

Article

Risk-Neutral Pricing Method of Options Based on Uncertainty Theory

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Abstract: In order to rationally deal with the belief degree, Liu proposed uncertainty theory and refined into a branch of mathematics based on normality, self-duality, sub-additivity and product axioms. Subsequently, Liu defined the uncertainty process to describe the evolution of uncertainty phenomena over time. This paper proposes a risk-neutral option pricing method under the assumption that the stock price is driven by Liu process, which is a special kind of uncertain process with a stationary independent increment. Based on uncertainty theory, the stock price's distribution and inverse distribution function under the risk-neutral measure are first derived. Then these two proposed functions are applied to price the European and American options, and verify the parity relationship of European call and put options.

Keywords: no-arbitrage market; risk-neutral option pricing; Liu process; uncertainty theory



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1. Introduction

The value of options comes from the difference between the benefits and costs they bring. Under normal circumstances, the cost of options is fixed, but the possible benefits are very uncertain and have great volatility. In other words, there is a huge asymmetry between the benefits and costs of options, option pricing theory studies this asymmetry, and then gives the option's "fair" price. Numerous research reports have confirmed that options not only have risk aversion, risk investment and price discovery functions, but options trading can also pass unpublished information into the spot market and play a role in improving information asymmetry in the spot market. Options theory has become an important part of modern finance and the study of options pricing theory is of great importance to academia, exchanges and over-the-counter markets, as well as the financial industry.

Under the Black–Scholes model framework [1], many scholars have done a lot of research on option pricing theory and obtained many useful results [2]. However, as the Black–Scholes formula is based on probability theory, some conclusions are not highly consistent with the market, and their practical applicability is not strong. This is because a central assumption of the Black–Scholes formula is that stock prices obey the Wiener process, which is described by a stochastic differential equation. It can only describe the continuous changes in asset value, but actual research has found that the stock price does not change continuously over time but there is a "jump" [3]. Many scholars have conducted research on the option pricing problem with jump in the price of the underlying asset. Using Lévy–Laplace transformation and the equivalence measure transformation, Eric Benhamou [4] obtained an European option pricing formula for the underlying asset driven by the Lévy process. Gong and Zhuang [5] introduced two pure jump Lévy processes into the double jump stochastic volatility model in which both the returns and volatility processes jump, and then proposed a option pricing formula. Huang and Wang [6,7]

studied a Lévy financial market under Knight uncertainty, and established an upper and lower bounds model.

The above studies on option pricing are conducted in the framework of stochastic differential equations. However, a large number of studies have shown that the distribution of stock returns does not conform to the assumption of normal probability distribution in stochastic differential equations and Liu [8] put forward a number of hypotheses to show that it is inappropriate to use stochastic differential equations to describe stock price changes. On the other hand, handling problems with probability requires a large number of sample data. However, there are often unexpected events in the real financial market, so that historical data cannot represent future trends. We have to invite some domain experts to evaluate their belief degree that each event will occur. Personal belief degrees always plays a very important role and influence behavioral decision making. Behavioral finance has made an in-depth exploration of this, but behavioral finance does not have a perfect mathematical theoretical framework. In order to rationally deal with this belief degree, Liu [9] proposed uncertainty theory and refined into a branch of mathematics based on normality, duality, sub-additivity and the product axiom. The essential difference between uncertainty theory and probability theory lies in the difference in product measure. The probability product measure is equal to the product of the measure and the product measure in the uncertainty theory is equal to the measure, whichever is smaller. In the real world, decisions are usually made in nondeterministic states, and both probability theory and uncertainty theory can be used to deal with nondeterministic phenomena. However, they belong to different branches of mathematics. Probability theory is used to deal with random phenomena of frequency, while uncertainty theory is used to deal with uncertain phenomena of belief degree. In recent years, uncertainty theory has been widely used in a number of different fields, forming important branches such as uncertain finance, uncertain statistics, and uncertain control. The study of uncertain process was started by Liu [10] for modelling the evolution of uncertain phenomena. An uncertain process is essentially a sequence of uncertain variables indexed by time. Liu process is a type of stationary independent increment process whose increments are normal uncertain variables. Subsequently, in order to integrate an uncertain process with respect to a canonical process, Liu proposed uncertain integral in [10]. In 2009, Liu [11] defined an uncertain differential equation driven by Liu process, and established an uncertain stock price model and then gave a pricing formula of standard European options by fair price principle. Based on this uncertain stock price model, Chen [12] proved American option pricing formulas, Peng and Yao [13] proved a stock model with mean-reverting process and the corresponding option pricing formulas. Geometric average and arithmetic average Asian option pricing formula are certified by Zhang, Liu [14] and Sun, Chen [15], respectively. Sun and Su [16] proposed a mean-reverting stock model with floating interest rate in the uncertain financial markets and then employed it to the European option and American option in the uncertain markets. In 2008, Liu [10] defined an uncertain renewal process to describe a discontinuous uncertain system, and Yao [17] established an uncertain differential equation with jumps, then gave a sufficient condition for this equation having a unique solution. Then Yu [18] proposed a stock model which was described by an uncertain differential equation with renewal process, and derived the pricing formulas for European call and put options with jumps. Ji and Zhou [19] proposed a stock model which contains both the positive jumps and the negative jumps, and they also proved European option pricing formulas.

The no-arbitrage equilibrium principle is the most fundamental research method in finance and is the basis of modern financial asset pricing, Yao [20] proposed a no-arbitrage theorem of Liu's stock model market in 2015 and gave a necessary and sufficient condition for no arbitrage in the uncertain financial market. However, the above articles about option pricing based on uncertainty theory did not consider the situation of risk-neutral financial market and the call and put options did not satisfy the classical parity relationship. Assuming that the uncertain market simulated by Liu's stock model is a no-arbitrage finance market, this paper presents a risk-neutral pricing method of options. This research makes

up for the lack of consideration of the arbitrage opportunities in uncertain financial markets, and further enriches and develops uncertain financial theory. In the rest of this paper, Section 1 will introduce some useful concepts and conclusions of uncertainty theory as needed. Section 3 introduces a risk-neutral pricing method of options and proved new distribution and inverse distribution for the Liu stock process in risk-neutral uncertainty measurement. Using these new distribution and inverse distribution of the stock process, Section 4 obtains the risk-neutral pricing formula of European and American options. Section 5 is the conclusion of this paper.

2. Preliminaries

From the mathematical viewpoint, Liu's uncertainty theory proposed by Liu [9] is essentially an alternative theory of measure. Probability theory is mainly used to resolve frequency problems, while uncertainty theories are mainly used to deal with reliability problems. They have different axiomatic foundations and different algorithms. In this section, we will introduce some important concepts and theorems of uncertainty theory such as uncertain measure, a fundamental concept in uncertainty theory.

Definition 1. (uncertain measure) [21]. Let Γ be a nonempty set, L be a σ -algebra over Γ . The uncertain measure is a set function M on the σ -algebra L which satisfies the following normality, duality, sub-additivity, and product measure axioms.

- (i) (normality). $M\{\Gamma\} = 1$, where Γ is the universal set.
- (ii) (duality). For any event Λ , $M\{\Lambda\} + M\{\Lambda^c\} = 1$.
- (iii) (sub-additivity). For every countable sequence of event $\Lambda_1, \Lambda_2, \dots$, we have

$$M\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} M\{\Lambda_i\}.$$

- (iv) (product measure). For every countable sequence of event $\Lambda_1, \Lambda_2, \dots$, we have

$$M\left\{\prod_{i=1}^{\infty} \Lambda_i\right\} = \bigwedge_{i=1}^{\infty} M\{\Lambda_i\}.$$

Uncertainty theory is a new axiomatic mathematical system based on the above four axioms. From Definition 1 we can see that product measure is the essential difference between uncertainty theory and probability theory. The probability product measure is equal to the product of the measure and the product measure in the uncertainty theory is equal to the measure, whichever is smaller.

Similar to random variable, Liu defined an uncertain variable ξ as a measurable function from an uncertainty space to the set of real numbers.

Definition 2. (uncertainty distribution) [21]. For an uncertain variable ξ , its uncertainty distribution is defined as

$$\Phi(x) = M\{\xi \leq x\}.$$

where x is an arbitrary real number.

Definition 3. (inverse uncertainty distribution) [21]. Assuming that ξ is an uncertain variable, its regular uncertain distribution function is $\Phi(x)$. Then the inverse uncertainty distribution of ξ is denoted as $\Phi^{-1}(\alpha)$, which is the inverse function of $\Phi(x)$.

Definition 4. (expected) [21]. For an uncertain variable ξ , its expected value is defined as

$$E[\xi] = \int_0^{+\infty} M\{\xi \geq r\} dr - \int_{-\infty}^0 M\{\xi \leq r\} dr.$$

From the perspective of uncertain distribution $\Phi(x)$, its expected value is defined as

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x)) dx - \int_{-\infty}^0 \Phi(x) dx,$$

and if $\Phi(x)$ has an inverse function $\Phi^{-1}(\alpha)$, then

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha) d\alpha.$$

Since uncertainty theory and probability theory have different axiomatic foundations, their algorithms are also different. In uncertainty theory, the calculation of expected value mainly uses the inverse uncertainty distribution.

In order to describe the evolution of uncertainty phenomena over time, Liu proposed uncertain process. An uncertain process is a function $X_t(\gamma)$ from $T \times (\Gamma, L, M)$ to the set of real numbers, where (Γ, L, M) is an uncertainty space and T is a totally ordered time set.

Definition 5. (Liu process) [21]. Liu process C_t is a special uncertain process which satisfies the following three conditions

- (i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous;
- (ii) The increments of C_t are stationary and independent;
- (iii) $C_{t+s} - C_t$ is a normal uncertain variable with the following uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(-\frac{\pi x}{\sqrt{3t}}\right)\right)^{-1}.$$

Definition 6. (Uncertain integral) [21]. For a closed interval $[a, b]$, define

$$\Delta = \max_{1 \leq i \leq k} |t_{i+1} - t_i|,$$

where $a = t_1 < t_2 < \dots < t_{k+1} = b$. The uncertain integral of X_t with respect to C_t is defined as

$$\int_a^b X_t dC_t = \lim_{\Delta \rightarrow 0} \sum_{i=1}^k X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i}).$$

for an uncertain process X_t and a canonical Liu process C_t .

Lemma 1. Following Ref. [21], let $f(x)$ be a Riemann integral function, then for any $a > 0$, Liu integral $Y = \int_a^b f(t) dC_t$ is a normal uncertain variable, and

$$Y \sim N\left(0, \int_0^a |f(t)| dt\right).$$

Lemma 2. Following [21], suppose $\xi_1, \xi_2, \dots, \xi_n$ are a set of independent uncertain variables, and $\Phi_1, \Phi_2, \dots, \Phi_n$ are their regular uncertainty distributions. For an uncertain variable $\xi = f(\xi_1, \xi_2, \dots, \xi_n)$, if $f(\cdot)$ is a continuous and strictly increasing function, its inverse uncertainty distribution is

$$\Psi^{-1}(\alpha) = f\left(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \dots, \Phi_n^{-1}(\alpha)\right).$$

Otherwise, if $f(\cdot)$ is a continuous and strictly decreasing function, its inverse uncertainty distribution is

$$\Psi^{-1}(\alpha) = f\left(\Phi_1^{-1}(1 - \alpha), \Phi_2^{-1}(1 - \alpha), \dots, \Phi_n^{-1}(1 - \alpha)\right).$$

3. Stock Model with Risk-Neutral Uncertainty Measure

Let (Γ, L, M) be an uncertainty space, representing a financial market, and M is called the objective uncertainty measure in this financial market. Furthermore, p_t is the bond price, and S_t is the stock price in this market; Liu [21] proposed a stock model as follows:

$$\begin{cases} dp_t = r_t p_t dt, \\ dS_t = \mu_t S_t dt + \sigma_t S_t dC_t, \quad t \in [0, T]. \end{cases} \quad (1)$$

here r_t is the risk-less interest rate, μ_t , σ_t are the drift and diffusion. C_t is a standard canonical Liu process.

Lemma 3. *If the uncertain stock model is described as (1), then the expected value of S_t under uncertain measure M is*

$$E(S_t) = \begin{cases} S_0 \theta_\sigma \exp\left(\int_0^t \mu_s ds\right), & \int_0^t |\sigma_s| ds < \frac{\pi}{\sqrt{3}}, \\ +\infty, & \int_0^t |\sigma_s| ds \geq \frac{\pi}{\sqrt{3}}. \end{cases} \quad (2)$$

where

$$\theta_\sigma = \frac{\sqrt{3} \int_0^t |\sigma_s| ds}{\sin\left(\sqrt{3} \int_0^t |\sigma_s| ds\right)}. \quad (3)$$

Proof. According to the stock process (1), the stock process S_t in uncertain market is

$$S_t = S_0 \exp\left(\int_0^t \mu_s ds + \int_0^t \sigma_s dC_s\right), \quad t \in [0, T]. \quad (4)$$

Let

$$\xi = \exp\left(\int_0^t \mu_s ds + \int_0^t \sigma_s dC_s\right),$$

then it is easy to verify that, when $x \leq 0$, uncertainty distribution $\Phi_\xi(x)$ of uncertain variable ξ is 0.

When $x > 0$,

$$\Phi_\xi(x) = M\{\xi \leq x\} = M\left\{\int_0^t \sigma_s dC_s \leq \ln x - \int_0^t \mu_s ds\right\},$$

according to Lemma 1, we can obtain

$$\Phi_\xi(x) = \left(1 + \exp\left(\frac{\pi\left(\int_0^t \mu_s ds - \ln x\right)}{\sqrt{3} \int_0^t |\sigma_s| ds}\right)\right)^{-1}, \quad (5)$$

and the inverse uncertainty distribution of ξ is

$$\Phi_\xi^{-1}(x) = \exp\left(\int_0^t \mu_s ds + \frac{\sqrt{3} \int_0^t |\sigma_s| ds}{\pi} \ln\left(\frac{\alpha}{1-\alpha}\right)\right), \quad 0 < \alpha < 1. \quad (6)$$

It can be obtained from the definition of uncertain expectation

$$E(S_t) = S_0 E(\xi) = S_0 \exp\left(\int_0^t \mu_s ds\right) \int_0^1 \left(\frac{\alpha}{1-\alpha}\right)^{\sqrt{3} \int_0^t |\sigma_s| ds / \pi} d\alpha,$$

as

$$\int_0^1 \left(\frac{\alpha}{1-\alpha}\right)^x d\alpha = \begin{cases} \frac{\pi x}{\sin(\pi x)}, & 0 < x < 1, \\ +\infty, & x \geq 1, \end{cases} \quad (7)$$

then we get the uncertain expectation

$$E(S_t) = \begin{cases} S_0 \exp\left(\int_0^t \mu_s ds\right) \frac{\sqrt{3} \int_0^t |\sigma_s| ds}{\sin(\sqrt{3} \int_0^t |\sigma_s| ds)}, & \int_0^t |\sigma_s| ds < \frac{\pi}{\sqrt{3}}, \\ +\infty, & \int_0^t |\sigma_s| ds \geq \frac{\pi}{\sqrt{3}}, \end{cases} \quad (8)$$

The proof is completed. $\square$

From Lemma 3, we can see that if the stock process is described by the Liu model (1), when $\int_0^t |\sigma_s| ds \geq \frac{\pi}{\sqrt{3}}$, we will not be able to predict the price of the stock as $E(S_t) = +\infty$. This also coincides with the fact that the longer the time, the more difficult to predict the stock price. Therefore, in the rest of this paper we only consider the short-term case when $\int_0^t |\sigma_s| ds < \frac{\pi}{\sqrt{3}}$.

For an uncertain financial market, according to the no-arbitrage theorem proposed by Yao in Ref [20], the equivalent condition to ensure a no-arbitrage market is that the stock drift coefficient μ_s be equal to the risk-free interest rate r_s . In other words, in a risk-neutral world, investors do not require risk compensation for risks, and all expected rates of return are risk-free interest rate. By discounting the risk-neutral expected value with the risk-free interest rate, the price of the option can be calculated. Then there is a risk-neutral uncertainty measure M^Q which satisfies

$$E^Q\left(\exp\left(-\int_0^t r_s ds\right) S_t\right) = S_0. \quad (9)$$

In the rest of this section, we are going to derive the uncertainty distribution as well as inverse uncertainty distribution of S_t under uncertainty measure M^Q .

Theorem 1. Assume the uncertain stock model is described by (1), then the uncertainty distribution of S_t under the risk-neutral uncertainty measure M^Q is

$$\Phi_S^Q(x) = \left(1 + \exp\left(\frac{\pi(\int_0^t r_s ds - \ln(\frac{\theta_\sigma}{S_0} x))}{\sqrt{3} \int_0^t |\sigma_s| ds}\right)\right)^{-1}, \quad (10)$$

and the inverse uncertainty distribution of S_t under the risk-neutral uncertainty measure M^Q is

$$\Phi_S^{Q,-1}(\alpha) = \frac{S_0}{\theta_\sigma} \exp\left(\int_0^t r_s ds + \frac{\sqrt{3} \int_0^t |\sigma_s| ds}{\pi} \ln\left(\frac{\alpha}{1-\alpha}\right)\right), \quad (11)$$

where θ_σ is defined as (3).

Proof. From the property of the risk-neutral uncertainty measure M^Q , we have

$$E^Q(S_t) = \exp\left(\int_0^t r_s ds\right) S_0,$$

and let

$$\xi = \exp\left(\int_0^t \mu_s ds + \int_0^t \sigma_s dC_s\right),$$

then

$$S_t = S_0 \xi,$$

and

$$E^Q(\xi) = \exp\left(\int_0^t r_s ds\right) = \frac{\sin(\sqrt{3} \int_0^t |\sigma_s| ds)}{\sqrt{3} \int_0^t |\sigma_s| ds} \exp\left(\int_0^t r_s ds - \int_0^t \mu_s ds\right) E(\xi)$$

$$= \frac{1}{\theta_\sigma} \exp\left(\int_0^t r_s ds - \int_0^t \mu_s ds\right) \int_0^{+\infty} (1 - \Phi_\xi(y)) dy,$$

here θ_σ is defined as (3), and let us make a transformation

$$y = \theta_\sigma \exp\left(\int_0^t \mu_s ds - \int_0^t r_s ds\right) x,$$

then

$$E^Q(\xi) = \int_0^{+\infty} \left(1 - \Phi\left(\theta_\sigma \exp\left(\int_0^t \mu_s ds - \int_0^t r_s ds\right) x\right)\right) dx,$$

contrasting with the definition of expectation, we get the uncertainty distribution of ξ under the risk-neutral uncertainty measure

$$\Phi_\xi^Q(x) = \Phi\left(\theta_\sigma \exp\left(\int_0^t \mu_s ds - \int_0^t r_s ds\right) x\right) = \left(1 + \exp\left(\frac{\pi\left(\int_0^t r_s ds - \ln(\theta_\sigma x)\right)}{\sqrt{3} \int_0^t |\sigma_s| ds}\right)\right)^{-1},$$

and the inverse uncertainty distribution of ξ under the risk-neutral uncertainty measure M^Q is

$$\Phi_\xi^{Q,-1}(\alpha) = \frac{1}{\theta_\sigma} \exp\left(\int_0^t r_s ds + \frac{\sqrt{3} \int_0^t |\sigma_s| ds}{\pi} \ln\left(\frac{\alpha}{1-\alpha}\right)\right),$$

this means $\theta_\sigma \xi$ is also a geometric Liu process with log-drift $\int_0^t r_s ds$ and log-diffusion $\int_0^t |\sigma_s| ds$.

Then the uncertainty distribution of S_t under the risk-neutral uncertainty measure is

$$\begin{aligned} \Phi_S^Q(x) &= M^Q\{S_t \leq x\} = M^Q\{\xi \leq \frac{x}{S_0}\} \\ &= \left(1 + \exp\left(\frac{\pi\left(\int_0^t r_s ds - \ln\left(\frac{\theta_\sigma}{S_0} x\right)\right)}{\sqrt{3} \int_0^t |\sigma_s| ds}\right)\right)^{-1}, \end{aligned}$$

and the inverse uncertainty distribution is

$$\Phi_S^{Q,-1}(\alpha) = \frac{S_0}{\theta_\sigma} \exp\left(\int_0^t r_s ds + \frac{\sqrt{3} \int_0^t |\sigma_s| ds}{\pi} \ln\left(\frac{\alpha}{1-\alpha}\right)\right).$$

The proof is completed. $\square$

4. Options Pricing with Risk-Neutral Uncertainty Measure

4.1. European Options

Theorem 2. In an uncertain financial market, assume that the underlying stock price model described by (1). Consider a European call option of which the strike price is K and the expiration date is T , then its price f_c under the risk-neutral uncertainty measure M^Q is

$$f_c = \exp\left(-\int_0^T r_t dt\right) \int_0^1 \left(\frac{S_0}{\theta_\sigma^T} \exp\left(\int_0^T r_t dt + \frac{\sqrt{3} \int_0^T |\sigma_t| dt}{\pi} \ln\left(\frac{\alpha}{1-\alpha}\right)\right) - K\right)^+ d\alpha. \quad (12)$$

here θ_σ^T is defined as (3) with $t = T$.

Proof. According the definition of European call option price by Liu [21],

$$f_c = E^Q\left(\exp\left(-\int_0^T r_t dt\right) (S_T - K)^+\right),$$

thus from Theorem 1 and Lemma 2, since $(S_T - K)^+$ is a monotonically increasing function of S_T , then the inverse uncertainty distribution of $(S_T - K)^+$ is

$$\Phi^{Q,-1}(\alpha) = \left(\frac{S_0}{\theta_\sigma^T} \exp \left(\int_0^T r_t dt + \frac{\sqrt{3} \int_0^T |\sigma_t| dt}{\pi} \ln \left(\frac{\alpha}{1-\alpha} \right) \right) - K \right)^+.$$

Then

$$\begin{aligned} f_c &= \exp \left(- \int_0^T r_t dt \right) E^Q \left((S_T - K)^+ \right), \\ &= \exp \left(- \int_0^T r_t dt \right) \int_0^1 \left(\frac{S_0}{\theta_\sigma^T} \exp \left(\int_0^T r_t dt + \frac{\sqrt{3} \int_0^T |\sigma_t| dt}{\pi} \ln \left(\frac{\alpha}{1-\alpha} \right) \right) - K \right)^+ d\alpha. \end{aligned}$$

The proof is completed. $\square$

Example 1. In an uncertain market, assume that the current selling price of a stock at time 0 is 40, the drift and diffusion of this stock is 0.06 and 0.25, respectively. The risk-less interest rate r is 0.08 per annum. Consider a European call option with this stock as the underlying asset—the expiration time T is 1 and the strike price K is 42. From the European call option pricing Formula (12) under the risk-neutral measure, we can calculate that the price of this European call option is 1.268.

Theorem 3. In an uncertain financial market, assume that the underlying stock price model described by (1). Consider a European put option whose strike price is K and expiration date is T , then its price f_p under the risk-neutral uncertainty measure M^Q is

$$f_p = \exp \left(- \int_0^T r_t dt \right) \int_0^1 \left(K - \frac{S_0}{\theta_\sigma^T} \exp \left(\int_0^T r_t dt + \frac{\sqrt{3} \int_0^T |\sigma_t| dt}{\pi} \ln \left(\frac{\alpha}{1-\alpha} \right) \right) \right)^+ d\alpha. \quad (13)$$

here θ_σ^T is defined as (3) with $t = T$.

Proof. According the definition of European put option price by Liu [21],

$$f_p = E^Q \left(\exp \left(- \int_0^T r_t dt \right) (K - S_T)^+ \right),$$

thus from Theorem 1 and Lemma 2, since $(K - S_T)^+$ is a monotonically decreasing function of S_T , then the inverse uncertainty distribution of $(K - S_T)^+$ is

$$\Psi^{Q,-1}(\alpha) = \left(K - \frac{S_0}{\theta_\sigma^T} \exp \left(\int_0^T r_t dt + \frac{\sqrt{3} \int_0^T |\sigma_t| dt}{\pi} \ln \left(\frac{1-\alpha}{\alpha} \right) \right) \right)^+.$$

Then we can get the price of European put option price (13) by replacing variables of integral similar to what we have done in Theorem 2.

$$\begin{aligned} f_p &= \exp \left(- \int_0^T r_t dt \right) E^Q \left((K - S_T)^+ \right) \\ &= \exp \left(- \int_0^T r_t dt \right) \int_0^1 \left(K - \frac{S_0}{\theta_\sigma^T} \exp \left(\int_0^T r_t dt + \frac{\sqrt{3} \int_0^T |\sigma_t| dt}{\pi} \ln \left(\frac{1-\alpha}{\alpha} \right) \right) \right)^+ d\alpha \\ &= \exp \left(- \int_0^T r_t dt \right) \int_0^1 \left(K - \frac{S_0}{\theta_\sigma^T} \exp \left(\int_0^T r_t dt + \frac{\sqrt{3} \int_0^T |\sigma_t| dt}{\pi} \ln \left(\frac{\alpha}{1-\alpha} \right) \right) \right)^+ d\alpha. \end{aligned}$$

The proof is completed. $\square$

Example 2. In an uncertain market, assume that the current selling price of a stock at time 0 is 40, the drift and diffusion of this stock is 0.06 and 0.25, respectively. The risk-less interest rate r is 0.08 per annum. Consider a European call option with this stock as the underlying asset, the expiration time T is 1 and the strike price K is 42. From the European put option pricing Formula (13) under the risk-neutral measure, we can calculate that the price of this European put option is 0.039.

According to the no-arbitrage pricing theory, the price of European call and put options should satisfy the parity formula. Therefore, Theorem 4 in the following will verify the price of call and put options obtained in Theorems 2 and 3, satisfying this parity relationship.

Theorem 4. European call option price (12) and put option price (13) under risk-neutral uncertainty measure satisfied the following parity relationship.

$$f_c + \exp\left(-\int_0^T r_t dt\right)K = f_p + S_0. \quad (14)$$

Proof. According to Theorems 2 and 3,

$$\begin{aligned} f_c - f_p &= \exp\left(-\int_0^T r_t dt\right) \int_0^1 \left(\frac{S_0}{\theta_\sigma^T} \exp\left(\int_0^T r_t dt + \frac{\sqrt{3} \int_0^T |\sigma_t| dt}{\pi} \ln\left(\frac{\alpha}{1-\alpha}\right)\right) - K \right)^+ d\alpha \\ &\quad - \exp\left(-\int_0^T r_t dt\right) \int_0^1 \left(K - \frac{S_0}{\theta_\sigma^T} \exp\left(\int_0^T r_t dt + \frac{\sqrt{3} \int_0^T |\sigma_t| dt}{\pi} \ln\left(\frac{\alpha}{1-\alpha}\right)\right) \right)^+ d\alpha. \end{aligned}$$

We discuss its value in two cases:

(i) When $\frac{S_0}{\theta_\sigma^T} \exp\left(\int_0^T r_t dt + \frac{\sqrt{3} \int_0^T |\sigma_t| dt}{\pi} \ln\left(\frac{\alpha}{1-\alpha}\right)\right) > K$,

$$\begin{aligned} f_c - f_p &= \exp\left(-\int_0^T r_t dt\right) \int_0^1 \left(\frac{S_0}{\theta_\sigma^T} \exp\left(\int_0^T r_t dt + \frac{\sqrt{3} \int_0^T |\sigma_t| dt}{\pi} \ln\left(\frac{\alpha}{1-\alpha}\right)\right) - K \right) d\alpha \\ &= \frac{S_0}{\theta_\sigma^T} \int_0^1 \left(\frac{\alpha}{1-\alpha} \right)^{\frac{\sqrt{3} \int_0^T |\sigma_t| dt}{\pi}} d\alpha - \exp\left(-\int_0^T r_t dt\right) K = S_0 - \exp\left(-\int_0^T r_t dt\right) K. \end{aligned}$$

(ii) When $\frac{S_0}{\theta_\sigma^T} \exp\left(\int_0^T r_t dt + \frac{\sqrt{3} \int_0^T |\sigma_t| dt}{\pi} \ln\left(\frac{\alpha}{1-\alpha}\right)\right) < K$,

$$\begin{aligned} f_c - f_p &= -\exp\left(-\int_0^T r_t dt\right) \int_0^1 \left(K - \frac{S_0}{\theta_\sigma^T} \exp\left(\int_0^T r_t dt + \frac{\sqrt{3} \int_0^T |\sigma_t| dt}{\pi} \ln\left(\frac{\alpha}{1-\alpha}\right)\right) \right) d\alpha \\ &= \exp\left(-\int_0^T r_t dt\right) \int_0^1 \left(\frac{S_0}{\theta_\sigma^T} \exp\left(\int_0^T r_t dt + \frac{\sqrt{3} \int_0^T |\sigma_t| dt}{\pi} \ln\left(\frac{\alpha}{1-\alpha}\right)\right) - K \right) d\alpha \\ &= S_0 - \exp\left(-\int_0^T r_t dt\right) K. \end{aligned}$$

So Equation (14) always holds.

The proof is completed. $\square$

4.2. American Options

Theorem 5. In an uncertain financial market, assume that the underlying stock price model described by (1). Consider an American call option whose strike price is K and expiration date is T , then its price f_c under the risk-neutral uncertainty measure M^Q is

$$f_c = \int_0^1 \left(\sup_{0 \leq t \leq T} \exp\left(-\int_0^t r_s ds\right) \left(\frac{S_0}{\theta_\sigma} \exp\left(\int_0^t r_s ds + \frac{\sqrt{3} \int_0^t |\sigma_s| ds}{\pi} \ln\left(\frac{\alpha}{1-\alpha}\right)\right) - K \right)^+ \right) d\alpha. \quad (15)$$

where θ_σ is also defined by (3).

Proof. In Ref [22], Chen defined the price of the American call option as

$$f_c = E^Q \left(\sup_{0 \leq t \leq T} \exp\left(-\int_0^t r_s ds\right) (S_t - K)^+ \right),$$

Since $\exp\left(-\int_0^t r_s ds\right) (S_t - K)^+$ is a monotonically increasing function of S_t , from Theorem 1 and Lemma 2, its inverse uncertainty distribution is

$$\Phi^{Q,-1}(\alpha) = \sup_{0 \leq t \leq T} \exp\left(-\int_0^t r_s ds\right) \left(\frac{S_0}{\theta_\sigma} \exp\left(\int_0^t r_s ds + \frac{\sqrt{3} \int_0^t |\sigma_s| ds}{\pi} \ln\left(\frac{\alpha}{1-\alpha}\right)\right) - K \right)^+.$$

Then

$$\begin{aligned} f_c &= E^Q \left(\sup_{0 \leq t \leq T} \exp\left(-\int_0^t r_s ds\right) (S_t - K)^+ \right) \\ &= \int_0^1 \left(\sup_{0 \leq t \leq T} \exp\left(-\int_0^t r_s ds\right) \left(\frac{S_0}{\theta_\sigma} \exp\left(\int_0^t r_s ds + \frac{\sqrt{3} \int_0^t |\sigma_s| ds}{\pi} \ln\left(\frac{\alpha}{1-\alpha}\right)\right) - K \right)^+ \right) d\alpha. \end{aligned}$$

The proof is completed. $\square$

Theorem 6. In an uncertain financial market, assume the underlying stock price model described by (1). Consider an American put option whose strike price is K and expiration date is T , then its price f_p under the risk-neutral uncertainty measure M^Q is

$$f_p = \int_0^1 \left(\sup_{0 \leq t \leq T} \exp\left(-\int_0^t r_s ds\right) \left(K - \frac{S_0}{\theta_\sigma} \exp\left(\int_0^t r_s ds + \frac{\sqrt{3} \int_0^t |\sigma_s| ds}{\pi} \ln\left(\frac{\alpha}{1-\alpha}\right)\right) \right)^+ \right) d\alpha. \quad (16)$$

where θ_σ is also defined by (3).

Proof. In Ref [22], Chen defined the price of American put option is

$$f_c = E^Q \left(\sup_{0 \leq t \leq T} \exp\left(-\int_0^t r_s ds\right) (K - S_t)^+ \right).$$

Since $\exp\left(-\int_0^t r_s ds\right) (K - S_t)^+$ is a monotonically decreasing function of S_t , from Theorem 1 and Lemma 2, its inverse uncertainty distribution is

$$\Phi^{Q,-1}(\alpha) = \sup_{0 \leq t \leq T} \exp\left(-\int_0^t r_s ds\right) \left(K - \frac{S_0}{\theta_\sigma} \exp\left(\int_0^t r_s ds + \frac{\sqrt{3} \int_0^t |\sigma_s| ds}{\pi} \ln\left(\frac{\alpha}{1-\alpha}\right)\right) \right)^+.$$

Then we can obtain the price of American put options (16) by replacing variables of integral similar to what we have done in Theorem 5.

The prove is completed. $\square$

5. Conclusions

This paper derived a risk-neutral pricing method of options for Liu's stock model in an uncertain market. The stock price process obtained by this method conforms to Yao's no-arbitrage theorem of uncertain financial markets and European call and put options satisfy the classical parity relationship. The results of this option pricing method

are consistent with the classic no-arbitrage principle, and provide a certain theoretical reference for investors to derivative pricing and make investment decisions. This article only discusses a risk-neutral pricing method of options in uncertain environments from the theoretical aspect. In future research, numerical simulations or actual data are needed to verify the feasibility of the theoretical model. On the other hand, we only discuss European and American option pricing without any additional conditions; there are many issues worthy of in-depth study, such as pricing issues of general options and various new options considering the distribution of dividends, transaction costs or credit risks.

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References

- Black, F.; Scholes, M. The pricing of options and corporate liabilities. *J. Political Econ.* **1973**, *3*, 637–654. [CrossRef]
- Prabakaran, S. Black Scholes Option Pricing Model—Brownian Motion Approach, Universidad del Norte-Business School. Available online: <http://dx.doi.org/10.2139/ssrn.2508148> (accessed on 10 October 2014).
- Lo, A.W.; MacKinlay, A.C. Stock market prices do not following random walks: Evidence from a simple specificities. *Rev. Financ. Stud.* **1988**, *1*, 41–46. [CrossRef]
- Benhamou, E. Option pricing with Lévy process. *Soc. Sci. Res. Netw. Electron. J.* **2003**, *1*, 1–22. [CrossRef]
- Gong, X.; Zhuang, X. Option pricing of a double jumps model driven by tempered stable Lévy process and its application. *J. Syst. Manag.* **2017**, *26*, 1089–1096.
- Huang, H.; Wang, X.; Zhang, Y. Option pricing under Knight uncertainty in Lévy market. *J. Math. Pract. Theory* **2016**, *46*, 87–92.
- Huang, H.; Wang, X. Option pricing under Knight uncertainty driven by Lévy Process. *China Sci. Pap.* **2017**, *12*, 120–125.
- Liu, B. Toward uncertain finance theory. *J. Uncertain. Anal. Appl.* **2013**, *1*, 1. [CrossRef]
- Liu, B. *Uncertainty Theory: A Branch of Mathematics for Modeling Human Uncertainty*; Springer: Berlin/Heidelberg, Germany, 2010.
- Liu, B. Fuzzy process, hybrid process and uncertain process. *J. Uncertain. Anal. Appl.* **2008**, *2*, 3–16.
- Liu, B. Some research problem in uncertainty theory. *J. Uncertain Syst.* **2009**, *2*, 3–10.
- Chen, X. American option pricing formula for uncertain financial market. *Eur. J. Oper. Res.* **2011**, *8*, 32–37.
- Peng, J.; Yao, K. A new option pricing model for stocks in uncertainty markets. *Int. J. Oper. Res.* **2011**, *8*, 18–26.
- Zhang, Z.; Liu, W. Geometric and average Asian option pricing for uncertain financial market. *J. Uncertain Syst.* **2014**, *8*, 317–320.
- Sun, J.; Chen, X. Asian option pricing formula for uncertain financial market. *J. Uncertain. Anal. Appl.* **2015**, *3*, 11. [CrossRef]
- Sun, Y.; Su, T. Mean-reverting stock model with floating interest rate in uncertain environment. *Fuzzy Optim. Decis. Mak.* **2017**, *16*, 235–255. [CrossRef]
- Yao, K. *Uncertain Renewal Process*, 2th ed.; Uncertainty Theory Laboratory: Beijing, China, 2021.
- Yu, X. A stock model with jumps for uncertain markets. *Int. J. Uncertain. Fuzziness Knowl. Based Syst.* **2012**, *20*, 421–432. [CrossRef]
- Ji, X.; Zhou, J. Option pricing for an uncertain stock model with jumps. *Soft Comput.* **2015**, *19*, 3323–3329. [CrossRef]
- Yao, K. A no-arbitrage theorem for uncertain stock model. *Fuzzy Optim. Decis. Mak.* **2015**, *14*, 227–242. [CrossRef]
- Liu, B. *Uncertainty Theory*, 5th ed.; Uncertainty Theory Laboratory: Beijing, China, 2021.
- Chen, X. *Theory of Uncertainty Finance*; Uncertainty Theory Laboratory: Beijing, China, 2016.