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Some Identities of the Degenerate Higher Order Derangement Polynomials and Numbers

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Abstract: Recently, Kim-Kim (J. Math. Anal. Appl. (2021), Vol. 493(1), 124521) introduced the λ-Sheffer sequence and the degenerate Sheffer sequence. In addition, Kim et al. (arXiv:2011.08535v1 17 November 2020) studied the degenerate derangement polynomials and numbers, and investigated some properties of those polynomials without using degenerate umbral calculus. In this paper, the y the degenerate derangement polynomials of order s ($s \in \mathbb{N}$) and give a combinatorial meaning about higher order derangement numbers. In addition, the author gives some interesting identities related to the degenerate derangement polynomials of order s and special polynomials and numbers by using degenerate Sheffer sequences, and at the same time derive the inversion formulas of these identities.

Keywords: derangement numbers and polynomials; degenerate derangement numbers and polynomials; Lah–Bell numbers and polynomials; the degenerate Sheffer sequence; the degenerate Bernoulli (Euler) polynomials; the degenerate Frobenius–Euler polynomials; the degenerate Daehee polynomials; the degenerate Bell polynomials

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1. Introduction

Beginning with Carlitz's degenerate Bernoulli polynomial and degenerate Euler polynomial [1], many scholars in the field of mathematics have been working on degenerate versions of special polynomials and numbers which include the degenerate Stirling numbers of the first and second kinds, the degenerate Bernstein polynomials, the degenerate Bell numbers and polynomials, the degenerate gamma function, the degenerate gamma random variables, degenerate coloring and so on [2–17]. They have been studied by various ways like combinatorial methods, umbral calculus techniques, generating functions, differential equations and probability theory, etc. We can find the motivation to study degenerate polynomials and numbers in the following real world examples. Suppose the probability of a baseball player getting a hit in a match is p. We wonder if the probability that the player will succeed in the 6th trial after failing 4 times in 5 trials is still p. We can see cases where the probability is less than p because of the psychological burden that the player must succeed in the 6th trial.

In particular, the umbral calculus, based on the modern idea of linear functions, linear operators and adjoints, began to build a rigorous foundation by Rota in the 1970s, primarily as symbolic techniques for the manipulation of numerical and polynomial sequences [18]. One of the important tools in the study of degenerate polynomials and numbers is the umbral calculus [16–21]. Kim Kim recently introduced the degenerate umbral calculus [15]. Furthermore, Kim et al. [13] studied the degenerate derangement polynomials of order s ($s \in \mathbb{N}$) and numbers, investigate some properties of those polynomials without using degenerate umbral calculus. Motivated by Kim et al.'s work, the author considers the degenerate derangement polynomials of order s ($s \in \mathbb{N}$) and give an example of the derangement number of order s in real-life. In addition, the author gives their connections with the degenerate derangement polynomials of order s and the well-known special polynomials and numbers.

First, we provide the definitions and properties required for this paper. Let n objects be labelled $1, 2, \ldots, n$. An arrangement or permutation in which object i is not placed in the

i-th place for any i is called a derangement. The number of derangements of an n-element set is called the nth derangement number and denoted by d_n. The nth derangement number is given by [13,22–24]

$$d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

We note that the generating function of the nth derangement number is given by [12,13,25]

$$\begin{aligned} \frac{1}{1-t}e^{-t} &= \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^k \right) \left(\sum_{m=0}^{\infty} t^m \right) \\ &= \sum_{n=0}^{\infty} \left(n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} d_n \frac{t^n}{n!}. \end{aligned} \tag{1}$$

From (1), Kim et al. naturally considered the derangement polynomials and degenerate derangement polynomials, respectively, which are given by [13,24]

$$\frac{1}{1-t}e^{-1}e^{xt} = \sum_{n=0}^{\infty} d_n(x) \frac{t^n}{n!}, \tag{2}$$

and

$$\frac{1}{1-t}e_\lambda^{-1}(t)e_\lambda^{x}(t) = \sum_{n=0}^{\infty} d_{n,\lambda}(x) \frac{t^n}{n!}. \tag{3}$$

When $x = 0$, $d_n(0) = d_n$, $n \geq 0$ is the n-th derangement numbers.

When $x = 0$, $d_{n,\lambda}(0) := d_{n,\lambda}$ is called the degenerate derangement numbers and $d_{0,\lambda} = 1$.

For any nonzero $\lambda \in \mathbb{R}$, the degenerate exponential function is defined by [1,3–13]

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}}, \quad e_\lambda(t) = (1+\lambda t)^{\frac{1}{\lambda}}.$$

By Taylor expansion, we get

$$e_\lambda^x(t) = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \tag{4}$$

where $(x)_{0,\lambda} = 1,\ (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda),\ (n \geq 1)$.

It was known [25] that

$$(1-t)^{-m} = \sum_{l=0}^{\infty} \binom{-m}{l} (-1)^l t^l = \sum_{l=0}^{\infty} < m >_l \frac{t^l}{l!}. \tag{5}$$

where $< x >_0 = 1$, $< x >_n = x(x+1)(x+2)\cdots(x+n-1)$, $(n \geq 1)$.

For $n \geq 0$, it is well known that the Stirling numbers of the first and second kind, respectively, are given by [1,4–7]

$$\frac{1}{k!}(\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n,k) \frac{t^n}{n!}, \tag{6}$$

and

$$\frac{1}{k!}(e^t-1)^k = \sum_{n=k}^{\infty} S_2(n,k) \frac{t^n}{n!}, \tag{7}$$

where $(x)_0 = 1,\ (x)_n = x(x-1)\dots(x-n+1),\ (n \geq 1)$.

Moreover, the degenerate Stirling numbers of the first and second kind, respectively, are given by [4–7]

$$\frac{1}{k!}\big(\log_\lambda(1+t)\big)^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!}, \quad (k \geq 0), \tag{8}$$

and

$$\frac{1}{k!}\big(e_\lambda(t)-1\big)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!}, \quad (k \geq 0). \tag{9}$$

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all power series in the variable t over $\mathbb{C}$ with

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \,\middle|\, a_k \in \mathbb{C} \right\}.$$

Let $\mathbb{P} = \mathbb{C}[x]$ and $\mathbb{P}^*$ be the vector space all linear functional on $\mathbb{P}$.

$$\mathbb{P}_n = \{\ P(x) \in \mathbb{C}[x] \mid degP(x) \leq n\},\ \ (n \geq 0).$$

Then $\mathbb{P}_n$ is an $(n+1)$-dimensional vector space over $\mathbb{C}$.

Recently, Kim-Kim [15] considered λ-linear functional and λ-differential operator as follows:

For $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathcal{F}$ and a fixed nonzero real number λ, each λ gives rise to the linear functional $\langle f(t) \mid \cdot \rangle_\lambda$ on $\mathbb{P}$, called λ-linear functional given by $f(t)$, which is defined by

$$\langle f(t) \mid (x)_{n,\lambda} \rangle_\lambda = a_n, \quad \text{for all } \ n \geq 0. \tag{10}$$

For $\lambda = 0$, we observe that the linear functional $\langle f(t) \mid \cdot \rangle_0$ agrees with the one in $\langle f(t) \mid x^n \rangle = a_k,\ (k \geq 0)$.

From $\langle f(t)g(t) \mid (x)_{n,\lambda} \rangle_\lambda = \langle f(t) \mid (g(t))_\lambda (x)_{n,\lambda} \rangle_\lambda$ and (10), we note that

$$\langle t^k \mid (x)_{n,\lambda} \rangle_\lambda = \langle 1 \mid (t^k)_\lambda (x)_{n,\lambda} \rangle_\lambda = \langle 1 \mid (n)_k (x)_{n-k,\lambda} \rangle_\lambda = n!\delta_{n,k}, \tag{11}$$

for all $n,\ k\ \geq 0$, where $\delta_{n,k}$ is the Kronecker's symbol.

From (11), for each $\lambda \in \mathbb{R}$, and each nonnegative integer k, the differential operator on $\mathbb{P}$ is given by [15]

$$(t^k)_\lambda (x)_{n,\lambda} = \begin{cases} (n)_k (x)_{n-k,\lambda}, & \text{if} \ \ k \leq n, \\ 0 & \text{if} \ \ k \geq n. \end{cases} \tag{12}$$

and for any power series $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathcal{F}$, $(f(t))_\lambda (x)_{n,\lambda} = \sum_{k=0}^{n} \binom{n}{k} a_k (x)_{n-k,\lambda}, (n \geq 0)$.

The order $o(f(t))$ of a power series $f(t)(\neq 0)$ is the smallest integer k for which the coefficient of t^k does not vanish. The series $f(t)$ is called invertible if $o(f(t)) = 0$. $f(t)$ is called a delta series if $o(f(t)) = 1$ and it has a compositional inverse $\overline{f}(t)$ of $f(t)$ with $\overline{f}(f(t)) = f(\overline{f}(t)) = t$.

Let $f(t)$ and $g(t)$ be a delta series and an invertible series, respectively, and $s_{n,\lambda}(x)$ be a degenerate polynomial of a degree n. Then there exists a unique sequences $s_{n,\lambda}(x)$ such that the orthogonality conditions [15]

$$\langle g(t)\big(f(t)\big)^k \mid s_{n,\lambda}(x) \rangle_\lambda = n!\delta_{n,k}, \quad (n,k \geq 0). \tag{13}$$

The sequences $s_{n,\lambda}(x)$ are called the λ-Sheffer sequences for $(g(t), f(t))$, which are denoted by $s_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$.

The sequence $s_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$ if and only if

$$\frac{1}{g(\overline{f}(t))} e_\lambda^x(\overline{f}(t)) = \sum_{k=0}^{\infty} \frac{s_{k,\lambda}(x)}{k!} t^k \quad (n, k \geq 0). \tag{14}$$

Assume that for each $\lambda \in \mathbb{R}^*$ of the set of nonzero real numbers, $s_{n,\lambda}(x)$ is λ-Sheffer for $(g_\lambda(t), f_\lambda(t))$. Assume also that $\lim_{\lambda\to 0} f_\lambda(t) = f(t)$ and $\lim_{\lambda\to 0} g_\lambda(t) = g(t)$, for some delta series $f(t)$ and an invertible series $g(t)$. Then $\lim_{\lambda\to 0} \overline{f}_\lambda(t) = \overline{f}(t)$, where is the compositional inverse of $f(t)$ with $\overline{f}(f(t)) = f(\overline{f}(t)) = t$. Let $\lim_{\lambda\to 0} s_{k,\lambda}(x) = s_k(x)$.

In this case, Kim-Kim called that the family $\{s_{n,\lambda}(x)\}_{\lambda\in\mathcal{R}-\{0\}}$ of λ-Sheffer sequences $s_{n,\lambda}$ are the degenerate (Sheffer) sequences for the Sheffer polynomial $s_n(x)$.

Let $s_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$ and $r_{n,\lambda}(x) \sim (h(t), g(t))_\lambda$, $(n \geq 0)$. Then

$$\begin{aligned} s_{n,\lambda}(x) &= \sum_{k=0}^{n} \mu_{n,k} r_{k,\lambda}(x), \quad (n \geq 0), \\ \text{where} \quad \mu_{n,k} &= \frac{1}{k!} \left\langle \frac{h(\overline{f}(t))}{g(\overline{f}(t))} (l(\overline{f}(t)))^k \mid (x)_{n,\lambda} \right\rangle_\lambda, \quad (n, k \geq 0). \end{aligned} \tag{15}$$

2. Degenerate Derangement Polynomials Order s Arising from Degenerate Sheffer Sequences

In this section, we consider the degenerate derangement polynomials of order s, and give a combinatorial meaning of these numbers and noble identities related to these polynomials and the well-known special polynomials and numbers arising from degenerate Sheffer sequences.

From (3), naturally, we can consider the degenerate derangement polynomials of order s (c.f. [13]) which is given by

$$\frac{1}{(1-t)^s} e_\lambda^{-s}(t) e_\lambda^x(t) = \sum_{n=0}^{\infty} d_{n,\lambda}^{(s)}(x) \frac{t^n}{n!}. \tag{16}$$

When $x = 0$, $d_{n,\lambda}^{(s)}(0) := d_{n,\lambda}^{(s)}$ is called the degenerate derangement numbers of order s and $d_{0,\lambda}^{(s)} = 1$. When $s = 1$, $d_{n,\lambda}^{(1)}(x) = d_{n,\lambda}(x)$, and $\lim_{\lambda\to 0} d_{n,\lambda}(0) = d_n$, $n \geq 0$.

From (16), we get

$$d_{n,\lambda}^{(s)}(x) = n! \sum_{l=0}^{n} \frac{(x-s)_{l,\lambda}}{l!}. \tag{17}$$

It is known (see [15]) that for $f(t), g(t) \in \mathcal{F}$,

$$\left\langle f(t)g(t) \middle| (x)_{n,\lambda} \right\rangle_\lambda = \sum_{k=0}^{n} \binom{n}{k} \left\langle f(t) \middle| (x)_{k,\lambda} \right\rangle_\lambda \left\langle g(t) \middle| (x)_{n-k,\lambda} \right\rangle_\lambda. \tag{18}$$

From (18), by the mathematical induction, for $f_1(t), f_2(t), \cdots, f_m(t) \in \mathcal{F}$, we get

$$\left\langle f_1(t) f_2(t) \cdots f_m(t) \middle| (x)_{n,\lambda} \right\rangle_\lambda = \sum_{i_1+\cdots+i_m=n} \binom{n}{i_1\ i_2 \cdots\ i_m} \left\langle f_1(t) \middle| (x)_{i_1,\lambda} \right\rangle_\lambda \cdots \left\langle f_m(t) \middle| (x)_{i_m,\lambda} \right\rangle_\lambda, \tag{19}$$

where

$$\binom{n}{i_1\ i_2 \cdots\ i_m} = \binom{n}{i_1} \binom{n-i_1}{i_2} \cdots \binom{n-i_1-i_2\cdots-i_{m-1}}{i_m} = \frac{n!}{i_1! i_2! \cdots i_m!}.$$

Therefore, from (19), we get

$$\begin{aligned}
&\left\langle \left(\frac{1}{(1-t)}e_\lambda^{-1}(t)\right)^s \middle| (x)_{n,\lambda} \right\rangle_\lambda \\
&= \sum_{i_1+\cdots+i_r=n}^{n} \binom{n}{i_1\, i_2 \cdots\, i_s} \left\langle \frac{1}{(1-t)}e_\lambda^{-1}(t) \middle| (x)_{i_1,\lambda} \right\rangle_\lambda \cdots \left\langle \frac{1}{(1-t)}e_\lambda^{-1}(t) \middle| (x)_{i_s,\lambda} \right\rangle_\lambda \\
&= \sum_{i_1+\cdots+i_r=n}^{n} \binom{n}{i_1\, i_2 \cdots\, i_s} d_{i_1,\lambda} d_{i_2,\lambda} \cdots d_{i_s,\lambda}.
\end{aligned} \tag{20}$$

From (20), we have

$$d_{n,\lambda}^{(s)} = \sum_{i_1+\cdots+i_r=n}^{n} \binom{n}{i_1\, i_2 \cdots\, i_s} d_{i_1,\lambda} d_{i_2,\lambda} \cdots d_{i_s,\lambda}. \tag{21}$$

From (21), when $\lambda \to 0$, we can give a combinatorial meaning about derangement numbers of order s in real-life.

Example 1. *Suppose n players play a card game randomly divided into s rooms. In addition, assume everyone wears a hat and hangs it on the entrance wall when entering a room. If all the lights suddenly turn out during the game, how many ways no one takes his hat when everyone comes out at same time?*

2.1. Connection with the Degenerate Lah–Bell Polynomials

The unsigned Lah number $L(n,k)$ counts the number of ways of all distributions of n balls, labelled $1, 2, \cdots, n$, among k unlabelled, contents-ordered boxes, with no box left empty and have an explicit formula [26,27]

$$\mathbf{L}(n,k) = \binom{n-1}{k-1}\frac{n!}{k!}. \tag{22}$$

From (22), the generating function of $L(n,k)$ is given by [6,23]

$$\frac{1}{k!}\left(\frac{t}{1-t}\right)^k = \sum_{n=k}^{\infty} L(n,k)\frac{t^n}{n!}, \quad (k \geq 0). \tag{23}$$

From (23), Kim-Kim naturally introduced the Lah–Bell polynomials and the degenerate Lah–Bell polynomials, respectively, which are given by [26,27]

$$e^{x\left(\frac{1}{1-t}-1\right)} = \sum_{n=0}^{\infty} B_n^L(x)\frac{t^n}{n!}, \qquad (n,\ k \geq 0).$$

and

$$e_\lambda^x\left(\frac{1}{1-t}-1\right) = \sum_{n=0}^{\infty} \mathbf{B}_{n,\lambda}^L(x)\frac{t^n}{n!}, \quad (n,\ k \geq 0). \tag{24}$$

When $x = 1$, $B_n^L = B_n^L(1)$ are called the Lah–Bell numbers.
When $x = 1$, $\mathbf{B}_{n,\lambda}^L := \mathbf{B}_{n,\lambda}^L(1)$ are called the n-th degenerate Lah–Bell numbers.
When $\lambda \to 0$, $\lim_{\lambda\to 0} \mathbf{B}_{n,\lambda}^L = \mathbf{B}_n^L$ are the n-th Lah–Bell numbers.

Theorem 1. *For $n \in \mathbb{N} \cup \{0\}$ and $s \in \mathbb{N}$, we have*

$$d_{n,\lambda}^{(s)}(x) = \sum_{k=0}^{n}\left(\sum_{l=k}^{n} L(l,k)(-1)^{l-k} d_{n-l,\lambda}^{(s)}\right)\mathbf{B}_{k,\lambda}^L(x). \tag{25}$$

As the inversion formula of (25), *we have*

$$\mathbf{B}^{L}_{n,\lambda}(x)=\sum_{k=0}^{n}\left(\sum_{l=k}^{n}\sum_{m=0}^{n-l}\sum_{j=0}^{s}\binom{n}{l}L(l,k)\binom{n-l}{m}\binom{s}{j}2^{s-j}(-1)^{j}<j>_{n-l-m}\mathbf{B}^{L}_{m,\lambda}(s)\right)d^{(s)}_{k,\lambda}(x). \quad (26)$$

Proof. From (14), (16) and (24), we consider the following two Sheffer sequence as follows:

$$d^{(s)}_{n,\lambda}(x)\sim\left((1-t)^{s}e^{s}_{\lambda}(t),\,t\right)_{\lambda}\quad\text{and}\quad\mathbf{B}^{L}_{n,\lambda}(x)\sim\left(1,\,\frac{t}{1+t}\right)_{\lambda}. \quad (27)$$

From (15), (16), (23) and (27), we have

$$d^{(s)}_{n,\lambda}(x)=\sum_{k=0}^{n}\mu_{n,k}\mathbf{B}^{L}_{k,\lambda}(x). \quad (28)$$

where

$$\begin{aligned}\mu_{n,k}&=\frac{1}{k!}\left\langle(1-t)^{-s}e^{-s}_{\lambda}(t)\left(\frac{t}{1+t}\right)^{k}\,\middle|\,(x)_{n,\lambda}\right\rangle_{\lambda}\\&=\sum_{l=k}^{n}L(l,k)(-1)^{l-k}\left\langle(1-t)^{-s}e^{-s}_{\lambda}(t)\,\middle|\,(x)_{n-l,\lambda}\right\rangle_{\lambda}\\&=\sum_{l=k}^{n}L(l,k)(-1)^{l-k}d^{(s)}_{n-l,\lambda}.\end{aligned} \quad (29)$$

Therefore, from (28) and (29) we have the identity (25).
To find the inversion formula of (25), from (15) and (27), we have

$$\mathbf{B}^{L}_{n,\lambda}(x)=\sum_{k=0}^{n}\widetilde{\mu}_{n,k}d^{(s)}_{k,\lambda}(x), \quad (30)$$

where, by using (5), (23) and (24)

$$\begin{aligned}\widetilde{\mu}_{n,k}&=\frac{1}{k!}\left\langle\left(\frac{1-2t}{1-t}\right)^{s}e^{s}_{\lambda}\left(\frac{t}{1-t}\right)\left(\frac{t}{1-t}\right)^{k}\,\middle|\,(x)_{n,\lambda}\right\rangle_{\lambda}\\&=\sum_{l=k}^{n}\binom{n}{l}L(l,k)\left\langle\left(\frac{1-2t}{1-t}\right)^{s}e^{s}_{\lambda}\left(\frac{t}{1-t}\right)\,\middle|\,(x)_{n-l,\lambda}\right\rangle_{\lambda}\\&=\sum_{l=k}^{n}\binom{n}{l}L(l,k)\sum_{m=0}^{n-l}\binom{n-l}{m}\mathbf{B}^{L}_{m,\lambda}(s)\left\langle\left(2-\frac{1}{1-t}\right)^{s}\,\middle|\,(x)_{n-l-m,\lambda}\right\rangle_{\lambda}\\&=\sum_{l=k}^{n}\binom{n}{l}L(l,k)\sum_{m=0}^{n-l}\binom{n-l}{m}\mathbf{B}^{L}_{m,\lambda}(s)\sum_{j=0}^{s}\binom{s}{j}2^{s-j}(-1)^{j}<j>_{n-l-m}.\end{aligned} \quad (31)$$

Therefore, from (30) and (31), we have the identity (26). □

2.2. Connection with the Degenerate r-Extended Lah–Bell Polynomials

The r-Lah number $L_r(n,k)$ counts the number of partitions of a set with $n+r$ elements into $k+r$ ordered blocks such that r distinguished elements have to be in distinct ordered blocks and an explicit formula of $L_r(n,k)$ (see [8,24,26,28–30]) given by

$$L_{r}(n,k)=\binom{n+2r-1}{k+2r-1}\frac{n!}{k!}\ (k\geq 0). \quad (32)$$

From (32), we have the generating function of $L_r(n,k)$ given by [28–30]

$$\frac{1}{k!}\left(\frac{1}{1-t}\right)^{2r}\left(\frac{t}{1-t}\right)^{k} = \sum_{n=k}^{\infty} L_r(n,k)\frac{t^n}{n!}, (k \geq 0). \tag{33}$$

Recently, Kim-Kim introduced the r-extended Lah–Bell polynomials, respectively, as follows [30]:

$$\left(\frac{1}{1-t}\right)^{2r} e^{x\left(\frac{1}{1-t}-1\right)} = \sum_{n=k}^{\infty} \mathbf{B}^L_{r,n}(x)\frac{t^n}{n!},\ (k \geq 0). \tag{34}$$

When $x = 1$, $\mathbf{B}^L_n = \mathbf{B}^L_n(1)$ and $\mathbf{B}^L_{r,n} = \mathbf{B}^L_{r,n}(1)$ are called the Lah–Bell numbers and r-extended Lah–Bell numbers respectively.

From (34), naturally, KL defined a degenerate r-extended Lah–Bell polynomials [31] by

$$\left(\frac{1}{1-t}\right)^{2r} e^x_\lambda\left(\frac{1}{1-t}-1\right) = \sum_{n=0}^{\infty} \mathbf{B}^L_{r,n,\lambda}(x)\frac{t^n}{n!}. \tag{35}$$

When $x = 1$, $\mathbf{B}^L_{r,n,\lambda} := \mathbf{B}^L_{r,n,\lambda}(1)$ is called the n-th degenerate r-extended Lah–Bell number. As $\lambda \to 0$, $\lim_{\lambda\to 0} \mathbf{B}^L_{r,n,\lambda} = \mathbf{B}^L_{r,n}$ is the n-th r-extended Lah–Bell number.

Theorem 2. *For $n \in \mathbb{N} \cup \{0\}$ and $r,\ s \in \mathbb{N}$, we have*

$$d^{(s)}_{n,\lambda}(x) = \sum_{k=0}^{n}\left(\sum_{l=k}^{n} L_r(l,k)(-1)^{l-k}\, d^{(s)}_{n-l,\lambda}\right)\mathbf{B}^L_{r,k,\lambda}(x). \tag{36}$$

As the inversion formula of (36), we have

$$\mathbf{B}^L_{r,n,\lambda}(x) = \sum_{k=0}^{n}\left(\sum_{l=k}^{n}\sum_{m=0}^{n-l}\sum_{j=0}^{s}\binom{n}{l}\binom{n-l}{m}\binom{s}{j}2^{s-j}(-1)^j < j >_{n-l-m} L_r(l,k)\mathbf{B}^L_{m,\lambda}(s)\right)d^{(s)}_{k,\lambda}(x). \tag{37}$$

Proof. From (14), (16) (35), we consider the following two degenerate Sheffer sequences.

$$d^{(s)}_{n,\lambda}(x) \sim \left((1-t)^s e^s_\lambda(t),\ t\right)_\lambda \quad \text{and} \quad \mathbf{B}^L_{r,n,\lambda}(x) \sim \left(\left(\frac{1}{1+t}\right)^{2r},\ \frac{t}{1+t}\right)_\lambda. \tag{38}$$

From (15), (16), (33) and (38), we have

$$d^{(s)}_{n,\lambda}(x) = \sum_{k=0}^{n} \mu_{n,k}\mathbf{B}^L_{r,k,\lambda}(x), \tag{39}$$

where

$$\begin{aligned}\mu_{n,k} &= \frac{1}{k!}\left\langle (1-t)^{-s}e^{-s}_\lambda(t)\left(\frac{1}{1+t}\right)^{2r}\left(\frac{t}{1+t}\right)^{k} \middle| (x)_{n,\lambda}\right\rangle_\lambda \\ &= \sum_{l=k}^{n} L_r(l,k)(-1)^{l-k}\left\langle (1-t)^{-s}e^{-s}_\lambda(t) \middle| (x)_{n-l,\lambda}\right\rangle_\lambda \\ &= \sum_{l=k}^{n} L_r(l,k)(-1)^{l-k}\, d^{(s)}_{n-l,\lambda}.\end{aligned} \tag{40}$$

Therefore, from (39) and (40), we have the identity (36).

To find the inversion formula of (36), from (15) and (38), we have

$$\mathbf{B}^{L}_{r,n,\lambda}(x) = \sum_{k=0}^{n} \widetilde{\mu}_{n,k} d^{(s)}_{k,\lambda}(x). \tag{41}$$

where, by using (5), (24) and (34)

$$\begin{aligned}
\widetilde{\mu}_{n,k} &= \frac{1}{k!}\left\langle \left(\frac{1-2t}{1-t}\right)^{s} e^{s}_{\lambda}\left(\frac{t}{1-t}\right)\left(\frac{1}{1-t}\right)^{2r}\left(\frac{t}{1-t}\right)^{k} \,\middle|\, (x)_{n,\lambda}(x)\right\rangle_{\lambda} \\
&= \sum_{l=k}^{n}\binom{n}{l} L_r(l,k)\left\langle \left(\frac{1-2t}{1-t}\right)^{s} e^{s}_{\lambda}\left(\frac{t}{1-t}\right) \,\middle|\, (x)_{n-l,\lambda}\right\rangle_{\lambda} \\
&= \sum_{l=k}^{n}\binom{n}{l} L_r(l,k)\sum_{m=0}^{n-l}\binom{n-l}{m}\mathbf{B}^{L}_{m,\lambda}(s)\left\langle \left(2-\frac{1}{1-t}\right)^{s} \,\middle|\, (x)_{n-l-m,\lambda}\right\rangle_{\lambda} \\
&= \sum_{l=k}^{n}\binom{n}{l} L_r(l,k)\sum_{m=0}^{n-l}\binom{n-l}{m}\mathbf{B}^{L}_{m,\lambda}(s)\sum_{j=0}^{s}\binom{s}{j}2^{s-j}(-1)^{j} < j >_{n-l-m}.
\end{aligned} \tag{42}$$

Therefore, from (41) and (42) we have the identity (37). □

2.3. Connection with the Degenerate Bernoulli Polynomials of Higher Order r

The degenerate Bernoulli polynomials of order r are given by the generating function [1,6,21] to be

$$\left(\frac{t}{e_{\lambda}(t)-1}\right)^{r} e^{x}_{\lambda}(t) = \sum_{n=0}^{\infty} \beta^{(r)}_{n,\lambda}(x)\frac{t^{n}}{n!}. \tag{43}$$

We note that $\beta^{(r)}_{n,\lambda} = \beta^{(r)}_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers of order r. From (20), we observe that

$$\begin{aligned}
&\left\langle \left(\frac{t}{e_{\lambda}(t)-1}\right)^{r} \,\middle|\, (x)_{n,\lambda}\right\rangle_{\lambda} \\
&\quad = \sum_{i_1+\cdots+i_r=n}^{n}\binom{n}{i_1 i_2\cdots i_r}\left\langle \frac{t}{e_{\lambda}(t)-1}\middle|(x)_{i_1,\lambda}\right\rangle_{\lambda}\cdots\left\langle \frac{t}{e_{\lambda}(t)-1}\middle|(x)_{i_r,\lambda}\right\rangle_{\lambda} \\
&\quad = \sum_{i_1+\cdots+i_r=n}^{n}\binom{n}{i_1 i_2\cdots i_r}\beta_{i_1,\lambda}\beta_{i_2,\lambda}\cdots\beta_{i_r,\lambda}.
\end{aligned} \tag{44}$$

Theorem 3. *For $n \in \mathbb{N}\cup\{0\}$ and $r, s \in \mathbb{N}$, we have*

$$\begin{aligned}
d^{(s)}_{n,\lambda}(x) = &\sum_{k=0}^{r-1}\Bigg(\frac{1}{k!}\sum_{l=0}^{n}\sum_{m=0}^{k}\sum_{j=0}^{n-l}\frac{(r-k)!l!}{(l+r-k)!}\binom{n}{l}\binom{k}{m}(-1)^{k-m}\binom{n-l}{j}(m)_{j,\lambda} \\
&S_{2,\lambda}(l+r-k,r-k)d^{(s)}_{n-l-j,\lambda}\Bigg)\beta^{(r)}_{k,\lambda}(x) + \sum_{k=r}^{n}\Bigg(\frac{n!}{k!(n-k+r)!} \\
&\sum_{l=0}^{r}\sum_{j=0}^{n-k+r}\binom{r}{l}(-1)^{r-l}(l)_{j,\lambda}\binom{n-k+r}{j}d^{(s)}_{n-k+r-j,\lambda}\Bigg)\beta^{(r)}_{k,\lambda}(x).
\end{aligned} \tag{45}$$

In particular, when $s=1$, as the inversion formula of (45), we have

$$\beta_{n,\lambda}^{(r)}(x)=\sum_{k=0}^{n}\left(\sum_{m=0}^{n-k}\frac{n!}{k!m!(n-k-m)!}(1)_{m,\lambda}(1-m+m^2\lambda)\right.\\ \left.\times\sum_{i_1+\cdots+i_r=n-k-m}^{n-k-m}\binom{n-k-m}{i_1 i_2\cdots i_r}\beta_{i_1,\lambda}\beta_{i_2,\lambda}\cdots\beta_{i_r,\lambda}\right)d_{k,\lambda}(x). \tag{46}$$

Proof. From (14), (16) and (43), we consider the following two degenerate Sheffer sequences.

$$d_{n,\lambda}^{(s)}(x)\sim\left((1-t)^s e_\lambda^s(t),\ t\right)_\lambda \quad\text{and}\quad \beta_{n,\lambda}^{(r)}(x)\sim\left(\left(\frac{e_\lambda(t)-1}{t}\right)^r,\ t\right)_\lambda. \tag{47}$$

From (15) and (47), we have

$$d_{n,\lambda}^{(s)}(x)=\sum_{k=0}^{n}\mu_{n,k}\beta_{k,\lambda}^{(r)}(x), \tag{48}$$

where

$$\mu_{n,k}=\frac{1}{k!}\left\langle\left(\frac{e_\lambda(t)-1}{t}\right)^r\left(\frac{1}{1-t}e_\lambda^{-1}(t)\right)^s t^k\,\middle|\,(x)_{n,\lambda}\right\rangle_\lambda. \tag{49}$$

For $r>n$, from (9), we note that

$$\frac{1}{(r-k)!}(e_\lambda(t)-1)^{r-k}=\sum_{l=r-k}^{\infty}S_{2,\lambda}(l,r-k)\frac{t^l}{l!}=\sum_{l=0}^{\infty}S_{2,\lambda}(l+r-k,r-k)\frac{t^{l+r-k}!}{(l+r-k)!}. \tag{50}$$

Now, by using (4), (16) and (50), we have

$$\begin{aligned}
\mu_{n,k}&=\frac{1}{k!}\left\langle\left(\frac{e_\lambda(t)-1}{t}\right)^r\left(\frac{1}{1-t}e_\lambda^{-1}(t)\right)^s t^k\,\middle|\,(x)_{n,\lambda}\right\rangle_\lambda\\
&=\frac{1}{k!}\left\langle\left(\frac{1}{1-t}e_\lambda^{-1}(t)\right)^s(e_\lambda(t)-1)^k(e_\lambda(t)-1)^{r-k}\,t^{-(r-k)}\,\middle|\,(x)_{n,\lambda}\right\rangle_\lambda\\
&=\frac{1}{k!}\sum_{l=0}^{n}\frac{(r-k)!}{(l+r-k)!}S_{2,\lambda}(l+r-k,r-k)\left\langle\left(\frac{1}{1-t}e_\lambda^{-1}(t)\right)^s(e_\lambda(t)-1)^k t^l\,\middle|\,(x)_{n,\lambda}\right\rangle_\lambda\\
&=\frac{1}{k!}\sum_{l=0}^{n}\frac{(r-k)!}{(l+r-k)!}S_{2,\lambda}(l+r-k,r-k)\binom{n}{l}\,l!\,\left\langle\left(\frac{1}{1-t}e_\lambda^{-1}(t)\right)^s(e_\lambda(t)-1)^k\,\middle|\,(x)_{n-l,\lambda}\right\rangle_\lambda\\
&=\frac{1}{k!}\sum_{l=0}^{n}\frac{(r-k)!}{(l+r-k)!}S_{2,\lambda}(l+r-k,r-k)\binom{n}{l}\,l!\,\sum_{m=0}^{k}\binom{k}{m}(-1)^{k-m}\\
&\qquad\times\left\langle\left(\frac{1}{1-t}e_\lambda^{-1}(t)\right)^s e_\lambda^m(t)\,\middle|\,(x)_{n-l,\lambda}\right\rangle_\lambda\\
&=\frac{1}{k!}\sum_{l=0}^{n}\frac{(r-k)!}{(l+r-k)!}S_{2,\lambda}(l+r-k,r-k)\binom{n}{l}\,l!\,\sum_{m=0}^{k}\binom{k}{m}(-1)^{k-m}\sum_{j=0}^{n-l}\binom{n-l}{j}(m)_{j,\lambda}d_{n-l-j,\lambda}^{(s)}.
\end{aligned} \tag{51}$$

For $r>n$ and $0\le k<r$, we have the same result when $r>n$.
For $r\le k$, we note that $k-r\ge 0$ and $n-k+r\ge 0$. Thus, we have

$$\begin{aligned}\mu_{n,k} &= \frac{1}{k!}\left\langle \left(\frac{e_\lambda(t)-1}{t}\right)^r \left(\frac{1}{1-t}e_\lambda^{-1}(t)\right)^s t^k \,\middle|\, (x)_{n,\lambda}\right\rangle_\lambda \\
&= \frac{1}{k!}\left\langle \left(\frac{1}{1-t}e_\lambda^{-1}(t)\right)^s (e_\lambda(t)-1)^r t^{k-r} \,\middle|\, (x)_{n,\lambda}\right\rangle_\lambda \\
&= \frac{1}{k!}\binom{n}{k-r}(k-r)!\left\langle \left(\frac{1}{1-t}e_\lambda^{-1}(t)\right)^s (e_\lambda(t)-1)^r \,\middle|\, (x)_{n-k+r,\lambda}\right\rangle_\lambda \\
&= \frac{n!}{k!(n-k+r)!}\sum_{l=0}^{r}\binom{r}{l}(-1)^{r-l}\left\langle \left(\frac{1}{1-t}e_\lambda^{-1}(t)\right)^s e_\lambda^l(t) \,\middle|\, (x)_{n-k+r,\lambda}\right\rangle_\lambda \\
&= \frac{n!}{k!(n-k+r)!}\sum_{l=0}^{r}\binom{r}{l}(-1)^{r-l}\sum_{j=0}^{n-k+r}(l)_{j,\lambda}\binom{n-k+r}{j}d^{(s)}_{n-k+r-j,\lambda}.\end{aligned} \tag{52}$$

Therefore, from (48), (49), (51) and (52), we have the identity (45).

In particular, when $s = 1$, to find the inversion formula of (45), from (4), (15), (44) and (47), we have

$$\beta^{(r)}_{n,\lambda}(x) = \sum_{k=0}^{n} \widetilde{\mu}_{n,k} d^{(s)}_{k,\lambda}(x), \tag{53}$$

where

$$\begin{aligned}\widetilde{\mu}_{n,k} &= \frac{1}{k!}\left\langle (1-t)e_\lambda(t)\left(\frac{t}{e_\lambda(t)-1}\right)^r t^k \,\middle|\, (x)_{n,\lambda}\right\rangle_\lambda \\
&= \binom{n}{k}\left\{\left\langle e_\lambda(t)\left(\frac{t}{e_\lambda(t)-1}\right)^r \,\middle|\, (x)_{n-k,\lambda}\right\rangle_\lambda - \left\langle te_\lambda(t)\left(\frac{t}{e_\lambda(t)-1}\right)^r \,\middle|\, (x)_{n-k,\lambda}\right\rangle_\lambda\right\} \\
&= \binom{n}{k}\left\{\sum_{m=0}^{n-k}\binom{n-k}{m}(1)_{m,\lambda} - \sum_{m=1}^{n-k}\binom{n-k}{m}m(1)_{m-1,\lambda}\right\}\left\langle\left(\frac{t}{e_\lambda(t)-1}\right)^r \,\middle|\, (x)_{n-k-m,\lambda}\right\rangle_\lambda \\
&= \binom{n}{k}\left\{\sum_{m=0}^{n-k}\binom{n-k}{m}(1)_{m,\lambda} - \sum_{m=0}^{n-k}\binom{n-k}{m}m(1)_{m-1,\lambda}\right\} \\
&\qquad \times \sum_{i_1+\cdots+i_r=n-k-m}^{n-k-m}\binom{n}{i_1 i_2\cdots i_r}\beta_{i_1,\lambda}\beta_{i_2,\lambda}\cdots\beta_{i_r,\lambda} \\
&= \binom{n}{k}\sum_{m=0}^{n-k}\binom{n-k}{m}(1)_{m,\lambda}(1-m+m^2\lambda)\sum_{i_1+\cdots+i_r=n-k-m}^{n-k-m}\binom{n-k-m}{i_1 i_2\cdots i_r}\beta_{i_1,\lambda}\beta_{i_2,\lambda}\cdots\beta_{i_r,\lambda}.\end{aligned} \tag{54}$$

Therefore, from (53) and (54), we have the identity (46). □

2.4. Connection with the Degenerate Frobenius–Euler Polynomials of Order r

Kim et al. introduced the degenerate Frobenius–Euler polynomials of order r [20] defined by

$$\left(\frac{1-u}{e_\lambda(t)-u}\right)^r e_\lambda^x(t) = \sum_{n=0}^{\infty} h^{(r)}_{n,\lambda}(x|u)\frac{t^n}{n!}, \quad (u \neq 1,\ u \in \mathbb{C},\ \ k \geq 0). \tag{55}$$

When $x = 0$, $h^{(r)}_{n,\lambda}(u) = h^{(r)}_{n,\lambda}(0|u)$ are called the degenerate Frobenius–Euler numbers of order r.

When $x = 0$ and $r = 1$, $h_{n,\lambda}(u) = h_{n,\lambda}(0|u)$ are called the degenerate Frobenius–Euler numbers.

When $u = -1$, the degenerate Euler polynomials of order r respectively are given by the generating function [1,21] to be

$$\left(\frac{2}{e_\lambda(t)+1}\right)^r e_\lambda^x(t) = \sum_{n=0}^{\infty} E^{(r)}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{56}$$

We note that $E^{(r)}_{n,\lambda} = E^{(r)}_{n,\lambda}(0)$ $(n \geq 0)$, are called the degenerate Euler numbers of order r.

When $x = 0$ and $r = 1$, $E_{n,\lambda} = E_{n,\lambda}(0)$ are called the degenerate Euler numbers.

From (20), in the same way as (44), we have

$$\left\langle \left(\frac{1-u}{e_\lambda(t)-u} \right)^r \middle| (x)_{n,\lambda} \right\rangle_\lambda = \sum_{i_1+\cdots+i_r=n}^{n} \binom{n}{i_1 i_2 \cdots i_r} h_{i_1,\lambda}(u) h_{i_2,\lambda}(u) \cdots h_{i_r,\lambda}(u), \tag{57}$$

and

$$\left\langle \left(\frac{2}{e_\lambda(t)-1} \right)^r \middle| (x)_{n,\lambda} \right\rangle_\lambda = \sum_{i_1+\cdots+i_r=n}^{n} \binom{n}{i_1 i_2 \cdots i_r} E_{i_1,\lambda} E_{i_2,\lambda} \cdots E_{i_r,\lambda}. \tag{58}$$

Theorem 4. *For $n \in \mathbb{N} \cup \{0\}$, $r, s \in \mathbb{N}$ we have*

$$d^{(s)}_{n,\lambda}(x) = \frac{1}{(1-u)^r} \sum_{k=0}^{n} \left(\sum_{l=0}^{n-k} \sum_{j=0}^{r} \binom{n}{k} \binom{n-k}{l} \binom{r}{j} (-u)^{r-j} (j)_{n-l,\lambda} d^{(s)}_{l,\lambda} \right) h^{(r)}_{k,\lambda}(x|u). \tag{59}$$

In particular, when $s = 1$, as the inversion formula of (59), we have

$$\begin{aligned} h^{(r)}_{n,\lambda}(x|u) = \sum_{k=0}^{n} \Bigg(& \sum_{m=0}^{n-k} \frac{n!}{k!m!(n-k-m)!} (1)_{m,\lambda} (1-m+m^2\lambda) \\ & \times \sum_{i_1+\cdots+i_r=n-k-m}^{n-k-m} \binom{n-k-m}{i_1 i_2 \cdots i_r} h_{i_1,\lambda}(u) h_{i_2,\lambda}(u) \cdots h_{i_r,\lambda}(u) \Bigg) d_{k,\lambda}(x). \end{aligned} \tag{60}$$

Proof. From (15), (16) and (55), we consider the following two degenerate Sheffer sequences.

$$d^{(s)}_{n,\lambda}(x) \sim \left((1-t)^s e^s_\lambda(t),\, t \right)_\lambda \quad \text{and} \quad h^{(r)}_{n,\lambda}(x|u) \sim \left(\left(\frac{e_\lambda(t)-u}{1-u} \right)^r,\, t \right)_\lambda. \tag{61}$$

By using (4), (15), (16) and (61), we have

$$d^{(s)}_{n,\lambda}(x) = \sum_{k=0}^{n} \mu_{n,k} h^{(r)}_{k,\lambda}(x|u), \tag{62}$$

where

$$\begin{aligned} \mu_{n,k} &= \frac{1}{k!} \left\langle \left(\frac{1}{1-t} e^{-1}_\lambda(t) \right)^s \left(\frac{e_\lambda(t)-u}{1-u} \right)^r t^k \middle| (x)_{n,\lambda} \right\rangle_\lambda \\ &= \binom{n}{k} \left\langle \left(\frac{1}{1-t} e^{-1}_\lambda(t) \right)^s \left(\frac{e_\lambda(t)-u}{1-u} \right)^r \middle| (x)_{n-k,\lambda} \right\rangle_\lambda \\ &= \frac{1}{(1-u)^r} \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} d^{(s)}_{l,\lambda} \left\langle (e_\lambda(t)-u)^r \middle| (x)_{n-l,\lambda} \right\rangle_\lambda \\ &= \frac{1}{(1-u)^r} \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} d^{(s)}_{l,\lambda} \sum_{j=0}^{r} \binom{r}{j} (-u)^{r-j} (j)_{n-l,\lambda}. \end{aligned} \tag{63}$$

Therefore, from (62) and (63), we get the identity (59).

In particular, when $s = 1$, to find the inversion formula of (59), by (4), (13), (15) and (61),

We have

$$h^{(r)}_{n,\lambda}(x|u) = \sum_{k=0}^{n} \widetilde{\mu}_{n,k} d^{(s)}_{k,\lambda}(x). \tag{64}$$

where,

$$\begin{aligned}
\widetilde{\mu}_{n,k} &= \frac{1}{k!}\left\langle (1-t)e_\lambda(t)\left(\frac{1-u}{e_\lambda(t)-u}\right)^r t^k \,\middle|\, (x)_{n,\lambda}\right\rangle_\lambda \\
&= \binom{n}{k}\left\{\left\langle e_\lambda(t)\left(\frac{1-u}{e_\lambda(t)-u}\right)^r \,\middle|\, (x)_{n-k,\lambda}\right\rangle_\lambda - \left\langle te_\lambda(t)\left(\frac{t}{e_\lambda(t)-1}\right)^r \,\middle|\, (x)_{n-k,\lambda}\right\rangle_\lambda\right\} \\
&= \binom{n}{k}\left\{\sum_{m=0}^{n-k}\binom{n-k}{m}(1)_{m,\lambda} - \sum_{m=1}^{n-k}\binom{n-k}{m}m(1)_{m-1,\lambda}\right\}\left\langle\left(\frac{1-u}{e_\lambda(t)-u}\right)^r \,\middle|\, (x)_{n-k-m,\lambda}\right\rangle_\lambda \\
&= \binom{n}{k}\left\{\sum_{m=0}^{n-k}\binom{n-k}{m}(1)_{m,\lambda} - \sum_{m=0}^{n-k}\binom{n-k}{m}m(1)_{m-1,\lambda}\right\} \\
&\qquad\times \sum_{i_1+\cdots+i_r=n-k-m}^{n-k-m}\binom{n-k-m}{i_1 i_2\cdots i_r}h_{i_1,\lambda}(u)h_{i_2,\lambda}(u)\cdots h_{i_r,\lambda}(u).
\end{aligned} \tag{65}$$

Therefore, from (64) and (65), we have the identity (60). □

When $u=-1$ in Theorem 3, we have the following corollary.

Corollary 1. *For $n \in \mathbb{N}\cup\{0\}$ and $r,\ s \in \mathbb{N}$, we have*

$$d_{n,\lambda}^{(s)}(x) = \frac{1}{2^r}\sum_{k=0}^{n}\left(\sum_{l=0}^{n-k}\sum_{j=0}^{r}\binom{n}{k}\binom{n-k}{l}\binom{r}{j}(j)_{n-l,\lambda}d_{l,\lambda}^{(s)}\right)E_{k,\lambda}^{(r)}(x). \tag{66}$$

In particular, when $s=1$, the inversion formula of (66), we have

$$\begin{aligned}
E_{n,\lambda}^{(r)}(x) &= \sum_{k=0}^{n}\left(\sum_{m=0}^{n-k}\frac{n!}{k!m!(n-k-m)!}(1)_{m,\lambda}(1-m+m^2\lambda)\right. \\
&\qquad\left.\times\sum_{i_1+\cdots+i_r=n-k-m}^{n-k-m}\binom{n-k-m}{i_1 i_2\cdots i_r}E_{i_1,\lambda}E_{i_2,\lambda}\cdots E_{i_r,\lambda}\right)d_{k,\lambda}(x).
\end{aligned}$$

2.5. Connection with the Degenerate Daehee Polynomials of Order r

The degenerate Daehee polynomials of order r [11] is defined by

$$\left(\frac{\log_\lambda(1+t)}{t}\right)^r(1+t)^x = \sum_{n=0}^{\infty}D_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}. \tag{67}$$

where $\log_\lambda(1+t) = \frac{1}{\lambda}((1+t)^\lambda - 1)$ and $\log_\lambda(e_\lambda(t)) = e_\lambda(\log_\lambda(t)) = t$. When $x=0$, $D_{n,\lambda}^{(r)} = D_{n,\lambda}^{(r)}(0)$ are called the degenerate Daehee numbers of order r.

Theorem 5. *For $n \in \mathbb{N}\cup\{0\}$ and $r,\ s \in \mathbb{N}$, we have*

$$d_{n,\lambda}^{(s)}(x) = \sum_{k=0}^{n}\left(\sum_{l=0}^{n}\sum_{m=k}^{n-l}\binom{n-l}{m}\frac{r!n!}{(l+r)!(n-l)!}S_{2,\lambda}(l+r,r)\,S_{2,\lambda}(m,k)d_{n-l-m,\lambda}^{(s)}\right)D_{k,\lambda}^{(r)}(x). \tag{68}$$

As the inversion formula of (68), we have

$$\begin{aligned}
D_{n,\lambda}^{(r)}(x) &= \sum_{k=0}^{n}\left(\sum_{l=0}^{n}\sum_{m=0}^{n-l}\sum_{j=0}^{s}\sum_{i=0}^{n-l-m}\sum_{v=0}^{s}\binom{n}{l}\binom{n-l}{m}\binom{s}{j}\binom{n-l-m}{i}\right. \\
&\qquad\left.\times\binom{s}{v}j!(n-l-m-i)!S_{1,\lambda}(l,k)D_{n,\lambda}^{(r)}S_{1,\lambda}(i,j)\delta_{n-l-m-i,v}\right)d_{k,\lambda}^{(s)}(x).
\end{aligned} \tag{69}$$

Proof. From (14), (16) and (67), we consider the following two degenerate Sheffer sequences.

$$d_{n,\lambda}^{(s)}(x) \sim \left((1-t)^s e_\lambda^s(t),\ t\right)_\lambda \quad \text{and} \quad D_{n,\lambda}^{(r)}(x) \sim \left(\left(\frac{e_\lambda(t)-1}{t}\right)^r,\ e_\lambda(t)-1\right)_\lambda. \tag{70}$$

In addition, from (9), we note that

$$\frac{1}{r!}(e_\lambda(t)-1)^r = \sum_{l=r}^{\infty} S_{2,\lambda}(l,r)\frac{t^l}{l!} = \sum_{l=0}^{\infty} S_{2,\lambda}(l+r,r)\frac{t^{l+r}!}{(l+r)!}. \tag{71}$$

Thus from (15), (16), (70) and (71), we have

$$d_{n,\lambda}^{(s)}(x) = \sum_{k=0}^{n} \mu_{n,k} D_{k,\lambda}^{(r)}(x). \tag{72}$$

where

$$\begin{aligned}
\mu_{n,k} &= \frac{1}{k!}\left\langle \left(\frac{1}{1-t}e_\lambda^{-1}(t)\right)^s \left(\frac{e_\lambda(t)-1}{t}\right)^r (e_\lambda(t)-1)^k \,\middle|\, (x)_{n,\lambda} \right\rangle_\lambda \\
&= \frac{r!}{k!}\sum_{l=0}^{n} S_{2,\lambda}(l+r,r)\frac{1}{(l+r)!}\left\langle \left(\frac{1}{1-t}e_\lambda^{-1}(t)\right)^s (e_\lambda(t)-1)^k t^l \,\middle|\, (x)_{n,\lambda} \right\rangle_\lambda \\
&= \frac{r!}{k!}\sum_{l=0}^{n} S_{2,\lambda}(l+r,r)\frac{1}{(l+r)!}\binom{n}{l}\, l!\, \left\langle \left(\frac{1}{1-t}e_\lambda^{-1}(t)\right)^s (e_\lambda(t)-1)^k \,\middle|\, (x)_{n-l,\lambda} \right\rangle_\lambda \\
&= \sum_{l=0}^{n} S_{2,\lambda}(l+r,r)\frac{r!n!}{(l+r)!(n-l)!}\sum_{m=k}^{n-l} S_{2,\lambda}(m,k)\binom{n-l}{m} d_{n-l-m,\lambda}^{(s)}.
\end{aligned} \tag{73}$$

Therefore, from (72) and (73), we get the identity (68).
To find the inversion formula of (68), from (8), (13), (15) and (67) we have

$$D_{k,\lambda}^{(r)}(x) = \sum_{k=0}^{n} \widetilde{\mu}_{n,k} d_{k,\lambda}^{(s)}(x). \tag{74}$$

where,

$$\begin{aligned}
\widetilde{\mu}_{n,k} &= \frac{1}{k!}\left\langle (1+t)^s(1+\log_\lambda(1+t))^s \left(\frac{\log_\lambda(1+t)}{t}\right)^r \left(\log_\lambda(1+t)\right)^k \,\middle|\, (x)_{n,\lambda} \right\rangle_\lambda \\
&= \sum_{l=0}^{n} S_{1,\lambda}(l,k)\binom{n}{l}\left\langle (1+t)^s\left(\log_\lambda(1+t)+1\right)^s \left(\frac{\log_\lambda(1+t)}{t}\right)^r \,\middle|\, (x)_{n-l,\lambda} \right\rangle_\lambda \\
&= \sum_{l=0}^{n}\binom{n}{l} S_{1,\lambda}(l,k)\sum_{m=0}^{n-l} D_{n,\lambda}^{(r)}\binom{n-l}{m}\left\langle (1+t)^s\left(\log(1+t)+1\right)^s \middle| (x)_{n-l-m,\lambda} \right\rangle_\lambda \\
&= \sum_{l=0}^{n}\binom{n}{l} S_{1,\lambda}(l,k)\sum_{m=0}^{n-l} D_{n,\lambda}^{(r)}\binom{n-l}{m}\sum_{j=0}^{s}\binom{s}{j}\left\langle (1+t)^s\left(\log_\lambda(1+t)\right)^j \middle| (x)_{n-l-m,\lambda} \right\rangle_\lambda \\
&= \sum_{l=0}^{n}\binom{n}{l} S_{1,\lambda}(l,k)\sum_{m=0}^{n-l} D_{n,\lambda}^{(r)}\binom{n-l}{m}\sum_{j=0}^{s}\binom{s}{j} j! \\
&\qquad \times \sum_{i=0}^{n-l-m} S_{1,\lambda}(i,j)\binom{n-l-m}{i}\sum_{v=0}^{s}\binom{s}{v}\left\langle t^v \middle| (x)_{n-l-m-i,\lambda} \right\rangle_\lambda \\
&= \sum_{l=0}^{n}\binom{n}{l} S_{1,\lambda}(l,k)\sum_{m=0}^{n-l} D_{n,\lambda}^{(r)}\binom{n-l}{m}\sum_{j=0}^{s}\binom{s}{j} j! \\
&\qquad \times \sum_{i=0}^{n-l-m} S_{1,\lambda}(i,j)\binom{n-l-m}{i}\sum_{v=0}^{s}\binom{s}{v}(n-l-m-i)!\,\delta_{n-l-m-i,v}.
\end{aligned} \tag{75}$$

□

2.6. Connection with the Degenerate Bell Polynomials

The Bell polynomials are defined by the generating function [4–6]

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} Bel_n(x)\frac{t^n}{n!}.$$

Kim-Kim introduced the degenerate Bell polynomial [4] given by

$$e_\lambda^x(e_\lambda(t)-1) = \sum_{l=0}^{\infty} Bel_{l,\lambda}(x)\frac{t^l}{l!}. \tag{76}$$

Theorem 6. *For $n \in \mathbb{N} \cup \{0\}$ and $r,\ s \in \mathbb{N}$, we have*

$$d_{n,\lambda}^{(s)}(x) = \sum_{k=0}^{n}\left(\sum_{l=k}^{n} S_{1,\lambda}(l,k)\binom{n}{l} d_{n-l,\lambda}^{(s)}\right) Bel_{k,\lambda}(x) \tag{77}$$

As the inversion formula of (77), we have

$$Bel_{n,\lambda}(x) = \sum_{k=0}^{n}\left(-\sum_{l=k}^{n}\binom{n}{l} S_{2,\lambda}(l,k)\sum_{j=0}^{n-l}\binom{n-l}{j}(s)_{j,\lambda}Bel_{n-l-j,\lambda}(s)\right) d_{k,\lambda}^{(s)}(x). \tag{78}$$

Proof. From (14), (16) and (76), we consider two degenerate Sheffer sequences as follows:

$$d_{n,\lambda}^{(s)}(x) \sim \big((1-t)^s e_\lambda^s(t),\ t\big)_\lambda \quad \text{and} \quad Bel_{n,\lambda}(x) \sim (1,\ \log_\lambda(1+t))_\lambda. \tag{79}$$

By using (8),(15), (16), (30) and (79), we have

$$d_{n,\lambda}^{(s)}(x) = \sum_{k=0}^{n} \mu_{n,k} Bel_{k,\lambda}(x), \tag{80}$$

where

$$\begin{aligned}\mu_{n,k} &= \frac{1}{k!}\left\langle (1-t)^{-s} e_\lambda^{-s}(t)(\log_\lambda(1+t))^k \,\middle|\, (x)_{n,\lambda}\right\rangle_\lambda \\ &= \sum_{l=k}^{n} S_{1,\lambda}(l,k)\binom{n}{l}\left\langle (1-t)^{-s}e_\lambda^{-s}(t) \,\middle|\, (x)_{n-l,\lambda}\right\rangle_\lambda = \sum_{l=k}^{n} S_{1,\lambda}(l,k)\binom{n}{l} d_{n-l,\lambda}^{(s)}.\end{aligned} \tag{81}$$

Therefore from (80) and (81), we get the identity (77).
To find inversion formula of (77), from (15) and (79) we have

$$Bel_{n,\lambda}(x) = \sum_{k=0}^{n} \widetilde{\mu}_{n,k} d_{k,\lambda}^{(s)}(x). \tag{82}$$

Thus, by using (4), (9) and (76), we have

$$\begin{aligned}\widetilde{\mu}_{n,k} &= \frac{1}{k!}\left\langle -e_\lambda^s(t)e_\lambda(e_\lambda^s(t)-1)(e_\lambda(t)-1)^k \,\middle|\, (x)_{n,\lambda}\right\rangle_\lambda \\ &= \sum_{l=k}^{n}\binom{n}{l} S_{2,\lambda}(l,k)\left\langle -e_\lambda^s(t)e_\lambda^s(e_\lambda(t)-1) \,\middle|\, (x)_{n-l,\lambda}\right\rangle_\lambda \\ &= -\sum_{l=k}^{n}\binom{n}{l} S_{2,\lambda}(l,k)\sum_{j=0}^{n-l}\binom{n-l}{j}(s)_{j,\lambda}\left\langle e_\lambda(e_\lambda(t)-1) \,\middle|\, (x)_{n-l-j,\lambda}\right\rangle_\lambda \\ &= -\sum_{l=k}^{n}\binom{n}{l} S_{2,\lambda}(l,k)\sum_{j=0}^{n-l}\binom{n-l}{j}(s)_{j,\lambda}Bel_{n-l-j,\lambda}(s).\end{aligned} \tag{83}$$

Therefore, from (82) and (83), we have the identity (78). □

2.7. Connection with the Falling Factorial Polynomials

Theorem 7. *For $n \in \mathbb{N} \cup \{0\}$, $r,\ s \in \mathbb{N}$ we have*

$$d_{n,\lambda}^{(s)}(x) = \sum_{k=0}^{n} \left(\sum_{l=k}^{n} S_{2,\lambda}(l,k) \binom{n}{l} d_{n-l,\lambda} \right) (x)_n,\ (n \geq 0). \tag{84}$$

Proof. Since $e_\lambda^x(\log(1+t)) = (1+t)^x = \sum_{n=0}^{\infty} (x)_n \frac{t^n}{n!}$, we have $(x)_n \sim (1, e_\lambda(t) - 1)_\lambda$. We consider the two degenerate Sheffer sequences as follows:

$$d_{n,\lambda}^{(s)}(x) \sim \big((1-t)e_\lambda(t),\ t\big)_\lambda \quad \text{and} \quad (x)_n \sim (1,\ e_\lambda(t)-1)_\lambda. \tag{85}$$

Thus, from (9), (15), (16) and (85), we have

$$d_{n,\lambda}^{(s)}(x) = \sum_{k=0}^{n} \mu_{n,k} (x)_k,\ (n \geq 0), \tag{86}$$

where

$$\begin{aligned} \mu_{n,k} &= \frac{1}{k!} \left\langle (1-t)^{-s} e_\lambda^{-s}(t) \Big(e_\lambda(t) - 1 \Big)^k \,\Big|\, (x)_{n,\lambda} \right\rangle_\lambda \\ &= \sum_{l=k}^{n} S_{2,\lambda}(l,k) \binom{n}{l} \left\langle (1-t)^{-s} e_\lambda^{-s}(t) \,\Big|\, (x)_{n-l,\lambda} \right\rangle_\lambda = \sum_{l=k}^{n} S_{2,\lambda}(l,k) \binom{n}{l} d_{n-l,\lambda}^{(s)}. \end{aligned} \tag{87}$$

Therefore, from (86) and (87), we have the identity (85). □

3. Conclusions

In this paper, the author considered the degenerate derangement polynomials of order s ($s \in \mathbb{N}$) and expressed the degenerate derangement numbers order s as the product of s degenerate derangement numbers (see (21)). Thus the author gave a combinatorial meaning about higher order derangement numbers. The author represented various expressions for the degenerate degenerate derangement polynomials of order s in terms of quite a few well-known special polynomials and numbers by using the degenerate Sheffer sequences. Here is the special polynomials and numbers: the Lah numbers and the degenerate Lah polynomials; the r-Lah numbers and the degenerate r- Lah polynomials; the degenerate Bernoulli polynomials of order r and the product of r degenerate Bernoulli numbers; the degenerate Frobenius–Euler polynomials of order r; the Stirling numbers of the first and second kind, and the degenerate Deahee polynomials of order r; the Stirling numbers of the first and second kind, and the degenerate Bell polynomials; the Stirling numbers of the second kind and the falling factorial.

The study of the degenerate version of the well-known special polynomials and numbers is applied to characterize properties in various fields of mathematics (see [2–14,31]). As one of our future projects, the author would like to continue to study degenerate versions of certain special polynomials and numbers.

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References

1. Carlitz, L. Degenerate Stirling, Bernoulli and Eulerian numbers. *Utilitas Math.* **1979**, *15*, 51–88.
2. Kierstead, H.A.; Mohar, B.; Špacapan, S.; Yang, D.; Zhu, X. The two-coloring number and degenerate coloring of planer graphs. *Siam J. Discret. Math.* **2009**, *23*, 1548–1560. [CrossRef]
3. Kim, D.S.; Kim, T. Degenerate Bernstein polynomials. *RACSAM* **2019**, *113*, 2913–2920. [CrossRef]
4. Kim, D.S.; Kim, T.; Kim, H.Y.; Kwon, J. Some identities of degenerate Bell polynomials. *Mathematics* **2020**, *8*, 40. [CrossRef]
5. Kim, H.K. Fully degenerate Bell polynomials associated with degenerate Poisson random variables. *Researchgate* **2021**. [CrossRef]
6. Kim, T.; Kim, D.S.; Lee, H.; Kwon, J. A note no some identities of new type degenerate Bell polynomials. *Mathematics* **2019**, *7*, 1086. [CrossRef]
7. Kim, T. A note on degenerate Stirling polynomials of the second kind. *Proc. Jangjeon Math. Soc.* **2017**, *20*, 319–331.
8. Kim, T.; Kim, D.S. Note on the Degenerate Gamma Function. *Russ. J. Math. Phys.* **2020**, *27*, 352–358. [CrossRef]
9. Kim, T.; Kim, D.S.; Kwon, J.; Lee, H. A note on degenerate gamma random variables. *Revista Edu.* **2020**, *388*, 39–44.
10. Kim, T.; Kim, D.S. An identity of symmetry for the degenerate Frobenius–Euler polynomials. *Math. Slovaca.* **2018**, *68*, 239–243. [CrossRef]
11. Kim, D.S.; Kim, T. Some identities of degenerate Daehee numbers arising from certain differential equations. *J. Nonlinear Sci. Appl.* **2017**, *10*, 744–751. [CrossRef]
12. Kim, T.; Kim, D.S. Some identities on derangement and degenerate derangement polynomials. In *Advances in Mathematical Inequalities and Applications, Trends Math.*; Birkhauser/Springer: Singapore, 2018; pp. 265–277.
13. Kim, T.; Kim, D.S.; Lee, H.; Jang, L.-C. A note on degenerate derangement polynomials and numbers. *arXiv* **2020**, arXiv:2011.08535v1.
14. Mohar, B.; Špacapan, S. Degenerate and star colorings of graphs an surfaces. *Eur. J. Comb.* **2012**, *33*, 340–349. [CrossRef]
15. Kim, D.S.; Kim, T. Degenerate Sheffer sequence and λ-Sheffer sequence. *J. Math. Anal. Appl.* **2021**, *493*, 124521. [CrossRef]
16. Kim, T.; Kim, D.S.; Kim, H.Y.; Lee, H.; Jang, L.-C. Degenerate Bell polynomials associated with umbral calculus. *J. Ineq. Appl.* **2020**, *2020*, 226. [CrossRef]
17. Kim, T.; Kim, D.S. Some identities of extended degenerate r-central Bell polynomials arising from umbral calculus. *Rev. R. Acad. Cienc. Exactas. Nat. Ser. A Mat. RACSAM* **2020**, *114*, 19. [CrossRef]
18. Ernst, T. Examples of a q-umbral calculus. *Adv. Stud. Contemp. Math. (Kyungshang)* **2008**, *16*, 1–22.
19. Roman, S. The umbral calculus. In *Pure and Applied Mathematics*; Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers]: New York, NY, USA, 1984.
20. Kim, T.; Mansour, T. Umbral calculus associated with Frobenius-type Eulerian polynomials. *Russ. J. Math. Phys.* **2014**, *21*, 484–493. [CrossRef]
21. Kim, D.S.; Kim, T. Some identities of Bernoulli and Euler polynomials arising from umbral calculus. *Adv. Stud. Contemp. Math. (Kyungshang)* **2013**, *23*, 159–171. [CrossRef]
22. Carlitz, L. The number of derangements of a sequence with given specification. *Fibonacci Quart.* **1978**, *16*, 255–258.
23. Jackson, D. M. Laguerre polynomials and derangements. *Math. Proc. Cambridge Philos. Soc.* **1976**, *80*, 213–214. [CrossRef]
24. Kim, T.; Kim, D.S.; Jang, G.-W.; Kwon, J. A note on some identities of derangement polynomials. *J. Inequal. Appl.* **2018**, *2018*, 40. [CrossRef] [PubMed]
25. Comtet, L. *Advanced Combinatorics. The Art of Finite and Infinite Expansions*, Revised and enlarged edition; D. Reidel Publishing Co.: Dordrecht, The Netherlands, 1974.
26. Kim, D.S.; Kim, T. Lah–Bell numbers and polynomials. *Proc. Jangjeon Math. Soc.* **2020**, *23*, 577–586.
27. Ma. Y.; Kim, D.S.; Kim, T.; Kim, H.Y.; Lee, H.S. Some identities of Lah–Bell polynomials. *Adv. Differ. Equ.* **2020**, *2020*, 510. [CrossRef]
28. Nyul, G.; Racz, G. The r-Lah numbers. *Discrete Math.* **2015**, *338*, 1660–1666. [CrossRef]
29. Nyul, G.; Racz, G. Sums of r-Lah numbers and r-Lah polynomials. *ARS Math. Contemp.* **2020**. [CrossRef]
30. Kim, D.S.; Kim, T. r-extended Lah–Bell numbers and polynomials associated with r-Lah numbers. *Proc. Jangjeon Math. Soc. 2* **2021**, *24*, 1–10.
31. Kim, H.K.; Lee, D.S. Some identities of degenerate r-extended Lah–Bell polynomials. *Proc. Jangjeon Math. Soc.* **2021**, *24*, 47–61.