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Note on the Hurwitz–Lerch Zeta Function of Two Variables

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Abstract: A number of generalized Hurwitz–Lerch zeta functions have been presented and investigated. In this study, by choosing a known extended Hurwitz–Lerch zeta function of two variables, which has been very recently presented, in a systematic way, we propose to establish certain formulas and representations for this extended Hurwitz–Lerch zeta function such as integral representations, generating functions, derivative formulas and recurrence relations. We also point out that the results presented here can be reduced to yield corresponding results for several less generalized Hurwitz–Lerch zeta functions than the extended Hurwitz–Lerch zeta function considered here. For further investigation, among possibly various more generalized Hurwitz–Lerch zeta functions than the one considered here, two more generalized settings are provided.

Keywords: beta function; gamma function; Pochhammer symbol; Hurwitz–Lerch zeta function; Hurwitz–Lerch zeta function of two variables; hypergeometric functions; confluent hypergeometric functions; Appell hypergeometric functions; Humbert hypergeometric functions of two variables; integral representations; generating functions; derivative formulas; recurrence relation

1. Introduction and Preliminaries

The generalized (or Hurwitz) zeta function $\zeta(s,\nu)$ is defined by (see, e.g., [1], pp. 24–27); see also ([2] Chapter XIII)

$$\zeta(s,\nu)=\sum_{n=0}^{\infty}\frac{1}{(n+\nu)^s}\quad\left(\Re(s)>1,\ \nu\in\mathbb{C}\setminus\mathbb{Z}_0^-\right), \tag{1}$$

which is a generalization of the Riemann zeta function $\zeta(s):=\zeta(s,1)$ (see, e.g., [1] Section 1.12). Apostol [3] showed that the following analytic continuation formula (see, e.g., ([1] p. 26, Equation (6)), ([4] Equation (5.1))

$$\zeta(1-s,\nu)=\frac{2\,\Gamma(s)}{(2\pi)^s}\sum_{n=1}^{\infty}\frac{\cos(\pi s/2-2\pi\nu n)}{n^s}\quad(0<\nu\leq 1,\,\Re(s)>1), \tag{2}$$

where $\Gamma(s)$ is the familiar Gamma function whose Euler's integral (see, e.g., [1] pp. 1–24) is

$$\Gamma(s)=\int_0^{\infty}e^{-u}\,u^{s-1}\,\mathrm{d}u\quad(\Re(s)>0), \tag{3}$$

and it can be derived from a known transformation formula for the Lerch zeta function

$$\phi(x,\nu,s)=\sum_{n=0}^{\infty}\frac{e^{2\pi inx}}{(n+\nu)^{s}}\quad\left(\Re(s)>1,\ \nu\in\mathbb{C}\setminus\mathbb{Z}_0^-,\ x\in\mathbb{R}\right). \tag{4}$$

Note that It is easy to read that $\phi(x,\nu,s)=\zeta(s,\nu)$ when $x\in\mathbb{Z}$. The Hurwitz–Lerch zeta function $\Phi\,(z,s,\nu)$ is defined by (see, e.g., [1] p. 27)

$$\Phi\,(z,s,\nu)=\sum_{n=0}^{\infty}\frac{z^n}{(n+\nu)^s}, \tag{5}$$

where $\nu\in\mathbb{C}\setminus\mathbb{Z}_0^-$. The $\Phi\,(z,s,\nu)$ in (5) converges for all $s\in\mathbb{C}$ when $|z|<1$ and for $\Re(s)>1$ when $|z|=1$. Here and elsewhere, let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ be the sets of positive integers, integers, real numbers, and complex numbers, respectively. Furthermore, let us denote $\mathbb{Z}_0^-:=\mathbb{Z}\setminus\mathbb{N}$ and $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$.

The various special cases of the Hurwitz–Lerch zeta function (5) including the Riemann zeta function $\zeta(s)$, the Hurwitz zeta function (1), and the Lerch zeta function (4) have been intensively studied and applied. We choose to take some examples: Adamchik and Srivastava ([5] Propostion 5) evaluated a series involving polygamma functions in terms of the Hurwitz–Lerch zeta function (5). Rassias and Yang [6] studied certain equivalent conditions of a reverse Hilbert-type integral inequality, for which, in an example, the generalized zeta function $\zeta(s,a)$ is shown to be related to a best possible constant factor (see also [7**?** ,8]). Recently, a number of generalizations of the Hurwitz–Lerch zeta function (5) have been actively investigated (see, e.g., [9–18] and the references cited therein). Furthermore, very recently, Choi and Parmar [19] have introduced and investigated the following two-variable extension of the Hurwitz–Lerch zeta function (5)

$$\Phi_{a,b,b';c}\,(x,y,s,\alpha)=\sum_{k,\ell=0}^{\infty}\frac{(a)_{k+\ell}\,(b)_k\,(b')_\ell}{(c)_{k+\ell}\,k!\,\ell!}\,\frac{x^k\,y^\ell}{(k+\ell+\alpha)^s}, \tag{6}$$

where $a,\ b,\ b'\in\mathbb{C}$ and $c,\ \alpha\in\mathbb{C}\setminus\mathbb{Z}_0^-$. The function $\Phi_{a,b,b';c}\,(x,y,s,\alpha)$ in (6) converges for all $s\in\mathbb{C}$ when $|x|<1$ and $|y|<1$, and for $\Re(s+c-a-b-b')>1$ when $|x|=1$ and $|y|=1$. Here $(\eta)_\nu$ is the Pochhammer symbol given (for $\eta,\ \nu\in\mathbb{C}$) by

$$\begin{aligned}(\eta)_\nu:&=\frac{\Gamma(\eta+\nu)}{\Gamma(\eta)}\qquad(\eta+\nu\in\mathbb{C}\setminus\mathbb{Z}_0^-)\\ &=\begin{cases}1 & (\nu=0),\\ \eta(\eta+1)\cdots(\eta+n-1) & (\nu=n\in\mathbb{N}).\end{cases}\end{aligned} \tag{7}$$

From (3) and (7), the following integral formula for the Pochhammer symbol

$$(s)_\nu=\frac{1}{\Gamma(s)}\int_0^\infty e^{-u}\,u^{s+\nu-1}\,\mathrm{d}u\quad(\Re(s+\nu)>0) \tag{8}$$

can be easily obtained.

Here, in a systematic way, we aim to establish certain formulas and representations for the extended Hurwitz–Lerch zeta function of two variables (6) such as integral representations, generating functions, derivative formulas and recurrence relations. We also point out that the results presented here can be reduced to produce corresponding results for several less generalized Hurwitz–Lerch zeta functions than the extended Hurwitz–Lerch zeta function (6). Further, two more generalized settings than (6) are provided.

2. Integral Representations for the Extended Hurwitz–Lerch Zeta Function of Two Variables

We begin by recalling a known integral representation of the extended Hurwitz–Lerch zeta function (6) (see [19] Theorem 1)

$$\Phi_{a,b,b';c}(x,y,s,\alpha) = \frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} e^{-\alpha u} F_1\left[a,b,b';c;xe^{-u},ye^{-u}\right] \mathrm{d}u, \tag{9}$$

which converges for $\min\{\Re(s),\Re(\alpha)\} > 0$ when $|x| \leq 1$ $(x \neq 1)$ and $|y| \leq 1$ $(y \neq 1)$, and for $\Re(s) > 1$ when $x = 1$ and $y = 1$. Here F_1 is the Appell hypergeometric function of two variables defined by (see, e.g., ([1] p. 224, Equation (6)); see also ([20] p. 22))

$$\begin{aligned} F_1\left[a,b,b';c;x,y\right] &= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}\,(b)_m\,(b')_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \\ &= \sum_{m=0}^{\infty} \frac{(a)_m\,(b)_m}{(c)_m}\, {}_2F_1\left[\begin{matrix} a+m,\, b'\,; \\ c+m\,; \end{matrix}\; y\right] \frac{x^m}{m!}, \end{aligned} \tag{10}$$

where $a,\, b,\, b' \in \mathbb{C},\, c \in \mathbb{C} \setminus \mathbb{Z}_0^-$ and whose convergence region is $\max\{\Re(x),\, \Re(y)\} < 1$. Here ${}_pF_q$ $(p,\, q \in \mathbb{N}_0)$ are the generalized hypergeometric functions (see, e.g., ([1] Chapters II and V); see also ([21] Section 1.5), [20,22–25]).

The following confluent form of the Appell hypergeometric function F_1 is recalled (see, e.g., [1] p. 225, Equation (21)); see also ([20] p. 22 et seq.)

$$\Phi_2[b,b';c;x,y] = \sum_{m,n=0}^{\infty} \frac{(b)_m\,(b')_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \quad (|x| < \infty,\ |y| < \infty). \tag{11}$$

We provide further integral representations of the extended Hurwitz–Lerch zeta function (6), asserted in the following theorem.

Theorem 1. *Each of the following integral representations holds.*

$$\begin{aligned} \Phi_{a,b,b';c}(x,y,s,\alpha) =& \frac{1}{\Gamma(s)\,\Gamma(b)\Gamma(b')} \int_0^\infty\int_0^\infty\int_0^\infty \Big(u^{s-1}\, v_1^{b-1}\, v_2^{b'-1} \exp(-\alpha u - v_1 - v_2) \\ &\times {}_1F_1\left[a;c;\, e^{-u}\,(xv_1 + yv_2)\right] \Big)\, \mathrm{d}u\, \mathrm{d}v_1\, \mathrm{d}v_2, \end{aligned} \tag{12}$$

where $\min\{\Re(s),\, \Re(\alpha),\, \Re(b),\, \Re(b')\} > 0;$

$$\begin{aligned} \Phi_{a,b,b';c}(x,y,s,\alpha) =& \frac{1}{\Gamma(s)\,\Gamma(a)\Gamma(b)\Gamma(b')} \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \Big(u^{s-1}\, v_1^{b-1}\, v_2^{b'-1}\, v_3^{a-1} \\ &\times \exp(-\alpha u - v_1 - v_2 - v_3)\, {}_0F_1\left[-;c;\, e^{-u}\,(xv_1 + yv_2)\, v_3\right] \Big) \mathrm{d}u\, \mathrm{d}v_1\, \mathrm{d}v_2\, \mathrm{d}v_3, \end{aligned} \tag{13}$$

where $\min\{\Re(s),\, \Re(\alpha),\, \Re(a),\, \Re(b),\, \Re(b')\} > 0;$

$$\begin{aligned} \Phi_{a,b,b';c}(x,y,s,\alpha) =& \frac{1}{\Gamma(s)\,\Gamma(a)} \int_0^\infty\int_0^\infty \Big(u^{s-1}\, v^{a-1} \exp(-\alpha u - v) \\ &\times \Phi_2\left[b,b';c;xe^{-u}v,\, ye^{-u}v\right] \Big)\, \mathrm{d}u\, \mathrm{d}v, \end{aligned} \tag{14}$$

where $\min\{\Re(s), \Re(\alpha), \Re(a)\} > 0$ *and* $\max\{\Re(x), \Re(y)\} < 1$;

$$\Phi_{a,b,b';c}(x,y,s,\alpha) = \frac{\Gamma(c)}{\Gamma(s)\Gamma(a)\Gamma(b)\Gamma(b')\Gamma(c-b-b')} \times \int_0^\infty \int_0^\infty \int_0^1 \int_0^1 \Big(u^{s-1} v^{a-1} \xi^{b-1} \eta^{b'-1} (1-\xi)^{c-b-1} \times (1-\eta)^{c-b-b'-1} \exp\left[-\alpha u - v + xe^{-u} v\xi + ye^{-u} v(1-\xi)\eta\right] \Big) \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}u\, \mathrm{d}v, \tag{15}$$

where $\min\{\Re(s), \Re(\alpha), \Re(a), \Re(b)\} > 0$ *and* $\Re(c-b) > \Re(b') > 0$;

$$\Phi_{a,b,b';c}(x,y,s,\alpha) = \frac{\Gamma(\delta)}{\Gamma(s)\Gamma(a)\Gamma(b)\Gamma(b')\Gamma(\delta-b-b')} \int_0^\infty \int_0^\infty \int_0^1 \int_0^1 \Big(u^{s-1} v^{a-1} \xi^{b-1} \eta^{b'-1} \times (1-\xi)^{\delta-b-1} (1-\eta)^{\delta-b-b'-1} \exp\left[-\alpha u - v + xe^{-u} v\xi + ye^{-u} v(1-\xi)\eta\right] \times {}_1F_1\left[c-\delta; c; -ze^{-u} v\xi - ye^{-u} v(1-\xi)\eta\right] \Big) \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}u\, \mathrm{d}v, \tag{16}$$

where $\min\{\Re(s), \Re(\alpha), \Re(a), \Re(b)\} > 0$ *and* $\Re(\delta-b) > \Re(b') > 0$;

$$\Phi_{a,b,b';c}(x,y,s,\alpha) = \frac{\Gamma(c)}{\Gamma(s)\Gamma(a)\Gamma(\delta)\Gamma(b')\Gamma(c-\delta-b')} \times \int_0^\infty \int_0^\infty \int_0^1 \int_0^1 \Big(u^{s-1} v^{a-1} \xi^{\delta-1} \eta^{b'-1} (1-\xi)^{c-\delta-1} (1-\eta)^{c-\delta-b'-1} \times \exp\left[-\alpha u - v + xe^{-u} v\xi + ye^{-u} v(1-\xi)\eta\right] {}_1F_1\left[\delta-b; \delta; -xe^{-u} v\xi\right] \Big) \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}u\, \mathrm{d}v, \tag{17}$$

where $\min\{\Re(s), \Re(\alpha), \Re(a), \Re(\delta)\} > 0$ *and* $\Re(c-\delta) > \Re(b') > 0$;

$$\Phi_{a,b,b';c}(x,y,s,\alpha) = \frac{\Gamma(c)}{\Gamma(s)\Gamma(a)\Gamma(\delta)\Gamma(b')\Gamma(c-\delta-b')} \int_0^\infty \int_0^\infty \int_0^1 \int_0^1 \Big(u^{s-1} v^{a-1} \xi^{\delta-1} \eta^{b'-1} \times (1-\xi)^{c-\delta-1} (1-\eta)^{c-\delta-b'-1} \exp\left[-\alpha u - v + xe^{-u} v\xi + ye^{-u} v(1-\xi)\eta\right] \times {}_1F_1\left[b-\delta; c-\delta-b'; xe^{-u} v(1-\xi)(1-\eta)\right] \Big) \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}u\, \mathrm{d}v, \tag{18}$$

where $\min\{\Re(s), \Re(\alpha), \Re(a), \Re(\delta)\} > 0$ *and* $\Re(c-\delta) > \Re(b') > 0$;

$$\Phi_{a,b,b';c}(x,y,s,\alpha) = \frac{\Gamma(c)}{\Gamma(s)\Gamma(a)\Gamma(\delta_1)\Gamma(\delta_2)\Gamma(c-\delta_1-\delta_2)} \int_0^\infty \int_0^\infty \int_0^1 \int_0^1 \Big(u^{s-1} v^{a-1} \xi^{\delta_1-1} \eta^{\delta_2-1} \times (1-\xi)^{c-\delta_1-1} (1-\eta)^{c-\delta_1-\delta_2-1} \exp\left[-\alpha u - v + xe^{-u} v\xi + ye^{-u} v(1-\xi)\eta\right] \times {}_1F_1\left[\delta_1-b; \delta_1; -xe^{-u} v\xi\right] {}_1F_1\left[\delta_2-b'; \delta_2; -ye^{-u} v(1-\xi)\eta\right] \Big) \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}u\, \mathrm{d}v, \tag{19}$$

where $\min\{\Re(s), \Re(\alpha), \Re(a), \Re(\delta_1)\} > 0$ *and* $\Re(c-\delta_1) > \Re(\delta_2) > 0$;

$$\Phi_{a,b,b';c}(x,y,s,\alpha) = \frac{\Gamma(c)}{\Gamma(s)\Gamma(a)\Gamma(\delta_1)\Gamma(\delta_2)\Gamma(c-\delta_1-\delta_2)} \times \int_0^\infty \int_0^\infty \int_0^1 \int_0^1 \Big(u^{s-1} v^{a-1} \xi^{\delta_1-1} \eta^{\delta_2-1} (1-\xi)^{c-\delta_1-1} (1-\eta)^{c-\delta_1-\delta_2-1} \times \exp\left[-\alpha u - v + xe^{-u} v\xi + ye^{-u} v(1-\xi)\eta\right] \times \Phi_2\left[b-\delta_1; b'-\delta_2; c; xe^{-u} v(1-\xi)(1-\eta), ye^{-u} v(1-\xi)(1-\eta)\right] \Big) \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}u\, \mathrm{d}v, \tag{20}$$

where $\min\{\Re(s), \Re(\alpha), \Re(a), \Re(\delta_1)\} > 0$ *and* $\Re(c-\delta_1) > \Re(\delta_2) > 0$;

$$\begin{aligned}\Phi_{a,b,b';c}(x,y,s,\alpha) =& \frac{\Gamma(c)}{\Gamma(s)\,\Gamma(a)\,\Gamma(b)\,\Gamma(b')\,\Gamma(c-b-b')}\\ &\times \int_0^\infty\int_0^1\int_0^{1-\eta} \Big(e^{-\alpha u}\, u^{s-1}\xi^{b-1}\eta^{b'-1}\,(1-\xi-\eta)^{c-b-b'-1}\\ &\times \left(1-x\xi e^{-u}-y\eta e^{-u}\right)^{-a}\Big)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}u,\end{aligned} \tag{21}$$

where $\min\{\Re(s), \Re(\alpha), \Re(b), \Re(b')\} > 0$ *and* $\Re(c-b-b') > 0$;

$$\begin{aligned}\Phi_{a,b,b';c}(x,y,s,\alpha) =& \frac{\Gamma(c)}{\Gamma(s)\,\Gamma(a)\,\Gamma(c-a)}\\ &\times \int_0^\infty\int_0^1 e^{-\alpha u}\, u^{s-1} v^{a-1}\,(1-v)^{c-a-1}\left(1-vxe^{-u}\right)^{-b}\left(1-vye^{-u}\right)^{-b'}\,\mathrm{d}v\,\mathrm{d}u,\end{aligned} \tag{22}$$

where $\min\{\Re(s), \Re(\alpha), \Re(a)\} > 0$ *and* $\Re(c-a) > 0$;

$$\begin{aligned}\Phi_{a,b,b';c}(x,y,s,\alpha) =& \frac{\Gamma(c)}{\Gamma(s)\,\Gamma(b)\,\Gamma(c-b)}\int_0^\infty\int_0^1 \Big(e^{-\alpha u}\, u^{s-1}\, v^{c-b-1}\,(1-v)^{b-1}\\ &\times \left(1-xe^{-u}\right)^{c-a-b}\left(1-vxe^{-u}\right)^{a-c}\,{}_2F_1\left(a,b';c-b;ye^{-u}v\right)\Big)\,\mathrm{d}v\,\mathrm{d}u,\end{aligned} \tag{23}$$

where $\min\{\Re(s), \Re(\alpha), \Re(b)\} > 0$ *and* $\Re(c-b) > 0$.

Proof. Using (8) in the series definition of F_1 in (9), we get

$$\begin{aligned}F_1\left[a,b,b';c;xe^{-u},ye^{-u}\right] =& \frac{1}{\Gamma(b)\,\Gamma(b')}\int_0^\infty\int_0^\infty \Big(e^{-v_1-v_2}\, v_1^{b-1}\, v_2^{b'-1}\\ &\times \sum_{m,n=0}^\infty \frac{(a)_{m+n}}{(c)_{m+n}}\,\frac{(xv_1e^{-u})^m}{m!}\,\frac{(yv_2e^{-u})^n}{n!}\Big)\mathrm{d}v_1\,\mathrm{d}v_2.\end{aligned} \tag{24}$$

Applying the following identity (see, e.g., [25] p. 52)

$$\sum_{m,n=0}^\infty f(m+n)\,\frac{x^m}{m!}\,\frac{y^n}{n!} = \sum_{N=0}^\infty f(N)\,\frac{(x+y)^N}{N!} \tag{25}$$

to the double series in the right side of (24), we have an integral representation of F_1

$$\begin{aligned}F_1\left[a,b,b';c;xe^{-u},ye^{-u}\right] =& \frac{1}{\Gamma(b)\,\Gamma(b')}\\ &\times \int_0^\infty\int_0^\infty e^{-v_1-v_2}\, v_1^{b-1}\, v_2^{b'-1}{}_1F_1\left[a\,;\,c\,;\,xv_1e^{-u}+yv_2e^{-u}\right]\,\mathrm{d}v_1\,\mathrm{d}v_2.\end{aligned} \tag{26}$$

Finally, using (26) in (9), we get (12).

Similarly, using (8), we have an integral representation of ${}_1F_1$

$${}_1F_1[a\,;\,c\,;\,x] = \frac{1}{\Gamma(a)}\int_0^\infty e^{-u}\,u^{a-1}\,{}_0F_1[-\,;\,c\,;xu]\,\mathrm{d}u \quad (\Re(a) > 0). \tag{27}$$

Applying (27) in (12), we get (13).

Using the following known integral formula (see [20] p. 282, Equation (27))

$$F_1\left[a,b,b';c;x,y\right]=\frac{1}{\Gamma(a)}\int\limits_0^\infty u^{a-1}\,e^{-u}\,\Phi_2\left[b,b';c;xu,yu\right]\,\mathrm{d}u, \tag{28}$$

where $\max\{\Re(x),\,\Re(y)\}<1$ and $\Re(a)>0$, in the integrand of (9) yields (14).

From the known integral representation of Φ_2 (see [26] Equation (4.2)) in (14), we obtain (15).

Similarly, applying the known integral representations ([26] Equation (4.10)–Equation (4.14)) of Φ_2 to (14), respectively, yields (16)–(20).

Applying known integral representations for F_1 (see, e.g., ([22] p. 76, Equation (1)); see also ([27] Equation (3.2)) and (see, e.g., ([22] p. 77, Equation (4)); see also ([27] Equation (3.1)) to (9), respectively, we obtain (21) and (22).

Using a known integral representation for F_1 (see [27] Equation (3.3)) in (9), we get (23). □

3. Generating Functions for the Extended Hurwitz–Lerch Zeta Function of Two Variables

Certain generating functions for the extended Hurwitz–Lerch zeta function (6) are given in the following theorem.

Theorem 2. *The following two formulas hold true:*

$$\sum_{m=0}^{\infty}\binom{\lambda+m-1}{m}\Phi_{\lambda+m,b,b';c}\,(x,y,s,\alpha)\,u^m=(1-u)^{-\lambda}\Phi_{\lambda,b,b';c}\left(\frac{x}{1-u},\frac{y}{1-u},s,\alpha\right) \tag{29}$$

and

$$\sum_{m=0}^{\infty}\binom{\lambda+m-1}{m}\Phi_{a,b,\lambda+m;c}\,(x,y,s,\alpha)\;u^m=(1-u)^{-\lambda}\Phi_{a,b,\lambda;c}\left(x,\frac{y}{1-u},s,\alpha\right). \tag{30}$$

Proof. We begin by recalling the generalized binomial theorem

$$\begin{aligned}(1-y)^{-\lambda}&=\sum_{m=0}^{\infty}(-1)^m\binom{-\lambda}{m}y^m\\&=\sum_{m=0}^{\infty}\frac{(\lambda)_m}{m!}\,y^m=\sum_{m=0}^{\infty}\binom{\lambda+m-1}{m}y^m,\end{aligned} \tag{31}$$

where $\lambda\in\mathbb{C}$ and $|y|<1$.

For simplicity's sake, let us denote the left side of (29) by $\mathcal{L}$. Then, by using (6), we have

$$\mathcal{L}=\sum_{m=0}^{\infty}\frac{(\lambda)_m}{m!}\sum_{k,\ell=0}^{\infty}\frac{(\lambda+m)_{k+\ell}\,(b)_k\,(b')_\ell}{(c)_{k+\ell}\,k!\,\ell!}\,\frac{x^k\,y^\ell}{(k+\ell+\alpha)^s}\,u^m.$$

In view of $(\lambda)_m\,(\lambda+m)_{k+\ell}=(\lambda)_{m+k+\ell}=(\lambda)_{k+\ell}\,(\lambda+k+\ell)_m$, changing the order of summations and using (31), we obtain

$$\begin{aligned}\mathcal{L}&=\sum_{k,\ell=0}^{\infty}\frac{(\lambda)_{k+\ell}\,(b)_k\,(b')_\ell}{(c)_{k+\ell}\,k!\,\ell!}\,\frac{x^k\,y^\ell}{(k+\ell+\alpha)^s}\sum_{m=0}^{\infty}\frac{(\lambda+k+\ell)_m}{m!}\,u^m\\&=(1-u)^{-\lambda}\sum_{k,\ell=0}^{\infty}\frac{(\lambda)_{k+\ell}\,(b)_k\,(b')_\ell}{(c)_{k+\ell}\,k!\,\ell!}\,\frac{x^k\,y^\ell}{(k+\ell+\alpha)^s}\,(1-u)^{-k-\ell}\end{aligned}$$

which, using the definition (6), leads to the right side of (29). Similarly, we can obtain (30).

We can also prove the generating relations here by using some known generating relations for F_1 (see [28] Equation (2.1)) and ([21] Equations (1.2)–(1.3)) in (9). The details of the proof are omitted here. □

4. Derivative Formulas for the Extended Hurwitz–Lerch Zeta Function of Two Variables

Certain derivative formulas for the extended Hurwitz–Lerch zeta function (6) are established in the following theorem.

Theorem 3. *Each of the following derivative formulas holds true for $m, n \in \mathbb{N}_0$:*

$$\frac{\partial^m}{\partial x^m} \Phi_{a,b,b';c}\,(x,y,s,\alpha) = \frac{(a)_m (b)_m}{(c)_m} \Phi_{a+m,b+m,b';c+m}\,(x,y,s,\alpha+m)\,, \tag{32}$$

$$\frac{\partial^n}{\partial y^n} \Phi_{a,b,b';c}\,(x,y,s,\alpha) = \frac{(a)_n\,(b')_n}{(c)_n} \Phi_{a+n,b,b'+n;c+n}\,(x,y,s,\alpha+n)\,, \tag{33}$$

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n} \Phi_{a,b,b';c}\,(x,y,s,\alpha) = \frac{(a)_{m+n}\,(b)_m\,(b')_n}{(c)_{m+n}} \Phi_{a+m+n,b+m,b'+n;c+m+n}\,(x,y,s,\alpha+m+n)\,. \tag{34}$$

Proof. Differentiating the series definition (6) with respect to the variable x under the double summations, which is valid under the conditions in (6), we have

$$\frac{\partial}{\partial x} \Phi_{a,b,b';c}\,(x,y,s,\alpha) = \sum_{\ell=0}^{\infty} \sum_{k=1}^{\infty} \frac{(a)_{k+\ell}\,(b)_k\,(b')_\ell}{(c)_{k+\ell}\,(k-1)!\,\ell!} \frac{x^{k-1}\,y^\ell}{(k+\ell+\alpha)^s}. \tag{35}$$

Putting $k-1=k'$ in (35) and cancelling the prime on k, we obtain

$$\frac{\partial}{\partial x} \Phi_{a,b,b';c}\,(x,y,s,\alpha) = \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \frac{(a)_{k+1+\ell}\,(b)_{k+1}\,(b')_\ell}{(c)_{k+1+\ell}\,k!\,\ell!} \frac{x^k\,y^\ell}{(k+\ell+1+\alpha)^s}. \tag{36}$$

Using $(\lambda)_{k+1} = \lambda\,(\lambda+1)_k$ $(k \in \mathbb{N}_0)$ in (36), we get

$$\frac{\partial}{\partial x} \Phi_{a,b,b';c}\,(x,y,s,\alpha) = \frac{ab}{c}\,\Phi_{a+1,b+1,b';c+1}\,(x,y,s,\alpha+1)\,. \tag{37}$$

Differentiating the right side of (37), successively, $m-1$ times, with respect to the variable x, we have (32). Similarly, we can obtain (33) and (34). The details are omitted. □

5. Recurrence Relations for the Extended Hurwitz–Lerch Zeta Function of Two Variables

Wang [29] presented a number of recurrence relations for F_1, which are chosen to give some recurrence relations for the extended Hurwitz–Lerch zeta function (6), asserted in Theorem 4.

Theorem 4. *Let $n \in \mathbb{N}_0$. Then the following recurrence relations are satisfied:*

$$\begin{aligned} \Phi_{a+n,b,b';c}\,(x,y,s,\alpha) =& \Phi_{a,b,b';c}\,(x,y,s,\alpha) + \frac{bx}{c} \sum_{m=1}^{n} \Phi_{a+m,b+1,b'+c+1}\,(x,y,s,\alpha) \\ &+ \frac{b'y}{c} \sum_{m=1}^{n} \Phi_{a+m,b,b'+1;c+1}\,(x,y,s,\alpha)\,; \end{aligned} \tag{38}$$

$$\begin{aligned}\Phi_{a-n,b,b';c}\left(x,y,s,\alpha\right) =&\Phi_{a,b,b';c}\left(x,y,s,\alpha\right) - \frac{bx}{c}\sum_{m=0}^{n-1}\Phi_{a-m,b+1,b';c+1}\left(x,y,s,\alpha\right)\\&- \frac{b'y}{c}\sum_{m=0}^{n-1}\Phi_{a-m,b,b'+1;c+1}\left(x,y,s,\alpha\right);\end{aligned} \tag{39}$$

$$\Phi_{a,b-n,b';c}\left(x,y,s,\alpha\right) = \sum_{m=0}^{n}\binom{n}{m}\frac{(a)_m}{(c)_m}\left(-y\right)^m\Phi_{a+m,b,b';c+m}\left(x,y,s,\alpha+m\right). \tag{40}$$

Here the involved empty sum in each identity is assumed to be nil.

Proof. Using the following recursion relation for F_1 ([29] Theorem 1, Equation (1)):

$$\begin{aligned}F_1\left[a+n,b,b';c;x,y\right] =&F_1\left[a,b,b';c;x,y\right] + \frac{bx}{c}\sum_{m=1}^{n}F_1\left[a+m,b+1,b';c+1;x,y\right]\\&+ \frac{b'y}{c}\sum_{m=1}^{n}F_1\left[a+m,b,b'+1;c+1;x,y\right]\end{aligned} \tag{41}$$

and ([29] Theorem 1, Equation (2)) in (9), respectively, we obtain (38) and (39).
Using the recurrence relation for F_1 in ([29] Theorem 4, Equation (12)) in (9), we find (40). □

Using six known recurrence relations for ${}_1F_1$ in (12), we establish six recurrence relations for the extended Hurwitz–Lerch zeta function (6), which are asserted in Theorem 5.

Theorem 5. *The following recurrence relations hold true:*

$$\begin{aligned}a\Phi_{a+1,b,b';c}\left(x,y,s,\alpha\right) =&(c-a)\,\Phi_{a-1,b,b';c}\left(x,y,s,\alpha\right) + (2a-c)\Phi_{a,b,b';c}\left(x,y,s,\alpha\right)\\&+ bz\Phi_{a,b+1,b';c}\left(x,y,s,\alpha+1\right) + b'y\Phi_{a,b,b'+1;c}\left(x,y,s,\alpha+1\right);\end{aligned} \tag{42}$$

$$\begin{aligned}c(c-1)\Phi_{a,b,b';c-1}\left(x,y,s,\alpha\right) =& c(c-1)\Phi_{a,b,b';c}\left(x,y,s,\alpha\right) + cbz\,\Phi_{a,b+1,b';c}\left(x,y,s,\alpha+1\right)\\&+ cb'y\,\Phi_{a,b,b'+1;c}\left(x,y,s,\alpha+1\right) + (a-c)bz\,\Phi_{a,b+1,b';c+1}\left(x,y,s,\alpha+1\right)\\&+ (a-c)b'y\Phi_{a,b,b'+1;c}\left(x,y,s,\alpha+1\right);\end{aligned} \tag{43}$$

$$(1+a-c)\,\Phi_{a,b,b';c}\left(x,y,s,\alpha\right) =a\,\Phi_{a+1,b,b';c}\left(x,y,s,\alpha\right) + (1-c)\,\Phi_{a,b,b';c-1}\left(x,y,s,\alpha\right); \tag{44}$$

$$\begin{aligned}c\,\Phi_{a,b,b';c}\left(x,y,s,\alpha\right) =&c\,\Phi_{a-1,b,b';c}\left(x,y,s,\alpha\right) + bz\,\Phi_{a,b+1,b';c+1}\left(x,y,s,\alpha+1\right)\\&+ b'y\,\Phi_{a,b,b'+1;c+1}\left(x,y,s,\alpha+1\right);\end{aligned} \tag{45}$$

$$\begin{aligned}ac\,\Phi_{a+1,b,b';c}\left(x,y,s,\alpha\right) =&ac\,\Phi_{a,b,b';c}\left(x,y,s,\alpha\right) + cbz\,\Phi_{a,b+1,b';c}\left(x,y,s,\alpha+1\right)\\&+ cb'y\,\Phi_{a,b,b'+1;c}\left(x,y,s,\alpha+1\right) + (a-c)bz\,\Phi_{a,b+1,b';c+1}\left(x,y,s,\alpha+1\right)\\&+ (a-c)b'y\,\Phi_{a,b,b'+1;c+1}\left(x,y,s,\alpha+1\right);\end{aligned} \tag{46}$$

$$\begin{aligned}(c-1)\,\Phi_{a,b,b';c-1}\left(x,y,s,\alpha\right) =&(a-1)\,\Phi_{a,b,b';c}\left(x,y,s,\alpha\right) + bz\,\Phi_{a,b+1,b';c}\left(x,y,s,\alpha+1\right)\\&+ b'y\,\Phi_{a,b,b'+1;c}\left(x,y,s,\alpha+1\right) + (c-a)\,\Phi_{a-1,b,b';c}\left(x,y,s,\alpha+1\right).\end{aligned} \tag{47}$$

Proof. Using the following known recurrence relation for ${}_1F_1$ (see [24] p. 19, Equation (2.2.1))

$$(c-a)\,{}_1F_1[a-1;c;x] + (2a-c+x)\,{}_1F_1[a;c;x] - a\,{}_1F_1[a+1;c;x] = 0$$

in (12), we obtain (42). Similarly, using the recurrence relations ([24] p. 19, Equations (2.2.2)–(2.2.6)) for ${}_1F_1$, respectively, we get (43)–(47). □

6. Symmetries and Conclusions

We can find some interesting identities from *symmetries* involved in certain definitions and formulas. From (6) and (10), we demonstrate the follow symmetric relations:

$$\Phi_{a,b,b';c}(x,y,s,\alpha) = \Phi_{a,b',b;c}(y,x,s,\alpha) \tag{48}$$

and

$$F_1\left[a,b,b';c;x,y\right] = F_1\left[a,b',b;c;y,x\right]. \tag{49}$$

Further, in view of the symmetric relation (48), each integral representation in Theorem 1 may yield another integral representation. For example, from (14) and (48), we have

$$\begin{aligned}\Phi_{a,b,b';c}(x,y,s,\alpha) =& \frac{1}{\Gamma(s)\,\Gamma(a)} \int_0^\infty\int_0^\infty \Big(u^{s-1}\, v^{a-1} \exp(-\alpha u - v) \\ & \times \Phi_2\left[b',b;c;y\,e^{-u}v,\ x\,e^{-u}v\right]\Big)\, \mathrm{d}u\, \mathrm{d}v,\end{aligned} \tag{50}$$

where $\min\{\Re(s),\Re(\alpha),\Re(a)\} > 0$ and $\max\{\Re(x),\Re(y)\} < 1$.

The extended Hurwitz–Lerch zeta function of two variables in (6) may be further generalized in various ways. Here we introduce two extensions, one of which is due to parametric increase and the other of which is due to variable addition:

$$\Phi^{p:q;k}_{\ell:m;n}\left[\begin{matrix}(a_p): & (b_q); & (c_k); \\ (\alpha_\ell): & (\beta_m); & (\gamma_n);\end{matrix}\ x,\ y,\ s,\ \alpha\right] = \sum_{u,v=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{u+v}\prod_{j=1}^{q}(b_j)_u\prod_{j=1}^{k}(c_j)_v}{\prod_{j=1}^{\ell}(\alpha_j)_{u+v}\prod_{j=1}^{m}(\beta_j)_u\prod_{j=1}^{n}(\gamma_j)_v}\, \frac{x^u\, y^v}{u!\, v!\, (u+v+\alpha)^s} \tag{51}$$

and

$$\begin{aligned}&\Phi^{(n)}\left[a,b_1,\ldots,b_n;c;x_1,\ldots,x_n,s,\alpha\right] \\ &= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}\,(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c)_{m_1+\cdots+m_n}\, m_1!\cdots m_n!}\, \frac{x_1^{m_1}\cdots x_n^{m_n}}{(m_1+\cdots+m_n+\alpha)^s}.\end{aligned} \tag{52}$$

Here, for convergence, the parameters and variables in (51) and (52) would be suitably restricted. Obviously,

$$\Phi^{1:1;1}_{1:0;0}\left[\begin{matrix}a: & b; & b'; \\ c: & -; & -;\end{matrix}\ x,\ y,\ s,\ \alpha\right] = \Phi_{a,b,b';c}(x,y,s,\alpha)$$

and

$$\Phi^{(2)}\left[a,b,b';c;x,y,s,\alpha\right] = \Phi_{a,b,b';c}(x,y,s,\alpha).$$

The extended Hurwitz–Lerch zeta function (6) can be specialized to yield several known generalizations of the Hurwitz–Lerch zeta function (5) (see, e.g., [19]). Thus, the results presented here can yield corresponding identities regarding several reduced cases of the extended Hurwitz–Lerch zeta function (6), which are still generalizations of the Hurwitz–Lerch zeta function (5).

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References

1. Erdélyi, A.; Magnus, W.; Oberhettinger, F.; Tricomi, F.G. *Higher Transcendental Functions*; McGraw-Hill Book Company: New York, NY, USA; Toronto, ON , Canada; London, UK, 1953; Volume I.
2. Whittaker, E.T.; Watson, G.N. *A Course of Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions; With an Account of the Principal Transcendental Functions*, 4th ed.; Cambridge University Press: Cambridge, UK, 1963.
3. Apostol, T.M. Remark on the Hurwitz zeta function. *Proc. Amer. Math. Soc.* **1951**, *5*, 690–693. [CrossRef]
4. Berndt, B.C. On the Hurwitz zeta-function. *Rocky Mountain J. Math.* **1972**, *2*, 151–158. [CrossRef]
5. Adamchik, V.S.; Srivastava, H.M. Some series of the zeta and related functions. *Analysis* **1998**, *2*, 131–144. [CrossRef]
6. Rassias, M.T.; Yang, B. On an equivalent property of a reverse Hilbert-type integral inequality related to the extended Hurwitz-zeta function. *J. Math. Inequal.* **2019**, *13*, 315–334. [CrossRef]

   [CrossRef]
7. Rassias, M.T.; Yang, B.; Raigorodskii, A. On a half-discrete Hilbert-type inequality in the whole plane with the kernel of hyperbolic secant function related to the Hurwitz zeta function. *J. Inequal. Appl.* **2020**, *2020*, 94. [CrossRef]
8. Rassias, M.T.; Yang, B.; Raigorodskii, A. On Hardy-type integral inequalities in the whole plane related to the extended Hurwitz-zeta function, In *Trigonometric Sums and their Applications*; Springer: Berlin/Heidelberg, Germany, 2020; pp. 229–259.
9. Choi, J.; Jang, D.S.; Srivastava, H.M. A generalization of the Hurwitz–Lerch Zeta function. *Integral Transform. Spec. Funct.* **2008**, *19*, 65–79.
10. Daman, O.; Pathan, M.A. A further generalization of the Hurwitz Zeta function. *Math. Sci. Res. J.* **2012**, *16*, 251–259.
11. Garg, M.; Jain, K.; Kalla, S.L. A further study of general Hurwitz–Lerch zeta function. *Algebras Groups Geom.* **2008**, *25*, 311–319.
12. Goyal, S.P.; Laddha, R.K. On the generalized Zeta function and the generalized Lambert function. *Ganita Sandesh* **1997**, *11*, 99–108.
13. Pathan, M.A.; Daman, O. On generalization of Hurwitz zeta function. *Non-Linear World J.* **2018**, submitted.
14. Lin, S.D.; Srivastava, H.M. Some families of the Hurwitz–Lerch Zeta functions and associated fractional derivative and other integral representations. *Appl. Math. Comput.* **2004**, 154, 725–733. [CrossRef]
15. Srivastava, H.M. A new family of the generalized Hurwitz–Lerch Zeta functions with applications. *Appl. Math. Inf. Sci.* **2014**, *8*, 1485–1500. [CrossRef]
16. Srivastava, H.M.; Jankov, D.; Pogány, T.K.; Saxena, R.K. Two-sided inequalities for the extended Hurwitz–Lerch Zeta function. *Comput. Math. Appl.* **2011**, *62*, 516–522. [CrossRef]
17. Srivastava, H.M.; Luo, M.-J.; Raina, R.K. New results involving a class of generalized Hurwitz–Lerch Zeta functions and their applications. *Turkish J. Anal. Number Theory* **2013**, *1*, 26–35. [CrossRef]
18. Srivastava, H.M.; Saxena, R.K.; Pogány, T.K.; Saxena, R. Integral and computational representations of the extended Hurwitz–Lerch Zeta function. *Integral Transforms Spec. Funct.* **2011**, *22*, 487–506. [CrossRef]
19. Choi, J.; Parmar, R.K. An extension of the generalized Hurwitz–Lerch zeta function of two variables. *Filomat* **2017**, *31*, 91–96. [CrossRef]
20. Srivastava, H.M.; Karlsson, P.W. *Multiple Gaussian Hypergeometric Series*; Ellis Horwood Limited: Chichester, UK, 1985.
21. Srivastava, H.M. Certain formulas involving Appell functions. *Rikkyo Daigaku Sugaku Zasshi* **1972**, *21*, 73–99.
22. Bailey, W.N. *Generalized Hypergeometric Series*; Cambridge Tracts in Mathematics and Mathematical Physics, Cambridge University Press: Cambridge, UK, 1935; Volume 32.
23. Milovanović, G.V.; Rassias, M.T. Some properties of a hypergeometric function which appear in an approximation problem. *J. Glob. Optim.* **2013**, *57*, 1173–1192. [CrossRef]

24. Slater, L.J. *Confluent Hypergeometric Functions*; Cambridge University Press: Cambridge, UK; London, UK; New York, NY, USA, 1960.
25. Srivastava, H.M.; Manocha, H.L. *A Treatise on Generating Functions*; Ellis Horwood Limited: Chichester, UK, 1984.
26. Choi, J.; Hasanov, A. Applications of the operator $H(\alpha, \beta)$ to the Humbert double hypergeometric functions. *Comput. Math. Appl.* **2011**, *61*, 663–671. [CrossRef]
27. Brychkov, Y.A.; Saad, N. Some formulas for the Appell function $F_1(a, b, b'; c; w, z)$. *Integral Transforms Spec. Funct.* **2012**, *11*, 793–802. [CrossRef]
28. Choi, J.; Agarwal, P. Certain generating functions involving Appell series. *Far East J. Math. Sci.* **2014**, *84*, 25–32.
29. Wang, X. Recursion formulas for Appell functions. *Integral Transforms Spec. Funct.* **2012**, *23*, 421–433. [CrossRef]

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