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Application of Asymptotic Homotopy Perturbation Method to Fractional Order Partial Differential Equation

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Abstract: In this article, we introduce a new algorithm-based scheme titled asymptotic homotopy perturbation method (AHPM) for simulation purposes of non-linear and linear differential equations of non-integer and integer orders. AHPM is extended for numerical treatment to the approximate solution of one of the important fractional-order two-dimensional Helmholtz equations and some of its cases. For probation and illustrative purposes, we have compared the AHPM solutions to the solutions from another existing method as well as the exact solutions of the considered problems. Moreover, it is observed that the symmetry or asymmetry of the solution of considered problems is invariant under the homotopy definition. Error estimates for solutions are also provided. The approximate solutions of AHPM are tabulated and plotted, which indicates that AHPM is effective and explicit.

Keywords: fractional order partial differential equation; caputo derivative; asymptotic homotopy perturbation method; AHPM

MSC: 35A22; 35A25; 35K57

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1. Introduction

In recent years, fractional calculus has made a great contribution to the fields of science and engineering due to its many applications in the fields of damping visco elasticity, biology, electronics, genetic algorithms, signal processing, robotic technology, traffic systems, telecommunication, chemistry, physics, and economics and finance. This has all been possible due to such mathematicians as Riemann, Liouville, Leibniz, Euler, Bernoulli, Wallis, and L' Hospital, who played an important role in the development of fractional calculus. In this regard, our research focuses on the following fractional order equations of Helmholtz, which are important in fractional calculus.

$$\frac{\partial^\alpha U(x, y)}{\partial x^\alpha} + \frac{\partial^2 U(x, y)}{\partial y^2} + \rho U(x, y) = \zeta(x, y) \quad (1)$$

and initial condition

$$U(0, y) = \zeta(y) \quad (2)$$

The Helmholtz equation is used in the study of physical problems consisting of partial differential equations in space-time, such as scattering problems in electromagnetism and acoustics in many areas, i.e., in aeronautics, marine technology geophysics, and optical problems. For further applications and studies about the concern problem, see the research study [1–3] and references therein.

In fact, in fractional calculus, many researchers have focused on the various schemes and aspects of partial differential equations and fractional order partial differential equations (FPDEs) as well, see [4–11]. In this regard, various types of techniques have been developed for numerical solutions of non-linear and linear differential equations of integer order. However, there are very few schemes that have been extended to find the solution of linear and nonlinear differential equations of fractional order; for reading, see [12–17]. In this article, we desire to contribute to and extend the recent technique asymptotic homotopy perturbation method (AHPM) for the solution of real-world problems. In 2019, AHPM [18] was used for the first time for the solution of the Zakharov–Kuznetsov equation. The main aim of our work is to introduce the proposed technique, which is easy to apply and more efficient than existing procedures. In this concern, we introduce and apply the asymptotic homotopy perturbation method (AHPM) to obtain the approximate solution of fractional order Helmholtz Equations (1) and (2). In addition, we compare the AHPM solution to the exact solution as well as to the HPETM solution.

2. Preliminaries

In this section, from the fractional calculus, we give some basic definitions for the reader, as follows:

Definition 1 ([19]). *Riemann–Liouville integral of fractional order $\alpha \in \mathbb{R}^+$ for the function $g \in L([0, 1], \mathbb{R})$ is given as:*

$$I_0^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds, \quad (3)$$

provided that integral exists (on right hand side).

Definition 2 ([19]). *For a real number $p \in \mathbb{R}$, a function $f: \mathbb{R} \rightarrow \mathbb{R}^+$ is said to be in the space C_p if it can be written as $f(t) = t^q f_1(t)$ with $q > p$, $f_1(t) \in C[0, \infty)$ such that $f(t) \in C_p^m$ if $f^{(m)} \in C_p$ for $m \in \mathbb{N} \cup \{0\}$.*

Definition 3 ([19]). *Caputo fractional derivative of a function $h \in C_{-1}^m$ with $m \in \mathbb{N} \cup \{0\}$ is provided as:*

$$D_t^\alpha h(t) = \begin{cases} I^{m-\alpha} f^{(m)}, & m-1 < \alpha \leq m, m \in \mathbb{N}, \\ \frac{d^m}{dt^m} h(t), & \alpha = m, m \in \mathbb{N}. \end{cases} \quad (4)$$

Definition 4 ([19]). *The two parameter Mittag-Leffler function is provided as:*

$$E_{\alpha, \beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\alpha + \beta)}. \quad (5)$$

If $\alpha = \beta = 1$ in eq(5), we obtain $E_{1,1}(t) = e^t$ and $E_{1,1}(-t) = e^{-t}$.

3. Construction of the Method (AHPM)

Here, in this section, we will discuss that the best way to establish AHPM procedure to solve fractional order problem in the following form

$$T(U(x, y)) + f(x, y) = 0 \quad (6)$$

$$B\left(U(x, y), \frac{\partial U(x, y)}{\partial x}\right) = 0 \quad (7)$$

where $T(U(x, y))$ is a differential operator that may consist of ordinary, partial, or time-fractional differential or space fractional derivative. $T(U(x, y))$ can be expressed for fractional model as follows:

$$\frac{\partial^\alpha U(x, y)}{\partial x^\alpha} + N(U(x, y)) + f(x, y) = 0 \quad (8)$$

and condition

$$B\left(U(x, y), \frac{\partial U(x, y)}{\partial x}\right) = 0, \quad (9)$$

where $\frac{\partial^\alpha}{\partial x^\alpha}$ denotes the Caputo derivative operator; N may be linear or non linear operator; B denotes a boundary operator; $U(x, y)$ is unknown exact solution of above equation; $f(x, y)$ denotes known function; and x, y denote special and temporal variables, respectively.

Let us construct a homotopy $\Phi(x, y; p) : \Omega \times [0, 1] \rightarrow R$ satisfies

$$\frac{\partial^\alpha \Phi(x, y; p)}{\partial x^\alpha} + f(x, y) - p[N(\Phi(x, y; p))] = 0, \quad (10)$$

where $p \in [0, 1]$ is said to be an embedding parameter. In this phase, the proposed deformation Equation (10) is an alternate form of the deformation equations as:

$$(1 - p)[L(\Phi(x, y; p)) - L(U_0(x, y)) + f(x, y)] + p[T(\Phi(x, y; p)) + f(x, y)] = 0, \quad (11)$$

$$(1 - p)[L(\Phi(x, y; p)) - L(U_0(x, y))] = ph[T(\Phi(x, y; p)) + f(x, y)], \quad (12)$$

and

$$(1 - p)[L(\Phi(x, y; p)) + f(x, y)] = H(p)[T(\Phi(x, y; p)) + f(x, y)], \quad (13)$$

in HPM, HAM, and OHAM proposed by Liao in [20], He in [21], and Marinca in [22], respectively.

Basically, according to homotopy definition, when $p = 0$ and $p = 1$ we have $\Phi(x, y; p) = U_0(x, y)$, $\Phi(x, y; p) = U(x, y)$.

Obviously, when the embedding parameter p varies from 0 to 1, the defined homotopy ensures the convergence of $\Phi(x, y; p)$ to the exact solution $U(x, y)$. Consider $\Phi(x, y; p)$ in the form

$$\Phi(x, y; p) = U_0(x, y) + \sum_{i=1}^{\infty} U_i(x, y)p^i \quad (14)$$

and assuming $N(\Phi(x, y; p))$ as follows

$$N(\Phi(x, y; p)) = B_1 N_0 + \sum_{i=1}^{\infty} \left(\sum_{m=0}^i B_{i+1-m} N_m \right) p^i, \quad B_1 + B_2 + B_3 + \dots = -1. \quad (15)$$

where $B_i = B_i(x, y; c_i)$, for $i = 1, 2, 3, \dots$ are arbitrary auxiliary functions, will be discussed later. Thus, if $p = 0$ and $p = 1$ in Equation (10), we have

$$\frac{\partial^\alpha U(x, y)}{\partial x^\alpha} + f(x) = 0, \text{ and } \frac{\partial^\alpha U(x, y)}{\partial x^\alpha} + N(U(x, y)) + g(x, y) = 0,$$

respectively.

It is obvious that the construction of introduced auxiliary function in Equation (15) is different from the auxiliary functions that are proposed in articles [20–22]. Hence, the procedure proposed in our paper is different from the procedures proposed by Liao, He, and Marinca in aforesaid papers [20–22] as well as optimal homotopy perturbation method (OHPM) in [23].

Furthermore, when we substitute Equations (14) and (15) in Equation (10) and equate like power of p , the obtained series of simpler linear problems are

$$p^0 : \frac{\partial^\alpha U_0(x, y)}{\partial x^\alpha} + f = 0,$$

$$p^1 : \frac{\partial^\alpha U_1(x, y)}{\partial x^\alpha} = B_1 N_0,$$

$$p^2 : \frac{\partial^\alpha U_2(x, y)}{\partial x^\alpha} = B_2 N_0 + B_1 N_1,$$

$$p^3 : \frac{\partial^\alpha U_3(x, y)}{\partial x^\alpha} = B_3 N_0 + B_2 N_1 + B_1 N_2,$$

and k th order iteration is

$$p^k : \frac{\partial^\alpha U_k(x, y)}{\partial x^\alpha} = \sum_{i=0}^{k-1} B_{k-i} N_i,$$

We obtain the series solutions by using the integral operator on both sides of the above simple fractional differential equation. The convergence of the series solution Equation (14) to the exact solution depends upon the auxiliary parameters (functions) $B_i(x, y; c_i)$. The choice of $B_i(x, y; c_i)$ is purely on the basis of terms that appear in non-linear section of the Equation (6). Equation (14) converges to the exact solution of Equation (6) at $p = 1$:

$$\tilde{U}(x, y) = U_0(x, y) + \sum_{k=1}^{\infty} U_k(x, y; c_i). \quad (16)$$

Particularly, we can truncate Equation (16) into finite m -terms to obtain the solution of nonlinear problem. The auxiliary convergence control constants c_1, c_2, c_3, \dots can be found by solving the system

$$R(\partial x_1, \partial y_1) = R(\partial x_2, \partial y_2) = R(\partial x_3, \partial y_3) = \dots = R(\partial x_k, \partial y_k) = 0, \quad \partial x_i, \partial y_i \in [0, 1]. \quad (17)$$

It can be verified to observe that HPM is only a case of Equation (10) when $p = -p$ and

$$N(\Phi(x, y; p)) = N_0 + \sum_{i=1}^{\infty} N_i P^i.$$

The HAM is also a case of Equation (10) when $p = ph$ and

$$N(\Phi(x, y; p)) = N_0 + \sum_{i=1}^{\infty} N_i P^i$$

The OHAM is also another case when

$$B_{k-1} = B_{k-2} + h_k(t, c_j) + \sum_{i=1}^{k-2} h_{k-(i+1)}(t, c_j) B_i, \text{ and } h_k(t, c_j) = c_k$$

in Equation (15), we obtain exactly the series problems that are obtained by OHAM after expanding and equating the like power of p in deformation equation. Furthermore, concerning the optimal homotopy asymptotic method (OHAM) mentioned in this manuscript and presented in [22], the version of OHAM proposed in 2008 was improved over time, and the most recent improvement, which also contains auxiliary functions, is presented in papers [24,25]. We also have improved the version of OHAM by introducing a very

new auxiliary function in Equation (15). This paper uses a new and more general form of auxiliary function:

$$N(\Phi(x, y; p)) = B_1 N_0 + \sum_{i=1}^{\infty} \left(\sum_{m=0}^i B_{i+1-m} N_m \right) p^i$$

that depends on arbitrary parameters B_1, B_2, B_3, \dots and is useful for adjusting and controlling the convergence of nonlinear part as well as linear part of the problem with simple way.

4. Applications of AHPM

Here, we apply AHPM to obtain the solutions of the following fractional order problems.

Example 1. Consider

$$\frac{\partial^\alpha U(x, y)}{\partial x^\alpha} + \frac{\partial^2 U(x, y)}{\partial y^2} - U(x, y) = 0, \quad (18)$$

with $1 < \alpha \leq 2$, and initial condition

$$U(0, y) = y.$$

By taking

$$N = \frac{\partial^2 U(x, y)}{\partial y^2} - U(x, y), \quad (19)$$

and the zero, 1st, 2nd, 3rd and 4th order problems are

$$\frac{\partial^\alpha U_0}{\partial x^\alpha} = 0, \quad U_0(0, y) = y$$

$$\frac{\partial^\alpha U_1}{\partial x^\alpha} = B_1 N_0, \quad U_1(0, y) = 0,$$

$$\frac{\partial^\alpha U_2}{\partial x^\alpha} = B_2 N_0 + B_1 N_1, \quad U_2(0, y) = 0,$$

$$\frac{\partial^\alpha U_3}{\partial x^\alpha} = B_3 N_0 + B_2 N_1 + B_1 N_2, \quad U_3(0, y) = 0,$$

$$\frac{\partial^\alpha U_4}{\partial x^\alpha} = B_4 N_0 + B_3 N_1 + B_2 N_2 + B_1 N_3, \quad U_4(0, y) = 0,$$

Solving the above equations, we get the zero, 1st, 2nd, 3rd, and 4th order solutions are

$$U_0(x, y) = y,$$

$$U_1(x, y) = -B_1 y \frac{x^\alpha}{\Gamma(\alpha + 1)},$$

$$U_2(x, y) = -B_2 y \frac{x^\alpha}{\Gamma(\alpha + 1)} + B_1^2 y \frac{x^{2\alpha}}{\Gamma(2\alpha + 1)},$$

$$U_3(x, y) = -B_3 y \frac{x^\alpha}{\Gamma(\alpha + 1)} + 2B_1 B_2 y \frac{x^{2\alpha}}{\Gamma(2\alpha + 1)} - B_1^3 y \frac{x^{3\alpha}}{\Gamma(3\alpha + 1)},$$

$$U_4(x, y) = -B_4 y \frac{x^\alpha}{\Gamma(\alpha + 1)} + (2B_1 B_3 + B_2^2) y \frac{x^{2\alpha}}{\Gamma(2\alpha + 1)} - 2B_1^2 B_2 y \frac{x^{3\alpha}}{\Gamma(3\alpha + 1)} + B_1^4 y \frac{x^{4\alpha}}{\Gamma(4\alpha + 1)},$$

and so on. By taking $B_1 = c_1, B_2 = c_2, B_3 = c_3, B_4 = c_4$ and consider

$$\tilde{U} = U_0 + U_1 + U_2 + U_3 + U_4.$$

The residual is

$$R = \frac{\partial^\alpha \tilde{U}(x, y)}{\partial x^\alpha} + \frac{\partial^2 \tilde{U}(x, y)}{\partial y^2} - \tilde{U}(x, y). \quad (20)$$

For constant, c_1, c_2, c_3 solving Equation (17) we get $c_1 = 1, c_2 = -0.6666, c_3 = 0.4482, c_4 = -1.7816$. Profile of solutions of the Example 1 have been provided in Tables 1 and 2 and Figure 1.

Example 2. Consider another model of FPDEs,

$$\frac{\partial^\alpha U(x, y)}{\partial x^\alpha} + \frac{\partial^2 U(x, y)}{\partial y^2} + 5U(x, y) = 0, \quad (21)$$

where $1 < \alpha \leq 2$, and initial condition is

$$U(0, y) = y.$$

Taking

$$N = \frac{\partial^2 U(x, y)}{\partial y^2} + 5U(x, y), \quad (22)$$

again consider the zero, 1st, 2nd, 3rd, and 4th order problems are

$$\frac{\partial^\alpha U_0}{\partial x^\alpha} = 0, \quad U_0(0, y) = y$$

$$\frac{\partial^\alpha U_1}{\partial x^\alpha} = B_1 N_0, \quad U_1(0, y) = 0,$$

$$\frac{\partial^\alpha U_2}{\partial x^\alpha} = B_2 N_0 + B_1 N_1, \quad U_2(0, y) = 0,$$

$$\frac{\partial^\alpha U_3}{\partial x^\alpha} = B_3 N_0 + B_2 N_1 + B_1 N_2, \quad U_3(0, y) = 0,$$

$$\frac{\partial^\alpha U_4}{\partial x^\alpha} = B_4 N_0 + B_3 N_1 + B_2 N_2 + B_1 N_3, \quad U_4(0, y) = 0,$$

Using the above equations, the zero, 1st, 2nd, 3rd, and 4th order solutions are

$$U_0(x, y) = y,$$

$$U_1(x, y) = 5B_1 y \frac{x^\alpha}{\Gamma(\alpha + 1)},$$

$$U_2(x, y) = 5B_2 y \frac{x^\alpha}{\Gamma(\alpha + 1)} + 25B_1^2 y \frac{x^{2\alpha}}{\Gamma(2\alpha + 1)},$$

$$U_3(x, y) = 5B_3 y \frac{x^\alpha}{\Gamma(\alpha + 1)} + 50B_1 B_2 y \frac{x^{2\alpha}}{\Gamma(2\alpha + 1)} + 125B_1^3 y \frac{x^{3\alpha}}{\Gamma(3\alpha + 1)},$$

$$U_4(x, y) = 5B_4y \frac{x^\alpha}{\Gamma(\alpha+1)} + 25(2B_1B_3 + B_2^2)y \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + 375B_1^2B_2y \frac{x^{3\alpha}}{\Gamma(3\alpha+1)} \\ + 625B_1^4y \frac{x^{4\alpha}}{\Gamma(4\alpha+1)},$$

and so on.

By taking $B_1 = c_1, B_2 = c_2, B_3 = c_3, B_4 = c_4$, and using

$$\tilde{U} = U_0 + U_1 + U_2 + U_3 + U_4,$$

the series solution of order four is

$$\tilde{U} = y + 5(c_1 + c_2 + c_3 + c_4)y \frac{x^\alpha}{\Gamma(\alpha+1)} + 25(c_1^2 + c_2^2 + 2c_1c_2 + 2c_1c_3) \\ y \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + 125(c_1^3 + 3c_1^2c_2)y \frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + 625c_1^4y \frac{x^{4\alpha}}{\Gamma(4\alpha+1)}.$$

For c_1, c_2, c_3 , and c_4 respectively, solving Equation (17) we get the constant values, $c_1 = -0.2, c_2 = -0.1, c_3 = -0.075$, and $c_4 = -0.05$. Profile of solutions of the Example 2 has been provided in Tables 3 and 4 and Figure 2.

Example 3. Consider another special model of FPDEs,

$$\frac{\partial^\alpha U(x, y)}{\partial x^\alpha} + \frac{\partial^2 U(x, y)}{\partial y^2} - 2U(x, y) = (12x^2 - 3x^4)\sin y, \quad (23)$$

where $1 < \alpha \leq 2$, and initial condition is

$$U(0, y) = 0.$$

taking

$$L = \frac{\partial^\alpha U(x, y)}{\partial x^\alpha} - (12x^2 - 3x^4)\sin y \quad (24)$$

and

$$N = \frac{\partial^\alpha U(x, y)}{\partial y^\alpha} - 2U(x, y) \quad (25)$$

$$\frac{\partial^\alpha U_0}{\partial x^\alpha} = 0, \quad U_0(0, y) = 0. \quad (26)$$

$$\frac{\partial^\alpha U_1}{\partial x^\alpha} = B_1 N_0, \quad U_1(0, y) = 0, \quad (27)$$

$$\frac{\partial^\alpha U_2}{\partial x^\alpha} = B_2 N_0 + B_1 N_1, \quad U_2(0, y) = 0, \quad (28)$$

$$\frac{\partial^\alpha U_3}{\partial x^\alpha} = B_3 N_0 + B_2 N_1 + B_1 N_2, \quad U_3(0, y) = 0, \quad (29)$$

Using the above Equation (26)–(29), the zero, 1st, 2nd, 3rd, and 4th order solutions are

$$U_0(x, y) = \left(24 \frac{x^{\alpha+2}}{\Gamma(\alpha+3)} - 72 \frac{x^{\alpha+4}}{\Gamma(\alpha+5)} \right) \sin(y),$$

$$U_1(x, y) = -B_1 \left(72 \frac{x^{2\alpha+2}}{\Gamma(2\alpha+3)} - 216 \frac{x^{2\alpha+4}}{\Gamma(2\alpha+5)} \right) \sin(y),$$

$$U_2(x, y) = -B_2 \left(\frac{72x^{2\alpha+2}}{\Gamma(2\alpha+3)} - \frac{216x^{2\alpha+4}}{\Gamma(2\alpha+5)} \right) \sin(y) + 3B_1^2 \left(\frac{72x^{3\alpha+2}}{\Gamma(3\alpha+3)} - \frac{216x^{3\alpha+4}}{\Gamma(3\alpha+5)} \right) \sin(y),$$

$$U_3(x, y) = -B_3 \left(\frac{72x^{2\alpha+2}}{\Gamma(2\alpha+3)} - \frac{216x^{2\alpha+4}}{\Gamma(2\alpha+5)} \right) \sin(y) + 6B_1B_2 \left(\frac{72x^{3\alpha+2}}{\Gamma(3\alpha+3)} - \frac{216x^{3\alpha+4}}{\Gamma(3\alpha+5)} \right) \sin(y)$$

$$-9B_1^3 \left(\frac{72x^{4\alpha+2}}{\Gamma(4\alpha+3)} - \frac{216x^{4\alpha+4}}{\Gamma(4\alpha+5)} \right) \sin(y),$$

The series solution of order three can be written as

$$\tilde{U} = U_0 + U_1 + U_2 + U_3.$$

The residual is

$$R = \frac{\partial^\alpha \tilde{U}(x, y)}{\partial x^\alpha} + \frac{\partial^2 \tilde{U}(x, y)}{\partial y^2} - 2\tilde{U}(x, y) - (12x^2 - 3x^4) \sin y. \quad (30)$$

Using Equation (30) and $B_1 = c_1$, $B_2 = c_2$, and $B_3 = c_3$, for these constants, again we solve Equation (17). Profile of solutions of the Example 3 has been provided in Tables 5 and 6 and Figure 3.

Table 1. Solution of Example 1 by AHPM taking $c_1 = 1$; $c_2 = -0.6666$; $c_3 = 0.4482$; $c_4 = -1.7816$; and $\alpha = 1.7, 1.8$, and 1.9 , for distinct values of x at $y = 0.01$.

x	AHPM ($\alpha = 1.7$)	AHPM ($\alpha = 1.8$)	AHPM ($\alpha = 1.9$)
0.05	0.010027164039	0.0100224080748	0.0100184659335
0.10	0.0100947256496	0.0100808753719	0.0100689828802
0.15	0.0101969542295	0.0101715408955	0.0101492689388
0.20	0.0103314950768	0.0102927831639	0.0102583683627
0.25	0.0104970602326	0.0104437304068	0.0103958031903
0.30	0.0106929289205	0.0106239262991	0.0105613761196
0.35	0.0109187473072	0.0108331937404	0.0107550889561
0.40	0.011174428065	0.0110715657551	0.0109771008868
0.45	0.0114600931854	0.0113392458535	0.0112277044737
0.50	0.0117760387022	0.0116365834877	0.0115073108446

Table 2. Comparison of AHPM and HPETM [1] results of example 1 in term of absolute error by taking $c_1 = 1$, $c_2 = -0.6666$, $c_3 = 0.4482$, $c_4 = -1.7816$, and $\alpha = 2$ for distinct values of (x, y) .

(x, y)	Exact Solution	AHPM Solution	Error (AHPM)	Error (HPETM [1])
(0.1, 0.1)	0.100500416806	0.100500419954	3.148×10^{-9}	4.4682560×10^{-8}
(0.2, 0.2)	0.204013351124	0.204013451861	1.007×10^{-7}	9.8994158×10^{-8}
(0.3, 0.3)	0.313601554239	0.313602319178	7.649×10^{-7}	9.8994158×10^{-7}
(0.4, 0.4)	0.432428948735	0.432432171974	3.223×10^{-6}	2.9025947×10^{-6}
(0.5, 0.5)	0.563812982603	0.563822818267	9.836×10^{-6}	3.8474635×10^{-6}
(0.6, 0.6)	0.711279130945	0.711303602181	2.447×10^{-5}	4.7800451×10^{-5}
(0.7, 0.7)	0.878618303942	0.878671186553	5.288×10^{-5}	5.6998309×10^{-5}
(0.8, 0.8)	1.06994795704	1.07005103358	1.031×10^{-4}	6.6061216×10^{-5}
(0.9, 0.9)	1.2897777469	1.28996342214	1.857×10^{-4}	7.4980409×10^{-5}
(1, 1)	1.54308063482	1.54339489437	3.143×10^{-4}	8.3745391×10^{-5}

Table 3. Solution of Example 2 by AHPM taking $c_1 = -0.2$; $c_2 = -0.1$; $c_3 = -0.075$; $c_4 = -0.05$; and $\alpha = 1.5, 1.7$, and 1.9 , for distinct values of x at $y = 0.001$.

x	AHPM($\alpha = 1.5$)	AHPM($\alpha = 1.7$)	AHPM($\alpha = 1.9$)
0.05	9.8219e−04	9.9156e−04	9.9608e−04
0.10	9.4995e−04	9.7267e−04	9.8539e−04
0.15	9.0881e−04	9.4578e−04	9.6849e−04
0.20	8.6099e−04	9.1206e−04	9.4573e−04
0.25	8.0790e−04	8.7232e−04	9.1738e−04
0.30	7.5062e−04	8.2722e−04	8.8368e−04
0.35	6.9002e−04	7.7735e−04	8.4488e−04
0.40	6.2683e−04	7.2322e−04	8.0122e−04
0.45	5.6171e−04	6.6532e−04	7.5297e−04
0.50	4.9524e−04	6.0411e−04	7.0037e−04

Table 4. Absolute error of Example 2 by AHPM taking $c_1 = -0.2$, $c_2 = -0.1$, $c_3 = -0.075$, $c_4 = -0.05$, and $\alpha = 2$ for distinct values of x at $y = 0.001$.

x	Exact Solution ($\alpha = 2$)	AHPMSolution ($\alpha = 2$)	Error
0.05	9.9376e−04	9.9734e−04	3.5880e−06
0.10	9.7510e−04	9.8939e−04	1.4284e−05
0.15	9.4428e−04	9.7616e−04	3.1882e−05
0.20	9.0166e−04	9.5770e−04	5.6044e−05
0.25	8.4778e−04	9.3408e−04	8.6304e−05
0.30	7.8331e−04	9.0538e−04	1.2207e−04
0.35	7.0907e−04	8.7171e−04	1.6265e−04
0.40	6.2597e−04	8.3319e−04	2.0722e−04
0.45	5.3505e−04	7.8994e−04	2.5489e−04
0.50	4.3745e−04	7.4213e−04	3.0468e−04

Table 5. Solution of Example 3 by AHPM taking $c_1 = 0.48075$; $c_2 = 0.1063055$; $c_3 = -0.15870555$; and $\alpha = 1.5, 1.7, 1.9$, for distinct values of x at $y = 0.005$.

x	AHPM($\alpha = 1.5$)	AHPM($\alpha = 1.7$)	AHPM($\alpha = 1.9$)
0.05	2.8921e−07	1.1950e−07	4.8972e−08
0.10	3.2885e−06	1.5558e−06	7.3135e−07
0.15	1.3677e−05	6.9909e−06	3.5572e−06
0.20	3.7693e−05	2.0325e−05	1.0931e−05
0.25	8.2927e−05	4.6558e−05	2.6117e−05
0.30	1.5822e−04	9.1722e−05	5.3223e−05
0.35	2.7362e−04	1.6285e−04	9.7182e−05
0.40	4.4034e−04	2.6795e−04	1.6375e−04
0.45	6.7076e−04	4.1599e−04	2.5948e−04
0.50	9.7838e−04	6.1687e−04	3.9175e−04

Table 6. Absolute error of of Example 3 by AHPM taking $c_1 = 0.48075$, $c_2 = 0.1063055$, $c_3 = -0.15870555$, and $\alpha = 2$ for distinct values of x at $y = 0.005$.

x	Exact Solution ($\alpha = 2$)	AHPMSolution ($\alpha = 2$)	Error
0.05	3.1250e−08	3.1250e−08	6.9749e−16
0.10	5.0000e−07	5.0000e−07	1.7859e−13
0.15	2.5312e−06	2.5312e−06	4.5783e−12
0.20	8.0000e−06	7.9999e−06	4.5749e−11
0.25	1.9531e−05	1.9531e−05	2.7282e−10
0.30	4.0500e−05	4.0499e−05	1.1738e−09
0.35	7.5031e−05	7.5027e−05	4.0316e−09
0.40	1.2800e−04	1.2799e−04	1.1743e−08
0.45	2.0503e−04	2.0500e−04	3.0158e−08
0.50	3.1250e−04	3.1243e−04	7.0133e−08

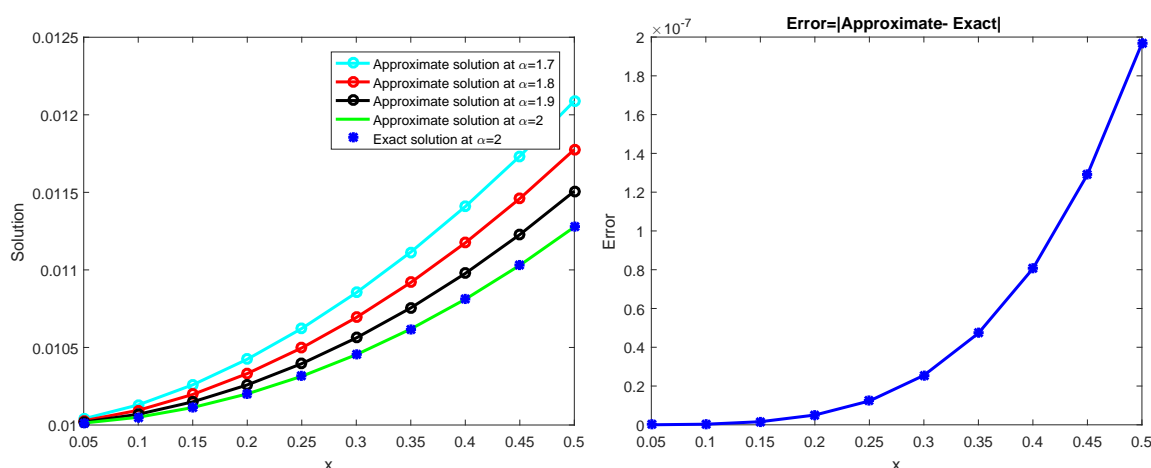


Figure 1. The left side figure shows the comparison of AHPM solution and exact solution of Example 1 for different values of α and x at $y = 0.01$. The right side figure shows the absolute error of AHPM solution and exact solution of Example 1 for different values of x and $\alpha = 2$ at $y = 0.01$.

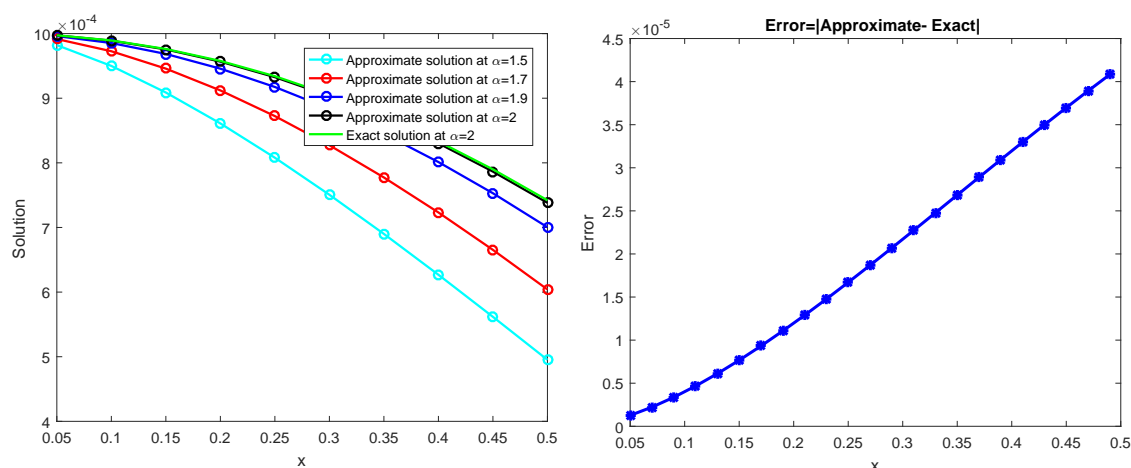


Figure 2. The left side figure shows the comparison of AHPM solution and exact solution of Example 2 for different values of α and x at $y = 0.001$. The right side figure shows the absolute error of AHPM solution and exact solution of Example 2 for different values of x and $\alpha = 2$ at $y = 0.001$.

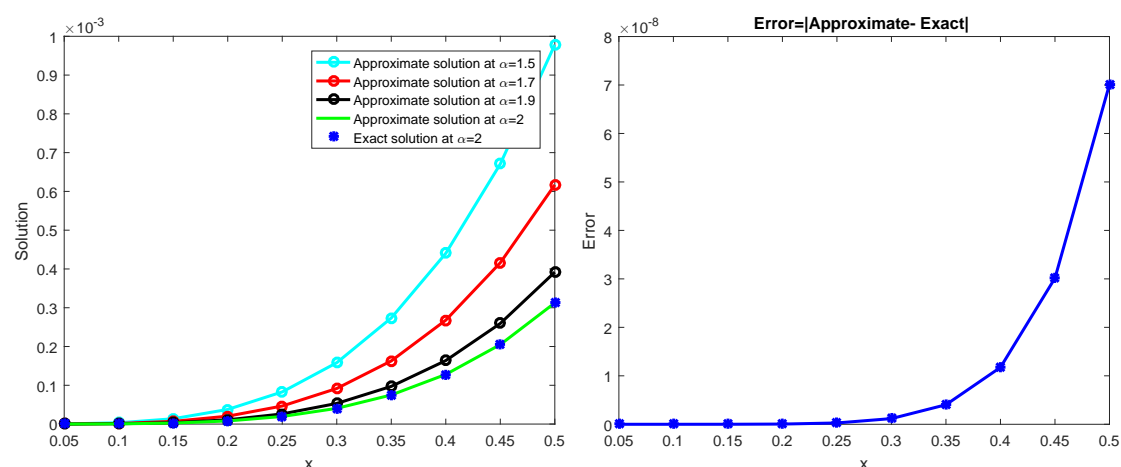


Figure 3. The left side figure shows the comparison of AHPM solution and exact solution of Example 3 for different values of α and x at $y = 0.005$. The right side figure shows the absolute error of AHPM solution and exact solution of Example 3 for different values of x and $\alpha = 2$ at $y = 0.005$.

5. Discussion and Conclusions

In this paper, we have applied a new recent procedure asymptotic homotopy perturbation method (AHPM) for the analytical solution of fractional order two-dimensional Helmholtz equations. The fractional derivatives are described in the Caputo sense. The procedure of AHPM is effective and accurate, as compared with existing analytical techniques. All the figures show that the symmetry or asymmetry of the solutions of the original problems is invariant by using the homotopy deformation Equation (10). All the computation work, tables, and figures are handled by Matlab software. The errors in the tables are very small. This reveals that the solutions obtained by AHPM are precise. From the tabulation and graphical section, it is observed that the results of AHPM are highly accurate. The high-accuracy solution of AHPM to the real-world problem (1) is one of the most important findings and reveals that AHPM can be used to find the approximate solution to fractional partial differential equations where analytical solutions may not exist, where the data provided prevents the direct application of existing analytical methods, or where existing analytical methods are time-consuming when huge data sets or complex functions are involved. AHPM is an extension and generalization of perturbation methods. The implementation of the proposed novel scheme for the numerical treatment of nonlinear coupled problems that emerge in numerous sectors of research and engineering is one of the proposed method's future directions.

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