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Approximations of Symmetric Functions on Banach Spaces with Symmetric Bases

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Abstract: This paper is devoted to studying approximations of symmetric continuous functions by symmetric analytic functions on a Banach space X with a symmetric basis. We obtain some positive results for the case when X admits a separating polynomial using a symmetrization operator. However, even in this case, there is a counter-example because the symmetrization operator is well defined only on a narrow, proper subspace of the space of analytic functions on X . For $X = c_0$, we introduce ε -slice G -analytic functions that have a behavior similar to G -analytic functions at points $x \in c_0$ such that all coordinates of x are greater than ε , and we prove a theorem on approximations of uniformly continuous functions on c_0 by ε -slice G -analytic functions.

Keywords: symmetric functions on Banach spaces; approximations by analytic functions; invariant means

MSC: 41A65; 46E15; 46G25; 05E05



Citation: Martsinkiv, M.; Zagorodnyuk, A. Approximations of symmetric functions on Banach spaces. *Symmetry* **2021**, *13*, 2318. <https://doi.org/10.3390/sym13122318>

Academic Editor: Alina Alb Lupas

Received: 18 November 2021

Accepted: 2 December 2021

Published: 4 December 2021

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1. Introduction and Preliminaries

Symmetric functions of infinitely many variables naturally appear in problems of statistical mechanics, particle physics, deep learning models, neural networks, and other branches of knowledge that proceed with big amounts of data that do not depend on ordering (see, e.g., [1,2]). In those applications, it is important to be able to approximate such functions by symmetric polynomials or analytic functions or maybe by some other “simple” functions [3]. Our goal is to find under which conditions such kinds of approximations are possible.

According to the Stone–Weierstrass theorem, every continuous function on a compact subset of a real Banach space can be uniformly approximated by continuous polynomials. However, if X is infinite-dimensional, we cannot extend this result to the case of closed bounded subsets (see, e.g., [4]). On the other hand, due to Kurzweil’s theorem [5], if X admits a separating polynomial, then every continuous function on X can be approximated by analytic functions uniformly on the whole space X . This result was extended to the case when X admits a separating analytic function by Boiso and Hajek in [6]. Some generalizations for complex spaces were obtained in [7,8]. In [4], Nemirovskii and Semenov proved that, if a symmetric function f on a closed ball of real ℓ_2 is smooth enough, then it can be approximated by symmetric polynomials uniformly on the ball. Here, “symmetric” means that it is invariant with respect to permutations of basis vectors. In this paper, we discuss the possibility of approximating symmetric continuous functions on a Banach space with symmetric basis by some special symmetric functions (in particular, polynomials or analytic functions).

Let X be a real or complex Banach space. We recall that a Schauder basis $\{e_n\}$ in X is symmetric if it is equivalent to the basis $\{e_{\sigma(n)}\}$ for every permutation σ of the set of

positive integers \mathbb{N} . The group of permutations S^∞ of \mathbb{N} acts on the Banach space X with a symmetric basis $\{e_n\}$ by

$$\sigma(x) = (x_{\sigma(1)}, \dots, x_{\sigma(n)}, \dots), \quad x = \sum_{n=1}^{\infty} x_n e_n = (x_1, \dots, x_n, \dots) \in X.$$

We also use a subgroup $S_0^\infty \subset S^\infty$ consisting of all finite permutations of \mathbb{N} . A permutation σ is finite if there is $m \in \mathbb{N}$ such that $\sigma(n) = n$ for every $n > m$. A function f on X is said to be symmetric if $f(\sigma(x)) = f(x)$ for every $x \in X$ and $\sigma \in S^\infty$. We know (see, e.g., [9], p. 114) that, on every Banach space with a symmetric basis, there exists an equivalent symmetric norm. Subsequently, we assume that the space X is endowed with the symmetric norm. If f is symmetric with respect to the subgroup S_0^∞ , we call it S_0^∞ -symmetric.

Spaces ℓ_p , $1 \leq p < \infty$ and c_0 are typical examples of spaces with symmetric bases, namely $\{e_n = (\underbrace{0, \dots, 0}_n, 1, 0, \dots)\}$. According to [4,10], we know that spaces ℓ_p , $1 \leq p < \infty$ support symmetric polynomials:

$$F_k(x) = \sum_{n=1}^{\infty} x_n^k, \quad k = [p], [p] + 1, \dots \quad (1)$$

We also know that polynomials form an algebraic basis in the algebra of all symmetric polynomials on ℓ_p , $1 \leq p < \infty$, where $[p]$ is the smallest integer greater than p . This means that every symmetric polynomial on ℓ_p can be uniquely represented as an algebraic combination of polynomials F_k , $k \geq [p]$. On the other hand, only constants are symmetric polynomials on c_0 . In [10], it is observed that a continuous function on a Banach space with a symmetric basis is symmetric if and only if it is S_0^∞ -symmetric. Note that the vectors $\{e_n = (\underbrace{0, \dots, 0}_n, 1, 0, \dots)\}$ do not form a basis in ℓ_∞ (because ℓ_∞ is non-separable), but we also can naturally define symmetric and S_0^∞ -symmetric functions on ℓ_∞ . In [11], it was shown that there are no nontrivial symmetric polynomials on ℓ_∞ while there are a lot of S_0^∞ -symmetric polynomials.

Let us recall that a mapping P from a normed space X to a normed space Y is an n -homogeneous polynomial if there is an n -linear map B on the n th Cartesian product X^n to Y such that $P(x) = B(x, \dots, x)$. A mapping $f: X \rightarrow Y$ is G -analytic if its restriction to any finite dimensional subspace is analytic. Every G -analytic mapping can be represented by its Taylor series expansion

$$f(x) = \sum_{n=0}^{\infty} f_n(x),$$

where f_n are n -homogeneous polynomials. If G -analytic map is continuous, then it is called analytic. For each analytic mapping f , the polynomials f_n are continuous and if f is a discontinuous G -analytic map, then at least one polynomial f_n must be discontinuous. Basically, we consider the case where the range space of f is the field of scalars. For details on polynomials and analytic mapping on Banach spaces, we refer the reader to [12,13]. The classical theory of symmetric polynomials can be found in [14]. Symmetric analytic functions on ℓ_p , algebras of symmetric functions and some generalizations were studied by many authors (see, e.g., [15–22]).

In Section 2, we consider the conditions when symmetric continuous functions on a real Banach space X with a symmetric basis can be approximated by symmetric analytic functions and symmetric polynomials. There are some positive results for the case when X admits a separating polynomial. In Section 3, we discuss the question about the approximation of Lipschitz symmetric functions by Lipschitz symmetric analytic functions. In Section 4, we consider the case of the space c_0 . Since c_0 does not support symmetric polynomials, we introduce symmetric ε -slice polynomials and ε -slice G -analytic functions and prove a theorem about approximation by such functions.

2. Kurzweil's Approximation and the Symmetrization Operator

A polynomial P on a real Banach space is said to be *separating* if $P(0) = 0$ and the minimum norm of P , $\inf_{\|x\|=1} |P(x)| > 0$. In [5], Kurzweil proved the following theorem.

Theorem 1. (Kurzweil) *Let X be any separable real Banach space that admits a separating polynomial, G be any open subset of X , and f be any continuous map from G to any real Banach space Y . Then, for every $\varepsilon > 0$, there exists an analytic map h from G to Y such that*

$$|f(x) - h(x)| < \varepsilon, \quad \text{for every } x \in G.$$

It is known [10] that, if an infinite-dimensional Banach space X has a symmetric basis and admits a separating polynomial, then it is isomorphic to ℓ_{2n} for some positive integer n . It is easy to see that

$$P(x) = \sum_{k=1}^{\infty} x_k^{2n}, \quad x = (x_1, x_2, \dots, x_k, \dots) \in \ell_{2n}$$

is a separating polynomial in ℓ_{2n} . Clearly, any finite-dimensional real Banach space admits a separating polynomial.

From the Kurzweil theorem, we can obtain the following corollary, which seems to be well-known.

Corollary 1. *Let g be a symmetric continuous function on \mathbb{R}^m . Then, g can be approximated by symmetric analytic functions uniformly on \mathbb{R}^m .*

Proof. Let us consider the following symmetrization operator on the space $C(\mathbb{R}^m)$ of continuous functions on \mathbb{R}^m . Set

$$\mathcal{S}_m(f)(x) = \frac{1}{m!} \sum_{\sigma \in S^m} f(\sigma(x)), \quad x \in \mathbb{R}^m,$$

where S^m is the group of permutations on the set $\{1, \dots, m\}$. Clearly, if $|f(x)| < \varepsilon$, $x \in \mathbb{R}^m$, then $|\mathcal{S}_m(f(x))| < \varepsilon$ on \mathbb{R}^m . In particular, if g is a symmetric continuous function and h is an analytic function on \mathbb{R}^m such that

$$|g(x) - h(x)| < \varepsilon, \quad x \in X,$$

then

$$|\mathcal{S}_m(g(x) - h(x))| = |g(x) - \mathcal{S}_m(h(x))| < \varepsilon, \quad x \in \mathbb{R}^m.$$

However, $\mathcal{S}_m(h(x)) = \mathcal{S}_m(h)(x)$ is a symmetric analytic function. \square

Let us consider how it is possible to extend the result of Corollary 1 for the infinite-dimensional case. In [18], a symmetrization operator on the space of continuous functions was constructed, bounded on bounded subsets on complex ℓ_p , $1 \leq p < \infty$. Using the same method, we prove a slightly improved version of this result.

For a topologically given group G , let us denote by $\mathcal{B}(G)$ the Banach algebra of all bounded complex functions on G and by $\mathcal{C}(G) \subset \mathcal{B}(G)$ the sub-algebra of continuous functions. A topological group G is said to be *amenable* if there exists an invariant mean φ on $\mathcal{B}(G)$, that is, a complex-valued positive linear functional φ such that $\|\varphi\| = 1$ and φ is invariant with respect to the actions by any element of G . It is well known (see e.g., [23], p. 89) that the group S_0^∞ is amenable. Let G be a subgroup of the group of all isometric operators on a complex Banach space Z . G is a topological group with respect to the topology of point-wise convergence on Z . Suppose that $V \subset Z$ is a G -symmetric subset, that is, $\sigma(z) \in V$ for all $z \in Z$ and $\sigma \in G$. Let A be a sub-algebra of bounded functions on V , $f \in A$ and $z \in Z$. We define a function $(f, z) \in \mathcal{B}(G)$ on by $(f, z)(\sigma) = f(\sigma(z))$. It is easy to see that, if f is continuous, then (f, z) is continuous too.

Theorem 2. (c.f. [18]) Let V be a S_0^∞ -symmetric subset of complex Banach space Z with a symmetric basis. There exists a continuous linear projection operator \mathcal{S} on the space of continuous bounded functions on V onto the space of S_0^∞ -symmetric bounded functions on V , $B_s(V)$ such that

$$\mathcal{S}(f)(z) = \lim_{\mathcal{U}} \frac{1}{m!} \sum_{\sigma \in S^m} f(\sigma(z)), \quad (2)$$

where \mathcal{U} is a free ultrafilter on the set of positive integers \mathbb{N} and $z \in Z$. Moreover, if f is uniformly continuous, then $\mathcal{S}(f)$ is uniformly continuous; if f is Lipschitz, then $\mathcal{S}(f)$ is Lipschitz; and if V is open and f is analytic and uniformly continuous on V , then $\mathcal{S}(f)$ is analytic on V .

Proof. In [18], it is observed that, according to [23], pp. 80, 147, there is an invariant mean on $\mathcal{C}(S_0^\infty)$ defined as

$$\varphi(g) = \lim_{\mathcal{U}} \frac{1}{m!} \sum_{\sigma \in S^m} g(\sigma), \quad g \in \mathcal{C}(S_0^\infty),$$

where \mathcal{U} is some free ultrafilter on the set of positive integers.

In ([18], Proposition 2.16), it is proved that, if φ is a continuous invariant mean of a sub-algebra $U \subset \mathcal{B}(G)$, where G is a subgroup of isometric operators on Z and A is a uniform algebra of functions on a G -symmetric subset $V \subset Z$ such that $(f, z) \in U$ for every $f \in A$ and $z \in V$, then there exists a continuous projection

$$\mathcal{S}_\varphi(f)(z) = \varphi(f, z)$$

that maps A onto a uniform algebra of bounded G -symmetric functions on V and $\|\mathcal{S}_\varphi(f)\| = 1$. If $G = S_0^\infty$, then we can set $\mathcal{S} = \mathcal{S}_\varphi$. Therefore, \mathcal{S} is a linear continuous operator that maps bounded continuous functions on V to bounded symmetric functions on V .

Let $\varepsilon > 0$ be given and let $\delta > 0$ be chosen such that if $\|x - y\| < \delta$; then, $|f(x) - f(y)| < \varepsilon$. Since $\|x - y\| < \delta$ implies $\|\sigma(x) - \sigma(y)\| < \delta$, it follows that

$$\left| \frac{1}{n!} \sum_{\sigma \in S^n} f(\sigma(x)) - \frac{1}{n!} \sum_{\sigma \in S^n} f(\sigma(y)) \right| < \varepsilon.$$

Consequently, $|\mathcal{S}(f)(x) - \mathcal{S}(f)(y)| \leq \varepsilon$. Thus, if f is uniformly continuous, then $\mathcal{S}(f)$ is uniformly continuous. By the similar reason, if f is a Lipschitz function; then $|f(x) - f(y)| \leq C\|x - y\|$ for some $C > 0$ and

$$\|f(\sigma(x)) - f(\sigma(y))\| \leq C\|\sigma(x) - \sigma(y)\| = C\|x - y\|, \quad \sigma \in S_0^\infty.$$

Hence, $\mathcal{S}(f)$ is Lipschitz. Since the mapping $\sigma: x \rightarrow \sigma(x)$ is linear, for every $\sigma \in S_0^\infty$, $\mathcal{S}(P)$ is an n -homogeneous polynomial if P is too. Thus, $\mathcal{S}(f)$ is analytic if f is analytic.

Finally, as we observed above, if $\mathcal{S}(f)$ is continuous and S_0^∞ -symmetric, then it is symmetric. \square

Let us notice that the operator \mathcal{S} depends on the choice of a free ultrafilter \mathcal{U} . We always suppose in the following that the ultrafilter is chosen and fixed.

In order to apply Theorem 2, we need to recall some definitions and results on real and complex analytic functions. Let Z be a complex Banach space and $f: Z \rightarrow \mathbb{C}$ be an analytic function on Z . Then, f can be represented by its Taylor's series expansion at any point $z_0 \in Z$

$$f(z + z_0) = \sum_{n=0}^{\infty} f_n(z), \quad (3)$$

where f_n are n -homogeneous polynomials. The space of all analytic functions on Z is denoted by $H(Z)$. Every analytic function is locally bounded and

$$\varrho_{z_0}(f) = \{\sup r: f \text{ is bounded in the open ball } B(z_0, r) \text{ of radius } r \text{ centered at } z_0\}$$

is called the radius of boundedness of f at z_0 . For the complex case, the radius of boundedness at z_0 is equal to the radius of uniform convergence $\rho_{z_0}(f)$, that is, the supremum of $r > 0$ such that the series (3) uniformly converges to $f(z + z_0)$ on $B(z_0, r)$, and

$$\rho_{z_0}(f) = \frac{1}{\limsup_{n \rightarrow \infty} \|f_n\|^{1/n}},$$

(see, e.g., [12], p. 166). Function f is a function of bounded type if $\rho_0(f) = \infty$, that is, f is bounded on all bounded subsets of Z . The space of analytic functions of bounded type on Z is denoted by $H_b(Z)$. It is well known (see, e.g., [24]) that, for any infinite dimensional Banach space (real or complex), there exists an analytic function on this space that is not bounded on some bounded subsets. Every analytic function f on a real Banach space X can be uniquely extended to an analytic function \tilde{f} on the complexification $X^{\mathbb{C}} = X \oplus iX$ of X [25], and for the radius of uniform convergence of \tilde{f} at any point $x + i0 \in X^{\mathbb{C}}$, $x \in X$, we have the following estimation [6]:

$$\frac{\rho_x(f)}{2e} \leq \rho_x(\tilde{f}).$$

In particular, if the radius of uniform convergence of f is ∞ , then $\tilde{f} \in H_b(X^{\mathbb{C}})$ and f is bounded on all bounded subsets of X . However, the inverse statement is not true because, in a real case, the radius of uniform convergence is not equal to the radius of boundedness. The following example shows that an analytic function f can be totally bounded on a real Banach space X but $\tilde{f} \notin H_b(X^{\mathbb{C}})$.

Example 1. (c.f. [26], Example 4) Let X be real c_0 or ℓ_p , $1 \leq p < \infty$ and

$$h(x) = \sum_{n=1}^{\infty} x_n^{2n}.$$

It is well known and easy to check that h is analytic on X and unbounded on bounded sequence $x_n = 2e_n$. Then, $g = e^h$ is unbounded on $(x_n)_n$ as well, but $f = 1/g$ is totally bounded on X , $|e^{-h(x)}| \leq 1$. On the other hand, $\tilde{f}(2ie_{2k+1}) = e^{2^{(2k+1)}} \rightarrow \infty$ as $k \rightarrow \infty$, where i is the imaginary unit. Thus, $\tilde{f} \notin H_b(X^{\mathbb{C}})$.

For a real Banach space X , we denote by $H_u(X)$ the space of all analytic functions on X with the radius of uniform convergence equals infinity. That is, $f \in H_u(X)$ if and only if $\tilde{f} \in H_b(X^{\mathbb{C}})$. Thus, every function $f \in H_u(X)$ can be considered the restriction of $\tilde{f} \in H_b(X^{\mathbb{C}})$ to X . Therefore, we can define the symmetrization operator \mathcal{S} on $H_u(X)$ as in (2) and $\mathcal{S}(f)$ is equal to the restriction of $\mathcal{S}(\tilde{f})$ to X .

Proposition 1. Let X be a real Banach space with a symmetric basis. Then, \mathcal{S} maps $H_u(X)$ to $H_u(X)$.

Proof. Since any ball in $X^{\mathbb{C}}$, centered at zero, is S_0^{∞} -symmetric and every function in $H_b(X^{\mathbb{C}})$ is uniformly continuous on the ball, \mathcal{S} maps $H_b(X^{\mathbb{C}})$ to itself by Theorem 2. Thus, $\mathcal{S}(\tilde{f}) \in H_b(X^{\mathbb{C}})$, that is, $f \in H_u(X)$. \square

Theorem 3. Let X be a real Banach space with a symmetric basis and $f: X \rightarrow \mathbb{R}$ be a symmetric continuous function. If f can be approximated by analytic functions in $H_u(X)$ uniformly on X , then

1. f can be approximated by symmetric analytic functions in $H_u(X)$ uniformly on X .
2. f can be approximated by symmetric polynomials uniformly on bounded subsets of X .

Proof. Let $\varepsilon > 0$ and h be a function in $H_u(X)$ such that

$$|f(x) - h(x)| < \varepsilon, \quad x \in X.$$

Let us apply the symmetrization operator \mathcal{S} to $f - h$. Since f is symmetric, $\mathcal{S}(f) = f$. Thus, $\mathcal{S}(f - h) = f - \mathcal{S}(h)$. Since $h \in H_u(X)$, $\tilde{h} \in H_b(X^{\mathbb{C}})$, and by Proposition 1, $\mathcal{S}(\tilde{h}) \in H_b(X^{\mathbb{C}})$. Hence, $\mathcal{S}(h)$, which is equal to the restriction of $\mathcal{S}(\tilde{h})$ to X belongs to $H_u(X)$. On the other hand, the inequality $|f(x) - h(x)| < \varepsilon$ for every $x \in X$ implies that $|f(\sigma(x)) - h(\sigma(x))| = |f(x) - h(\sigma(x))| < \varepsilon$ for every $x \in X$ and permutation σ . Thus, for the average over any finite number of permutations $\sigma_1, \dots, \sigma_m$ we have

$$\left| f(x) - \frac{1}{m} \sum_{j=1}^m h(\sigma_j(x)) \right| < \varepsilon.$$

Hence,

$$|f(x) - \mathcal{S}(h)(x)| < \varepsilon.$$

□

Corollary 2. *There exists a uniformly continuous symmetric function f on ℓ_2 that cannot be approximated by functions in $H_u(\ell_2)$ uniformly on ℓ_2 . No symmetric non-constant continuous function on c_0 can be approximated by functions in $H_u(c_0)$ uniformly on c_0 .*

Proof. In ([4], Proposition 3) was constructed a uniformly continuous symmetric function f on ℓ_2 that cannot be approximated by symmetric polynomials uniformly on the unit ball B_{ℓ_2} . Therefore, by Theorem 3, it cannot be approximated by functions in $H_u(\ell_{2n})$.

As we know, there is no symmetric polynomial on c_0 (excepting constants) and there are no nontrivial symmetric analytic functions. □

Note that, if a symmetric continuous function f on X is bounded on bounded subsets and h is an analytic function on X such that $|f(x) - h(x)| < \varepsilon$ for some $\varepsilon > 0$ and all $x \in X$, then h must be bounded on bounded subsets of X , and applying the symmetrization operator, we can conclude that $|f(x) - \mathcal{S}(h)(x)| < \varepsilon$, $x \in X$. However, $\mathcal{S}(h)(x)$ is not necessarily analytic if $h \notin H_u(X)$.

Example 2. Let $f(x) = e^{-h(x)}$, $x \in c_0$ be the function constructed in Example 1. Since f is bounded on c_0 ,

$$\mathcal{S}(f)(x) = \lim_{\mathcal{U}} \frac{1}{m!} \sum_{\sigma \in S^m} f(\sigma(x))$$

is well-defined for every ultrafilter \mathcal{U} . Let us calculate $\mathcal{S}(f)(\lambda e_1) = \mathcal{S}(f)(\lambda, 0, 0, \dots)$ for some $\lambda \in \mathbb{R}$.

$$\mathcal{S}(f)(\lambda e_1) = \lim_{m \rightarrow \infty} \frac{e^{-\lambda^2} + e^{-\lambda^4} + \dots + e^{-\lambda^{2m}}}{m}.$$

We can see that

$$\mathcal{S}(f)(\lambda e_1) = \begin{cases} 1 & \text{if } |\lambda| < 1 \\ e^{-1} & \text{if } |\lambda| = 1 \\ 0 & \text{if } |\lambda| > 1. \end{cases}$$

Thus, $\mathcal{S}(f)$ is discontinuous even on the one-dimensional subspace $\mathbb{R}e_1$.

3. Lipschitz Symmetric Functions

From Corollary 2, we can see that there is a poor amount of symmetric analytic functions on Banach spaces with symmetric bases, which is not sufficient for the uniform approximation of any symmetric continuous functions. In contrast, in this section, we construct a lot of Lipschitz symmetric functions on Banach spaces with symmetric bases.

Let X be a Banach space with a symmetric basis $\{e_n\}$ and a symmetric norm $\|\cdot\|$. Evidently, the function $x \mapsto \|x\|$ is symmetric and 1-Lipschitz because, from the triangle inequality, we have

$$\left| \|x\| - \|y\| \right| \leq \|x - y\|, \quad x, y \in X.$$

Moreover, if p is a continuous seminorm on X , then $p(x) \leq C\|x\|$ for some constant $C > 0$ and p is a C -Lipschitz function.

The following proposition allows us to construct more symmetric Lipschitz mappings.

Proposition 2. *Let γ be a λ -Lipschitz map on \mathbb{R} with $\gamma(0) = 0$. Then,*

$$A_\gamma(x) := \sum_{n=1}^{\infty} \gamma(x_n) e_n$$

is a λ -Lipschitz map from X to X and for every symmetric Lipschitz function f on X , $f \circ A_\gamma$ is a symmetric Lipschitz function.

Proof. We can see that

$$\|A_\gamma(x) - A_\gamma(y)\| \leq \left\| \sum_{n=1}^{\infty} |\gamma(x_n) - \gamma(y_n)| e_n \right\| \leq \lambda \left\| \sum_{n=1}^{\infty} |x_n - y_n| e_n \right\| = \lambda \|x - y\|.$$

Clearly, if f is symmetric and Lipschitz, then $f \circ A_\gamma$ is symmetric and Lipschitz. \square

The question about the approximation of Lipschitz functions by Lipschitz analytic functions on real Banach spaces was considered by many authors (see [27–30]). In particular, in [29], the following theorem is proved.

Theorem 4. (Azagra, Fry, Keener) *Let X be a real separable Banach space that admits a separating polynomial. Then, there exists a number $C \geq 1$ such that, for every Lipschitz function $f: X \rightarrow \mathbb{R}$ and for every $\varepsilon > 0$, there exists a Lipschitz, real analytic function $g: X \rightarrow \mathbb{R}$ such that $|f(x) - g(x)| \leq \varepsilon$ for all $x \in X$, and $\text{Lip}(g) \leq C \text{Lip}(f)$.*

From Theorem 2 and Proposition 1, we have the following corollary.

Corollary 3. *Let X be a real Banach space with a symmetric basis and f be a Lipschitz function on X . If f can be approximated by Lipschitz analytic functions that are in $H_u(X)$, then f can be approximated by symmetric Lipschitz analytic functions.*

4. Approximations of Symmetric Functions on c_0

The Kurzweil theorem was extended by Boiso and Hajek in [6] for a larger class of Banach spaces.

Let X be a real separable Banach space. According to [6], a real function d defined on X is uniformly analytic and separating if d satisfies the following conditions:

1. d is a real analytic function on X with radius of uniform convergence at any point $x \in X$ greater than or equal to R_d , for some $R_d > 0$.
2. $\emptyset \neq \{x \in X: d(x) < \alpha\} \subset B_X$ for some $\alpha \in \mathbb{R}$, where B_X is the unit ball of X .

A separating polynomial, of course, is a uniformly analytic and separating function. In [6], it is shown that any closed subspace of c_0 admits a uniformly analytic and separating function. In particular,

$$d(x) = \sum_{n=1}^{\infty} x_n^{2n}$$

is uniformly analytic and separating.

Theorem 5. (Boiso, Hajek [6]) Let X be a real separable Banach space admitting a uniformly analytic and separating function, O be an open set of X , and f be a uniformly continuous mapping defined on O and with values in a closed convex set C of an arbitrary Banach space Y . Then, for every $\varepsilon > 0$, there exists an analytic mapping h defined on O and having its values in C such that

$$\|f(x) - h(x)\| < \varepsilon, \quad \text{for any } x \in O.$$

Corollary 4. Let f be a uniformly continuous symmetric real-valued function on real c_0 . Then, for positive integer m and every $\varepsilon > 0$, there exists an analytic function h_m such that

$$|f(x) - h_m(x)| < \varepsilon$$

and $h_m(\sigma(x)) = h_m(x)$ for every $\sigma \in S^m$, $x \in c_0$.

Proof. Set

$$h_m(x) = \mathcal{S}_m(h)(x) = \frac{1}{m!} \sum_{\sigma \in S^m} h(\sigma(x)),$$

where h is as in Theorem 5. \square

We already know that there is a lot of symmetric continuous and even Lipschitz functions on c_0 while there are no nontrivial symmetric analytic functions. Thus, we have a natural question: What kind of simple symmetric functions can be convenient for approximations of symmetric continuous functions on real c_0 ?

For a given $\varepsilon > 0$, we denote by \mathcal{J}_ε the following mapping from c_0 to itself

$$\mathcal{J}_\varepsilon\left(\sum_{n=1}^{\infty} x_n e_n\right) = \sum_{n=1}^{\infty} y_n e_n,$$

where

$$y_n = \begin{cases} 0 & \text{if } |x_n| < \varepsilon \\ x_n - \varepsilon & \text{if } x_n \geq \varepsilon \\ x_n + \varepsilon & \text{if } -x_n \geq \varepsilon. \end{cases}$$

It is clear that \mathcal{J}_ε is nonlinear but

$$\mathcal{J}_\varepsilon\left(\sum_{n=1}^{\infty} x_n e_n\right) = \sum_{n=1}^{\infty} \mathcal{J}_\varepsilon(x_n e_n),$$

and the range of \mathcal{J}_ε is in the subspace of finite sequences c_{00} .

Proposition 3. The mapping \mathcal{J}_ε is 1-Lipschitz. If $\varepsilon \rightarrow 0$, then $\mathcal{J}_\varepsilon(x) \rightarrow x$ uniformly on c_0 .

Proof. Let $x, v \in c_0$, then $\|\mathcal{J}_\varepsilon(x_n e_n) - \mathcal{J}_\varepsilon(v_n e_n)\| < |x_n - v_n|$ and $\|\mathcal{J}_\varepsilon(x) - \mathcal{J}_\varepsilon(v)\| \leq \|x - v\|$.

By the definition of \mathcal{J}_ε and the norm of c_0 , we have

$$\|\mathcal{J}_\varepsilon(x) - x\| \leq \varepsilon \quad \text{for every } x \in c_0.$$

\square

Definition 1. Let f be a function on the real space c_0 and $\varepsilon > 0$. Denote by $f_\varepsilon := f \circ \mathcal{J}_\varepsilon$.

Clearly, if f is λ -Lipschitz, then f_ε is λ -Lipschitz, and if f is uniformly continuous, then f_ε is uniformly continuous as well.

Proposition 4. If f is uniformly continuous on c_0 , then $f_\varepsilon \rightarrow f$ uniformly on c_0 as $\varepsilon \rightarrow 0$.

Proof. For a given $\varepsilon_0 > 0$, let δ be such that $|f(x) - f(y)| < \varepsilon_0$ whenever $\|x - y\| < \delta$. Let $0 < \varepsilon < \delta$. Then, $\|x - \mathcal{J}_\varepsilon(x)\| < \delta$, and $|f(x) - f(\mathcal{J}_\varepsilon(x))| < \varepsilon_0$ for every $x \in c_0$. \square

Let h be a function on c_{00} . We denote by $h^{(\varepsilon)}$ the following function on c_0 , $h^{(\varepsilon)}(x) = h(\mathcal{J}_\varepsilon(x))$. Note that the definition of $h^{(\varepsilon)}$ looks very similar to the definition of f_ε . The principal difference is that f is defined on c_0 but h is defined on c_{00} . It allows us to have some kind of extension $h^{(\varepsilon)}$ of any function h on c_{00} . For example, there are no symmetric polynomials on c_0 but a lot of symmetric polynomials on c_{00} .

It is clear that, if $h^{(\varepsilon)}$ uniformly converges to a function f on c_0 , then $h^{(\varepsilon)} = f_\varepsilon$ and h is the restriction of f to c_{00} .

Theorem 6. Let h be a function on c_{00} such that its restriction to any finite dimensional subspace is continuous. Then, $h^{(\varepsilon)}$ is continuous with respect to the norm of c_0 .

Proof. Let $u = (u_1, \dots, u_n, \dots) \in c_0$ and $(x^{(m)})_m$ be a sequence in c_0 approaching u as $m \rightarrow \infty$. Let $\|u\| > \varepsilon$. Then, we can assume that $\|x^{(m)}\| > \varepsilon$ for all m . There is a finite subset $\mathcal{N} \subset \mathbb{N}$ such that $|u_n| > \varepsilon$ for every $n \in \mathcal{N}$.

Let $v = (v_1, \dots, v_n, \dots)$ be such that $v_n = 0$ if $n \in \mathcal{N}$ and $v_n = u_n$ if $n \notin \mathcal{N}$. In the same manner, we define $(y^{(m)})_m$ such that $y_n^{(m)} = 0$ if $n \in \mathcal{N}$ and $y_n^{(m)} = x_n^{(m)}$ if $n \notin \mathcal{N}$. Clearly, $y^{(m)} \rightarrow v$ as $m \rightarrow \infty$. That is, there is a number $m_0 \in \mathbb{N}$ such that, for every $m \geq m_0$, $|y_n^{(m)}| < \varepsilon$, $n \in \mathbb{N}$. Thus, for $m > m_0$ only coordinates $x_n^{(m)}$, $n \in \mathcal{N}$ may have absolute values that are greater than ε . Hence, \mathcal{J}_ε maps u and the sequence $x^{(m)}$ for $m > m_0$ to the finite dimensional subspace $V_{\mathcal{N}}$, spanned on $\{e_k\}_{k \in \mathcal{N}}$. Since h is continuous on $V_{\mathcal{N}}$ and $h^{(\varepsilon)} = h \circ \mathcal{J}_\varepsilon$, we have $h^{(\varepsilon)}(x^{(m)}) \rightarrow h^{(\varepsilon)}(u)$ as $m \rightarrow \infty$. Therefore, $h^{(\varepsilon)}$ is continuous at u .

If $\|u\| < \varepsilon$, then $\|x^{(m)}\| < \varepsilon$ for a big enough m . For this case, $\mathcal{J}_\varepsilon(x^{(m)}) = 0$ and $\mathcal{J}_\varepsilon(u) = 0$, and thus, h is trivially continuous at u .

Let us consider the case $\|u\| = \varepsilon$. Then, $|u_n| \leq \varepsilon$ and there are finite subsets only \mathcal{N}_+ and \mathcal{N}_- in \mathbb{N} such that $u_n = \varepsilon$ if $n \in \mathcal{N}_+$, and $u_n = -\varepsilon$ if $n \in \mathcal{N}_-$. As above, we can observe that $x_n^{(m)} \rightarrow \varepsilon$ if $n \in \mathcal{N}_+$, $x_n^{(m)} \rightarrow -\varepsilon$ if $n \in \mathcal{N}_-$, and $\lim_{m \rightarrow \infty} |x_n^{(m)}| < \varepsilon$ otherwise. By the definition of \mathcal{J}_ε , we can see that $\mathcal{J}_\varepsilon(u) = 0$ and $\mathcal{J}_\varepsilon(x^{(m)}) \rightarrow 0$ as $m \rightarrow \infty$. Thus, $h^{(\varepsilon)}$ is continuous at u . \square

Definition 2. Let P be a polynomial and f be a G -analytic function on c_{00} . We say that $P^{(\varepsilon)}$ is an ε -slice polynomial and $f^{(\varepsilon)}$ is an ε -slice G -analytic function on c_0 .

Corollary 5. Every ε -slice G -analytic function on c_0 is continuous.

Note that, if P is a symmetric non-constant polynomial on c_{00} , then it must be discontinuous with respect to the norm of c_0 . Therefore, P is unbounded on a bounded set of c_{00} . For example, polynomials F_k , $k \in \mathbb{N}$, defined by (1) are unbounded on the bounded sequence $u_n = (\underbrace{a, \dots, a}_n, 0, \dots)$ for any $a > 0$. By Corollary 5, $P^{(\varepsilon)}$ is continuous but still

unbounded on some bounded set of c_0 . In particular, F_k is unbounded on (u_n) for $a > \varepsilon$.

Let $x \in c_0$ and $\varepsilon > 0$. Denote by $K_\varepsilon(x)$ the cardinality of the maximal subset $\mathcal{N} = \{j_1, j_2, \dots\} \subset \mathbb{N}$ such that $|x_{j_k}| > \varepsilon$. Let

$$U_\varepsilon^m = \{x \in c_0 : K_\varepsilon(x) \leq m\}.$$

Clearly, K_ε is a symmetric \mathbb{N} -valued function, U_ε^m is a symmetric subset of c_0 , and

$$c_0 = \bigcup_{m \in \mathbb{N}} U_\varepsilon^m.$$

We need a technical result that may be interesting itself.

Lemma 1. Let V_m be a real or complex m -dimensional linear space and

$$g(t) = \sum_{n=0}^{\infty} g_n(t), \quad t = (t_1, \dots, t_m) \in V_m$$

be a symmetric analytic function on V_m , where g_n are n -homogeneous polynomials. Then, there exists an extension of g to a symmetric analytic function g^{ext} on ℓ_1 .

Proof. Let us consider first the complex case $V_m = \mathbb{C}^m$ and ℓ_1 is over the field of complex numbers. Let (P_1, \dots, P_n, \dots) be an algebraic basis of homogeneous polynomials, $\deg P_n = n$, $n \in \mathbb{N}$ in the algebra of all symmetric polynomials on ℓ_1 . Then the restrictions of P_1, \dots, P_m to V_m , which we denote by $P_1^{(m)}, \dots, P_m^{(m)}$, form an algebraic basis in the algebra of symmetric polynomials on V_m . Thus, for every g_n , there is a polynomial q_n on \mathbb{C}^m such that

$$g_n(x) = q_n(P_1^{(m)}(x), \dots, P_m^{(m)}(x)).$$

We set

$$g_n^{ext}(y) = q_n(P_1(y), \dots, P_m(y)), \quad y = (y_1, \dots, y_n, \dots) \in \ell_1.$$

Thus, q_n are n -homogeneous symmetric polynomials on ℓ_1 . Define

$$g^{ext}(y) = \sum_{n=0}^{\infty} g_n^{ext}(y), \quad y \in \ell_1.$$

Since g^{ext} is a series of continuous homogeneous polynomials, we just have to check that this series converges for every $y \in \ell_1$. In ([15], Lemma 1.1), it is proved that, for any algebraic basis, the map

$$\mathbb{C}^m \ni x \mapsto (P_1^{(m)}(x), \dots, P_m^{(m)}(x)) \in \mathbb{C}^m$$

is onto \mathbb{C}^m . Let $y \in \ell_1$ and $\xi_1 = P_1(y), \dots, \xi_m = P_m(y)$. Then, there is $x_y \in V_m$ such that $P_1^{(m)}(x_y) = \xi_1, \dots, P_m^{(m)}(x_y) = \xi_m$. That is $g^{ext}(y) = g(x_y)$ and so g^{ext} is well-defined on ℓ_1 .

Let now $V_m = \mathbb{R}^m$. Denote by \tilde{g} the analytic extension of g to \mathbb{C}^m . Then, \tilde{g}^{ext} is the extension of \tilde{g} to the complex ℓ_1 . Thus, g^{ext} is the restriction of \tilde{g}^{ext} to \mathbb{R}^m . \square

Theorem 7. Let f be a symmetric uniformly continuous function on c_0 . Then, for every $\varepsilon > 0$, the function f_ε can be approximated by symmetric ε -slice G -analytic functions uniformly on every U_ε^m , $m \in \mathbb{N}$.

Proof. If $x \in U_\varepsilon^m$ for some $\varepsilon > 0$, then $y = \mathcal{J}_\varepsilon(x)$ has a finite support (i_1, \dots, i_m) . Since f_ε is symmetric, $f_\varepsilon(x)$ depends only of $\{y_{i_1}, \dots, y_{i_m}\}$ and does not depend of the order or particular support. That is, if $y' = \mathcal{J}_\varepsilon(x')$ has a support (j_1, \dots, j_m) and $y_{i_k} = y'_{j_k}$, $k = 1, \dots, m$, then $f_\varepsilon(x) = f_\varepsilon(x')$. Thus, we can define a function on \mathbb{R}^m ,

$$\psi_m(t_1, \dots, t_m) = f_\varepsilon((t_1 + \varepsilon)e_1, \dots, (t_m + \varepsilon)e_m)$$

and $\psi_m(y_{i_1}, \dots, y_{i_m}) = f_\varepsilon(x)$, $x \in U_\varepsilon^m$. Since ψ_m is symmetric, from Corollary 1, it follows that, for every $\varepsilon_0 > 0$, there exists a symmetric analytic function $g^{(m)}$ on \mathbb{R}^m such that $|\psi_m(t) - g^{(m)}(t)| < \varepsilon_0$, $t \in \mathbb{R}^m$. Let $h^{(m)}$ be the restriction to c_{00} of the extension $(g^{(m)})^{ext}$ as in Lemma 1. Clearly, $(g^{(m)})^{ext}$ is a G -analytic function on c_{00} . Then,

$$|f_\varepsilon(x) - h^{(m)}(\mathcal{J}_\varepsilon(x))| < \varepsilon_0$$

for every $x \in U_\varepsilon^m$ and $h^{(m)}(\mathcal{J}_\varepsilon(x)) = (h^{(m)})^{(\varepsilon)}$ is an ε -slice G -analytic function. \square

5. Conclusions

The paper is devoted to studying approximations of symmetric continuous functions by symmetric analytic functions on Banach spaces with symmetric bases. We can see that, for many cases, such an approximation is impossible, even if every continuous function on the Banach space X can be approximated by analytic ones. It may happen because the symmetrization operator \mathcal{S} is well defined only on the subspace $H_u(X)$ of the space of analytic functions on X . The space $H_u(X)$ can be characterized by the property that the analytic extension \tilde{f} of any function $f \in H_u(X)$ to the complexification $X^{\mathbb{C}}$ of X is bounded on all bounded subsets of $X^{\mathbb{C}}$.

If $X = c_0$, then we have no symmetric analytic functions and symmetric continuous function on c_0 cannot be approximated by functions in $H_u(X)$. For this case, we introduce ε -slice G -analytic functions, which have behaviors similar to G -analytic functions at points $x \in c_0$ such that all coordinates of x are greater than ε . The last theorem delivers us some kind of approximation of uniformly continuous functions on c_0 by ε -slice G -analytic functions.

Author Contributions: Conceptualization, A.Z.; supervision, A.Z.; investigation, M.M. and A.Z.; writing—original draft preparation M.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the National Research Foundation of Ukraine, 2020.02/0025, 0121U111037.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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