, \tag{2}$$

and

$$(\mathbb{D}_c^{\alpha,\rho} f)(t) - \lambda(\mathbb{D}_c^{\beta,\rho} f)(t) = g(t),\ 0 < \alpha < 1,\ \rho > 0, \tag{3}$$

where $\mathbb{D}_c^{\alpha,\rho} f$ denotes the left generalized α order Liouville–Caputo fractional derivative for f with the parameter ρ (see Definition 2).

Next, we study to Cauchy problem for nonlinear equations as follows:

$$(\mathbb{D}_c^{\alpha,\rho} f)(t) = g(t, f(t)),\ 0 < \alpha < 1, \quad f(0) = f_0, \tag{4}$$

and show the existence and uniqueness of solutions via Banach fixed point theorem and Schaefer's fixed point theorem and obtain the generalized Hyers–Ulam-Rassias stability via an extended Gronwall's inequality.

2. Preliminaries

Let $C(I, \mathbb{R})$ be the Banach space of all continuous functions from I into $\mathbb{R}$ with the norm $\|y\|_C := \sup\{|y(x)| : x \in I\}$.

Definition 1. *(see [35], Definition 2.1) Let $\alpha > 0, t, \beta \in \mathbb{R}$. $\mathbb{E}_\alpha(t) := \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n+1)}$ is called the standard Mittag–Leffler function. $\mathbb{E}_{\alpha,\beta}(t) := \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n+\beta)}$ is called two-parameter Mittag–Leffler function.*

Definition 2. *(see [37], Definition 5) Let $f : [0, +\infty) \to \mathbb{R}$. The Liouville–Caputo generalized derivative of the function f is expressed in the form*

$$(\mathbb{D}_c^{\alpha,\rho} f)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (\frac{t^\rho - s^\rho}{\rho})^{-\alpha} f'(s) \frac{ds}{s^{1-\rho}},\ t > 0,$$

where the order $\alpha \in (0, 1)$, $\rho > 0$, and $\Gamma(\cdot)$ is the Gamma function.

Definition 3. *(see [36]) Let $0 < \alpha < 1$, $\rho > 0$. The generalized left fractional integrals of the function f is expressed in the form*

$$ {}_a(I^{\alpha,\rho} f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} f(s)) \frac{ds}{s^{1-\rho}},\ t > a,\ \rho > 0, $$

Theorem 4. *(see [36], Corollary 3.3) Let $\alpha \in (0,1)$, $\rho > 0$. The ρ–Laplace transform of the function of the generalized fractional derivative in the Liouville–Caputo sense is expressed in the following form:*

$$ \mathcal{L}_\rho\{(\mathbb{D}_c^{\alpha,\rho} f)(t)\}(s) = s^\alpha \mathcal{L}_\rho\{f(t)\}(s) - s^{\alpha-1} f_0,\ \rho > 0,\ \ f(0) = f_0. \tag{5} $$

The ρ–Laplace transform of the function f is given in the form

$$ \mathcal{L}_\rho\{f(t)\}(s) = \int_0^\infty e^{-s\frac{t^\rho}{\rho}} f(t) \frac{dt}{t^{1-\rho}},\ \rho > 0, $$

and

$$ \mathcal{L}_\rho\{f(t)\}(s) = \mathcal{L}\{f((\rho t)^{\frac{1}{\rho}})\}(s),\ \rho > 0, \tag{6} $$

where $\mathcal{L}\{f\}$ is the usual Laplace transform of f.

Definition 5. *(see [36], Definition 2.9) Let f and g be two functions which are piecewise continuous at each interval $[0,T]$ and of exponential order. The $\rho-$convolution of f and g is given by*

$$ (f *_\rho g)(t) = \int_0^t f((t^\rho - s^\rho)^{\frac{1}{\rho}}) g(s) \frac{ds}{s^{1-\rho}},\ \rho > 0. \tag{7} $$

Theorem 6. *(see [36], Theorem 2.11) Let f and g be two functions which are piecewise continuous at each interval $[0,T]$ and of exponential order $e^{c\frac{t^\rho}{\rho}}$. Then,*

$$ f *_\rho g = g *_\rho f,\ \rho > 0, $$

and

$$ \mathcal{L}_\rho\{f *_\rho g\}(s) = \mathcal{L}_\rho\{f\}\mathcal{L}_\rho\{g\}(s). $$

Lemma 7. *Let $Re(\alpha) > 0, \rho > 0$, and $|\frac{\lambda}{s^\alpha}| < 1$.*

(i) $\mathcal{L}_\rho\{1\}(s) = \frac{1}{s},\ s > 0.$

(ii) $\mathcal{L}_\rho\{e^{\lambda\frac{t^\rho}{\rho}}\}(s) = \frac{1}{s-\lambda}.$

Proof. It is easy to check the following facts:

(i) $\mathcal{L}_\rho\{1\}(s) = \int_0^\infty e^{-s\frac{t^\rho}{\rho}} \frac{dt}{t^{1-\rho}} = -\frac{1}{s}\int_0^\infty e^{-s\frac{t^\rho}{\rho}} d(-s\frac{t^\rho}{\rho}) = \frac{1}{s}.$

(ii) $\mathcal{L}_\rho\{e^{\lambda\frac{t^\rho}{\rho}}\}(s) = \int_0^\infty e^{\lambda\frac{t^\rho}{\rho}} e^{-s\frac{t^\rho}{\rho}} \frac{dt}{t^{1-\rho}} = \frac{1}{s-\lambda}.$

The proof is complete. □

Lemma 8. *(see [36], Lemma 3.4) Let $Re(\alpha) > 0, \rho > 0$, and $|\frac{\lambda}{s^\alpha}| < 1$.*

(iii) $\mathcal{L}_\rho\{\mathbb{E}_\alpha(\lambda(\frac{t^\rho}{\rho})^\alpha)\}(s) = \frac{s^{\alpha-1}}{s^\alpha-\lambda}.$

(iv) $\mathcal{L}_\rho\{(\frac{t^\rho}{\rho})^{\alpha-1})\mathbb{E}_{\alpha,\alpha}(\lambda(\frac{t^\rho}{\rho})^\alpha)\}(s) = \frac{1}{s^\alpha-\lambda}.$

From Lemma 8, we derive the following result.

Lemma 9. *Let $Re(\alpha) > 0, \rho > 0$, and $|\frac{\lambda}{s^\alpha}| < 1$. Then,*

$$\mathcal{L}_\rho\{(\frac{t^\rho}{\rho})^{\beta-1})\mathbb{E}_{\alpha,\beta}(\lambda(\frac{t^\rho}{\rho})^\alpha)\}(s) = \frac{s^{\alpha-\beta}}{s^\alpha - \lambda}.$$

Proof. One can see

$$\begin{aligned}
\mathcal{L}_\rho\{(\frac{t^\rho}{\rho})^{\beta-1})\mathbb{E}_{\alpha,\beta}(\lambda(\frac{t^\rho}{\rho})^\alpha)\}(s) &= \mathcal{L}_\rho\{(\frac{t^\rho}{\rho})^{\beta-1}\sum_{k=0}^{\infty}\frac{(\lambda(\frac{t^\rho}{\rho})^\alpha)^k}{\Gamma(\alpha k+\beta)}\}(s) \\
&= \mathcal{L}_\rho\{\sum_{k=0}^{\infty}\frac{\lambda^k t^{\alpha\rho k+\rho\beta-\rho}}{\Gamma(\alpha k+\beta)\rho^{\alpha k+\beta-1}}\}(s) \\
&= \sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(\alpha k+\beta)\rho^{\alpha k+\beta-1}}\mathcal{L}_\rho\{t^{\alpha\rho k+\rho\beta-\rho}\}(s) \\
&= \sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(\alpha k+\beta)\rho^{\alpha k+\beta-1}}\rho^{\alpha k+\beta-1}\frac{\Gamma(1+k\alpha+\beta-1)}{s^{1+k\alpha+\beta-1}} \\
&= \sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(\alpha k+\beta)}\frac{\Gamma(k\alpha+\beta)}{s^{k\alpha+\beta}} \\
&= \sum_{k=0}^{\infty}\frac{\lambda^k}{s^{k\alpha+\beta}} \\
&= \frac{1}{s^\beta}\sum_{k=0}^{\infty}(\frac{\lambda}{s^\alpha})^k \\
&= \frac{s^{\alpha-\beta}}{s^\alpha-\lambda}.
\end{aligned}$$

The proof is finished. □

Motivated by [34,38], we introduce the following definitions.

Definition 10. *The fractional differential equation $\Psi(g, f, (\mathbb{D}_c^{\alpha_1,\rho} f), \ldots, (\mathbb{D}_c^{\alpha_n,\rho} f)) = 0$ has Hyers–Ulam stability if there exists a real number $K > 0$, such that for a given $\epsilon > 0$ and for each solution $f \in C([0, T], \mathbb{R})$ of the inequality*

$$\Psi(g, f, (\mathbb{D}_c^{\alpha_1,\rho} f), \ldots, (\mathbb{D}_c^{\alpha_n,\rho} f)) \le \epsilon$$

there exists a solution $f_a \in C([0, T], \mathbb{R})$ of the differential equation such that $|f(t) - f_a(t)| \le K\epsilon$.

Definition 11. *The fractional differential equation $\Psi(g, f, (\mathbb{D}_c^{\alpha_1,\rho} f), \ldots, (\mathbb{D}_c^{\alpha_n,\rho} f)) = 0$ has generalized Ulam–Hyers–Rassias stable with respect to $G \in C([0, T], \mathbb{R})$ if there exists a real numbr $c_G > 0$, such that for each solution $f \in C([0, T], \mathbb{R})$ of the inequality*

$$\Psi(g, f, (\mathbb{D}_c^{\alpha_1,\rho} f), \ldots, (\mathbb{D}_c^{\alpha_n,\rho} f)) \le G(t), \;\; t \in [0, T]$$

there exists a solution $f_a \in C([0, T], \mathbb{R})$ of the differential equation such that $|f(t) - f_a(t)| \le c_G G(t)$.

Lemma 12. *([39], Corollary 2.4) Let $\alpha, \rho > 0$, $x(t)$, $a(t)$ be nonnegative functions and $b(t)$ be nonnegative and nondecreasing function for $t \in [t_0, T)$, $T > 0$, $b(t) \le M$, where M is a constant. If*

$$x(t) \le a(t) + b(t)\int_{t_0}^{t}(\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1}x(\tau)\frac{d\tau}{\tau^{1-\rho}},$$

then

$$x(t) \leq a(t)\mathbb{E}_a\left(b(t)\Gamma(\alpha)(\frac{t^\rho - t_0^\rho}{\rho})^\alpha\right),\ t \in [t_0, T).$$

3. Hyers–Ulam Stability for Linear Problems

Theorem 13. *Let $0 < \alpha < 1, \rho > 0$ and $g(t)$ be a given real continuous function on $[0, \infty)$. If a function $f : [0, \infty) \to \mathbb{R}$ satisfies the following inequality*

$$|(\mathbb{D}_c^{\alpha,\rho} f)(t) - g(t)| \leq \varepsilon,\ \ f(0) = f_0 \in \mathbb{R},$$

for each $t \geq 0$ and $\varepsilon > 0$, then there exists a solution $f_a : [0, \infty) \to \mathbb{R}$ of (1) such that

$$|f(t) - f_a(t)| \leq \frac{t^{\rho\alpha}}{\Gamma(\alpha+1)\rho^\alpha}\varepsilon. \tag{8}$$

Proof. Let

$$F(t) = (\mathbb{D}_c^{\alpha,\rho} f)(t) - g(t),\ t \geq 0. \tag{9}$$

Taking the ρ-Laplace transform of (9) via Theorem 4, we have

$$\begin{aligned} \mathcal{L}_\rho\{F(t)\}(s) &= \mathcal{L}_\rho\{(\mathbb{D}_c^{\alpha,\rho} f)(t) - g(t)\}(s) \\ &= \mathcal{L}_\rho\{(\mathbb{D}_c^{\alpha,\rho} f)(t)\}(s) - \mathcal{L}_\rho\{g(t)\}(s) \\ &= s^\alpha \mathcal{L}_\rho\{f(t)\}(s) - s^{\alpha-1} f_0 - \mathcal{L}_\rho\{g(t)\}(s), \end{aligned} \tag{10}$$

where $\mathcal{L}_\rho\{F(\cdot)\}$ denotes the ρ-Laplace transform of the function F. From (10), one has

$$\begin{aligned} &\mathcal{L}_\rho\{f(t)\}(s) \\ =\ & \frac{1}{s} f_0 + \frac{1}{s^\alpha}\mathcal{L}_\rho\{g(t)\}(s) + \frac{1}{s^\alpha}\mathcal{L}_\rho\{F(t)\}(s) \\ =\ & f_0\mathcal{L}_\rho\{1\}(s) + \frac{1}{\Gamma(\alpha)}\mathcal{L}_\rho\{(\frac{t^\rho}{\rho})^{\alpha-1} *_\rho g(t)\}(s) + \frac{1}{\Gamma(\alpha)}\mathcal{L}_\rho\{(\frac{t^\rho}{\rho})^{\alpha-1} *_\rho F(t)\}(s). \end{aligned}$$

Set

$$\begin{aligned} f_a(t) &= f_0 + \frac{1}{\Gamma(\alpha)}(\frac{t^\rho}{\rho})^{\alpha-1} *_\rho g(t) \\ &= f_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} g(s)\frac{ds}{s^{1-\rho}}. \end{aligned} \tag{11}$$

Taking the ρ–Laplace transform of (11), one has

$$\begin{aligned} &\mathcal{L}_\rho\{f_a(t)\}(s) \\ =\ & \frac{1}{s} f_0 + \frac{1}{\Gamma(\alpha)}\mathcal{L}_\rho\left\{\int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} g(s)\frac{ds}{s^{1-\rho}}\right\}(s) \\ =\ & \frac{1}{s} f_0 + \frac{1}{\Gamma(\alpha)}\mathcal{L}_\rho\left\{(\frac{t^\rho}{\rho})^{\alpha-1} *_\rho g(t)\right\}(s) \\ =\ & \frac{1}{s} f_0 + \mathcal{L}_\rho\left\{\frac{1}{\Gamma(\alpha)}(\frac{t^\rho}{\rho})^{\alpha-1}\right\}(s) \cdot \mathcal{L}_\rho\{g(t)\}(s) \\ =\ & \frac{1}{s} f_0 + \frac{1}{s^\alpha} \cdot \mathcal{L}_\rho\{g(t)\}(s). \end{aligned} \tag{12}$$

Note that

$$\begin{aligned}\mathcal{L}_\rho\{(\mathbb{D}_c^{\alpha,\rho} f_a)(t)\}(s) &= s^\alpha \mathcal{L}_\rho\{f_a(t)\}(s) - s^{\alpha-1} f_0 \\ &= s^{\alpha-1} f_0 + \mathcal{L}_\rho\{g(t)\}(s) - s^{\alpha-1} f_0 \\ &= \mathcal{L}_\rho\{g(t)\}(s),\end{aligned}$$

which yields that $f_a(\cdot)$ is a solution of Equation (1), since, according to the one-to-one transformation of $\mathcal{L}$ in (6), we can get that $\mathcal{L}_\rho$ is the one-to-one transformation.

From (11) and (12), we have

$$\begin{aligned}\mathcal{L}_\rho\{f(t) - f_a(t)\}(s) &= \frac{1}{s} f_0 + \frac{1}{s^\alpha}\mathcal{L}_\rho\{g(t)\}(s) + \frac{1}{s^\alpha}\mathcal{L}_\rho\{F(t)\}(s) - \frac{1}{s} f_0 - \frac{1}{s^\alpha}\cdot\mathcal{L}_\rho\{g(t)\}(s) \\ &= \frac{1}{s^\alpha}\mathcal{L}_\rho\{F(t)\}(s) \\ &= \mathcal{L}_\rho\left\{\frac{1}{\Gamma(\alpha)}(\frac{t^\rho}{\rho})^{\alpha-1}\right\}(s)\cdot\mathcal{L}_\rho\{F(t)\}(s) \\ &= \mathcal{L}_\rho\left\{\frac{1}{\Gamma(\alpha)}(\frac{t^\rho}{\rho})^{\alpha-1} *_\rho F(t)\right\}(s) \\ &= \mathcal{L}_\rho\left\{\int_0^t \frac{1}{\Gamma(\alpha)}(\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} F(s)\frac{ds}{s^{1-\rho}}\right\}(s).\end{aligned}$$

This implies that

$$f(t) - f_a(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} F(s)\frac{ds}{s^{1-\rho}}.$$

Thus,

$$\begin{aligned}&|f(t) - f_a(t)| \\ =\ & \frac{1}{\Gamma(\alpha)}\left|\int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} F(s)\frac{ds}{s^{1-\rho}}\right| \\ \leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t \left|(\frac{t^\rho - s^\rho}{\rho})^{\alpha-1}\frac{1}{s^{1-\rho}}\right| |F(s)| ds \\ \leq\ & \frac{\varepsilon}{\Gamma(\alpha)}\int_0^t \left|(\frac{t^\rho - s^\rho}{\rho})^{\alpha-1}\frac{1}{s^{1-\rho}}\right| ds \\ =\ & \frac{\varepsilon}{\Gamma(\alpha)}\int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1}\frac{1}{s^{1-\rho}} ds \\ \leq\ & -\frac{\varepsilon}{\Gamma(\alpha)}\int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} d\frac{t^\rho - s^\rho}{\rho} \\ =\ & \frac{t^{\rho\alpha}}{\Gamma(\alpha+1)\rho^\alpha}\varepsilon.\end{aligned}$$

The proof is complete. □

Remark 14. *From Definition 10, (8) shows that (1) is Hyers–Ulam stable with the constant* $K = \frac{T^{\rho\alpha}}{\Gamma(\alpha+1)\rho^\alpha}$ *provided that* $0 \leq t \leq T$. *(1) is not Hyers–Ulam stable if* $t = \infty$.

Theorem 15. *Let* $0 < \alpha < 1, \rho > 0$, $\lambda \in \mathbb{R}$, *and* $g(t)$ *be a given real continuous function on* $[0, \infty)$. *If a function* $f : [0, \infty) \to \mathbb{R}$ *satisfies the following inequality:*

$$|(\mathbb{D}_c^{\alpha,\rho} f)(t) - \lambda f(t) - g(t)| \leq \varepsilon, \ \ f(0) = f_0 \in \mathbb{R},$$

for each $t \geq 0$ and $\varepsilon > 0$, then there exists a solution $f_a : [0, \infty) \to \mathbb{R}$ of (2) such that

$$|f(t) - f_a(t)| \leq (\frac{t^\rho}{\rho})^\alpha \mathbb{E}_{\alpha,\alpha+1}(|\lambda|(\frac{t^\alpha}{\rho})^\alpha)\varepsilon. \tag{13}$$

Proof. Let

$$F_1(t) = (\mathbb{D}_c^{\alpha,\rho} f)(t) - \lambda f(t) - g(t),\ t \geq 0. \tag{14}$$

Taking the ρ-Laplace transform of (14) via Theorem 4, we have

$$\begin{aligned} \mathcal{L}_\rho\{F_1(t)\}(s) &= \mathcal{L}_\rho\{(\mathbb{D}_c^{\alpha,\rho} f)(t) - \lambda f(t) - g(t)\}(s) \\ &= \mathcal{L}_\rho\{(\mathbb{D}_c^{\alpha,\rho} f)(t)\}(s) - \lambda \mathcal{L}_\rho\{f(t)\}(s) - \mathcal{L}_\rho\{g(t)\}(s) \\ &= s^\alpha \mathcal{L}_\rho\{f(t)\}(s) - s^{\alpha-1} f_0 - \lambda \mathcal{L}_\rho\{f(t)\}(s) - \mathcal{L}_\rho\{g(t)\}(s), \end{aligned} \tag{15}$$

where $\mathcal{L}_\rho\{F_1(\cdot)\}$ denotes the ρ-Laplace transform of the function F_1.

From (15), one has

$$\begin{aligned} &\mathcal{L}_\rho\{f(t)\}(s) \\ =\ &\frac{s^{\alpha-1}}{s^\alpha - \lambda} f_0 + \frac{1}{s^\alpha - \lambda} \mathcal{L}_\rho\{g(t)\}(s) + \frac{1}{s^\alpha - \lambda} \mathcal{L}_\rho\{F_1(t)\}(s) \\ =\ &f_0 \mathcal{L}_\rho\{\mathbb{E}_\alpha(\lambda(\frac{t^\rho}{\rho})^\alpha)\}(s) + \mathcal{L}_\rho\{(\frac{t^\rho}{\rho})^{\alpha-1} \mathbb{E}_{\alpha,\alpha}(\lambda(\frac{t^\rho}{\rho})^\alpha)\}(s) \cdot \mathcal{L}_\rho\{g(t)\}(s) \\ &+ \mathcal{L}_\rho\{(\frac{t^\rho}{\rho})^{\alpha-1} \mathbb{E}_{\alpha,\alpha}(\lambda(\frac{t^\rho}{\rho})^\alpha)\}(s) \cdot \mathcal{L}_\rho\{F_1(t)\}(s) \\ =\ &f_0 \mathcal{L}_\rho\{\mathbb{E}_\alpha(\lambda(\frac{t^\rho}{\rho})^\alpha)\}(s) + \mathcal{L}_\rho\{(\frac{t^\rho}{\rho})^{\alpha-1} \mathbb{E}_{\alpha,\alpha}(\lambda(\frac{t^\rho}{\rho})^\alpha) *_\rho g(t)\}(s) \\ &+ \mathcal{L}_\rho\{(\frac{t^\rho}{\rho})^{\alpha-1} \mathbb{E}_{\alpha,\alpha}(\lambda(\frac{t^\rho}{\rho})^\alpha) *_\rho F_1(t)\}(s) \\ =\ &f_0 \mathcal{L}_\rho\{\mathbb{E}_\alpha(\lambda(\frac{t^\rho}{\rho})^\alpha)\}(s) + \mathcal{L}_\rho\{\int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} \mathbb{E}_{\alpha,\alpha}(\lambda(\frac{t^\rho - t^\rho}{\rho})^\alpha) \cdot g(t) \frac{ds}{s^{1-\rho}}\} \\ &+ \mathcal{L}_\rho\{\int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} \mathbb{E}_{\alpha,\alpha}(\lambda(\frac{t^\rho - s^\rho}{\rho})^\alpha) \cdot F_1(t) \frac{ds}{s^{1-\rho}}\}. \end{aligned} \tag{16}$$

Set

$$f_a(t) = f_0 \mathbb{E}_\alpha(\lambda(\frac{t^\rho}{\rho})^\alpha) + \int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} \mathbb{E}_{\alpha,\alpha}(\lambda(\frac{t^\rho - t^\rho}{\rho})^\alpha) \cdot g(t) \frac{ds}{s^{1-\rho}}. \tag{17}$$

Taking the ρ-Laplace transform of (17), one has

$$\begin{aligned} &\mathcal{L}_\rho\{f_a(t)\}(s) \\ =\ &f_0 \mathcal{L}_\rho\{\mathbb{E}_\alpha(\lambda(\frac{t^\rho}{\rho})^\alpha)\}(s) + \mathcal{L}_\rho\{\int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} \mathbb{E}_{\alpha,\alpha}(\lambda(\frac{t^\rho - t^\rho}{\rho})^\alpha) \cdot g(t) \frac{ds}{s^{1-\rho}}\} \\ =\ &f_0 \mathcal{L}_\rho\{\mathbb{E}_\alpha(\lambda(\frac{t^\rho}{\rho})^\alpha)\}(s) + \mathcal{L}_\rho\{(\frac{t^\rho}{\rho})^{\alpha-1} \mathbb{E}_{\alpha,\alpha}(\lambda(\frac{t^\rho}{\rho})^\alpha) *_\rho g(t)\}(s) \\ =\ &f_0 \mathcal{L}_\rho\{\mathbb{E}_\alpha(\lambda(\frac{t^\rho}{\rho})^\alpha)\}(s) + \mathcal{L}_\rho\{(\frac{t^\rho}{\rho})^{\alpha-1} \mathbb{E}_{\alpha,\alpha}(\lambda(\frac{t^\rho}{\rho})^\alpha)\}(s) \cdot \mathcal{L}_\rho\{g(t)\}(s) \\ =\ &\frac{s^{\alpha-1}}{s^\alpha - \lambda} f_0 + \frac{1}{s^\alpha - \lambda} \mathcal{L}_\rho\{g(t)\}(s). \end{aligned} \tag{18}$$

By Definition 4 and (18), we obtain

$$\begin{aligned}
& \mathcal{L}_\rho\{(\mathbb{D}_c^{\alpha,\rho} f_a)(t) - \lambda f_a(t)\}(s) \\
= \; & s^\alpha \mathcal{L}_\rho\{f_a(t)\}(s) - s^{\alpha-1} f_0 - \lambda \mathcal{L}_\rho\{f_a(t)\}(s) \\
= \; & (s^\alpha - \lambda)\frac{s^{\alpha-1}}{s^\alpha - \lambda} f_0 + (s^\alpha - \lambda)\frac{1}{s^\alpha - \lambda}\mathcal{L}_\rho\{g(t)\}(s) - s^{\alpha-1} f_0 \\
= \; & \mathcal{L}_\rho\{g(t)\}(s),
\end{aligned}$$

which yields that f_a is a solution of Equation (2) , since $\mathcal{L}_\rho$ is one-to-one.

From (16) and (18), we have

$$\begin{aligned}
& \mathcal{L}_\rho\{f(t) - f_a(t)\}(s) \\
= \; & f_0 \mathcal{L}_\rho\{\mathbb{E}_\alpha(\lambda(\frac{t^\rho}{\rho})^\alpha)\}(s) + \mathcal{L}_\rho\{\int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\frac{t^\rho - s^\rho}{\rho})^\alpha) \cdot g(t)\frac{ds}{s^{1-\rho}}\} \\
& +\mathcal{L}_\rho\{\int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\frac{t^\rho - s^\rho}{\rho})^\alpha) \cdot F_1(s)\frac{ds}{s^{1-\rho}}\} \\
& -f_0 \mathcal{L}_\rho\{\mathbb{E}_\alpha(\lambda(\frac{t^\rho}{\rho})^\alpha)\}(s) - \mathcal{L}_\rho\{\int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\frac{t^\rho - s^\rho}{\rho})^\alpha) \cdot g(t)\frac{ds}{s^{1-\rho}}\} \\
= \; & \mathcal{L}_\rho\{\int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\frac{t^\rho - s^\rho}{\rho})^\alpha) \cdot F_1(s)\frac{ds}{s^{1-\rho}}\}.
\end{aligned}$$

This implies that

$$f(t) - f_a(t) = \int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\frac{t^\rho - s^\rho}{\rho})^\alpha) \cdot F_1(s)\frac{ds}{s^{1-\rho}}.$$

Thus,

$$\begin{aligned}
& |f(t) - f_a(t)| \\
= \; & \left|\int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\frac{t^\rho - s^\rho}{\rho})^\alpha) \cdot F_1(s)\frac{ds}{s^{1-\rho}}\right| \\
\leq \; & \int_0^t \left|(\frac{t^\rho - s^\rho}{\rho})^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\frac{t^\rho - s^\rho}{\rho})^\alpha)\right| |F_1(s)|\frac{ds}{s^{1-\rho}} \\
\leq \; & \varepsilon \int_0^t \left|(\frac{t^\rho - s^\rho}{\rho})^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\frac{t^\rho - s^\rho}{\rho})^\alpha)\right| \frac{ds}{s^{1-\rho}} \\
\leq \; & -\varepsilon \int_0^t \left|(\frac{t^\rho - s^\rho}{\rho})^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\frac{t^\rho - s^\rho}{\rho})^\alpha)\right| d(\frac{t^\rho - s^\rho}{\rho}) \\
= \; & -\varepsilon \int_0^t \left|(\frac{t^\rho - s^\rho}{\rho})^{\alpha-1}\sum_{k=0}^{\infty}\frac{(\lambda(\frac{t^\rho - s^\rho}{\rho})^\alpha)^k}{\Gamma(\alpha k + \alpha)}\right| d(\frac{t^\rho - s^\rho}{\rho}) \\
= \; & -\varepsilon \sum_{k=0}^{\infty}\frac{|\lambda|^k}{\Gamma(\alpha k + \alpha)}\int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha k + \alpha - 1} d(\frac{t^\rho - s^\rho}{\rho}) \\
= \; & \varepsilon \sum_{k=0}^{\infty}\frac{|\lambda|^k}{\Gamma(\alpha k + \alpha + 1)}(\frac{t^\rho}{\rho})^{\alpha k + \alpha} \\
= \; & (\frac{t^\rho}{\rho})^\alpha \mathbb{E}_{\alpha,\alpha+1}(|\lambda|(\frac{t^\alpha}{\rho})^\alpha)\varepsilon.
\end{aligned}$$

The proof is complete. □

Remark 16. *From Definition 10, (13) shows that (2) is Hyers–Ulam stable with the constant* $K = \frac{T^{\alpha\rho}}{\rho^{\alpha(k+1)}}\mathbb{E}_{\alpha,\alpha+1}(|\lambda|T^{\alpha\rho})$ *provided that* $0 \leq t \leq T$*. (2) is not Hyers–Ulam stable if* $t = \infty$*.*

Remark 17. *Let $0 < \alpha < 1, \rho > 0$, $\lambda \in \mathbb{R}$, and $g(t)$ be a given real continuous function on $[0, \infty)$. If a function $f : [0, \infty) \to \mathbb{R}$ satisfies the following inequality*

$$|(\mathbb{D}_c^{\alpha,\rho} f)(t) - \lambda f(t) - g(t)| \leq G(t), \;\; f(0) = f_0 \in \mathbb{R},$$

then

$$|F_1(t)| \leq G(t)$$

for each $t \geq 0$ and some function $G(t) > 0$, where F_1 is defined in (14).

From Theorem 15, there exists a solution $f_a : [0, \infty) \to \mathbb{R}$ of (2) such that

$$\begin{aligned} |f(t) - f_a(t)| &= |\int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} \mathbb{E}_{\alpha,\alpha}(\lambda(\frac{t^\rho - s^\rho}{\rho})^\alpha) \cdot F_1(s) \frac{ds}{s^{1-\rho}} \\ &\leq \int_0^t \left| (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} \mathbb{E}_{\alpha,\alpha}(\lambda(\frac{t^\rho - s^\rho}{\rho})^\alpha) \right| |F_1(s)| \frac{ds}{s^{1-\rho}} \\ &\leq G(t) \int_0^t \left| (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} \mathbb{E}_{\alpha,\alpha}(\lambda(\frac{t^\rho - s^\rho}{\rho})^\alpha) \right| \frac{ds}{s^{1-\rho}} \\ &\leq -G(t) \int_0^t \left| (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} \mathbb{E}_{\alpha,\alpha}(\lambda(\frac{t^\rho - s^\rho}{\rho})^\alpha) \right| d(\frac{t^\rho - s^\rho}{\rho}) \\ &\leq G(t) \int_0^{\frac{t^\rho}{\rho}} z^{\alpha-1} \mathbb{E}_{\alpha,\alpha}(\lambda z^\alpha) dz \\ &= (\frac{t^\rho}{\rho})^\alpha \mathbb{E}_{\alpha,\alpha+1}(|\lambda|(\frac{t^\rho}{\rho})^\alpha) G(t). \end{aligned}$$

By Definition 11, (13) shows that (2) is generalized Hyers–Ulam-Rassias stable with the constant $c_G = (\frac{T^\rho}{\rho})^\alpha \mathbb{E}_{\alpha,\alpha+1}(|\lambda|(\frac{T^\rho}{\rho})^\alpha)$ for all $t \in [0, T]$.

Theorem 18. *Let $0 < \beta < \alpha < 1, \rho > 0$, $\lambda \in \mathbb{R}$, and $g(t)$ be a given real continuous function on $[0, \infty)$. If a function $f : [0, \infty) \to \mathbb{R}$ satisfies the following inequality*

$$|(\mathbb{D}_c^{\alpha,\rho} f)(t) - \lambda(\mathbb{D}_c^{\beta,\rho} f)(t) - g(t)| \leq \varepsilon, \;\; f(0) = f_0 \in \mathbb{R},$$

for each $t \geq 0$ and $\varepsilon > 0$, then there exists a solution $f_a : [0, \infty) \to \mathbb{R}$ of (3) such that

$$|f(t) - f_a(t)| \leq (\frac{t^\rho}{\rho})^\alpha \mathbb{E}_{\alpha-\beta,\alpha+1}(|\lambda|(\frac{t^\rho}{\rho})^{\alpha-\beta})\varepsilon. \tag{19}$$

Proof. Let

$$F_2(t) = (\mathbb{D}_c^{\alpha,\rho} f)(t) - \lambda(\mathbb{D}_c^{\beta,\rho} f)(t) - g(t), t \geq 0. \tag{20}$$

Taking the ρ-Laplace transform of (20) via Theorem 4, we have

$$\begin{aligned} \mathcal{L}_\rho\{F_2(t)\}(s) &= \mathcal{L}_\rho\{(\mathbb{D}_c^{\alpha,\rho} f)(t) - \lambda(\mathbb{D}_c^{\beta,\rho} f)(t) - g(t)\}(s) \\ &= \mathcal{L}_\rho\{(\mathbb{D}_c^{\alpha,\rho} f)(t)\}(s) - \lambda\mathcal{L}_\rho\{(\mathbb{D}_c^{\beta,\rho} f)(t)\}(s) - \lambda\mathcal{L}_\rho\{f(t)\}(s) - \mathcal{L}_\rho\{g(t)\}(s) \\ &= s^\alpha \mathcal{L}_\rho\{f(t)\}(s) - s^{\alpha-1} f_0 - \lambda s^\beta \mathcal{L}_\rho\{f(t)\}(s) + \lambda s^{\beta-1} f_0 - \mathcal{L}_\rho\{g(t)\}(s) \\ &= (s^\alpha - \lambda s^\beta)\mathcal{L}_\rho\{f(t)\}(s) - (s^{\alpha-1} - \lambda s^{\beta-1}) f_0 - \mathcal{L}_\rho\{g(t)\}(s), \end{aligned} \tag{21}$$

where $\mathcal{L}_\rho\{F_2(\cdot)\}$ denotes the ρ-Laplace transform of the function F_2.

Note

$$\frac{s^{\alpha-1}}{s^\alpha - \lambda s^\beta} = \frac{s^{\alpha-\beta-1}}{s^{\alpha-\beta} - \lambda} = \mathcal{L}_\rho\{\mathbb{E}_{\alpha-\beta}(\lambda(\frac{t^\rho}{\rho})^{\alpha-\beta})\}(s),$$

$$\frac{s^{\beta-1}}{s^\alpha-\lambda s^\beta} = \frac{s^{-1}}{s^{\alpha-\beta}-\lambda} = \mathcal{L}_\rho\{(\frac{t^\rho}{\rho})^{\alpha-\beta}\mathbb{E}_{\alpha-\beta,\alpha-\beta+1}(\lambda(\frac{t^\rho}{\rho})^{\alpha-\beta})\}(s),$$

and

$$\frac{1}{s^\alpha-\lambda s^\beta} = \frac{s^{-\beta}}{s^{\alpha-\beta}-\lambda} = \mathcal{L}_\rho\{(\frac{t^\rho}{\rho})^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(\frac{t^\rho}{\rho})^{\alpha-\beta})(s).$$

From (21), one has

$$\begin{aligned}
&\mathcal{L}_\rho\{f(t)\}(s)\\
=\;& \frac{s^{\alpha-1}-\lambda s^{\beta-1}}{s^\alpha-\lambda s^\beta}f_0+\frac{1}{s^\alpha-\lambda s^\beta}L_\rho\{g(t)\}(s)+\frac{1}{s^\alpha-\lambda s^\beta}\mathcal{L}_\rho\{F_2(t)\}(s)\\
=\;& \frac{s^{\alpha-1}}{s^\alpha-\lambda s^\beta}f_0-\lambda\frac{s^{\beta-1}}{s^\alpha-\lambda s^\beta}f_0+\frac{1}{s^\alpha-\lambda s^\beta}\mathcal{L}_\rho\{g(t)\}(s)+\frac{1}{s^\alpha-\lambda s^\beta}\mathcal{L}_\rho\{F_2(t)\}(s)\\
=\;& f_0\mathcal{L}_\rho\{\mathbb{E}_{\alpha-\beta}(\lambda(\frac{t^\rho}{\rho})^{\alpha-\beta})\}(s)-\lambda f_0\mathcal{L}_\rho\{(\frac{t^\rho}{\rho})^{\alpha-\beta}\mathbb{E}_{\alpha-\beta,\alpha-\beta+1}(\lambda(\frac{t^\rho}{\rho})^{\alpha-\beta})\}(s)\\
&+\mathcal{L}_\rho\{(\frac{t^\rho}{\rho})^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(\frac{t^\rho}{\rho})^{\alpha-\beta}*_\rho g(t))(s)\\
&+\mathcal{L}_\rho\{(\frac{t^\rho}{\rho})^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(\frac{t^\rho}{\rho})^{\alpha-\beta}*_\rho F_2(t))(s)\\
=\;& f_0\mathcal{L}_\rho\{\mathbb{E}_{\alpha-\beta}(\lambda(\frac{t^\rho}{\rho})^{\alpha-\beta})\}(s)-\lambda f_0\mathcal{L}_\rho\{(\frac{t^\rho}{\rho})^{\alpha-\beta}\mathbb{E}_{\alpha-\beta,\alpha-\beta+1}(\lambda(\frac{t^\rho}{\rho})^{\alpha-\beta})\}(s)\\
&+\mathcal{L}_\rho\bigg\{\int_0^t(\frac{t^\rho-x^\rho}{\rho})^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(\frac{t^\rho-x^\rho}{\rho})^{\alpha-\beta}\cdot g(t))\frac{dx}{x^{1-\rho}}\bigg\}(s)\\
&+\mathcal{L}_\rho\bigg\{\int_0^t(\frac{t^\rho-x^\rho}{\rho})^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(\frac{t^\rho-x^\rho}{\rho})^{\alpha-\beta}\cdot F_2(t))\frac{dx}{x^{1-\rho}}\bigg\}(s).
\end{aligned}\tag{22}$$

Set

$$\begin{aligned}
f_a(t) =\;& f_0\mathbb{E}_{\alpha-\beta}(\lambda(\frac{t^\rho}{\rho})^{\alpha-\beta})-\lambda f_0(\frac{t^\rho}{\rho})^{\alpha-\beta}\mathbb{E}_{\alpha-\beta,\alpha-\beta+1}(\lambda(\frac{t^\rho}{\rho})^{\alpha-\beta})(s)\\
&+\int_0^t(\frac{t^\rho-x^\rho}{\rho})^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(\frac{t^\rho-x^\rho}{\rho})^{\alpha-\beta}\cdot g(t))\frac{dx}{x^{1-\rho}}.
\end{aligned}\tag{23}$$

Taking the ρ-Laplace transform of (23), one has

$$\begin{aligned}
&\mathcal{L}_\rho\{f_a(t)\}(s)\\
=\;& f_0L_\rho\{\mathbb{E}_{\alpha-\beta}(\lambda(\frac{t^\rho}{\rho})^{\alpha-\beta})\}(s)-\lambda f_0\mathcal{L}_\rho\bigg\{(\frac{t^\rho}{\rho})^{\alpha-\beta}\mathbb{E}_{\alpha-\beta,\alpha-\beta+1}(\lambda(\frac{t^\rho}{\rho})^{\alpha-\beta})(s)\bigg\}(s)\\
&+\mathcal{L}_\rho\bigg\{\int_0^t(\frac{t^\rho-x^\rho}{\rho})^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(\frac{t^\rho-x^\rho}{\rho})^{\alpha-\beta}\cdot g(t))\frac{dx}{x^{1-\rho}}\bigg\}(s)\\
=\;& f_0\mathcal{L}_\rho\{\mathbb{E}_{\alpha-\beta}(\lambda(\frac{t^\rho}{\rho})^{\alpha-\beta})\}(s)-\lambda f_0\mathcal{L}_\rho\{((\frac{t^\rho}{\rho})^{\alpha-\beta}\mathbb{E}_{\alpha-\beta,\alpha-\beta+1}(\lambda(\frac{t^\rho}{\rho})^{\alpha-\beta})(s)\\
&+\mathcal{L}_\rho\{(\frac{t^\rho}{\rho})^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(\frac{t^\rho}{\rho})^{\alpha-\beta}*_\rho g(t))(s)\\
=\;& \frac{s^{\alpha-1}}{s^\alpha-\lambda s^\beta}f_0-\lambda\frac{s^{\beta-1}}{s^\alpha-\lambda s^\beta}f_0+\frac{1}{s^\alpha-\lambda s^\beta}\mathcal{L}_\rho\{g(t)\}(s)\\
=\;& \frac{s^{\alpha-1}-\lambda s^{\beta-1}}{s^\alpha-\lambda s^\beta}f_0+\frac{1}{s^\alpha-\lambda s^\beta}\mathcal{L}_\rho\{g(t)\}(s).
\end{aligned}\tag{24}$$

By Definition 4 and (24), we obtain

$$\begin{aligned}
& \mathcal{L}_\rho\{(\mathbb{D}_c^{\alpha,\rho} f_a)(t) - \lambda(\mathbb{D}_c^{\beta,\rho} f_a)(t)\} \\
= \; & s^\alpha \mathcal{L}_\rho\{f_a(t)\}(s) - s^{\alpha-1} f_0 - \lambda s^\beta \mathcal{L}_\subset\{f_a(t)\}(s) + \lambda s^{\beta-1} f_0 \\
= \; & (s^\alpha - \lambda s^\beta)\frac{s^{\alpha-1} - \lambda s^{\beta-1}}{s^\alpha - \lambda s^\beta} f_0 + (s^\alpha - \lambda s^\beta)\frac{1}{s^\alpha - \lambda s^\beta}\mathcal{L}_\rho\{g(t)\}(s) - (s^{\alpha-1} - \lambda s^{\beta-1}) f_0 \\
= \; & \mathcal{L}_\rho\{g(t)\}(s),
\end{aligned}$$

which yields that f_a is a solution of Equation (3), since $\mathcal{L}_\rho$ is one-to-one.

From (22) and (24), we have

$$\begin{aligned}
\mathcal{L}_\rho\{f(t) - f_a(t)\}(s) \; = \; & \frac{s^{\alpha-1} - \lambda s^{\beta-1}}{s^\alpha - \lambda s^\beta} f_0 + \frac{1}{s^\alpha - \lambda s^\beta}\mathcal{L}_\rho\{g(t)\}(s) + \frac{1}{s^\alpha - \lambda s^\beta}\mathcal{L}_\rho\{F_2(t)\}(s) \\
& - \frac{s^{\alpha-1} - \lambda s^{\beta-1}}{s^\alpha - \lambda s^\beta} f_0 - \frac{1}{s^\alpha - \lambda s^\beta}\mathcal{L}_\subset\{g(t)\}(s). \\
= \; & \frac{1}{s^\alpha - \lambda s^\beta}\mathcal{L}_\subset\{F_2(t)\}(s) \\
= \; & \mathcal{L}_\rho\left\{ \int_0^t (\frac{t^\rho - x^\rho}{\rho})^{\alpha-1} \mathbb{E}_{\alpha-\beta,\alpha}(\lambda(\frac{t^\rho - x^\rho}{\rho})^{\alpha-\beta} \cdot F_2(t))\frac{dx}{x^{1-\rho}} \right\}(s).
\end{aligned}$$

This implies that

$$f(t) - f_a(t) = \int_0^t (\frac{t^\rho - x^\rho}{\rho})^{\alpha-1} \mathbb{E}_{\alpha-\beta,\alpha}(\lambda(\frac{t^\rho - x^\rho}{\rho})^{\alpha-\beta} \cdot F_2(t))\frac{dx}{x^{1-\rho}},$$

so

$$\begin{aligned}
& |f(t) - f_a(t)| \\
= \; & \left| \int_0^t (\frac{t^\rho - x^\rho}{\rho})^{\alpha-1} \mathbb{E}_{\alpha-\beta,\alpha}(\lambda(\frac{t^\rho - x^\rho}{\rho})^{\alpha-\beta}) \cdot F_2(t)\frac{dx}{x^{1-\rho}} \right| \\
\leq \; & \int_0^t \left| (\frac{t^\rho - x^\rho}{\rho})^{\alpha-1} \mathbb{E}_{\alpha-\beta,\alpha}(\lambda(\frac{t^\rho - x^\rho}{\rho})^{\alpha-\beta}) \right| |F_2(s)| \frac{ds}{s^{1-\rho}} \\
\leq \; & \varepsilon \int_0^t \left| (\frac{t^\rho - x^\rho}{\rho})^{\alpha-1} \mathbb{E}_{\alpha-\beta,\alpha}(\lambda(\frac{t^\rho - x^\rho}{\rho})^{\alpha-\beta}) \right| \frac{ds}{s^{1-\rho}} \\
\leq \; & -\varepsilon \int_0^t \left| (\frac{t^\rho - x^\rho}{\rho})^{\alpha-1} \mathbb{E}_{\alpha-\beta,\alpha}(\lambda(\frac{t^\rho - x^\rho}{\rho})^{\alpha-\beta}) \right| d(\frac{t^\rho - s^\rho}{\rho}) \\
= \; & -\varepsilon \int_0^t \left| (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} \sum_{k=0}^{\infty} \frac{(\lambda(\frac{t^\rho - s^\rho}{\rho})^{\alpha-\beta})^k}{\Gamma((\alpha-\beta)k+\alpha)} \right| d(\frac{t^\rho - s^\rho}{\rho}) \\
\leq \; & (\frac{t^\rho}{\rho})^\alpha \mathbb{E}_{\alpha-\beta,\alpha+1}(|\lambda|(\frac{t^\rho}{\rho})^{\alpha-\beta})\varepsilon.
\end{aligned}$$

The proof is complete. □

Remark 19. *From Definition 10, (19) shows that (3) is Hyers–Ulam stable with the constant* $K = (\frac{T^\rho}{\rho})^\alpha \mathbb{E}_{\alpha-\beta,\alpha+1}(|\lambda|(\frac{T^\rho}{\rho})^{\alpha-\beta})$ *provided that* $0 \leq t \leq T$.

Remark 20. *If* $\beta = 0$*, then* $(\mathbb{D}_c^{\alpha,\rho} f)(t) - \lambda(\mathbb{D}_c^{\beta,\rho} f)(t) = g(t)$ *coincides with* $(\mathbb{D}_c^{\alpha,\rho} f)(t) - \lambda f(t) = g(t)$*, and* $(\frac{t^\rho}{\rho})^\alpha \mathbb{E}_{\alpha-\beta,\alpha+1}(|\lambda|(\frac{t^\rho}{\rho})^{\alpha-\beta})$ *coincides with* $(\frac{t^\rho}{\rho})^\alpha \mathbb{E}_{\alpha,\alpha+1}(|\lambda|(\frac{t^\rho}{\rho})^{\alpha})\varepsilon$*, so Theorem 18 generalizes Theorem 15.*

Remark 21. *Let* $0 < \beta < \alpha < 1, \rho > 0$, $\lambda \in \mathbb{R}$, *and* $g(t)$ *be a given real continuous function on* $[0, \infty)$. *If a function* $f : [0, \infty) \to \mathbb{R}$ *satisfies the following inequality*

$$|(\mathbb{D}_c^{\alpha,\rho} f)(t) - \lambda(\mathbb{D}_c^{\beta,\rho} f)(t) - g(t)| \leq G(t), \;\; f(0) = f_0 \in \mathbb{R},$$

then

$$|F_2(t)| \leq G(t)$$

for each $t \geq 0$ *and some function* $G(t) > 0$, *where* F_2 *is defined in* (14) .

From Theorem 18, there exists a solution $f_a : [0, \infty) \to \mathbb{R}$ *of* (2) *such that*

$$|f(t) - f_a(t)| \;\; \leq \;\; \frac{t^{\alpha\rho}}{\rho^{(\alpha-\beta)k+\alpha}} \mathbb{E}_{\alpha-\beta,\alpha+1}(|\lambda| t^{\rho(\alpha-\beta)}) G(t) \tag{25}$$

By Definition 11, (25) *shows* (3) *is generalized Hyers–Ulam–Rassias stable with the constant* $c_G = \frac{T^{\alpha\rho}}{\rho^{(\alpha-\beta)k+\alpha}} \mathbb{E}_{\alpha-\beta,\alpha+1}(|\lambda| T^{\rho(\alpha-\beta)})$ *for all* $t \in [0, T]$.

4. Existence and Stability Results for the Nonlinear Equation

We introduce the following conditions:
$[A1]$: $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous.
$[A2]$: There exists a $L > 0$ such that

$$|g(t, y) - g(t, f)| \leq L|y - f|, \;\; \forall\, y, f \in \mathbb{R}, t \in [0, T].$$

$[A3]$: There exists a constant $L_g > 0$ such that

$$|g(t, f)| \leq L_g(1 + |f|)$$

for each $t \in [0, T]$ and all $t \in \mathbb{R}$.

Theorem 22. *Let* $0 < \alpha < 1$, $\rho > 0$. *Assume that* $[A1]$ *and* $[A2]$ *hold. If* $L(\frac{T^\rho}{\rho})^\alpha < 1$, *then* (4) *has a unique solution on* $[0, T]$.

Proof. Consider $\Lambda : C([0, T], \mathbb{R}) \to C([0, T], \mathbb{R})$ as follows:

$$(\Lambda f)(t) = f_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} g(s, f(s)) \frac{ds}{s^{1-\rho}}, \; t \in [0, T]. \tag{26}$$

Note that Λ is well defined because of $[A1]$.
For all $f_1, f_2 \in C([0, T], \mathbb{R})$ and all $t \in [0, T]$, using $[A2]$, we have

$$\begin{aligned} |(\Lambda f_1)(t) - (\Lambda f_2)(t)| &\leq \int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} |g(s, f_1(s)) - g(s, f_2(s))| \frac{ds}{s^{1-\rho}} \\ &\leq L \int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} |f_1(s) - f_2(s)| \frac{ds}{s^{1-\rho}} \\ &= L\|f_1 - f_2\|_C (\frac{t^\rho}{\rho})^\alpha \\ &\leq L(\frac{T^\rho}{\rho})^\alpha \|f_1 - f_2\|_C, \end{aligned}$$

which implies

$$\|\Lambda f_1 - \Lambda f_2\|_C \leq L(\frac{T^\rho}{\rho})^\alpha \|f_1 - f_2\|_C.$$

From the condition $L(\frac{T^\rho}{\rho})^\alpha < 1$, Λ is a contraction mapping, and, by applying the Banach contraction mapping principle, we know that the operator Λ has a unique fixed point on $[0, T]$. □

Next, we show that the existence of solutions for (4) via Schaefer's fixed point theorem.

Theorem 23. *Assume that* $[A1]$ *and* $[A3]$ *hold. Then,* (4) *has at least one solution.*

Proof. Consider Λ as in (26). We divide our proof into several steps.

Step 1. Λ is continuous.

Let $\{f_n\}$ be a sequence such that $f_n \to f$ in $C([0, T], \mathbb{R})$. For all $t \in [0, T]$, we get

$$\begin{aligned}
|\Lambda f_n(t) - \Lambda f(t)| &= \frac{1}{\Gamma(\alpha)} \left| \int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} g(s, f_n(s)) \frac{ds}{s^{1-\rho}} - \int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} g(s, f(s)) \frac{ds}{s^{1-\rho}} \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} |g(s, f_n(s)) - g(s, f(s))| \frac{ds}{s^{1-\rho}} \\
&\leq \|g(\cdot, f_n) - g(\cdot, f)\|_C \frac{1}{\Gamma(\alpha)} \int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} \frac{ds}{s^{1-\rho}} \\
&\leq \frac{1}{\Gamma(\alpha)} (\frac{T^\rho}{\rho})^\alpha \|g(\cdot, f_n) - g(\cdot, f)\|_C.
\end{aligned}$$

This shows that Λ is continuous since $\|g(\cdot, f_n) - g(\cdot, f)\|_C \to 0$ when $n \to \infty$.

Step 2. Λ maps bounded sets into bounded sets of $C([0, T], \mathbb{R})$.

Indeed, we prove that for all $r > 0$, there exists a $k > 0$ such that for every $f \in B_r = \{f \in C([0, T], \mathbb{R}) : \|f\|_C \leq r\}$, we have $\|\Lambda f\|_C \leq k$. In fact, for any $t \in [0, T]$, from $[A3]$, we have

$$\begin{aligned}
|\Lambda f(t)| &\leq |f_0| + \frac{1}{\Gamma(\alpha)} \left| \int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} g(s, f(s)) \frac{ds}{s^{1-\rho}} \right| \\
&\leq |f_0| + \frac{1}{\Gamma(\alpha)} \int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} |g(s, f(s))| \frac{ds}{s^{1-\rho}} \\
&\leq |f_0| + \frac{L_g(1 + \|f\|_C)}{\Gamma(\alpha)} \int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} \frac{ds}{s^{1-\rho}} \\
&\leq |f_0| + \frac{L_g(1 + r)}{\Gamma(\alpha)} (\frac{T^\rho}{\rho})^\alpha,
\end{aligned}$$

which implies that

$$\|\Lambda f\|_C \leq |f_0| + + \frac{L_g(1 + r)}{\Gamma(\alpha)} (\frac{T^\rho}{\rho})^\alpha := k.$$

Step 3. P maps bounded sets into equicontinuous sets in $C([0, T], \mathbb{R})$.

Let $t_1, t_2 \in [0, T]$, with $0 \leq t_1 < t_2 \leq T$, $f \in B_r$. From $[A3]$, we have

$$\begin{aligned}
& |\Lambda f(t_1) - \Lambda f(t_2)| \\
= & \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_1} (\frac{t_1^\rho - s^\rho}{\rho})^{\alpha-1} g(s, f(s)) \frac{ds}{s^{1-\rho}} - \int_0^{t_2} (\frac{t_2^\rho - s^\rho}{\rho})^{\alpha-1} g(s, f(s)) \frac{ds}{s^{1-\rho}} \right| \\
\leq & \frac{L_g(1+r)}{\Gamma(\alpha)} \left(\int_0^{t_1} \left[(\frac{t_1^\rho - s^\rho}{\rho})^{\alpha-1} - (\frac{t_2^\rho - s^\rho}{\rho})^{\alpha-1} \right] \frac{ds}{s^{1-\rho}} + \int_{t_1}^{t_2} (\frac{t_2^\rho - s^\rho}{\rho})^{\alpha-1} \frac{ds}{s^{1-\rho}} \right) \\
\leq & \frac{L_g(1+r)}{\Gamma(\alpha)\alpha\rho^\alpha} \left(t_1^{\alpha\rho} - t_2^{\alpha\rho} - 2(t_2^\rho - t_1^\rho)^\alpha \right).
\end{aligned}$$

Then, the right-hand side of the above inequality tends to zero as $t_1 \to t_2$. Thus, Λ is equicontinuous.

We can conclude that Λ is completely continuous from Steps 1–3 with the Arzela–Ascoli theorem.

Step 4. A priori bounds.

Now, we show that the set $E(\Lambda) = \{f \in C([0, T], \mathbb{R}) : f = \iota f \text{ for some } \iota \in (0, 1)\}$ is bounded. Let $f \in E(\Lambda)$. Then, $f = \iota \Lambda f$ for some $\iota \in (0, 1)\}$. For each $t \in [0, T]$, we have

$$\begin{aligned}
|f(t)| & \leq f_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} |g(s, f(s))| \frac{ds}{s^{1-\rho}} \\
& \leq f_0 + \frac{L_g}{\Gamma(\alpha)} \int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} |1 + f(s)| \frac{ds}{s^{1-\rho}} \\
& \leq f_0 + \frac{L_g}{\Gamma(\alpha+1)\rho^\alpha} t^{\alpha\rho} + \frac{L_g}{\Gamma(\alpha)} \int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} |f(s)| \frac{ds}{s^{1-\rho}}.
\end{aligned}$$

By Lemma 12, we obtain

$$\begin{aligned}
|f(t)| & \leq (f_0 + \frac{L_g}{\Gamma(\alpha+1)\rho^\alpha} t^{\alpha\rho}) \mathbb{E}_\alpha (L_g (\frac{t^\rho}{\rho})^\alpha) \\
& \leq (f_0 + \frac{L_g}{\Gamma(\alpha+1)\rho^\alpha} T^{\alpha\rho}) \mathbb{E}_\alpha (L_g (\frac{T^\rho}{\rho})^\alpha) < \infty.
\end{aligned}$$

Then, the set $E(\Lambda)$ is bounded.

Schaefer's fixed point theorem guarantees that Λ has a fixed point, which is a solution of (4). The proof is finished. □

For the sake of discussion, the following inequality is given

$$|(\mathbb{D}_c^{\alpha,\rho} f)(t) - g(t, f(t))| \leq G(t), \ \forall\, t \in [0, T]. \tag{27}$$

In the following, we consider (4) and (27) to discuss the generalized Ulam–Hyers–Rassias stability. We need the following condition.

$[A4]$: Let $G \in C([0, T], \mathbb{R}_+)$ be an increasing function and there exists $\lambda_G > 0$ such that

$$\int_0^t (s^\rho)'(t^\rho - s^\rho)^{\alpha-1} G(s) ds \leq \lambda_G G(t), \ \ \forall\, t \in [0, T].$$

Theorem 24. *Assumptions $[A1]$, $[A2]$, and $[A4]$ hold. If $L(\frac{T^\rho}{\rho})^\alpha < 1$, then (4) is generalized Ulam–Hyers–Rassias stable with respect to G on $[0, T]$.*

Proof. Let $f \in C([0,T],\mathbb{R})$ be a solution of (4). From Theorem 22,

$$\begin{cases} (\mathbb{D}_c^{\alpha,\rho} f)(t) = g(t, f(t)),\ 0<\alpha<1,\ t\in[0,T], \\ f(0) = f_0 \end{cases} \tag{28}$$

has the unique solution

$$h(t) = f_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} g(s,h(s))\frac{ds}{s^{1-\rho}}.$$

Integrating the inequality (27) from 0 to t and using the condition $[A4]$, we have

$$\begin{aligned} & |f(t) - f_0 - \frac{1}{\Gamma(\alpha)}\int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} g(s,f(s))\frac{ds}{s^{1-\rho}}| \\ \leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} G(s)\frac{ds}{s^{1-\rho}} \\ =\ & \frac{1}{\Gamma(\alpha)\rho^\alpha}\int_0^t (s^\rho)'(t^\rho - s^\rho)^{\alpha-1} G(s)ds \\ \leq\ & \frac{1}{\Gamma(\alpha)\rho^\alpha}\lambda_G G(t). \end{aligned}$$

Thus,

$$\begin{aligned} & |f(t) - h(t)| \\ \leq\ & \left| f(t) - f_0 - \frac{1}{\Gamma(\alpha)}\int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} g(s,h(s))\frac{ds}{s^{1-\rho}} \right| \\ \leq\ & \left| f(t) - f_0 - \frac{1}{\Gamma(\alpha)}\int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} g(s,f(s))\frac{ds}{s^{1-\rho}} + \frac{1}{\Gamma(\alpha)}\int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} g(s,f(s))\frac{ds}{s^{1-\rho}} \right. \\ & \left. -\frac{1}{\Gamma(\alpha)}\int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} g(s,h(s))\frac{ds}{s^{1-\rho}} \right| \\ \leq\ & \left| f(t) - f_0 - \frac{1}{\Gamma(\alpha)}\int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} g(s,f(s))\frac{ds}{s^{1-\rho}} \right| \\ & +\frac{1}{\Gamma(\alpha)}\int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} |g(s,f(s)) - g(s,h(s))|\frac{ds}{s^{1-\rho}} \\ \leq\ & \frac{1}{\Gamma(\alpha)\rho^\alpha}\lambda_G G(t) + \frac{L}{\Gamma(\alpha)}\int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} |f(s) - h(s)|\frac{ds}{s^{1-\rho}}. \end{aligned}$$

From Lemma 12, we obtain

$$\begin{aligned} |f(t) - h(t)| &\leq (\frac{\lambda_G}{\Gamma(\alpha)\rho^\alpha})\mathbb{E}_\alpha(\frac{L}{\Gamma(\alpha)\rho^\alpha}\Gamma(\alpha)t^{\rho\alpha})G(t) \\ &\leq (\frac{\lambda_G}{\Gamma(\alpha)\rho^\alpha})\mathbb{E}_\alpha(L(\frac{T^\rho}{\rho})^\alpha)G(t),\ t\in[0,T]. \end{aligned}$$

Set $K^* = (\frac{\lambda_G}{\Gamma(\alpha)\rho^\alpha})\mathbb{E}_\alpha(L(\frac{T^\rho}{\rho})^\alpha)$. One has

$$|f(t) - h(t)| \leq K^* G(t),\ t\in[0,T].$$

From Definition 11, (4) is generalized Ulam–Hyers–Rassias stable with respect to G on $[0,T]$. The proof is complete. □

5. An Example

In this section, an example is given to illustrate our main results.

Example 25. *We consider the following fractional problem*

$$(\mathbb{D}^{\frac{1}{2},2}f)(t) = \frac{e^{-2t}}{5}\frac{|f(t)|}{1+|f(t)|},\ t \in [0,3], \tag{29}$$

and the inequality

$$\left|(\mathbb{D}^{\frac{1}{2},2}f)(t) - \frac{e^{-2t}}{5}\frac{|f(t)|}{1+|f(t)|}\right| \leq G(t),\ t \in [0,3].$$

Set $\alpha = \frac{1}{2}$, $\rho = 2$, $T = 3$ *and* $g(t,f) = \frac{e^{-2t}}{5}\frac{|f|}{1+|f|}$, $(t,f) \in [0,3] \times \mathbb{R}$. *For all* $t \in [0,3]$ *and* $f, q \in \mathbb{R}$,

$$|g(t,f(t)) - g(t,q(t))| \leq \frac{e^{-2t}}{5}||f(t)| - |q(t)|| \leq \frac{1}{5}|f(t) - q(t)|.$$

Set $L = \frac{1}{5}$. *Then,* $L(\frac{T^\rho}{\rho})^\alpha = \frac{1}{5}\sqrt{\frac{3^2}{2}} = \frac{3\sqrt{2}}{10} < 1$.
Let $G(t) = e^t, t \in [0,3]$ *and* $\lambda_G = \frac{3}{2} > 0$. *Note*

$$\int_0^t (s^\rho)'(t^\rho - s^\rho)^{\alpha-1}G(s)ds = \int_0^t 2s(t^2 - s^2)^{-\frac{1}{2}}e^s ds \leq \frac{1}{2}te^t \leq \lambda_G G(t),\ \ \forall t \in [0,3].$$

Thus, all the assumptions in Theorem 22 and Theorem 24 being satisfied, our results can be applied to the problem (29).

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