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Integral Inequalities for Generalized Harmonically Convex Functions in Fuzzy-Interval-Valued Settings

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Abstract: It is a well-known fact that convex and non-convex fuzzy mappings play a critical role in the study of fuzzy optimization. Due to the behavior of its definition, the idea of convexity also plays a significant role in the subject of inequalities. The concepts of convexity and symmetry have a tight connection. We may use whatever we learn from both the concepts, owing to the significant correlation that has developed between both in recent years. In this paper, we introduce a new class of harmonically convex fuzzy-interval-valued functions which is known as harmonically h-convex fuzzy-interval-valued functions (abbreviated as harmonically h-convex F-I-V-Fs) by means of fuzzy order relation. This fuzzy order relation is defined level-wise through Kulisch–Miranker order relation defined on interval space. Some properties of this class are investigated. BY using fuzzy order relation and h-convex F-I-V-Fs, Hermite–Hadamard type inequalities for harmonically are developed via fuzzy Riemann integral. We have also obtained some new inequalities for the product of harmonically h-convex F-I-V-Fs. Moreover, we establish Hermite–Hadamard–Fej'er inequality for harmonically h-convex F-I-V-Fs via fuzzy Riemann integral. These outcomes are a generalization of a number of previously known results, as well as many new outcomes can be deduced as a result of appropriate parameter "θ" and real valued function "∇" selections. For the validation of the main results, we have added some nontrivial examples. We hope that the concepts and techniques of this study may open new directions for research.

Keywords: harmonically h-convex fuzzy interval-valued function; fuzzy Riemannian integral; Hermite-Hadamard inequality; Hermite-Hadamard Fejér inequality

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1. Introduction

Convex analysis has contributed significantly to the advancement of applied and pure research. The study and distinction between several directions of the classical notion of convexity has received considerable interest in recent decades. A variety of convex function extensions and generalizations have recently been discovered, see [1–5] and the references therein for more information. In the classical approach, a real valued function $\Psi: K \to \mathbb{R}$ is called convex if

$$\Psi\big((1-\xi)w+\xi y\big) \le (1-\xi)\Psi(w)+\xi\Psi(y), \tag{1}$$

for all $w, y \in K, \xi \in [0,1]$.

The concept of convexity in the context of integral problems is a fascinating field of study. As a result, several inequalities have been proposed as convex function applications. Among these, the Hermite–Hadamard inequality ($H{\cdot}H$ inequality) is a fascinating convex analytic result. The $H{\cdot}H$ inequality [6,7] is defined as follows for the convex function $\Psi: K \to \mathbb{R}$ on an interval $K = [u, \nu]$:

$$\Psi\left(\frac{u+\nu}{2}\right) \le \frac{1}{\nu-u}\int_u^\nu \Psi(w)dw \le \frac{\Psi(u)+\Psi(\nu)}{2}, \tag{2}$$

for all $u, \nu \in K$.

Various applications have been discovered resulting from a geometrical interpretation; see for example [8–10]. In several works [11–13], we can find analysis on the generalization of this disparity. Set-valued analysis is a generalization of the interval analysis. The topic arose as a response to the interval uncertainty that can be found in various computational or mathematical models of deterministic real-world problems.

The Archimedes method, which is used to calculate the circumference of a circle, is one of the earliest examples of an interval enclosure, see [14]. In 1966, Moore [15] published the first manuscript on interval analysis. Stimulated by this work the theory and application of interval arithmetic's started to be further investigated. We can mention applications such as robotics, computer graphics, chemical and structural engineering, economics, behavioral ecology, constraint fulfillment, signal processing and global optimization, neural network output optimization, that have explored the use of interval analysis [16,17].

Researchers have looked at various major inequalities as a result of the aforementioned applications, including the Jensen's inequality, $H{\cdot}H$ inequality, and Ostrowski inequality, among others. Chalco-Cano et al. [18,19] used the Hukuhara derivative for interval-valued functions to generate Ostrowski type inequalities. The Minkowski and Beckenbach inequalities were established by Romn-Flores et al. [20], see also [20–22] for additional inequalities. The $H{\cdot}H$ inequality was proposed by Sadowska [23]. Other studies can be found in [23,24].

Zhao et al. used extended fractional integrals to prove the $H{\cdot}H$ inequality for interval-valued approximately h-convex functions in [25]. Kamran et al. [26] developed the $H{\cdot}H$ inequality by means of the notion of interval-valued generalized p-convex functions. Khan et al. introduced new classes of convex and generalized convex F-I-V-F, and derived fractional $H{\cdot}H$ type and $H{\cdot}H$ type inequalities for convex F-I-V-F [27], h-convex F-I-V-F [28], (h_1, h_2)-connvex F-I-V-F [29], (h_1, h_2)-preinvex F-I-V-F [30], log-h-convex F-I-V-Fs [31], log-s-convex F-I-V-Fs in the second sense [32], and the references therein. We refer the readers for further analysis to literature on the applications and properties of fuzzy-interval, and inequalities and generalized convex fuzzy mappings, see [33–53] and the references therein.

In this paper, we established the Hermite–Hadamard type inequality for harmonically h-convex F-I-V-Fs via fuzzy Riemann integral. We also established Hermite–Hadamard Fej'er inequality via the fuzzy Riemann integral. Moreover, we have discussed some new and classical inequalities as exceptional cases.

2. Preliminary Concepts

In this section, we recall some basic preliminary notions, definitions, and results.

We define interval as,

$[\omega_*, \omega^*] = \{w \in \mathbb{R}: \omega_* \le w \le \omega^* \text{ and } \omega_*, \omega^* \in \mathbb{R}\}$, where $\omega_* \le \omega^*$.

We write $\text{len}\,[\omega_*, \omega^*] = \omega^* - \omega_*$, and if $\text{len}\,[\omega_*, \omega^*] = 0$, then $[\omega_*, \omega^*]$ is called degenerate. Hereafter, all intervals will be non-degenerate. The collection of all closed and bounded intervals of $\mathbb{R}$ is defined as $\mathcal{K}_C = \{[\omega_*, \omega^*]: \omega_*, \omega^* \in \mathbb{R}/\text{and}/\omega_* \le \omega^*\}$. If $\omega_* \ge 0$, then $[\omega_*, \omega^*]$ is called positive interval. The set of all positive interval is denoted by $\mathcal{K}_C{}^+$ and defined as $\mathcal{K}_C{}^+ = \{[\omega_*, \omega^*]: [\omega_*, \omega^*] \in \mathcal{K}_C \text{ and } \omega_* \geq 0\}$.

Now we look at some of the properties of intervals using arithmetic operations. Let $[\varrho_*, \varrho^*], [s_*, s^*] \in \mathcal{K}_C$ and $\rho \in \mathbb{R}$. Then, we have

$$[\varrho_*, \varrho^*] + [s_*, s^*] = [\varrho_* + s_*, \varrho^* + s^*],$$

$$[\varrho_*, \varrho^*] \times [s_*, s^*] = \begin{bmatrix} min\{\varrho_* s_*, \varrho^* s_*, \varrho_* s^*, \varrho^* s^*\}, \\ max\{\varrho_* s_*, \varrho^* s_*, \varrho_* s^*, \varrho^* s^*\} \end{bmatrix}$$

$$\rho \cdot [\varrho_*, \varrho^*] = \begin{cases} [\rho\varrho_*, \rho\varrho^*] & \text{if } \rho > 0, \\ \{0\} & \text{if } \rho = 0 \\ [\rho\varrho^*, \rho\varrho_*] & \text{if } \rho < 0. \end{cases}$$

For $[\varrho_*, \varrho^*], [s_*, s^*] \in \mathcal{K}_C$, the inclusion " $\subseteq$ " is defined by
$[\varrho_*, \varrho^*] \subseteq [s_*, s^*]$, if and only if $s_* \leq \varrho_*$, $\varrho^* \leq s^*$.

Remark 1. *The relation " $\leq_I$ " is defined on $\mathcal{K}_C$ by*

$$[\varrho_*, \varrho^*] \leq_I [s_*, s^*] \text{ if and only if } \varrho_* \leq s_*, \varrho^* \leq s^*, \tag{3}$$

for all $[\varrho_*, \varrho^*], [s_*, s^*] \in \mathcal{K}_C$, and it is an order relation, see [45].

Moore [24] initially proposed the concept of Riemann integral for I-V-F, which is defined as follows:

Theorem 1. [14] *If $\Psi: [u, \nu] \subset \mathbb{R} \to \mathcal{K}_C$ is an I-V-F such that $\Psi(w) = [\Psi_*(w), \Psi^*(w)]$, then Ψ is Riemann integrable over $[u, \nu]$ if and only if, Ψ_* and Ψ^* are both Riemann integrable over $[u, \nu]$ such that*

$$(IR)\int_u^\nu \Psi(w)dw = \left[(R)\int_u^\nu \Psi_*(w)dw, (R)\int_u^\nu \Psi^*(w)dw\right].$$

Let $\mathbb{R}$ be the set of real numbers. A mapping $\zeta: \mathbb{R} \to [0,1]$ called the membership function distinguishes a fuzzy subset set $\mathcal{A}$ of $\mathbb{R}$. This representation is found to be acceptable in this study. The notation $\mathbb{F}(\mathbb{R})$ also stands for the collection of all fuzzy subsets of $\mathbb{R}$.

A real fuzzy interval ζ is a fuzzy set in $\mathbb{R}$ with the following properties:

(1) ζ is normal, i.e., there exists $w \in \mathbb{R}$ such that $\zeta(w) = 1$;
(2) ζ is upper semi continuous, i.e., for given $w \in \mathbb{R}$, for every $w \in \mathbb{R}$ there exists $\varepsilon > 0$, there exists $\delta > 0$ such that $\zeta(w) - \zeta(y) < \varepsilon$ for all $y \in \mathbb{R}$ with $|w - y| < \delta$.
(3) ζ is fuzzy convex, i.e., $\zeta((1-\xi)w + \xi y) \geq min(\zeta(w), \zeta(y))$, $\forall\, w, y \in \mathbb{R}$ and $\xi \in [0,1]$;
(4) ζ is compactly supported, i.e., $cl\{w \in \mathbb{R} |\, \zeta(w) > 0\}$ is compact.

The collection of all real fuzzy intervals is denoted by $\mathbb{F}_0$.

For $\theta \in [0,1]$, θ-levels $[\zeta]^\theta$ is a nonempty compact convex set of $\mathbb{R}$. This is represented by

$$[\zeta]^\theta = \{w \in \mathbb{R} |\, \zeta(w) \geq \theta\},$$

from these definitions, we have

$$[\zeta]^\theta = [\zeta_*(\theta), \zeta^*(\theta)],$$

where

$$\zeta_*(\theta) = inf\{w \in \mathbb{R} |\, \zeta(w) \geq \theta\},$$

$$\zeta^*(\theta) = sup\{w \in \mathbb{R} |\, \zeta(w) \geq \theta\}.$$

Thus, a real fuzzy interval ζ can be identified by a parametrized triples

$$\{(\zeta_*(\theta), \zeta^*(\theta), \theta) : \theta \in [0,1]\}.$$

The two end point functions, $\zeta_*(\theta)$ and $\zeta^*(\theta)$, are used to characterize a real fuzzy interval.

Proposition 1. [42] *Let $\zeta, \Theta \in \mathbb{F}_0$, the fuzzy order relation " $\preccurlyeq$ " given on $\mathbb{F}_0$ by*

$$\zeta \preccurlyeq \Theta, \text{ if and only if, } [\zeta]^\theta \leq_I [\Theta]^\theta \text{ for all } \theta \in (0,1],$$

is a partial order relation.

We will now look at some of the properties of fuzzy intervals using arithmetic operations. Let $\zeta, \Theta \in \mathbb{F}_0$ and $\rho \in \mathbb{R}$. Then, we have

$$[\zeta \tilde{+} \Theta]^\theta = [\zeta]^\theta + [\Theta]^\theta, \quad (4)$$

$$[\zeta \tilde{\times} \Theta]^\theta = [\zeta]^\theta \times [\Theta]^\theta, \quad (5)$$

$$[\rho \cdot \zeta]^\theta = \rho \cdot [\zeta]^\theta \quad (6)$$

For $\psi \in \mathbb{F}_0$, such that $\zeta = \Theta \tilde{+} \psi$, we have the existence of the Hukuhara difference of ζ and Θ, which we call the H-difference of ζ and Θ, and is denoted by $\zeta \tilde{-} \Theta$. The H-difference exists, then

$$(\psi)^*(\theta) = (\zeta \tilde{-} \Theta)^*(\theta) = \zeta^*(\theta) - \Theta^*(\theta),$$
$$(\psi)_*(\theta) = (\zeta \tilde{-} \Theta)_*(\theta) = \zeta_*(\theta) - \Theta_*(\theta). \quad (7)$$

Definition 1. [42] *A fuzzy-interval-valued map $\widetilde{\Psi} : K \subset \mathbb{R} \to \mathbb{F}_0$ is called F-I-V-F. For each $\theta \in (0,1]$, whose θ-levels define the family of I-V-Fs $\Psi_\theta : K \subset \mathbb{R} \to \mathcal{K}_C$ are given by $\Psi_\theta(w) = [\Psi_*(w,\theta), \Psi^*(w,\theta)]$ for all $w \in K$. Here, for each $\theta \in (0,1]$, the end point real functions $\Psi_*(.,\theta), \Psi^*(.,\theta) : K \to \mathbb{R}$ are called lower and upper functions of $\widetilde{\Psi}(w)$.*

The following conclusions can be drawn from the literature [42–44]:

Definition 2. *Let $\widetilde{\Psi} : [u,v] \subset \mathbb{R} \to \mathbb{F}_0$ be an F-I-V-F. Then, fuzzy integral of $\widetilde{\Psi}$ over $[u,v]$, denoted by $(FR)\int_u^v \widetilde{\Psi}(w)dw$, is given level-wise by*

$$\left[(FR)\int_u^v \widetilde{\Psi}(w)dw\right]^\theta = (IR)\int_u^v \Psi_\theta(w)dw = \left\{\int_u^v \Psi(w,\theta)dw : \Psi(w,\theta) \in \mathcal{R}_{([u,v],\theta)}\right\}, \quad (8)$$

for all $\theta \in (0,1]$, where $\mathcal{R}_{([u,v],\theta)}$ denotes the collection of Riemannian integrable functions of I-V-Fs. The F-I-V-F $\widetilde{\Psi}$ is FR-integrable over $[u,v]$ if $(FR)\int_u^v \widetilde{\Psi}(w)dw \in \mathbb{F}_0$. *Note that, if $\Psi_*(w,\theta), \Psi^*(w,\theta)$ are Lebesgue-integrable, then Ψ is fuzzy Aumann-integrable function over $[u,v]$, see* [21,27,31].

Theorem 2. *Let $\widetilde{\Psi} : [u,v] \subset \mathbb{R} \to \mathbb{F}_0$ be a F-I-V-F, whose θ-levels define the family of I-V-Fs $\Psi_\theta : [u,v] \subset \mathbb{R} \to \mathcal{K}_C$ are given by $\Psi_\theta(w) = [\Psi_*(w,\theta), \Psi^*(w,\theta)]$ for all $w \in [u,v]$ and for all $\theta \in (0,1]$. Then, $\widetilde{\Psi}$ is FR-integrable over $[u,v]$ if and only if, $\Psi_*(w,\theta)$* and *$\Psi^*(w,\theta)$ are both R-integrable over $[u,v]$. Moreover,* if *$\widetilde{\Psi}$ is FR-integrable over $[u,v]$, then*

$$\left[(FR)\int_u^v \widetilde{\Psi}(w)dw\right]^\theta = \left[(R)\int_u^v \Psi_*(w,\theta)dw, (R)\int_u^v \Psi^*(w,\theta)dw\right]$$
$$= (IR)\int_u^v \Psi_\theta(w)dw, \quad (9)$$

for all $\theta \in (0,1]$. For all $\theta \in (0,1]$, $\mathcal{FR}_{([u,v],\theta)}$ denotes the collection of all FR-integrable F-I-V-Fs over $[u,v]$.

Definition 3. [46] *A set* $K = [u, v] \subset \mathbb{R}^+ = (0, \infty)$ *is said to be convex set, if, for all* $w, y \in K, \xi \in [0, 1]$, *we have*

$$\frac{wy}{\xi w + (1-\xi)y} \in K. \tag{10}$$

Definition 4. [46] *The relation* $\Psi: [u, v] \to \mathbb{R}^+$ *is called a harmonically convex function on* $[u, v]$ *if*

$$\Psi\left(\frac{wy}{\xi w + (1-\xi)y}\right) \le (1-\xi)\Psi(w) + \xi\Psi(y), \tag{11}$$

for all $w, y \in [u, v], \xi \in [0, 1]$, *where* $\Psi(w) \ge 0$ *for all* $w \in [u, v]$. *If expression* (11) *is reversed, then* Ψ *is called harmonically concave F-I-V-F on* $[u, v]$, *such that*

$$\Psi\left(\frac{wy}{\xi w + (1-\xi)y}\right) \ge (1-\xi)\Psi(w) + \xi\Psi(y).$$

Definition 5. [47] *The positive real-valued function* $\Psi: [u, v] \to \mathbb{R}^+$ *is called harmonically* h-*convex function on* $[u, v]$ *if*

$$\Psi\left(\frac{wy}{\xi w + (1-\xi)y}\right) \le h(1-\xi)\Psi(w) + h(\xi)\Psi(y), \tag{12}$$

for all $w, y \in [u, v], \xi \in [0, 1]$, *where* $\Psi(w) \ge 0$ *for all* $w \in [u, v]$ *and* $h: [0,1] \subseteq [u, v] \to \mathbb{R}^+$ *such that* $h \not\equiv 0$. *If expression* (12) *is reversed, then* Ψ *is called harmonically* h-*concave function on* $[u, v]$, *such that*

$$\Psi\left(\frac{wy}{\xi w + (1-\xi)y}\right) \ge h(1-\xi)\Psi(w) + h(\xi)\Psi(y).$$

The set of all harmonically h-convex (harmonically h-concave) functions is denoted by

$$HSX([u, v], \mathbb{R}^+, h)\ \big(HSV([u, v], \mathbb{R}^+, h)\big).$$

Definition 6. [28] *The F-I-V-F* $\widetilde{\Psi}: [u, v] \to \mathbb{F}_0$ *is called* h-*convex* F-I-V-F *on* $[u, v]$ *if*

$$\widetilde{\Psi}(\,(1-\xi)w + \xi y) \preccurlyeq h(1-\xi)\widetilde{\Psi}(w)\tilde{+}h(\xi)\widetilde{\Psi}(y), \tag{13}$$

for all $w, y \in [u, v], \xi \in [0, 1]$, where $\Psi(w) \ge 0$ *for all* $w \in [u, v]$ *and* $h: [0, 1] \subseteq [u, v] \to \mathbb{R}^+$, *such that* $h \not\equiv 0$. *If expression* (13) *is reversed, then* $\widetilde{\Psi}$ *is called* h-*concave F-I-V-F on* $[u, v]$. *The set of all* h-*convex* (h-concave) *F-I-V-F is denoted by*

$$FSX([u, v], \mathbb{F}_0, h)\ \big(FSV([u, v], \mathbb{F}_0, h)\big).$$

Definition 7. [33] *The F-I-V-F* $\widetilde{\Psi}: [u, v] \to \mathbb{F}_0$ *is called harmonically convex F-I-V-F on* $[u, v]$ *if*

$$\widetilde{\Psi}\left(\frac{wy}{\xi w + (1-\xi)y}\right) \preccurlyeq (1-\xi)\widetilde{\Psi}(w)\tilde{+}\xi\widetilde{\Psi}(y), \tag{14}$$

for all $w, y \in [u, v], \xi \in [0, 1]$, where $\widetilde{\Psi}(w) \succcurlyeq \tilde{0}$, for all $w \in [u, v]$. *If expression* (14) *is reversed, then* $\widetilde{\Psi}$ *is called harmonically concave F-I-V-F on* $[u, v]$.

Definition 8. *The F-I-V-F* $\widetilde{\Psi}: [u, v] \to \mathbb{F}_0$ *is called harmonically* h-*convex F-I-V-F on* $[u, v]$ *if*

$$\widetilde{\Psi}\left(\frac{wy}{\xi w+(1-\xi)y}\right) \preccurlyeq h(1-\xi)\widetilde{\Psi}(w)\widetilde{+}h(\xi)\widetilde{\Psi}(y), \tag{15}$$

for all $w, y \in [u, v], \xi \in [0,1]$, *where* $\widetilde{\Psi}(w) \succcurlyeq \tilde{0}$, *for all* $w \in [u, v]$ *and* $h: [0,1] \subseteq [u, v] \to \mathbb{R}^+$ *such that* $h \not\equiv 0$. *If expression (15) is reversed, then* $\widetilde{\Psi}$ *is called harmonically* h*-concave F-I-V-F on* $[u, v]$. *The set of all harmonically* h*-convex (harmonically* h*-concave) F-I-V-F is denoted by*

$$HFSX([u,v], \mathbb{F}_0, h)\ \big(HFSV([u,v], \mathbb{F}_0, h)\big).$$

Theorem 3. *Let* $[u, v]$ *be harmonically convex set, and let* $\widetilde{\Psi}: [u, v] \to \mathbb{F}_C(\mathbb{R})$ *be a F-I-V-F, whose* θ*-levels define the family of I-V-Fs* $\Psi_\theta: [u, v] \subset \mathbb{R} \to \mathcal{K}_C^+ \subset \mathcal{K}_C$ *are given by*

$$\Psi_\theta(w) = [\Psi_*(w,\theta), \Psi^*(w,\theta)], \forall\, w \in [u,v]. \tag{16}$$

for all $w \in [u, v]$, $\theta \in [0,1]$. *Then,* $\widetilde{\Psi} \in HFSX([u,v], \mathbb{F}_0, h)$, *if and only if, for all* $\in [0,1]$, $\Psi_*(w,\theta)$, $\Psi^*(w,\theta) \in HSX([u,v], \mathbb{R}^+, h)$.

Proof. The proof is similar to the proof of Theorem 2.12, see [29]. □

Example 1. *Let us consider the F-I-V-Fs* $\widetilde{\Psi}: [0,2] \to \mathbb{F}_C(\mathbb{R})$ *defined by,*

$$\widetilde{\Psi}(w)(\partial) = \begin{cases} \dfrac{\partial}{\sqrt{w}}\ \partial \in [0, \sqrt{w}] \\ \dfrac{2-\partial}{2\sqrt{w}}\ \partial \in (\sqrt{w}, 2\sqrt{w}] \\ \quad 0 \text{ otherwis.} \end{cases}$$

Then, for each $\theta \in [0,1]$, we have $\Psi_\theta(w) = \left[\theta\sqrt{w}, (2-\theta)\sqrt{w}\right]$. Since $\Psi_*(w,\theta)$, $\Psi^*(w,\theta) \in HSX([u,v], \mathbb{R}^+, h)$, with $h(\xi) = \xi$, for each $\theta \in [0,1]$, then $\widetilde{\Psi} \in HFSX([u,v], \mathbb{F}_0, h)$.

Remark 2. *If* $h(\xi) = \xi$, *then Definition 8 reduces to the Definition 7.*

If $\Psi_*(w,\alpha) = \Psi^*(w,\alpha)$ with $\alpha = 1$, then the harmonically h-convex F-I-V-F reduces to the classical harmonically h-convex function, see [47].

If $\Psi_*(w,\alpha) = \Psi^*(w,\alpha)$ with $\alpha = 1$ and $h(\xi) = \xi^s$ with $s \in (0,1)$, then the harmonically h-convex F-I-V-F reduces to the classical harmonically s-convex function, see [47].

If $\Psi_*(w,\alpha) = \Psi^*(w,\alpha)$ with $\alpha = 1$ and $h(\xi) = \xi$, then the harmonically h-convex F-I-V-F reduces to the classical harmonically convex function, see [46].

If $\Psi_*(w,\alpha) = \Psi^*(w,\alpha)$ with $\alpha = 1$ and $h(\xi) = 1$, then the harmonically h-convex F-I-V-F reduces to the classical harmonically P-function, see [47].

3. Fuzzy-Interval Hermite-Hadamard Inequalities

In this section, we prove two types of inequalities. First one is $H{\cdot}H$ and their variant forms, and the second one is $H{\cdot}H$ Fejér inequalities for convex F-I-V-Fs where the integrands are F-I-V-Fs.

Theorem 4. *Let* $\widetilde{\Psi} \in HFSX([u,v], \mathbb{F}_0, h)$, *whose* θ*-levels define the family of I-V-Fs* $\Psi_\theta: [u,v] \subset \mathbb{R} \to \mathcal{K}_C^+$ *that are given by* $\Psi_\theta(w) = [\Psi_*(w,\theta), \Psi^*(w,\theta)]$ *for all* $w \in [u,v]$, $\theta \in [0,1]$. *If* $h\left(\frac{1}{2}\right) \neq 0$ *and* $\widetilde{\Psi} \in \mathcal{FR}_{([u,v],\theta)}$, *so that*

$$\frac{1}{2\,h\left(\frac{1}{2}\right)}\widetilde{\Psi}\left(\frac{2uv}{u+v}\right) \preccurlyeq \frac{uv}{v-u}\int_u^v \frac{\widetilde{\Psi}(w)}{w^2}\,dw \preccurlyeq \left[\widetilde{\Psi}(u)\,\widetilde{+}\,\widetilde{\Psi}(v)\right]\int_0^1 h(\xi)\,d\xi. \tag{17}$$

If $\widetilde{\Psi} \in HFSV([u,v],\mathbb{F}_0,h)$, then

$$\frac{1}{2\,h\left(\frac{1}{2}\right)}\widetilde{\Psi}\left(\frac{2uv}{u+v}\right) \succcurlyeq \frac{uv}{v-u}\int_u^v \frac{\widetilde{\Psi}(w)}{w^2}\,dw \succcurlyeq \left[\widetilde{\Psi}(u)\,\widetilde{+}\,\widetilde{\Psi}(v)\right]\int_0^1 h(\xi)\,d\xi. \tag{18}$$

Proof. Let $\widetilde{\Psi} \in HFSX([u,v],\mathbb{F}_0,h)$. Then, by hypothesis, we can write

$$\frac{1}{h\left(\frac{1}{2}\right)}\widetilde{\Psi}\left(\frac{2uv}{u+v}\right) \preccurlyeq \widetilde{\Psi}\left(\frac{uv}{\xi u+(1-\xi)v}\right)\widetilde{+}\widetilde{\Psi}\left(\frac{uv}{(1-\xi)u+\xi v}\right).$$

Therefore, for each $\theta \in [0,1]$, we have

$$\frac{1}{h\left(\frac{1}{2}\right)}\Psi_*\left(\frac{2uv}{u+v},\theta\right) \le \Psi_*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right) + \Psi_*\left(\frac{uv}{(1-\xi)u+\xi v},\theta\right),$$
$$\frac{1}{h\left(\frac{1}{2}\right)}\Psi^*\left(\frac{2uv}{u+v},\theta\right) \le \Psi^*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right) + \Psi^*\left(\frac{uv}{(1-\xi)u+\xi v},\theta\right).$$

Then

$$\frac{1}{h\left(\frac{1}{2}\right)}\int_0^1 \Psi_*\left(\frac{2uv}{u+v},\theta\right)d\xi \le \int_0^1 \Psi_*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right)d\xi + \int_0^1 \Psi_*\left(\frac{uv}{(1-\xi)u+\xi v},\theta\right)d\xi$$
$$\frac{1}{h\left(\frac{1}{2}\right)}\int_0^1 \Psi^*\left(\frac{2uv}{u+v},\theta\right)d\xi \le \int_0^1 \Psi^*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right)d\xi + \int_0^1 \Psi^*\left(\frac{uv}{(1-\xi)u+\xi v},\theta\right)d\xi$$

It follows that

$$\frac{1}{2\,h\left(\frac{1}{2}\right)}\Psi_*\left(\frac{2uv}{u+v},\theta\right) \le \frac{uv}{v-u}\int_u^v \frac{\Psi_*(w,\theta)}{w^2}\,dw,$$
$$\frac{1}{2\,h\left(\frac{1}{2}\right)}\Psi^*\left(\frac{2uv}{u+v},\theta\right) \le \frac{uv}{v-u}\int_u^v \frac{\Psi^*(w,\theta)}{w^2}\,dw.$$

That is

$$\frac{1}{2\,h\left(\frac{1}{2}\right)}\left[\Psi_*\left(\frac{2uv}{u+v},\theta\right),\Psi^*\left(\frac{2uv}{u+v},\theta\right)\right] \le_I \frac{uv}{v-u}\left[\int_u^v \frac{\Psi_*(w,\theta)}{w^2}\,dw,\ \int_u^v \frac{\Psi^*(w,\theta)}{w^2}\,dw\right].$$

Thus,

$$\frac{1}{2\,h\left(\frac{1}{2}\right)}\,\widetilde{\Psi}\left(\frac{2uv}{u+v}\right) \preccurlyeq \frac{uv}{v-u}\,(FR)\int_u^v \frac{\widetilde{\Psi}(w)}{w^2}\,dw. \tag{19}$$

In a similar way as above, we have

$$\frac{uv}{v-u}\,(FR)\int_u^v \frac{\widetilde{\Psi}(w)}{w^2}\,dw \preccurlyeq \left[\widetilde{\Psi}(u)\,\widetilde{+}\,\widetilde{\Psi}(v)\right]\int_0^1 h(\xi)\,d\xi. \tag{20}$$

Combining (19) and (20), we have

$$\frac{1}{2\,h\left(\frac{1}{2}\right)}\widetilde{\Psi}\left(\frac{2uv}{u+v}\right) \preccurlyeq \frac{uv}{v-u}\int_u^v \frac{\widetilde{\Psi}(w)}{w^2}\,dw \preccurlyeq \left[\widetilde{\Psi}(u)\,\widetilde{+}\,\widetilde{\Psi}(v)\right]\int_0^1 h(\xi)\,d\xi.$$

Hence, we obtain the required result. □

Example 2. *We consider $h(\xi)=\xi$, for $\xi \in [0,1]$, and the FIVFs $\widetilde{\Psi}: [0,2] \to \mathbb{F}_C(\mathbb{R})$, as in Example 1. Then, for each $\theta \in [0,1]$, we have $\Psi_\theta(w) = \left[\theta\sqrt{w}, (2-\theta)\sqrt{w}\right]$ is a harmonically h-convex FIVFs. Since, $\Psi_*(w,\theta) = \theta\sqrt{w}$, $\Psi^*(w,\theta) = (2-\theta)\sqrt{w}$. We now compute the following:*

$$\frac{1}{2\,h\left(\frac{1}{2}\right)}\;\Psi_*\left(\frac{2uv}{u+v},\theta\right) \leq \frac{uv}{v-u}\int_u^v \frac{\Psi_*(w,\theta)}{w^2}dw$$

$$\leq [\Psi_*(u,\theta) + \Psi_*(v,\theta)]\int_0^1 h(\xi)\,d\xi.$$

$$\frac{1}{2\,h\left(\frac{1}{2}\right)}\;\Psi_*\left(\frac{2uv}{u+v},\theta\right) = \Psi_*(0,\theta) = 0,$$

$$\frac{uv}{v-u}\int_u^v \frac{\Psi_*(w,\theta)}{w^2}dw = \frac{0}{2}\int_0^2 \frac{\theta\sqrt{w}}{w^2}dw = 0,$$

$$[\Psi_*(u,\theta) + \Psi_*(v,\theta)]\int_0^1 h(\xi)\,d\xi = \frac{\theta}{\sqrt{2}},$$

for all $\theta \in [0,1]$. *That means*

$$0 \leq 0 \leq \frac{\theta}{\sqrt{2}}.$$

Similarly, it can be easily shown that

$$\frac{1}{2\,h\left(\frac{1}{2}\right)}\;\Psi^*\left(\frac{2uv}{u+v},\theta\right) \leq \frac{uv}{v-u}\int_u^v \frac{\Psi^*(w,\theta)}{w^2}dw \leq [\Psi^*(u,\theta) + \Psi^*(v,\theta)]\int_0^1 h(\xi)\,d\xi.$$

for all $\theta \in [0,1]$, such that

$$\frac{1}{2\,h\left(\frac{1}{2}\right)}\;\Psi^*\left(\frac{2uv}{u+v},\theta\right) = \Psi_*(0,\theta) = 0,$$

$$\frac{uv}{v-u}\int_u^v \frac{\Psi^*(w,\theta)}{w^2}dw = \frac{0}{2}\int_0^2 \frac{(2-\theta)\sqrt{w}}{w^2}dw = 0,$$

$$[\Psi^*(u,\theta) + \Psi^*(v,\theta)]\int_0^1 h(\xi)\,d\xi = \frac{(2-\theta)}{\sqrt{2}}.$$

From which, we have

$$0 \leq 0 \leq \frac{(2-\theta)}{\sqrt{2}},$$

that is

$$[0,0] \leq_I [0,0] \leq_I \frac{1}{\sqrt{2}}[\theta,(2-\theta)], \text{ for all } \theta \in [0,1].$$

Hence,

$$\frac{1}{2\,h\left(\frac{1}{2}\right)}\widetilde{\Psi}\left(\frac{2uv}{u+v}\right) \preccurlyeq \frac{uv}{v-u}\int_u^v \frac{\widetilde{\Psi}(w)}{w^2}dw \preccurlyeq \left[\widetilde{\Psi}(u) \,\widetilde{+}\, \widetilde{\Psi}(v)\right]\int_0^1 h(\xi)\,d\xi.$$

Remark 3. *If* $h(\xi) = \xi^s$, *where* $s \in (0,1)$, *then Theorem 4 reduces to the result for the harmonically s-convex fuzzy-interval-valued function, see* [28]:

$$2^{s-1}\,\widetilde{\Psi}\left(\frac{2uv}{u+v}\right) \preccurlyeq \frac{uv}{v-u}\;(FR)\int_u^v \frac{\widetilde{\Psi}(w)}{w^2}dw \preccurlyeq \frac{1}{s+1}\left[\widetilde{\Psi}(u) \,\widetilde{+}\, \widetilde{\Psi}(v)\right].$$

If $h(\xi) = \xi$, then Theorem 4 reduces to the result for the harmonically convex fuzzy-interval-valued function, see [28]:

$$\widetilde{\Psi}\left(\frac{2uv}{u+v}\right) \preccurlyeq \frac{uv}{v-u}\;(FR)\int_u^v \frac{\widetilde{\Psi}(w)}{w^2}dw \preccurlyeq \frac{\widetilde{\Psi}(u) \,\widetilde{+}\, \widetilde{\Psi}(v)}{2}.$$

If $h(\xi) \equiv 1$, then Theorem 4 reduces to the result for the harmonically P fuzzy-interval-valued function, see [28]:

$$\frac{1}{2}\widetilde{\Psi}\left(\frac{2uv}{u+v}\right) \preccurlyeq \frac{uv}{v-u}\ (FR)\int_u^v \frac{\widetilde{\Psi}(w)}{w^2}dw \preccurlyeq \widetilde{\Psi}(u)\ \widetilde{+}\ \widetilde{\Psi}(v).$$

If $\Psi_*(w,\theta) = \Psi^*(w,\theta)$ with $\theta = 1$, then Theorem 4 reduces to the result for the classical harmonically h-convex function, see [47]:

$$\frac{1}{2\,h\left(\frac{1}{2}\right)}\ \Psi\left(\frac{2uv}{u+v}\right) \le \frac{uv}{v-u}\ (R)\int_u^v \frac{\Psi(w)}{w^2}dw \le [\Psi(u) +\ \Psi(v)]\int_0^1 h(\xi)\,d\xi.$$

If $\Psi_*(w,\theta) = \Psi^*(w,\theta)$ with $\theta = 1$ and $h(\xi) = \xi^s$, then Theorem 4 reduces to the result for the classical harmonically s-convex function, see [47]:

$$2^{s-1}\,\Psi\left(\frac{2uv}{u+v}\right) \le \frac{uv}{v-u}\ (R)\int_u^v \frac{\Psi(w)}{w^2}dw \le \frac{1}{s+1}[\Psi(u) +\ \Psi(v)].$$

If $\Psi_*(w,\theta) = \Psi^*(w,\theta)$ with $\theta = 1$ and $h(\xi) = \xi$, then Theorem 4 reduces to the result for the classical harmonically convex function, see [46]:

$$\Psi\left(\frac{2uv}{u+v}\right) \le \frac{uv}{v-u}\ (R)\int_u^v \frac{\Psi(w)}{w^2}dw \le \frac{\Psi(u)+\ \Psi(v)}{2}.$$

If $\Psi_*(w,\theta) = \Psi^*(w,\theta)$ with $\theta = 1$ and $h(\xi) \equiv 1$, then Theorem 4 reduces to the result for the classical harmonically P function, see [47]:

$$\frac{1}{2}\Psi\left(\frac{2uv}{u+v}\right) \le \frac{uv}{v-u}\ (R)\int_u^v \frac{\Psi(w)}{w^2}dw \le \Psi(u) +\ \Psi(v).$$

Theorem 5. *Let* $\widetilde{\Psi} \in HFSX([u,v],\mathbb{F}_0,h)$ *with* $h\left(\frac{1}{2}\right) \neq 0$, *whose* θ*-levels define the family of I-V-Fs* $\Psi_\theta\colon [u,v] \subset \mathbb{R} \to \mathcal{K}_C^+$ *that are given by* $\Psi_\theta(w) = [\Psi_*(w,\theta), \Psi^*(w,\theta)]$ *for all* $w \in [u,v]$, $\theta \in [0,1]$. *If* $\widetilde{\Psi} \in \mathcal{FR}_{([u,v],\theta)}$, *so that*

$$\frac{1}{4\left[h\left(\frac{1}{2}\right)\right]^2}\ \widetilde{\Psi}\left(\frac{2uv}{u+v}\right) \preccurlyeq \rhd_2 \preccurlyeq \frac{uv}{v-u}\ (FR)\int_u^v \frac{\widetilde{\Psi}(w)}{w^2}dw \preccurlyeq \rhd_1 \preccurlyeq \left[\widetilde{\Psi}(u)\ \widetilde{+}\ \widetilde{\Psi}(v)\right]\left[\frac{1}{2} + h\left(\frac{1}{2}\right)\right]\int_0^1 h(\xi)\,d\xi, \tag{21}$$

where

$$\rhd_1 = \left[\frac{\widetilde{\Psi}(u)\ \widetilde{+}\ \widetilde{\Psi}(v)}{2}\ \widetilde{+}\ \widetilde{\Psi}\left(\frac{2uv}{u+v}\right)\right]\int_0^1 h(\xi)\,d\xi,$$

$$\rhd_2 = \frac{1}{4\,h\left(\frac{1}{2}\right)}\left[\widetilde{\Psi}\left(\frac{4uv}{u+3v}\right)\ \widetilde{+}\ \widetilde{\Psi}\left(\frac{4uv}{3u+v}\right)\right],$$

and $\rhd_1 = \left[\rhd_{1*}, \rhd_1{}^*\right]$, $\rhd_2 = \left[\rhd_{2*}, \rhd_2{}^*\right]$.

If $\widetilde{\Psi} \in HFSV([u,v],\mathbb{F}_0,h)$, then inequality (21) is reversed.

Proof. Take $\left[u, \frac{2uv}{u+v}\right]$, so that

$$\frac{1}{h\left(\frac{1}{2}\right)}\widetilde{\Psi}\left(\frac{u\frac{4uv}{u+v}}{\xi u+(1-\xi)\frac{2uv}{u+v}} + \frac{u\frac{4uv}{u+v}}{(1-\xi)u+\xi\frac{2uv}{u+v}}\right) \preccurlyeq \widetilde{\Psi}\left(\frac{u\frac{2uv}{u+v}}{\xi u+(1-\xi)\frac{2uv}{u+v}}\right)\widetilde{+}\widetilde{\Psi}\left(\frac{u\frac{2uv}{u+v}}{(1-\xi)u+\xi\frac{2uv}{u+v}}\right).$$

Therefore, for every $\theta \in [0,1]$, yields

$$\frac{1}{h\left(\frac{1}{2}\right)}\Psi_*\left(\frac{u\frac{4uv}{u+v}}{\xi u+(1-\xi)\frac{2uv}{u+v}} + \frac{u\frac{4uv}{u+v}}{(1-\xi)u+\xi\frac{2uv}{u+v}}, \theta\right) \le \Psi_*\left(\frac{u\frac{2uv}{u+v}}{\xi u+(1-\xi)\frac{2uv}{u+v}}, \theta\right) +$$

$$\frac{1}{h\left(\frac{1}{2}\right)}\Psi^*\left(\frac{u\frac{4uv}{u+v}}{\xi u+(1-\xi)\frac{2uv}{u+v}} + \frac{u\frac{4uv}{u+v}}{(1-\xi)u+\xi\frac{2uv}{u+v}}, \theta\right) \le \Psi^*\left(\frac{u\frac{2uv}{u+v}}{\xi u+(1-\xi)\frac{2uv}{u+v}}, \theta\right) +$$

In consequence, we obtain

$$\frac{1}{4\,h\left(\frac{1}{2}\right)}\Psi_{*}\left(\frac{4uv}{u+3v},\theta\right) \le \frac{uv}{v-u}\int_{u}^{\frac{2uv}{u+v}}\frac{\Psi_{*}(w,\theta)}{w^{2}}dw\,,$$

$$\frac{1}{4h\left(\frac{1}{2}\right)}\Psi^{*}\left(\frac{4uv}{u+3v},\theta\right) \le \frac{uv}{v-u}\int_{u}^{\frac{2uv}{u+v}}\frac{\Psi^{*}(w,\theta)}{w^{2}}dw\,.$$

That is

$$\frac{1}{4h\left(\frac{1}{2}\right)}\left[\Psi_{*}\left(\frac{4uv}{u+3v},\theta\right),\Psi^{*}\left(\frac{4uv}{u+3v},\theta\right)\right] \le_{I} \frac{uv}{v-u}\left[\int_{u}^{\frac{2uv}{u+v}}\frac{\Psi_{*}(w,\theta)}{w^{2}}dw\,,\int_{u}^{\frac{2uv}{u+v}}\frac{\Psi^{*}(w,\theta)}{w^{2}}dw\right].$$

It follows that

$$\frac{1}{4h\left(\frac{1}{2}\right)}\widetilde{\Psi}\left(\frac{4uv}{u+3v}\right) \preccurlyeq \frac{uv}{v-u}\int_{u}^{\frac{2uv}{u+v}}\frac{\widetilde{\Psi}(w)}{w^{2}}dw. \tag{22}$$

In a similar way as above, we have

$$\frac{1}{4\,h\left(\frac{1}{2}\right)}\widetilde{\Psi}\left(\frac{4uv}{3u+v}\right) \preccurlyeq \frac{uv}{v-u}\int_{\frac{2uv}{u+v}}^{v}\frac{\widetilde{\Psi}(w)}{w^{2}}dw. \tag{23}$$

Combining (22) and (23), we can write

$$\frac{1}{4h\left(\frac{1}{2}\right)}\left[\widetilde{\Psi}\left(\frac{4uv}{u+3v}\right)\widetilde{+}\widetilde{\Psi}\left(\frac{4uv}{3u+v}\right)\right] \preccurlyeq \frac{uv}{v-u}\int_{u}^{v}\frac{\widetilde{\Psi}(w)}{w^{2}}dw. \tag{24}$$

Therefore, for every $\theta \in [0,1]$, by using Theorem 4, we have

$$\begin{aligned}
\frac{1}{4\left[h\left(\frac{1}{2}\right)\right]^{2}}\Psi_{*}\left(\frac{2uv}{u+v},\theta\right) &\le \frac{1}{4\left[h\left(\frac{1}{2}\right)\right]^{2}}\left[h\left(\frac{1}{2}\right)\Psi_{*}\left(\frac{4uv}{u+3v},\theta\right)+h\left(\frac{1}{2}\right)\Psi_{*}\left(\frac{4uv}{3u+v},\theta\right)\right],\\
\frac{1}{4\left[h\left(\frac{1}{2}\right)\right]^{2}}\Psi^{*}\left(\frac{2uv}{u+v},\theta\right) &\le \frac{1}{4\left[h\left(\frac{1}{2}\right)\right]^{2}}\left[h\left(\frac{1}{2}\right)\Psi^{*}\left(\frac{4uv}{u+3v},\theta\right)+h\left(\frac{1}{2}\right)\Psi^{*}\left(\frac{4uv}{3u+v},\theta\right)\right],\\
&= \rhd_{2*},\\
&= \rhd_{2}^{*},\\
&\le \frac{uv}{v-u}\int_{u}^{v}\frac{\Psi_{*}(w,\theta)}{w^{2}}dw\,,\\
&\le \frac{uv}{v-u}\int_{u}^{v}\frac{\Psi^{*}(w,\theta)}{w^{2}}dw\,,\\
&\le \left[\frac{\Psi_{*}(u,\theta)+\Psi_{*}(v,\theta)}{2}+\Psi_{*}\left(\frac{2uv}{u+v},\theta\right)\right]\int_{0}^{1}h(\xi)\,d\xi\,,\\
&\le \left[\frac{\Psi^{*}(u,\theta)+\Psi^{*}(v,\theta)}{2}+\Psi^{*}\left(\frac{2uv}{u+v},\theta\right)\right]\int_{0}^{1}h(\xi)\,d\xi\,,\\
&= \rhd_{1*},\\
&= \rhd_{1}^{*},\\
&\le \left[\frac{\Psi_{*}(u,\theta)+\Psi_{*}(v,\theta)}{2}+h\left(\frac{1}{2}\right)\left(\Psi_{*}(u,\theta)+\Psi_{*}(v,\theta)\right)\right]\int_{0}^{1}h(\xi)\,d\xi\,,\\
&\le \left[\frac{\Psi^{*}(u,\theta)+\Psi^{*}(v,\theta)}{2}+h\left(\frac{1}{2}\right)\left(\Psi^{*}(u,\theta)+\Psi^{*}(v,\theta)\right)\right]\int_{0}^{1}h(\xi)\,d\xi\,,\\
&= \left[\Psi_{*}(u,\theta)+\Psi_{*}(v,\theta)\right]\left[\frac{1}{2}+h\left(\frac{1}{2}\right)\right]\int_{0}^{1}h(\xi)\,d\xi\,,\\
&= \left[\Psi^{*}(u,\theta)+\Psi^{*}(v,\theta)\right]\left[\frac{1}{2}+h\left(\frac{1}{2}\right)\right]\int_{0}^{1}h(\xi)\,d\xi\,,
\end{aligned}$$

that is

$$\frac{1}{4\left[h\left(\frac{1}{2}\right)\right]^{2}}\widetilde{\Psi}\left(\frac{2uv}{u+v}\right) \preccurlyeq \rhd_{2} \preccurlyeq \frac{uv}{v-u}\,(FR)\int_{u}^{v}\frac{\widetilde{\Psi}(w)}{w^{2}}dw \preccurlyeq \rhd_{1} \preccurlyeq \left[\widetilde{\Psi}(u)\,\widetilde{+}\,\widetilde{\Psi}(v)\right]\left[\frac{1}{2}+h\left(\frac{1}{2}\right)\right]\int_{0}^{1}h(\xi)\,d\xi.$$

□

Theorem 6. *Let* $\widetilde{\Psi} \in HFSX([u,\upsilon],\mathbb{F}_0,h_1)$ *and* $\widetilde{\mathcal{P}} \in HFSX([u,\upsilon],\mathbb{F}_0,h_2)$ *, whose* θ *-levels* $\Psi_\theta,\mathcal{P}_\theta : [u,\upsilon] \subset \mathbb{R} \to \mathcal{K}_C^+$ *are defined by* $\Psi_\theta(w) = [\Psi_*(w,\theta),\Psi^*(w,\theta)]$ *and* $\mathcal{P}_\theta(w) = [\mathcal{P}_*(w,\theta),\mathcal{P}^*(w,\theta)]$ *for all* $w \in [u,\upsilon]$*,* $\theta \in [0,1]$*, respectively. If* $\widetilde{\Psi} \,\widetilde{\times}\, \widetilde{\mathcal{P}} \in \mathcal{FR}_{([u,\upsilon],\theta)}$*, then*

$$\frac{u\upsilon}{\upsilon-u}\,(FR)\int_u^\upsilon \frac{\widetilde{\Psi}(w)\,\widetilde{\times}\,\widetilde{\mathcal{P}}(w)}{w^2}dw \\ \preccurlyeq \widetilde{\mathcal{M}}(u,\upsilon)\int_0^1 h_1(\xi)\,h_2(\xi)\,d\xi \,\widetilde{+}\, \widetilde{\mathcal{N}}(u,\upsilon)\int_0^1 h_1(\xi)\,h_2(1-\xi)\,d\xi,$$

where $\widetilde{\mathcal{M}}(u,\upsilon) = \widetilde{\Psi}(u)\,\widetilde{\times}\,\widetilde{\mathcal{P}}(u) \,\widetilde{+}\, \widetilde{\Psi}(\upsilon)\,\widetilde{\times}\,\widetilde{\mathcal{P}}(\upsilon)$, $\widetilde{\mathcal{N}}(u,\upsilon) = \widetilde{\Psi}(u)\,\widetilde{\times}\,\widetilde{\mathcal{P}}(\upsilon) \,\widetilde{+}\, \widetilde{\Psi}(\upsilon)\,\widetilde{\times}\,\widetilde{\mathcal{P}}(u)$, $\mathcal{M}_\theta(u,\upsilon) = [\mathcal{M}_*((u,\upsilon),\theta),\mathcal{M}^*((u,\upsilon),\theta)]$ *and* $\mathcal{N}_\theta(u,\upsilon) = [\mathcal{N}_*((u,\upsilon),\theta),\mathcal{N}^*((u,\upsilon),\theta)]$.

Proof. Since $\widetilde{\Psi}$ and $\widetilde{\mathcal{P}}$ are harmonically h_1 and h_2-convex F-I-V-Fs then, for each $\theta \in [0,1]$ we have

$$\Psi_*\left(\frac{u\upsilon}{(1-\xi)u+\xi\upsilon},\theta\right) \le h_1(\xi)\Psi_*(u,\theta) + h_1(1-\xi)\Psi_*(\upsilon,\theta),$$
$$\Psi^*\left(\frac{u\upsilon}{(1-\xi)u+\xi\upsilon},\theta\right) \le h_1(\xi)\Psi^*(u,\theta) + h_1(1-\xi)\Psi^*(\upsilon,\theta),$$

and

$$\mathcal{P}_*\left(\frac{u\upsilon}{(1-\xi)u+\xi\upsilon},\theta\right) \le h_2(\xi)\,\mathcal{P}_*(u,\theta) + h_2(1-\xi)\,\mathcal{P}_*(\upsilon,\theta),$$
$$\mathcal{P}^*\left(\frac{u\upsilon}{(1-\xi)u+\xi\upsilon},\theta\right) \le h_2(\xi)\,\mathcal{P}^*(u,\theta) + h_2(1-\xi)\,\mathcal{P}^*(\upsilon,\theta).$$

From the definition of harmonically h-convexity of F-I-V-Fs it follows that $\widetilde{\Psi}(w) \succcurlyeq \tilde{0}$ and $\widetilde{\mathcal{P}}(w) \succcurlyeq \tilde{0}$, so that

$$\Psi_*\left(\frac{u\upsilon}{(1-\xi)u+\xi\upsilon},\theta\right) \times \mathcal{P}_*\left(\frac{u\upsilon}{(1-\xi)u+\xi\upsilon},\theta\right) \\ \le \left(h_1(\xi)\Psi_*(u,\theta) + h_1(1-\xi)\Psi_*(\upsilon,\theta)\right)\left(h_2(\xi)\,\mathcal{P}_*(u,\theta) + h_2(1-\xi)\,\mathcal{P}_*(\upsilon,\theta)\right) \\ = \Psi_*(u,\theta)\times\mathcal{P}_*(u,\theta)[h_1(\xi)\,h_2(\xi)] + \Psi_*(\upsilon,\theta)\times\mathcal{P}_*(\upsilon,\theta)[h_1(1-\xi)\,h_2(1-\xi) \\ +\Psi_*(u,\theta)\mathcal{P}_*(\upsilon,\theta)\,h_1(\xi)\,h_2(\xi) + \Psi_*(\upsilon,\theta)\times\mathcal{P}_*(u,\theta)\,h_1(1-\xi)\,h_2(\xi)\,,$$

$$\Psi^*\left(\frac{u\upsilon}{(1-\xi)u+\xi\upsilon},\theta\right) \times \mathcal{P}^*\left(\frac{u\upsilon}{(1-\xi)u+\xi\upsilon},\theta\right) \\ \le \left(h_1(\xi)\Psi^*(u,\theta) + h_1(1-\xi)\Psi^*(\upsilon,\theta)\right)\left(h_2(\xi)\,\mathcal{P}^*(u,\theta) + h_2(1-\xi)\,\mathcal{P}^*(\upsilon,\theta)\right) \\ = \Psi^*(u,\theta)\times\mathcal{P}^*(u,\theta)[h_1(\xi)\,h_2(\xi)] + \Psi^*(\upsilon,\theta)\times\mathcal{P}^*(\upsilon,\theta)[h_1(1-\xi)\,h_2(1- \\ +\Psi^*(u,\theta)\times\mathcal{P}^*(\upsilon,\theta)\,h_1(\xi)\,h_2(1-\xi) + \Psi^*(\upsilon,\theta)\times\mathcal{P}^*(u,\theta)\,h_1(1-\xi)\,h_2(\xi)$$

Integrating both sides of above inequality over [0, 1] results

$$\int_0^1 \Psi_*\left(\frac{u\upsilon}{(1-\xi)u+\xi\upsilon},\theta\right) \times \mathcal{P}_*\left(\frac{u\upsilon}{(1-\xi)u+\xi\upsilon},\theta\right) = \frac{u\upsilon}{\upsilon-u}\int_u^\upsilon \frac{\Psi_*(w,\theta)\times\mathcal{P}_*(w,\theta)}{w^2}dw \\ \le \left(\Psi_*(u,\theta)\times\mathcal{P}_*(u,\theta) + \Psi_*(\upsilon,\theta)\times\mathcal{P}_*(\upsilon,\theta)\right)\int_0^1 h_1(\xi)\,h_2(\xi)\,d\xi \\ +\left(\Psi_*(u,\theta)\times\mathcal{P}_*(\upsilon,\theta) + \Psi_*(\upsilon,\theta)\times\mathcal{P}_*(u,\theta)\right)\int_0^1 h_1(\xi)\,h_2(1-\xi)\,d\xi$$

$$\int_0^1 \Psi^*\left(\frac{u\upsilon}{(1-\xi)u+\xi\upsilon},\theta\right) \times \mathcal{P}^*\left(\frac{u\upsilon}{(1-\xi)u+\xi\upsilon},\theta\right) = \frac{u\upsilon}{\upsilon-u}\int_u^\upsilon \frac{\Psi^*(w,\theta)\times\mathcal{P}^*(w,\theta)}{w^2}dw \\ \le \left(\Psi^*(u,\theta)\times\mathcal{P}^*(u,\theta) + \Psi^*(\upsilon,\theta)\times\mathcal{P}^*(\upsilon,\theta)\right)\int_0^1 h_1(\xi)\,h_2(\xi)\,d\xi \\ +\left(\Psi^*(u,\theta)\times\mathcal{P}^*(\upsilon,\theta) + \Psi^*(\upsilon,\theta)\times\mathcal{P}^*(u,\theta)\right)\int_0^1 h_1(\xi)\,h_2(1-\xi)\,d$$

It follows that,

$$\frac{uv}{v-u}\int_u^v \Psi_*(w,\theta)\times \mathcal{P}_*(w,\theta)\,dw \le \mathcal{M}_*\big((u,v),\theta\big)\int_0^1 h_1(\xi)\,h_2(\xi)\,d\xi$$
$$+\mathcal{N}_*\big((u,v),\theta\big)\int_0^1 h_1(\xi)\,h_2(1-\xi)\,d\xi,$$
$$\frac{uv}{v-u}\int_u^v \Psi^*(w,\theta)\times \mathcal{P}^*(w,\theta)\,dw \le \mathcal{M}^*\big((u,v),\theta\big)\int_0^1 h_1(\xi)\,h_2(\xi)\,d\xi$$
$$+\mathcal{N}^*\big((u,v),\theta\big)\int_0^1 h_1(\xi)\,h_2(1-\xi)\,d\xi,$$

that is

$$\frac{uv}{v-u}\left[\int_u^v \Psi_*(w,\theta)\times \mathcal{P}_*(w,\theta)dw\,,\int_u^v \Psi^*(w,\theta)\times \mathcal{P}^*(w,\theta)\,dw\right]$$
$$\le_I \left[\mathcal{M}_*\big((u,v),\theta\big),\mathcal{M}^*\big((u,v),\theta\big)\right]\int_0^1 h_1(\xi)\,h_2(\xi)\,d\xi$$
$$+\left[\mathcal{N}_*\big((u,v),\theta\big),\mathcal{N}^*\big((u,v),\theta\big)\right]\int_0^1 h_1(\xi)\,h_2(1-\xi)\,d\xi.$$

Thus,

$$\frac{uv}{v-u}\,(FR)\int_u^v \frac{\widetilde{\Psi}(w)\widetilde{\times}\widetilde{\mathcal{P}}(w)}{w^2}dw \preccurlyeq \widetilde{\mathcal{M}}(u,v)\int_0^1 h_1(\xi)\,h_2(\xi)\,d\xi\,\widetilde{+}\,\widetilde{\mathcal{N}}(u,v)\int_0^1 h_1(\xi)\,h_2(1-\xi)\,d\xi.$$

□

Theorem 7. *Let* $\widetilde{\Psi}\in HFSX([u,v],\mathbb{F}_0,h_1)$, $\widetilde{\mathcal{P}}\in HFSX([u,v],\mathbb{F}_0,h_2)$, *whose* θ *-levels* $\Psi_\theta,\mathcal{P}_\theta:[u,v]\subset\mathbb{R}\to\mathcal{K}_C^+$ *that are defined by* $\Psi_\theta(w)=[\Psi_*(w,\theta),\Psi^*(w,\theta)]$ *and* $\mathcal{P}_\theta(w)=[\mathcal{P}_*(w,\theta),\mathcal{P}^*(w,\theta)]$ *for all* $w\in[u,v]$, $\theta\in[0,1]$, *respectively. If* $h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\neq 0$ *and* $\widetilde{\Psi}\,\widetilde{\times}\,\widetilde{\mathcal{P}}\in\mathcal{FR}_{([u,v],\theta)}$, *so that*

$$\frac{1}{2\,h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\,\widetilde{\Psi}\left(\frac{2uv}{u+v}\right)\widetilde{\times}\,\widetilde{\mathcal{P}}\left(\frac{2uv}{u+v}\right)$$
$$\preccurlyeq \frac{uv}{v-u}\,(FR)\int_u^v \frac{\widetilde{\Psi}(w)\widetilde{\times}\widetilde{\mathcal{P}}(w)}{w^2}dw+\widetilde{\mathcal{M}}(u,v)\int_0^1 h_1(\xi)\,h_2(1-\xi)\,d\xi\,\widetilde{+}\,\widetilde{\mathcal{N}}(u,v)\int_0^1 h_1(\xi)\,h_2(\xi)\,d\xi,$$

where $\widetilde{\mathcal{M}}(u,v)=\widetilde{\Psi}(u)\,\widetilde{\times}\,\widetilde{\mathcal{P}}(u)\,\widetilde{+}\,\widetilde{\Psi}(v)\,\widetilde{\times}\,\widetilde{\mathcal{P}}(v)$, $\widetilde{\mathcal{N}}(u,v)=\widetilde{\Psi}(u)\,\widetilde{\times}\,\widetilde{\mathcal{P}}(v)\,\widetilde{+}\,\widetilde{\Psi}(v)\,\widetilde{\times}\,\widetilde{\mathcal{P}}(u)$, $\mathcal{M}_\theta(u,v)=\left[\mathcal{M}_*\big((u,v),\theta\big),\mathcal{M}^*\big((u,v),\theta\big)\right]$ *and* $\mathcal{N}_\theta(u,v)=\left[\mathcal{N}_*\big((u,v),\theta\big),\mathcal{N}^*\big((u,v),\theta\big)\right]$.

Proof. By hypothesis, for each $\theta\in[0,1]$, we have

$$\Psi_*\left(\frac{2uv}{u+v},\theta\right)\times\mathcal{J}_*\left(\frac{2uv}{u+v},\theta\right)$$
$$\Psi^*\left(\frac{2uv}{u+v},\theta\right)\times\mathcal{J}^*\left(\frac{2uv}{u+v},\theta\right)$$

$$\leq h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\begin{bmatrix}\Psi_*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right)\times\mathcal{J}_*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right)\\+\Psi_*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right)\times\mathcal{J}_*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right)\end{bmatrix}$$

$$+\ h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\begin{bmatrix}\Psi_*\left(\frac{uv}{(1-\xi)u+\xi v},\theta\right)\times\mathcal{J}_*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right)\\+\Psi_*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right)\times\mathcal{J}_*\left(\frac{uv}{(1-\xi)u+\xi v},\theta\right)\end{bmatrix},$$

$$\leq h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\begin{bmatrix}\Psi^*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right)\times\mathcal{J}^*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right)\\+\Psi^*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right)\times\mathcal{J}^*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right)\end{bmatrix}$$

$$+\ h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\begin{bmatrix}\Psi^*\left(\frac{uv}{(1-\xi)u+\xi v},\theta\right)\times\mathcal{J}^*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right)\\+\Psi^*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right)\times\mathcal{J}^*\left(\frac{uv}{(1-\xi)u+\xi v},\theta\right)\end{bmatrix},$$

$$\leq h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\begin{bmatrix}\Psi_*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right)\times\mathcal{J}_*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right)\\+\Psi_*\left(\frac{uv}{(1-\xi)u+\xi v},\theta\right)\times\mathcal{J}_*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right)\end{bmatrix}$$

$$+\ h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\begin{bmatrix}\left(h_1(\xi)\,\Psi_*(u,\theta)+h_1(1-\xi)\,\Psi_*(v,\theta)\right)\\\times\left(h_2(1-\xi)\,\mathcal{J}_*(u,\theta)+h_2(\xi)\,\mathcal{J}_*(v,\theta)\right)\\+\left(h_1(1-\xi)\,\Psi_*(u,\theta)+h_1(\xi)\,\Psi_*(v,\theta)\right)\\\times\left(h_2(\xi)\,\mathcal{J}_*(u,\theta)+h_2(1-\xi)\,\mathcal{J}_*(v,\theta)\right)\end{bmatrix},$$

$$\leq h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\begin{bmatrix}\Psi^*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right)\times\mathcal{J}^*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right)\\+\Psi^*\left(\frac{uv}{\xi u+(1-\xi)v}\ \theta\right)\times\mathcal{J}^*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right)\end{bmatrix}$$

$$+\ h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\begin{bmatrix}\left(h_1(\xi)\,\Psi^*(u,\theta)+h_1(1-\xi)\,\Psi^*(v,\theta)\right)\\\times\left(h_2(1-\xi)\,\mathcal{J}^*(u,\theta)+h_2(\xi)\,\mathcal{J}^*(v,\theta)\right)\\+\left(h_1(1-\xi)\,\Psi^*(u,\theta)+h_1(\xi)\,\Psi^*(v,\theta)\right)\\\times\left(h_2(\xi)\,\mathcal{J}^*(u,\theta)+h_2(1-\xi)\,\mathcal{J}^*(v,\theta)\right)\end{bmatrix},$$

$$=h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\begin{bmatrix}\Psi_*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right)\times\mathcal{J}_*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right)\\+\Psi_*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right)\times\mathcal{J}_*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right)\end{bmatrix}$$

$$+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\begin{bmatrix}\{h_1(\xi)\,h_2(\xi)+h_1(1-\xi)\,h_2(1-\xi)\}\mathcal{N}_*\left((u,v),\theta\right)\\+\{h_1(\xi)\,h_2(1-\xi)+h_1(1-\xi)\,h_2(\xi)\}\mathcal{M}_*\left((u,v),\theta\right)\end{bmatrix},$$

$$=h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\begin{bmatrix}\Psi^*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right)\times\mathcal{J}^*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right)\\+\Psi^*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right)\times\mathcal{J}^*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right)\end{bmatrix}$$

$$+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\begin{bmatrix}\{h_1(\xi)\,h_2(\xi)+h_1(1-\xi)\,h_2(1-\xi)\}\mathcal{N}^*\left((u,v),\theta\right)\\+\{h_1(\xi)\,h_2(1-\xi)+h_1(1-\xi)\,h_2(\xi)\}\mathcal{M}^*\left((u,v),\theta\right)\end{bmatrix},$$

Integrating over $[0,1]$, gives

$$\frac{1}{2\,h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\ \Psi_*\left(\frac{2uv}{u+v},\theta\right)\times\mathcal{J}_*\left(\frac{2uv}{u+v},\theta\right)\le\frac{1}{v-u}\ (R)\int_u^v\Psi_*(w,\theta)\times\mathcal{J}_*(w,\theta)d$$
$$+\mathcal{M}_*\big((u,v),\theta\big)\int_0^1 h_1(\xi)\,h_2(1-\xi)\,d\xi$$
$$+\mathcal{N}_*\big((u,v),\theta\big)\int_0^1 h_1(\xi)\,h_2(\xi)\,d\xi,$$
$$\frac{1}{2\,h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\ \Psi^*\left(\frac{2uv}{u+v},\theta\right)\times\mathcal{J}^*\left(\frac{2uv}{u+v},\theta\right)\le\frac{1}{v-u}\ (R)\int_u^v\Psi^*(w,\theta)\times\mathcal{J}^*(w,\theta)$$
$$+\mathcal{M}^*\big((u,v),\theta\big)\int_0^1 h_1(\xi)\,h_2(1-\xi)\,d\xi$$
$$+\mathcal{N}^*\big((u,v),\theta\big)\int_0^1 h_1(\xi)\,h_2(\xi)\,d\xi\,,$$

that is

$$\frac{1}{2\,h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\ \widetilde{\Psi}\left(\frac{2uv}{u+v}\right)\widetilde{\times}\,\widetilde{\mathcal{P}}\left(\frac{2uv}{u+v}\right)$$

$$\preccurlyeq\frac{uv}{v-u}\ (FR)\int_u^v\frac{\widetilde{\Psi}(w)\widetilde{\times}\widetilde{\mathcal{P}}(w)}{w^2}dw+\widetilde{\mathcal{M}}(u,v)\int_0^1 h_1(\xi)\,h_2(1-$$
$$\xi)\,d\xi\,\widetilde{+}\,\widetilde{\mathcal{N}}(u,v)\int_0^1 h_1(\xi)\,h_2(\xi)\,d\xi.$$

The theorem has been proved. □

We now discuss an inequality related with the right part of the classical $H\cdot H$ Fejér inequality for harmonically h-convex F-I-V-Fs through a fuzzy order relation, called second fuzzy $H\cdot H$ Fejér inequality.

Theorem 8. *(Second fuzzy $H\cdot H$ Fejér inequality) Let $\widetilde{\Psi}\in HFSX([u,v],\mathbb{F}_0,h)$, whose θ-levels define the family of I-V-Fs $\Psi_\theta\colon[u,v]\subset\mathbb{R}\to\mathcal{K}_C^+$ that are given by $\Psi_\theta(w)=[\Psi_*(w,\theta),\Psi^*(w,\theta)]$ for all $w\in[u,v]$, $\theta\in[0,1]$. If $\widetilde{\Psi}\in\mathcal{FR}_{([u,v],\theta)}$ and $\nabla\colon[u,v]\to\mathbb{R},\nabla\left(\frac{1}{\frac{1}{u}+\frac{1}{v}-\frac{1}{w}}\right)=\nabla(w)\ge0$, then*

$$\frac{uv}{v-u}\ (FR)\int_u^v\frac{\widetilde{\Psi}(w)}{w^2}\nabla(w)dw\preccurlyeq\left[\widetilde{\Psi}(u)\,\widetilde{+}\,\widetilde{\Psi}(v)\right]\int_0^1 h(\xi)\,\nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right)d\xi. \tag{25}$$

If $\widetilde{\Psi}\in HFSV([u,v],\mathbb{F}_0,h)$, then inequality (25) is reversed such that

$$\frac{uv}{v-u}\ (FR)\int_u^v\frac{\widetilde{\Psi}(w)}{w^2}\nabla(w)dw\succcurlyeq\left[\widetilde{\Psi}(u)\,\widetilde{+}\,\widetilde{\Psi}(v)\right]\int_0^1 h(\xi)\,\nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right)d\xi.$$

Proof. Let Ψ be a h-convex F-I-V-F. Then, for each $\theta\in[0,1]$, we have

$$\begin{aligned}&\Psi_*\left(\frac{uv}{(1-\xi)u+\xi v},\theta\right)\nabla\left(\frac{uv}{(1-\xi)u+\xi v}\right)\\&\qquad\le\big(h(\xi)\,\Psi_*(u,\theta)+h(1-\xi)\,\Psi_*(v,\theta)\big)\nabla\left(\frac{uv}{(1-\xi)u+\xi v}\right),\\&\Psi^*\left(\frac{uv}{(1-\xi)u+\xi v},\theta\right)\nabla\left(\frac{uv}{(1-\xi)u+\xi v}\right)\\&\qquad\le\big(h(\xi)\,\Psi^*(u,\theta)+h(1-\xi)\,\Psi^*(v,\theta)\big)\nabla\left(\frac{uv}{(1-\xi)u+\xi v}\right).\end{aligned}\tag{26}$$

Similarly, we can write

$$\begin{aligned}
&\Psi_*\left(\frac{uv}{\xi u+(1-\xi)v},\Theta\right)\nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right)\\
&\qquad\le\left(h(1-\xi)\,\Psi_*(u,\Theta)+h(\xi)\,\Psi_*(\nu,\Theta)\right)\nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right),\\
&\Psi^*\left(\frac{uv}{\xi u+(1-\xi)v},\Theta\right)\nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right)\\
&\qquad\le\left(h(1-\xi)\,\Psi^*(u,\Theta)+h(\xi)\,\Psi^*(\nu,\Theta)\right)\nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right).
\end{aligned}\tag{27}$$

After adding (26) and (27), and integrating over $[0,1]$, we obtain

$$\begin{aligned}
&\int_0^1\Psi_*\left(\frac{uv}{(1-\xi)u+\xi v},\Theta\right)\nabla\left(\frac{uv}{(1-\xi)u+\xi v}\right)d\xi\\
&+\int_0^1\Psi_*\left(\frac{uv}{\xi u+(1-\xi)v},\Theta\right)\nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right)d\xi\\
&\le\int_0^1\left[\begin{array}{l}\Psi_*(u,\Theta)\left\{\begin{array}{c}h(\xi)\nabla\left(\frac{uv}{(1-\xi)u+\xi v}\right)\\+h(1-\xi)\,\nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right)\end{array}\right\}\\+\Psi_*(\nu,\Theta)\left\{\begin{array}{c}h(1-\xi)\nabla\left(\frac{uv}{(1-\xi)u+\xi v}\right)\\+h(\xi)\,\nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right)\end{array}\right\}\end{array}\right]d\xi,\\
&\int_0^1\Psi^*\left(\frac{uv}{(1-\xi)u+\xi v},\Theta\right)\nabla\left(\frac{uv}{(1-\xi)u+\xi v}\right)d\xi\\
&+\int_0^1\Psi^*\left(\frac{uv}{\xi u+(1-\xi)v},\Theta\right)\nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right)d\xi\\
&\le\int_0^1\left[\begin{array}{l}\Psi^*(u,\Theta)\left\{\begin{array}{c}h(\xi)\nabla\left(\frac{uv}{(1-\xi)u+\xi v}\right)\\+h(1-\xi)\,\nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right)\end{array}\right\}\\+\Psi^*(\nu,\Theta)\left\{\begin{array}{c}h(1-\xi)\nabla\left(\frac{uv}{(1-\xi)u+\xi v}\right)\\+h(\xi)\,\nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right)\end{array}\right\}\end{array}\right]d\xi,
\end{aligned}$$

$$\begin{aligned}
&=2\Psi_*(u,\Theta)\int_0^1h(\xi)\nabla\left(\frac{uv}{(1-\xi)u+\xi v}\right)d\xi\\
&+2\Psi_*(\nu,\Theta)\int_0^1h(\xi)\,\nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right)d\xi,\\
&=2\Psi^*(u,\Theta)\int_0^1h(\xi)\nabla\left(\frac{uv}{(1-\xi)u+\xi v}\right)d\xi\\
&+2\Psi^*(\nu,\Theta)\int_0^1h(\xi)\,\nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right)d\xi.
\end{aligned}$$

Since ∇ is symmetric, then

$$\begin{aligned}
&=2[\Psi_*(u,\Theta)+\Psi_*(\nu,\Theta)]\int_0^1h(\xi)\,\nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right)d\xi,\\
&=2[\Psi^*(u,\Theta)+\Psi^*(\nu,\Theta)]\int_0^1h(\xi)\,\nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right)d\xi.
\end{aligned}\tag{28}$$

Therefore, results

$$\begin{aligned}
&\int_0^1 \Psi_*(\xi u+(1-\xi)\nu,\theta)\nabla\left(\frac{uv}{(1-\xi)u+\xi v}\right)d\xi \\
&\qquad = \int_0^1 \Psi_*\big((1-\xi)u+\xi\nu,\theta\big)\nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right)d\xi \\
&\qquad = \frac{uv}{\nu-u}\int_u^\nu \Psi_*(w,\theta)\nabla(w)dw \\
&\int_0^1 \Psi^*\big((1-\xi)u+\xi\nu,\theta\big)\nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right)d\xi \\
&\qquad = \int_0^1 \Psi^*(\xi u+(1-\xi)\nu,\theta)\nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right)d\xi \\
&\qquad = \frac{uv}{\nu-u}\int_u^\nu \Psi^*(w,\theta)\nabla(w)dw.
\end{aligned} \tag{29}$$

From (28) and (29), we have

$$\begin{aligned}
&\frac{uv}{\nu-u}\int_u^\nu \Psi_*(w,\theta)\nabla(w)dw \\
&\qquad \le [\Psi_*(u,\theta)+\Psi_*(\nu,\theta)]\int_0^1 h(\xi)\,\nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right)d\xi, \\
&\frac{uv}{\nu-u}\int_u^\nu \Psi^*(w,\theta)\nabla(w)dw \\
&\qquad \le [\Psi^*(u,\theta)+\Psi^*(\nu,\theta)]\int_0^1 h(\xi)\,\nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right)d\xi,
\end{aligned}$$

that is

$$\left[\frac{uv}{\nu-u}\int_u^\nu \Psi_*(w,\theta)\nabla(w)dw\,,\frac{uv}{\nu-u}\int_u^\nu \Psi^*(w,\theta)\nabla(w)dw\right]$$

$$\le_I [\Psi_*(u,\theta)+\Psi_*(\nu,\theta),\Psi^*(u,\theta)+\Psi^*(\nu,\theta)]\int_0^1 h(\xi)\,\nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right)d\xi,$$

and hence

$$\frac{uv}{\nu-u}\,(FR)\int_u^\nu \frac{\widetilde{\Psi}(w)}{w^2}\nabla(w)dw \preccurlyeq \left[\widetilde{\Psi}(u)\,\widetilde{+}\,\widetilde{\Psi}(\nu)\right]\int_0^1 h(\xi)\,\nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right)d\xi,$$

concludes the proof. □

Next, we construct first $H{\cdot}H$ Fej´er inequality for harmonically h-convex F-I-V-F, which generalizes first $H{\cdot}H$ Fej´er inequality for harmonically convex function.

Theorem 9. *(First fuzzy fractional $H-H$ Fejér inequality) Let $\widetilde{\Psi}\in HFSX([u,v],\mathbb{F}_0,h)$, whose θ-levels define the family of I-V-Fs $\Psi_\theta:[u,v]\subset\mathbb{R}\to\mathcal{K}_C^+$ that are given by $\Psi_\theta(w)=[\Psi_*(w,\theta),\Psi^*(w,\theta)]$ for all $w\in[u,v]$, $\theta\in[0,1]$. If $\widetilde{\Psi}\in\mathcal{FR}_{([u,\nu],\theta)}$ and $\nabla:[u,v]\to\mathbb{R}$, $\nabla\left(\frac{1}{\frac{1}{u}+\frac{1}{v}-\frac{1}{w}}\right)=\nabla(w)\ge 0$, so that*

$$\frac{1}{2h(\frac{1}{2})}\,\widetilde{\Psi}\left(\frac{2u\nu}{u+\nu}\right)\int_u^\nu \frac{\widetilde{\Psi}(w)}{w^2}dw \preccurlyeq (FR)\int_u^\nu \frac{\widetilde{\Psi}(w)}{w^2}\nabla(w)dw \tag{30}$$

If $\widetilde{\Psi}\in HFSV([u,v],\mathbb{F}_0,h)$, then inequality (30) is reversed such that

$$\frac{1}{2h(\frac{1}{2})}\,\widetilde{\Psi}\left(\frac{2u\nu}{u+\nu}\right)\int_u^\nu \frac{\widetilde{\Psi}(w)}{w^2}dw \succcurlyeq (FR)\int_u^\nu \frac{\widetilde{\Psi}(w)}{w^2}\nabla(w)dw.$$

Proof. Since Ψ is a harmonically h-convex, then for $\theta\in[0,1]$, we have

$$\begin{aligned} \Psi_*\left(\frac{2uv}{u+v},\theta\right) &\le h(\frac{1}{2})\left(\Psi_*\left(\frac{uv}{(1-\xi)u+\xi v},\theta\right)+\Psi_*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right)\right) \\ \Psi^*\left(\frac{2uv}{u+v},\theta\right) &\le h\left(\frac{1}{2}\right)\left(\Psi^*\left(\frac{uv}{(1-\xi)u+\xi v},\theta\right)+\Psi^*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right)\right). \end{aligned} \tag{31}$$

By multiplying (31) by $\nabla\left(\frac{uv}{(1-\xi)u+\xi v}\right)=\nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right)$ and integrating it by ξ over $[0,1]$, yields

$$\begin{aligned} &\Psi_*\left(\frac{2uv}{u+v},\theta\right)\int_0^1\nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right)d\xi \\ &\quad\le h\left(\frac{1}{2}\right)\left(\begin{array}{l}\int_0^1\Psi_*\left(\frac{uv}{(1-\xi)u+\xi v},\theta\right)\nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right)d\xi \\ +\int_0^1\Psi_*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right)\nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right)d\xi\end{array}\right) \\ &\Psi^*\left(\frac{2uv}{u+v},\theta\right)\int_0^1\nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right)d\xi \\ &\quad\le h\left(\frac{1}{2}\right)\left(\begin{array}{l}\int_0^1\Psi^*\left(\frac{uv}{(1-\xi)u+\xi v},\theta\right)\nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right)d\xi \\ +\int_0^1\Psi^*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right)\nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right)d\xi\end{array}\right) \end{aligned} \tag{32}$$

Therefore, results

$$\begin{aligned} &\int_0^1\Psi_*\left(\frac{uv}{(1-\xi)u+\xi v},\theta\right)\nabla\left(\frac{uv}{(1-\xi)u+\xi v}\right)d\xi \\ &\qquad=\int_0^1\Psi_*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right)\nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right)d\xi, \\ &\qquad=\frac{uv}{v-u}\int_u^v\Psi_*(w,\theta)\nabla(w)dw, \\ &\int_0^1\Psi^*\left(\frac{uv}{\xi u+(1-\xi)v},\theta\right)\nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right)d\xi \\ &\qquad=\int_0^1\Psi^*\left(\frac{uv}{(1-\xi)u+\xi v},\theta\right)\nabla\left(\frac{uv}{(1-\xi)u+\xi v}\right)d\xi, \\ &\qquad=\frac{uv}{v-u}\int_u^v\Psi^*(w,\theta)\nabla(w)dw. \end{aligned} \tag{33}$$

From (32) and (33), results

$$\Psi_*\left(\frac{2uv}{u+v},\theta\right)\le\frac{2h\left(\frac{1}{2}\right)}{\int_u^v\nabla(w)dw}\int_u^v\Psi_*(w,\theta)\nabla(w)dw,$$

$$\Psi^*\left(\frac{2uv}{u+v},\theta\right)\le\frac{2h\left(\frac{1}{2}\right)}{\int_u^v\nabla(w)dw}\int_u^v\Psi^*(w,\theta)\nabla(w)dw,$$

from which, we have

$$\begin{aligned} &\left[\Psi_*\left(\frac{2uv}{u+v},\theta\right),\quad\Psi^*\left(\frac{2uv}{u+v},\theta\right)\right] \\ &\qquad\le_I\frac{2h\left(\frac{1}{2}\right)}{\int_u^v\nabla(w)dw}\left[\int_u^v\Psi_*(w,\theta)\nabla(w)dw,\ \int_u^v\Psi^*(w,\theta)\nabla(w)dw\right], \end{aligned}$$

that is

$$\frac{1}{2h\left(\frac{1}{2}\right)}\,\widetilde{\Psi}\left(\frac{2uv}{u+v}\right)\int_u^v \frac{\widetilde{\Psi}(w)}{w^2}dw \preccurlyeq (FR)\int_u^v \frac{\widetilde{\Psi}(w)}{w^2}\nabla(w)dw,$$

is completes the proof. □

Remark 4. *If* $\nabla(w) = 1$, *then from Theorems 10 and 11, we obtain inequality* (17).

If $h(\xi) = \xi$, then from Theorems 10 and 11, we obtain results for harmonically convex F-I-V-Fs, see [28].

If $\Psi_*(w,\theta) = \Psi^*(w,\theta)$ with $\theta = 1$ and $h(\xi) = \xi$, then Theorems 10 and 11 reduce to the classical first and second classical $H{\cdot}H$ Fejér inequality for classical harmonically convex function.

4. Conclusions and Future Plan

In this paper, we proposed an approach for the Hermite–Hadamard type integral inequalities via harmonically h-convex F-I-V-Fs. The main findings include some new bounds of integral mean of harmonically h-convex F-I-V-Fs with error estimations via fuzzy Riemann integrals. We also proved Fej´er type inequality with the same argument. We find in the literature several papers with classical integrals and fundamental concepts. These papers aim to provide new estimations and optimal approaches for harmonically h-convex F-I-V-Fs. The main idea of this paper is that we can obtain new results by using fuzzy integrals for harmonically h-convex F-I-V-Fs calculus. In future, we will explore this concept via the fuzzy Riemann–Liouville nd Hes fractional integrals.

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