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On Existence Theorems to Symmetric Functional Set-Valued Differential Equations

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Abstract: In this paper, we consider functional set-valued differential equations in their integral representations that possess integrals symmetrically on both sides of the equations. The solutions have values that are the nonempty compact and convex subsets. The main results contain a Peano type theorem on the existence of the solution and a Picard type theorem on the existence and uniqueness of the solution to such equations. The proofs are based on sequences of approximations that are constructed with appropriate Hukuhara differences of sets. An estimate of the magnitude of the solution's values is provided as well. We show the closeness of the unique solutions when the equations differ slightly.

Keywords: set-valued differential equations; existence theorems; approximation sequences; Gronwall–Bellman inequality



Citation: Malinowski, M.T. On Existence Theorems to Symmetric Functional Set-Valued Differential Equations. *Symmetry* **2021**, *13*, 1219. <https://doi.org/10.3390/sym13071219>

Academic Editor: Alina Alb Lupas

Received: 7 June 2021
Accepted: 1 July 2021
Published: 7 July 2021

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1. Introduction

In this paper, we study symmetric functional set-valued equations of the form:

$$X(t) + \int_{t_0}^t F(s, X_s) ds = \chi_0(0) + \int_{t_0}^t G(s, X_s) ds \quad \text{for } t \in [t_0, t_0 + T], \quad (1)$$

with initial condition

$$X_{t_0} = \chi_0.$$

These equations are called symmetric because the integrals appear symmetrically on both sides of the equation and cannot be reduced to one integral, because they are sets and not numbers. The equations of type (1) are functional because X_s, X_{t_0}, χ_0 are functions; they are set-valued because each mapping in (1) has values that are sets and sets are also both integrals.

To more accurately describe the meaning of symbols and the Equation (1) itself, let us introduce and explain some notations. Thus, the number t_0 can be interpreted as the present moment, T is the length of the time horizon. By $\text{CompConv}(\mathbb{R}^d)$ we denote the family of nonempty compact and convex subsets of \mathbb{R}^d . In $\text{CompConv}(\mathbb{R}^d)$, we consider the Hausdorff–Pompeiu metric H

$$H(A, B) := \max \left\{ \sup_{x \in A} \inf_{y \in B} \|x - y\|, \sup_{y \in B} \inf_{x \in A} \|x - y\| \right\}, \quad A, B \in \text{CompConv}(\mathbb{R}^d),$$

where $\|\cdot\|$ stands for the Euclidean norm in \mathbb{R}^d . It is well known that the metric space $(\text{CompConv}(\mathbb{R}^d), H)$ possesses nice properties; in particular it is complete, separable and locally compact. If in the set $\text{CompConv}(\mathbb{R}^d)$ we introduce the operation of adding sets and multiplying by a number,

$$A + B := \{x + y \mid x \in A, y \in B\}, \quad k \cdot A := \{k \cdot x \mid x \in A\}$$

for $A, B \in \text{CompConv}(\mathbb{R}^d)$ and $k \in \mathbb{R}$,

then unfortunately we will not obtain a linear structure due to the problem with the existence of the opposite element. In the same way, there is a problem with the existence of the difference of the two sets, which is the direct cause of the impossibility of reducing two set-valued integrals in (1) to one integral. In symmetric equations of type (1), one has to use the concept of the Hukuhara difference of two sets and in the main part of the paper it will have to be assumed that differences of this type exist. Let us recall that if for two sets A and B there is a third set C , such that $A = B + C$, then the set C is called the Hukuhara difference of sets A and B , and we denote it by $A \ominus B$. Let us note (see [1]) that for $A, B, C, D \in \text{CompConv}(\mathbb{R}^d)$,

- (P1) $H(A + C, B + C) = H(A, B)$;
- (P2) $H(A + B, C + D) \leq H(A, C) + H(B, D)$;
- (P3) if there exist $A \ominus B$ and $C \ominus D$ then $H(A \ominus B, C \ominus D) \leq H(A, C) + H(B, D)$.

Let $C_\theta = C([- \theta, 0], \text{CompConv}(\mathbb{R}^d))$ denote the set of all H -continuous set-valued mappings acting from $[- \theta, 0]$ to $\text{CompConv}(\mathbb{R}^d)$, where θ is a positive real number. In C_θ , we consider metric ρ defined as $\rho(\chi_1, \chi_2) = \sup_{u \in [- \theta, 0]} H(\chi_1(u), \chi_2(u))$ for $\chi_1, \chi_2 \in C_\theta$. The function χ_0 , which appears in (1), belongs to C_θ . Moreover, X_s from (1) is an element of C_θ as well, and it is understood as $X_s(u) = X(s + u)$ for $u \in [- \theta, 0]$, where s is fixed from $I := [t_0, t_0 + T]$ and $X \in C(J, \text{CompConv}(\mathbb{R}^d))$, $J := [t_0 - \theta, t_0 + T]$. Thus, the initial condition of Equation (1) is to be interpreted as having the time t_0 history, which is described by χ_0 .

The mappings F and G in (1) are set-valued, $F, G: I \times C_\theta \rightarrow \text{CompConv}(\mathbb{R}^d)$, and their integrals are the Aumann set-valued integrals (see [2]), that is, for a mapping $V: I \rightarrow \text{CompConv}(\mathbb{R}^d)$,

$$\int_I V(t)dt := \left\{ \int_I v(t)dt \mid v \in S(V) \right\},$$

where $S(V)$ is the set of integrable selectors of V and this set is nonempty. The Aumann integral has the following properties (see [1]):

- (P4) $\int_I V(t)dt \in \text{CompConv}(\mathbb{R}^d)$;
- (P5) $\int_a^b V(t)dt = \int_a^c V(t)dt + \int_c^b V(t)dt$ if $a \leq c \leq b$;
- (P6) $H\left(\int_I V(t)dt, \int_I W(t)dt\right) \leq \int_I H(V(t), W(t))dt$ if V, W are integrable set-valued mappings.

Note that by putting $F \equiv \{0\}$ in Equation (1), it takes the form:

$$X(t) = \chi_0(0) + \int_{t_0}^t G(s, X_s)ds \quad \text{for } t \in I \tag{2}$$

with initial condition

$$X_{t_0} = \chi_0,$$

and this integral equation for continuous G is equivalent, in terms of the identity of the solution sets, to the Cauchy problem for the functional set-valued differential equations,

$$\begin{cases} X'(t) &= G(t, X_t), \quad t \in I, \\ X_{t_0} &= \chi_0, \end{cases}$$

where the notation $'$ denotes the Hukuhara derivative of set-valued functions (see [3]). Some studies on the last differential problem were conducted, for example, in [4–8], where research was carried out on equations of the type (2) and the obtained results relate to their existence and uniqueness, the comparison method and stability, the approximation of the solution and data dependence, the distance between two solutions, the nonuniform practical stability and the nonuniform boundedness of the solution.

In this paper, the range of the considered Equation (1) becomes wider than that of Equation (2). Equation (1) also has the advantage of being more general than the equations of the following form:

$$X(t) + \int_{t_0}^t F(s, X_s) ds = \chi_0(0) \quad \text{for } t \in I, \quad (3)$$

with initial condition

$$X_{t_0} = \chi_0.$$

The latter equations (3) are under certain conditions equivalent to the functional set-valued differential equations with the so-called second type Hukuhara derivative, that is,

$$\begin{cases} X^*(t) &= (-1) \cdot F(t, X_t), \quad t \in I, \\ X_{t_0} &= \chi_0, \end{cases}$$

where the symbol $*$ stands for the second type Hukuhara derivative of set-valued functions. The set-valued differential equations with the second type Hukuhara derivative were considered in [9–11] and then in, for example, [12]. The solutions to both Equations (2) and (3) show very different geometric properties. Namely, if for the solution X of Equation (2) we consider the diameter of the set $X(t)$, then this diameter is a non-decreasing function of the variable t . For solutions of equations of the type (3), the situation is the opposite, that is, this diameter is a non-increasing function of the variable t . The equations considered in this work have the advantage of covering both of the previously mentioned cases, allowing the monotone nature of the diameter of the set $X(t)$ to change.

Although the start of the study of set-valued differential and integral equations dates back to the 1960s, this issue is still relevant, as indicated by the monograph [1] and the recently published literature, see for example, [13–19]. Moreover, these equations have been applied to modeling important socio-biological and medical tasks, such as diagnosing cancer [20,21].

At the end of this section of the paper we provide the well-known Gronwall–Bellman Lemma of analytic integral inequality, which we will often use in Section 3.

Lemma 1 (Gronwall–Bellman inequality, [22,23]). *Let α , β and f be real-valued functions defined on interval I . Let β and f be continuous. Suppose that the negative part of α is integrable on every closed and bounded subinterval of I .*

(a) *If β is non-negative and if f satisfies the integral inequality*

$$f(t) \leq \alpha(t) + \int_{t_0}^t \beta(s) f(s) ds \quad \text{for } t \in I,$$

then

$$f(t) \leq \alpha(t) + \int_{t_0}^t \alpha(s) \beta(s) \exp\left(\int_s^t \beta(r) dr\right) ds \quad \text{for } t \in I.$$

(b) *If, in addition, the function α is non-decreasing, then*

$$f(t) \leq \alpha(t) \exp\left(\int_{t_0}^t \beta(s) ds\right) \quad \text{for } t \in I.$$

2. Existence of at Least One Solution

First, we define how we understand the solution to Equation (1).

Definition 1. *An H -continuous set-valued mapping $X: J \rightarrow \text{CompConv}(\mathbb{R}^d)$ is said to be a solution to Equation (1), if $X(t) = \chi_0(t - t_0)$ for every $t \in [t_0 - \theta, t_0]$ and $X(t)$ verifies equality*

$$X(t) + \int_{t_0}^t F(s, X_s) ds = \chi_0(0) + \int_{t_0}^t G(s, X_s) ds \quad \text{for every } t \in I = [t_0, t_0 + T].$$

Let us notice that, for jointly continuous mappings F and G and continuous mapping X , the compositions $F \circ X$ and $G \circ X$ are also continuous, and as a consequence the integrals in (1) are well-defined.

We start the presentation of the results concerning symmetric functional set-valued integral equations by formulating the Peano type theorem on the existence of at least one solution. The conditions that have been imposed on the mappings F and G in the equation are their continuity, especially in the functional variable, and their integrable boundedness.

Theorem 1. *Suppose that $F, G \in C(I \times C_\theta, \text{CompConv}(\mathbb{R}^d))$, $\chi_0 \in C_\theta$ and there are integrable mappings $m_F, m_G: I \rightarrow [0, \infty)$ such that*

$$H(F(t, \chi), \{0\}) \leq m_F(t) \quad \text{and} \quad H(G(t, \chi), \{0\}) \leq m_G(t) \quad \text{for every } (t, \chi) \in I \times C_\theta.$$

Let $I_k^n := [t_0 + \frac{k-1}{n}T, t_0 + \frac{k}{n}T]$ for $n \in \mathbb{N}$, $k = 1, 2, \dots, n$. Assume that the sequence $\{X^n\}_{n=1}^\infty, X^n: J \rightarrow \text{CompConv}(\mathbb{R}^d)$ described as

$$X^1(t) = \begin{cases} \chi_0(t - t_0), & t \in [t_0 - \theta, t_0], \\ \chi_0(0), & t \in I, \end{cases}$$

and for $n \in \{2, 3, \dots\}$

$$X^n(t) = \begin{cases} \chi_0(t - t_0), & t \in [t_0 - \theta, t_0], \\ \chi_0(0), & t \in I_1^n, \\ \left[\chi_0(0) + \int_{t_0}^{t-\frac{T}{n}} G(s, X_s^n) ds \right] \ominus \int_{t_0}^{t-\frac{T}{n}} F(s, X_s^n) ds, & t \in I_2^n \cup I_3^n \cup \dots \cup I_n^n \end{cases}$$

can be defined. Then the symmetric functional set-valued integral Equation (1) admits at least one solution.

Proof. Let us fix $n \geq 2$ and $u, v \in I$, $u < v$. If $u, v \in [t_0 - \theta, t_0]$ or $u, v \in I_1^n$ then $H(X^n(u), X^n(v)) = 0$. If $u, v \in I_2^n \cup I_3^n \cup \dots \cup I_n^n$ then

$$H(X^n(u), X^n(v)) = H\left(\left[\chi_0(0) + \int_{t_0}^{u-\frac{T}{n}} G(s, X_s^n) ds \right] \ominus \int_{t_0}^{u-\frac{T}{n}} F(s, X_s^n) ds, \left[\chi_0(0) + \int_{t_0}^{v-\frac{T}{n}} G(s, X_s^n) ds \right] \ominus \int_{t_0}^{v-\frac{T}{n}} F(s, X_s^n) ds \right).$$

By virtue of properties (P3), (P1), (P5) and (P6), we obtain:

$$\begin{aligned} H(X^n(u), X^n(v)) &\leq H\left(\int_{t_0}^{u-\frac{T}{n}} G(s, X_s^n) ds, \int_{t_0}^{v-\frac{T}{n}} G(s, X_s^n) ds \right) \\ &\quad + H\left(\int_{t_0}^{u-\frac{T}{n}} F(s, X_s^n) ds, \int_{t_0}^{v-\frac{T}{n}} F(s, X_s^n) ds \right) \\ &\leq \int_{u-\frac{T}{n}}^{v-\frac{T}{n}} H(\{0\}, G(s, X_s^n)) ds + \int_{u-\frac{T}{n}}^{v-\frac{T}{n}} H(\{0\}, F(s, X_s^n)) ds. \end{aligned}$$

Now, by the integrable boundedness assumption,

$$H(X^n(u), X^n(v)) \leq \int_{u-\frac{T}{n}}^{v-\frac{T}{n}} m_G(s) ds + \int_{u-\frac{T}{n}}^{v-\frac{T}{n}} m_F(s) ds.$$

Thus, if we let $v - u \searrow 0$ then

$$H(X^n(u), X^n(v)) \rightarrow 0, \quad \text{uniformly in } n.$$

Hence, we infer that $\{X^n\}$ is equicontinuous. In particular, $X^n \in C(J, \text{CompConv}(\mathbb{R}^d))$ for each $n \in \mathbb{N}$.

Further, notice that for every $n \in \mathbb{N}$ and for $t \in [t_0 - \theta, t_0] \cup I_1^n$ it holds that $H(X^n(t), \{0\}) \leq \rho(\chi_0, \mathbf{0})$, where $\mathbf{0} \equiv \{0\}$. Considering $t \in I_2^n \cup I_3^n \cup \dots \cup I_n^n$, we get

$$\begin{aligned} H(X^n(t), \{0\}) &= H\left(\left[\chi_0(0) + \int_{t_0}^{t-\frac{T}{n}} G(s, X_s^n) ds\right] \ominus \int_{t_0}^{t-\frac{T}{n}} F(s, X_s^n) ds, \{0\}\right) \\ &\leq H(\chi_0(0), \{0\}) + \int_{t_0}^{t-\frac{T}{n}} H(G(s, X_s^n), \{0\}) ds \\ &\quad + \int_{t_0}^{t-\frac{T}{n}} H(F(s, X_s^n), \{0\}) ds \\ &\leq H(\chi_0(0), \{0\}) + \int_{t_0}^{t-\frac{T}{n}} m_G(s) ds + \int_{t_0}^{t-\frac{T}{n}} m_F(s) ds \\ &\leq \rho(\chi_0, \mathbf{0}) + \int_I (m_G(s) + m_F(s)) ds. \end{aligned}$$

This leads us to the conclusion that $\{X^n\}$ is uniformly bounded.

Invoking the Arzela–Ascoli Theorem, we infer that there is a subsequence $\{X^{k_n}\}$ and an H -continuous function $X: J \rightarrow \text{CompConv}(\mathbb{R}^d)$ such that

$$\sup_{t \in J} H(X^{k_n}(t), X(t)) \longrightarrow 0, \quad \text{as } n \rightarrow \infty.$$

We shall show that the limit mapping X is a solution of Equation (1). Notice that $X(t) = \chi_0(t - t_0)$ if $t \in [t_0 - \theta, t_0]$. So the initial condition is met. It remains to be shown whether X satisfies equality (1) for every $t \in I$. Let us observe that for $t \in I$,

$$\begin{aligned} &H\left(X(t) + \int_{t_0}^t F(s, X_s) ds, \chi_0(0) + \int_{t_0}^t G(s, X_s) ds\right) \\ &\leq H(X(t), X^{k_n}(t)) + H\left(\int_{t_0}^{t-\frac{T}{k_n}} F(s, X_s^{k_n}) ds, \int_{t_0}^{t-\frac{T}{k_n}} F(s, X_s) ds\right) \\ &\quad + H\left(\int_{t-\frac{T}{k_n}}^t F(s, X_s) ds, \{0\}\right) + H\left(\int_{t_0}^{t-\frac{T}{k_n}} G(s, X_s^{k_n}) ds, \int_{t_0}^{t-\frac{T}{k_n}} G(s, X_s) ds\right) \\ &\quad + H\left(\int_{t-\frac{T}{k_n}}^t G(s, X_s) ds, \{0\}\right). \end{aligned}$$

Due to (P6) and the integrable boundedness of F and G , we obtain

$$\begin{aligned} &H\left(X(t) + \int_{t_0}^t F(s, X_s) ds, \chi_0(0) + \int_{t_0}^t G(s, X_s) ds\right) \\ &\leq H(X(t), X^{k_n}(t)) + \int_{t_0}^{t-\frac{T}{k_n}} H(F(s, X_s^{k_n}), F(s, X_s)) ds \\ &\quad + \int_{t-\frac{T}{k_n}}^t m_F(s) ds + \int_{t_0}^{t-\frac{T}{k_n}} H(G(s, X_s^{k_n}), G(s, X_s)) ds \\ &\quad + \int_{t-\frac{T}{k_n}}^t m_G(s) ds \\ &\leq H(X(t), X^{k_n}(t)) + \int_I H(F(s, X_s^{k_n}), F(s, X_s)) ds \\ &\quad + \int_I H(G(s, X_s^{k_n}), G(s, X_s)) ds \\ &\quad + \int_{t-\frac{T}{k_n}}^t (m_F(s) + m_G(s)) ds. \end{aligned}$$

Since

$$\begin{aligned}
 H(X(t), X^{k_n}(t)) &\longrightarrow 0 \text{ as } n \rightarrow \infty, \\
 \int_I H(F(s, X_s^{k_n}), F(s, X_s)) ds &\longrightarrow 0 \text{ as } n \rightarrow \infty, \\
 \int_I H(G(s, X_s^{k_n}), G(s, X_s)) ds &\longrightarrow 0 \text{ as } n \rightarrow \infty,
 \end{aligned}$$

and

$$\int_{t-\frac{T}{kn}}^t (m_F(s) + m_G(s)) ds \longrightarrow 0 \text{ as } n \rightarrow \infty,$$

we infer that $H(X(t) + \int_{t_0}^t F(s, X_s) ds, \chi_0(0) + \int_{t_0}^t G(s, X_s) ds) = 0$ for every $t \in I$. This means that X satisfies equality (1) and the proof is completed. \square

By the magnitude of the set $A \in \text{CompConv}(\mathbb{R}^d)$ we understand the number $\sup_{x \in A} \|x\|$, and it is easy to see that $\sup_{x \in A} \|x\| = H(A, \{0\})$. We obtain a certain estimate of the magnitude of the solution values $X(t)$ in the following theorem.

Proposition 1. *Suppose that the assumptions of Theorem 1 hold. Then, each solution X to Equation (1) verifies*

$$\sup_{t \in J} H(X(t), \{0\}) \leq \rho(\chi_0, \mathbf{0}) + \int_I (m_F(s) + m_G(s)) ds.$$

The above property allows us to conclude that all solutions X of Equation (1) are uniformly bounded. Moreover, similarly to the proof of Theorem 1, it is possible to find that any sequence of solutions is equicontinuous. The application of the Arzela–Ascoli Theorem in locally compact spaces gives us the opportunity to formulate another theorem:

Proposition 2. *Suppose that the assumptions of Theorem 1 are satisfied. Then the set of all solutions to Equation (1) is a compact subset of the space $C(J, \text{CompConv}(\mathbb{R}^d))$ endowed with the supremum metric.*

3. Existence of a Unique Solution

In this part of the work, we focus on justifying the existence of a unique solution and examining properties of the solution. The solution $X: J \rightarrow \text{CompConv}(\mathbb{R}^d)$ to Equation (1) is said to be unique if for every $t \in J$ the equality $X(t) = Y(t)$ is true for any other solution $Y: J \rightarrow \text{CompConv}(\mathbb{R}^d)$ to Equation (1). The following theorem is of the Picard type, therefore we will assume that the mappings F and G in the functional variable satisfy the Lipschitz condition.

Theorem 2. *Assume that $F, G \in C(I \times C_\theta, \text{CompConv}(\mathbb{R}^d))$, $\chi_0 \in C_\theta$. Suppose there exists a positive constant L such that*

$$\max\{H(F(t, \chi_1), F(t, \chi_2)), H(G(t, \chi_1), G(t, \chi_2))\} \leq L\rho(\chi_1, \chi_2).$$

for every $t \in I$ and every $\chi_1, \chi_2 \in C_\theta$. Let the sequence $\{X^n\}_{n=0}^\infty, X^n: I \rightarrow \text{CompConv}(\mathbb{R}^d)$ be described as

$$X^0(t) = \begin{cases} \chi_0(t - t_0), & t \in [t_0 - \theta, t_0], \\ \chi_0(0), & t \in I, \end{cases}$$

and for $n \in \{1, 2, \dots\}$

$$X^n(t) = \begin{cases} \chi_0(t - t_0), & t \in [t_0 - \theta, t_0], \\ \left[\chi_0(0) + \int_{t_0}^t G(s, X_s^{n-1}) ds \right] \ominus \int_{t_0}^t F(s, X_s^{n-1}) ds, & t \in I, \end{cases}$$

can be defined. Then Equation (1) possesses only one solution.

Proof. In the proof, we shall use Picard's approximation sequence described in the assumptions. It is easy to observe that $X^n \in C(J, \text{CompConv}(\mathbb{R}^d))$ for every $n \in \{0, 1, 2, \dots\}$. We begin with preliminary calculations regarding the distance between consecutive terms of $\{X^n\}$. Notice, at first, that for $t \in I$,

$$H(G(t, X_t^0), \{0\}) \leq H(G(t, X_t^0), G(t, \mathbf{0})) + H(G(t, \mathbf{0}), \{0\}).$$

Using the Lipschitz condition and the continuity of G , we get

$$\begin{aligned} H(G(t, X_t^0), \{0\}) &\leq L\rho(X_t^0, \mathbf{0}) + H(G(t, \mathbf{0}), \{0\}) \\ &\leq L\rho(X_t^0, \mathbf{0}) + \sup_{t \in I} H(G(t, \mathbf{0}), \{0\}) < \infty. \end{aligned}$$

A similar estimate we obtain for F , that is,

$$H(F(t, X_t^0), \{0\}) \leq L\rho(X_t^0, \mathbf{0}) + \sup_{t \in I} H(F(t, \mathbf{0}), \{0\}) < \infty.$$

Let us denote $M = 2L\rho(X_t^0, \mathbf{0}) + \sup_{t \in I} H(G(t, \mathbf{0}), \{0\}) + \sup_{t \in I} H(F(t, \mathbf{0}), \{0\})$ and observe that for $t \in I$, due to (P3),

$$\begin{aligned} H(X^1(t), X^0(t)) &= H\left(\left[\chi_0(0) + \int_{t_0}^t G(s, X_s^0) ds\right] \ominus \int_{t_0}^t F(s, X_s^0) ds, \chi_0(0)\right) \\ &\leq H\left(\chi_0(0) + \int_{t_0}^t G(s, X_s^0) ds, \chi_0(0)\right) + H\left(\int_{t_0}^t F(s, X_s^0) ds, \{0\}\right). \end{aligned}$$

Further, by (P1) and (P6),

$$\begin{aligned} H(X^1(t), X^0(t)) &\leq H\left(\int_{t_0}^t G(s, X_s^0) ds, \{0\}\right) + H\left(\int_{t_0}^t F(s, X_s^0) ds, \{0\}\right) \\ &\leq \int_{t_0}^t H(G(s, X_s^0), \{0\}) ds + \int_{t_0}^t H(F(s, X_s^0), \{0\}) ds \\ &\leq M(t - t_0). \end{aligned}$$

Then note that, for $n \in \{2, 3, 4, \dots\}$ and $t \in I$, we have

$$\begin{aligned} H(X^n(t), X^{n-1}(t)) &\leq H\left(\int_{t_0}^t G(s, X_s^{n-1}) ds, \int_{t_0}^t G(s, X_s^{n-2}) ds\right) \\ &\quad + H\left(\int_{t_0}^t F(s, X_s^{n-1}) ds, \int_{t_0}^t F(s, X_s^{n-2}) ds\right) \\ &\leq \int_{t_0}^t H(G(s, X_s^{n-1}), G(s, X_s^{n-2})) ds \\ &\quad + \int_{t_0}^t H(F(s, X_s^{n-1}), F(s, X_s^{n-2})) ds. \end{aligned}$$

Now by the Lipschitz continuity assumption of G and F ,

$$H(X^n(t), X^{n-1}(t)) \leq 2L \int_{t_0}^t \rho(X_s^{n-1}, X_s^{n-2}) ds.$$

Since

$$\begin{aligned} \rho(X_s^{n-1}, X_s^{n-2}) &= \sup_{u \in [-\theta, 0]} H(X^{n-1}(s+u), X^{n-2}(s+u)) \\ &= \sup_{u \in [s-\theta, s]} H(X^{n-1}(u), X^{n-2}(u)), \end{aligned}$$

we have

$$H(X^n(t), X^{n-1}(t)) \leq 2L \int_{t_0}^t \sup_{u \in [s-\theta, s]} H(X^{n-1}(u), X^{n-2}(u)) ds.$$

Thus, we can infer that, for every $n \in \{1, 2, 3, \dots\}$ and $t \in I$,

$$H(X^n(t), X^{n-1}(t)) \leq \frac{M}{2L} \cdot \frac{[2L(t-t_0)]^n}{n!}. \tag{4}$$

This implies

$$\sup_{t \in J} H(X^n(t), X^{n-1}(t)) \leq \frac{M}{2L} \cdot \frac{[2LT]^n}{n!}, \tag{5}$$

and consequently for natural k, ℓ such that $k > \ell$,

$$\begin{aligned} \sup_{t \in J} H(X^k(t), X^\ell(t)) &\leq \sum_{n=\ell}^{k-1} \sup_{t \in J} H(X^{n+1}(t), X^n(t)) \\ &\leq \frac{M}{2L} \sum_{n=\ell}^{k-1} \frac{[2LT]^{n+1}}{(n+1)!}. \end{aligned}$$

Now it is obvious that

$$\lim_{k, \ell \rightarrow \infty} \sup_{t \in J} H(X^k(t), X^\ell(t)) = 0,$$

and $\{X^n(\cdot)\}$ is a Cauchy sequence in the complete metric space $C(J, \text{CompConv}(\mathbb{R}^d))$ endowed with supremum metric. Therefore, there is $X \in C(J, \text{CompConv}(\mathbb{R}^d))$ such that

$$\lim_{n \rightarrow \infty} \sup_{t \in J} H(X^n(t), X(t)) = 0.$$

The limit mapping X is a solution to Equation (1). We shall show it now. Let us begin with an observation that the initial condition is satisfied, that is, $X(t) = \chi_0(t - t_0)$ because $X^n(t) = \chi_0(t - t_0)$ for each $n \in \{0, 1, 2, \dots\}$ and each $t \in [t_0 - \theta, t_0]$. The next step is to show that $H(X(t) + \int_{t_0}^t F(s, X_s) ds, \chi_0(0) + \int_{t_0}^t G(s, X_s) ds) = 0$ for $t \in I$. Notice that

$$\begin{aligned} &H\left(X(t) + \int_{t_0}^t F(s, X_s) ds, \chi_0(0) + \int_{t_0}^t G(s, X_s) ds\right) \\ &\leq H\left(X(t) + \int_{t_0}^t F(s, X_s) ds, X^n(t) + \int_{t_0}^t F(s, X_s^{n-1}) ds\right) \\ &\quad + H\left(\chi_0(0) + \int_{t_0}^t G(s, X_s^{n-1}) ds, \chi_0(0) + \int_{t_0}^t G(s, X_s) ds\right), \end{aligned}$$

and therefore

$$\begin{aligned}
 & H\left(X(t) + \int_{t_0}^t F(s, X_s)ds, \chi_0(0) + \int_{t_0}^t G(s, X_s)ds\right) \\
 & \leq H(X(t), X^n(t)) + \int_{t_0}^t H(F(s, X_s), F(s, X_s^{n-1}))ds \\
 & \quad + \int_{t_0}^t H(G(s, X_s^{n-1}), G(s, X_s))ds \\
 & \leq H(X(t), X^n(t)) + 2L \int_{t_0}^t \rho(X_s, X_s^{n-1})ds \\
 & \leq H(X(t), X^n(t)) + 2LT \sup_{u \in J} H(X(u), X^{n-1}(u)).
 \end{aligned}$$

Since the right-hand side of the inequality above converges to zero for every $t \in I$, we obtain the desired property, that is,

$$H\left(X(t) + \int_{t_0}^t F(s, X_s)ds, \chi_0(0) + \int_{t_0}^t G(s, X_s)ds\right) = 0 \quad \text{for } t \in I.$$

To see that the solution X is the unique solution to Equation (1), let us assume that $Y: J \rightarrow \text{CompConv}(\mathbb{R}^d)$ is another solution to Equation (1). The fulfillment of the initial condition by X and Y implies that $Y(t) = X(t)$ for $t \in [t_0 - \theta, t_0]$. We shall show that $Y(t) = X(t)$ also for $t \in I$. Indeed, for $t \in I$,

$$\begin{aligned}
 \sup_{u \in [t_0, t]} H(Y(u), X(u)) &= \sup_{u \in [t_0, t]} H\left(\left[\chi_0(0) + \int_{t_0}^u G(s, Y_s)ds\right] \ominus \int_{t_0}^u F(s, Y_s)ds, \right. \\
 & \quad \left. \left[\chi_0(0) + \int_{t_0}^u G(s, X_s)ds\right] \ominus \int_{t_0}^u F(s, X_s)ds\right) \\
 &\leq \sup_{u \in [t_0, t]} \int_{t_0}^u H(G(s, Y_s), G(s, X_s))ds \\
 & \quad + \sup_{u \in [t_0, t]} \int_{t_0}^u H(F(s, Y_s), F(s, X_s))ds \\
 &\leq \int_{t_0}^t H(G(s, Y_s), G(s, X_s))ds \\
 & \quad + \int_{t_0}^t H(F(s, Y_s), F(s, X_s))ds.
 \end{aligned}$$

By the Lipschitz continuity assumption,

$$\begin{aligned}
 \sup_{u \in [t_0, t]} H(Y(u), X(u)) &\leq 2L \int_{t_0}^t \rho(Y_s, X_s)ds = 2L \int_{t_0}^t \sup_{u \in [s-\tau, s]} H(Y(u), X(u))ds \\
 &\leq 2L \int_{t_0}^t \sup_{u \in [t_0, s]} H(Y(u), X(u))ds.
 \end{aligned}$$

Now, by the Gronwall–Bellman inequality (Lemma 1), we arrive at

$$\sup_{u \in [t_0, t]} H(Y(u), X(u)) \leq 0 \quad \text{for every } t \in I,$$

and this leads us straightforwardly to $H(Y(t), X(t)) = 0$ for $t \in I$. Hence, the uniqueness of the solution X is proved. □

As in the previous part of this article, we will deal with the issue of estimating the magnitude of the solution value, assuming the Lipschitz character of the mappings F and G .

Proposition 3. *Let the assumptions of Theorem 2 be satisfied. Let X denote the unique solution to Equation (1). Then,*

$$\sup_{t \in J} H(X(t), \{0\}) \leq \left((1 + 2LT)\rho(\chi_0, \mathbf{0}) + \int_I H(G(s, \mathbf{0}), \{0\}) ds + \int_I H(F(s, \mathbf{0}), \{0\}) ds \right) e^{2LT}.$$

Proof. It is obvious that $\sup_{t \in [t_0 - \theta, t_0]} H(X(t), \{0\}) = \rho(\chi_0, \mathbf{0})$. For $t \in I = [t_0, t_0 + T]$

$$\begin{aligned} \sup_{u \in [t_0, t]} H(X(u), \{0\}) &= \sup_{u \in [t_0, t]} H\left(\left[\chi_0(0) + \int_{t_0}^u G(s, X_s) ds\right] \ominus \int_{t_0}^u F(s, X_s) ds, \{0\}\right) \\ &\leq H(\chi_0(0), \{0\}) + \int_{t_0}^t H(G(s, X_s), \{0\}) ds \\ &\quad + \int_{t_0}^t H(F(s, X_s), \{0\}) ds \\ &\leq \rho(\chi_0, \mathbf{0}) + 2L \int_{t_0}^t \rho(X_s, \mathbf{0}) ds \\ &\quad + \int_{t_0}^t H(G(s, \mathbf{0}), \{0\}) ds + \int_{t_0}^t H(F(s, \mathbf{0}), \{0\}) ds \\ &\leq \rho(\chi_0, \mathbf{0}) + 2L \int_{t_0}^t \rho(X_s, \mathbf{0}) ds \\ &\quad + \int_I H(G(s, \mathbf{0}), \{0\}) ds + \int_I H(F(s, \mathbf{0}), \{0\}) ds. \end{aligned}$$

Since

$$\begin{aligned} \rho(X_s, \mathbf{0}) &= \sup_{u \in [s - \theta, s]} H(X(u), \{0\}) \\ &\leq \sup_{u \in [t_0 - \theta, s]} H(X(u), \{0\}) \\ &\leq \sup_{u \in [t_0 - \theta, t_0]} H(X(u), \{0\}) + \sup_{u \in [t_0, s]} H(X(u), \{0\}), \end{aligned}$$

for $s \geq t_0$, we have

$$\begin{aligned} \sup_{u \in [t_0, t]} H(X(u), \{0\}) &\leq (1 + 2LT)\rho(\chi_0, \mathbf{0}) + 2L \int_{t_0}^t \sup_{u \in [t_0, s]} H(X(u), \{0\}) ds \\ &\quad + \int_I H(G(s, \mathbf{0}), \{0\}) ds + \int_I H(F(s, \mathbf{0}), \{0\}) ds. \end{aligned}$$

Applying the Gronwall–Bellman inequality, we get

$$\begin{aligned} \sup_{u \in [t_0, t]} H(X(u), \{0\}) &\leq \left((1 + 2LT)\rho(\chi_0, \mathbf{0}) + \int_I H(G(s, \mathbf{0}), \{0\}) ds + \int_I H(F(s, \mathbf{0}), \{0\}) ds \right) e^{2L(t-t_0)} \text{ for } t \in I. \end{aligned}$$

Putting this result together with the initial observation in the proof, we obtain the assertion. □

In the next step, we will analyze the behavior of the solution to Equation (1) in a situation where, instead of the initial history χ_0 , we use a history χ_0^ε that is slightly different from χ_0 in the sense that the distance between these two histories is small. It is desirable that the solutions corresponding to these two initial histories also differ only slightly. Otherwise, the theory of equations of type (1) would not be well-posed. The following result guarantees the expected property.

Proposition 4. *Let $F, G: I \times C_\theta \rightarrow \text{CompConv}(\mathbb{R}^d)$ and $\chi_0, \chi_0^\varepsilon \in C_\theta$. Suppose F, G, χ_0 and F, G, χ_0^ε , both systems of data, satisfy the assumptions of Theorem 2. Then,*

$$\sup_{t \in J} H(X^\varepsilon(t), X(t)) \leq \rho(\chi_0^\varepsilon, \chi_0)(1 + 2LT)e^{2LT},$$

where X denotes the unique solution to (1) with data F, G, χ_0 and X^ε symbolizes the unique solution to (1) with data F, G, χ_0^ε .

Proof. First, it is obvious that

$$\sup_{t \in [t_0 - \theta, t_0]} H(X^\varepsilon(t), X(t)) = \rho(\chi_0^\varepsilon, \chi_0).$$

For $t \in [t_0, t_0 + T]$, we can write

$$\begin{aligned} \sup_{u \in [t_0, t]} H(X^\varepsilon(u), X(u)) &= \sup_{u \in [t_0, t]} H\left([\chi_0^\varepsilon(0) + \int_{t_0}^u F(s, X_s^\varepsilon) ds] \ominus \int_{t_0}^u G(s, X_s^\varepsilon) ds, \right. \\ &\quad \left. [\chi_0(0) + \int_{t_0}^u F(s, X_s) ds] \ominus \int_{t_0}^u G(s, X_s) ds\right) \\ &\leq H(\chi_0^\varepsilon(0), \chi_0(0)) + \int_{t_0}^t H(F(s, X_s^\varepsilon), F(s, X_s)) ds \\ &\quad + \int_{t_0}^t H(G(s, X_s^\varepsilon), G(s, X_s)) ds. \end{aligned}$$

The Lipschitz type assumption imposed on F and G leads us to

$$\begin{aligned} \sup_{u \in [t_0, t]} H(X^\varepsilon(u), X(u)) &\leq H(\chi_0^\varepsilon(0), \chi_0(0)) + 2L \int_{t_0}^t \rho(X_s^\varepsilon, X_s) ds \\ &\leq \rho(\chi_0^\varepsilon, \chi_0) + 2L \int_{t_0}^t \rho(X_s^\varepsilon, X_s) ds. \end{aligned}$$

Going forward,

$$\begin{aligned} \sup_{u \in [t_0, t]} H(X^\varepsilon(u), X(u)) &\leq \rho(\chi_0^\varepsilon, \chi_0) + 2L \int_{t_0}^t \sup_{u \in [s - \theta, s]} H(X^\varepsilon(u), X(u)) ds \\ &\leq \rho(\chi_0^\varepsilon, \chi_0) + 2L \int_{t_0}^t \sup_{u \in [t_0 - \theta, t_0]} H(X^\varepsilon(u), X(u)) ds \\ &\quad + 2L \int_{t_0}^t \sup_{u \in [t_0, s]} H(X^\varepsilon(u), X(u)) ds \\ &\leq \rho(\chi_0^\varepsilon, \chi_0)(1 + 2LT) + 2L \int_{t_0}^t \sup_{u \in [t_0, s]} H(X^\varepsilon(u), X(u)) ds. \end{aligned}$$

So now, using the Gronwall–Bellman inequality we arrive at

$$\sup_{u \in [t_0, t]} H(X^\varepsilon(t), X(t)) \leq \rho(\chi_0^\varepsilon, \chi_0)(1 + 2LT)e^{2L(t-t_0)} \quad \text{for } t \in I,$$

and this allows us to formulate the assertion. □

The question arises of whether a similar property to the above can be obtained when the coefficients F and G of Equation (1) change slightly. The affirmative answer is given by the following statement.

Proposition 5. *Let $F, G, F^\epsilon, G^\epsilon: I \times C_\theta \rightarrow \text{CompConv}(\mathbb{R}^d)$ and $\chi_0 \in C_\theta$ be such that*

$$\max\{H(F^\epsilon(t, \chi), F(t, \chi)), H(G^\epsilon(t, \chi), G(t, \chi))\} \leq \epsilon \text{ for all } (t, \chi) \in I \times C_\theta,$$

where ϵ is a positive real number. Suppose F, G, χ_0 and $F^\epsilon, G^\epsilon, \chi_0$, both systems of data, satisfy the assumptions of Theorem 2. Then,

$$\sup_{t \in J} H(X^\epsilon(t), X(t)) \leq 2Te^{2LT}\epsilon,$$

where X denotes the unique solution to (1) with data F, G, χ_0 and X^ϵ symbolizes the unique solution to (1) with data $F^\epsilon, G^\epsilon, \chi_0$.

Proof. Let us mention that $\sup_{t \in [t_0 - \theta, t_0]} H(X^\epsilon(t), X(t)) = 0$. For $t \in [t_0, t_0 + T]$, we have

$$\begin{aligned} \sup_{u \in [t_0, t]} H(X^\epsilon(u), X(u)) &= \sup_{u \in [t_0, t]} H\left([\chi_0(0) + \int_{t_0}^u F^\epsilon(s, X_s^\epsilon) ds] \ominus \int_{t_0}^u G^\epsilon(s, X_s^\epsilon) ds, \right. \\ &\quad \left. [\chi_0(0) + \int_{t_0}^u F(s, X_s) ds] \ominus \int_{t_0}^u G(s, X_s) ds\right) \\ &\leq \int_{t_0}^t H(F^\epsilon(s, X_s^\epsilon), F(s, X_s)) ds + \int_{t_0}^t H(G^\epsilon(s, X_s^\epsilon), G(s, X_s)) ds \\ &\leq \int_{t_0}^t H(F^\epsilon(s, X_s^\epsilon), F(s, X_s^\epsilon)) ds + \int_{t_0}^t H(F(s, X_s^\epsilon), F(s, X_s)) ds \\ &\quad + \int_{t_0}^t H(G^\epsilon(s, X_s^\epsilon), G(s, X_s^\epsilon)) ds + \int_{t_0}^t H(G(s, X_s^\epsilon), G(s, X_s)) ds \end{aligned}$$

Applying the assumptions, we get

$$\sup_{u \in [t_0, t]} H(X^\epsilon(u), X(u)) \leq 2(t - t_0)\epsilon + 2L \int_{t_0}^t \sup_{u \in [t_0, s]} H(X^\epsilon(u), X(u)) ds.$$

Invoking the Gronwall–Bellman inequality again, we get

$$\sup_{u \in [t_0, t]} H(X^\epsilon(t), X(t)) \leq 2Te^{2L(t-t_0)}\epsilon \text{ for } t \in [t_0, t_0 + T],$$

which leads us to the assertion. □

4. Conclusions

In this paper, we consider set-valued functional equations with set-valued integrals on both sides of the equation,

$$X(t) + \int_{t_0}^t F(s, X_s) ds = \chi_0(0) + \int_{t_0}^t G(s, X_s) ds \text{ for } t \in [t_0, t_0 + T],$$

with initial condition

$$X_{t_0} = \chi_0.$$

This form of an equation cannot be reduced to an equation with only one integral. The sets $X(t)$, being the values at the time t of the solution X , for such equations do not have

to have a monotonically changing diameter, unlike the equations with the integral on one side of the equation only. The first main results obtained in the paper are the justification of the existence of solutions for the above set-valued functional equations, assuming the continuity and integrally boundedness of the mappings F and G . In this case, we also found that the set of all solutions has the topological compactness property. Then we proved the existence and uniqueness of the solution under a stronger assumption than continuity, namely that F and G satisfy the Lipschitz condition. In this setting, we justified that the solutions show continuity according to the initial condition and the coefficients, F and G , of the equation. These properties are crucial from the point of view of the well-posedness of the theory of such equations, as well as from the perspective of their future applications in practice, where the initial values and relationships governing the dynamics of modeled phenomena may be burdened with a slight ε -error. This paper can be a good theoretical basis for future use in the modeling of real life tasks by researchers specializing in mathematical modeling, for example, in diagnosing cancer [20,21]. Future research may also concern theoretical problems. The justification for the existence of solutions to Equation (1) in a more general case than the case of continuous coefficients F and G would be good. The existence and uniqueness of the solution to the problem (1), assuming a weaker condition than Lipschitz continuity in the functional variable, would also be in the range of interests.

Funding: This research received no external funding.

Conflicts of Interest: The author declares no conflict of interest.

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