

Poisson–Lie Groups and Gauge Theory

Catherine Meusburger 

Department Mathematik, Friedrich-Alexander-Universität Erlangen-Nürnberg, Cauerstraße 11,
91058 Erlangen, Germany; catherine.meusburger@math.fau.de

Abstract: We review Poisson–Lie groups and their applications in gauge theory and integrable systems from a mathematical physics perspective. We also comment on recent results and developments and their applications. In particular, we discuss the role of quasitriangular Poisson–Lie groups and dynamical r -matrices in the description of moduli spaces of flat connections and the Chern–Simons gauge theory.

Keywords: Poisson–Lie groups; Lie bialgebras and r -matrices; integrable models; Poisson homogeneous spaces



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1. Poisson–Lie Groups

This review article originated from a lecture series on Poisson–Lie groups given at the 12th International ICMAT Summer School on Geometry, Mechanics and Control in Santiago de Compostela. It is meant to provide an introduction that is accessible to physicists and mathematicians with no background on this topic, while at the same time covering current results.

Due to the large body of work on Poisson–Lie groups and their numerous applications, it is impossible to do justice to all the work on this topic. For this reason, we focus on certain aspects and omit many others. For each result, we either cite the work where it was first discovered or, where appropriate and available, refer the reader to a textbook or review article and the references therein.

Short and accessible introductions to the topic with emphasis on different aspects are given in lecture notes by A. G. Reyman [1], by Y. Kosmann-Schwarzbach [2], by M. Audin [3], which also contain detailed lists of references. A good introduction to Poisson geometry and Poisson–Lie groups is the textbook [4]. Other textbooks that cover Poisson–Lie groups are [5,6]. For a textbook presenting Poisson–Lie groups from the point of view of integrable systems, see [7]. For an accessible introduction to dynamical r -matrices, see the textbook [8].

Motivation

A Poisson–Lie group is a Lie group that is also a Poisson manifold in such a way that its multiplication is a Poisson map. Poisson–Lie groups arise in the description of integrable systems and in the context of two- and three-dimensional gauge theories. They can be viewed as the classical counterparts of quantum group symmetries in certain quantum systems, such as quantum integrable systems and quantised Chern–Simons or BF-Theories. Both, quantum groups and Poisson–Lie groups, admit an infinitesimal description in terms of Lie algebras with additional structure that is the infinitesimal counterpart of the Poisson structure. Such a Lie algebra is called a Lie bialgebra.

One motivation for Poisson–Lie groups arises from Lie group actions on Poisson manifolds. Phase spaces of physical systems are usually Poisson manifolds, and Lie group actions on these manifolds arise from physical or gauge symmetries of these systems. This includes integrable systems and constrained Hamiltonian systems.

The functions that are invariant under such Lie group actions play a distinguished role. They correspond to functions on their orbit space and Poisson commute with the Hamiltonians for the Lie group actions. In this sense, they are conserved quantities of the system. If the group action under consideration represents a gauge symmetry, the group action relates different points on an extended, non-physical phase space that correspond to the same physical state of the system. In this case, the invariant functions represent functions on the gauge invariant phase space, the physical observables of the theory.

The functions that are invariant under a Lie group action form a subalgebra of the algebra of all functions on the Poisson manifold. The question is if they form a *Poisson subalgebra*, or, in other words, if the Poisson bracket of two invariant functions is again invariant. This must hold in particular for the conserved quantities of an integrable system and for the physical observables of a theory with gauge symmetries. Poisson actions of Poisson–Lie groups are a natural setting in which this is the case.

Definition 1.

1. A **Poisson manifold** is a smooth manifold M together with a map

$$\{ , \} : C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M),$$

the **Poisson bracket**, that satisfies

- (a) **antisymmetry**: $\{f_1, f_2\} = -\{f_2, f_1\}$ for all $f_1, f_2 \in C^\infty(M)$.
- (b) **Leibniz identity**: $\{f_1 \cdot f_2, f_3\} = \{f_1, f_3\}f_2 + \{f_2, f_3\}f_1$.
- (c) **Jacobi identity**: $\{\{f_1, f_2\}, f_3\} + \{\{f_2, f_3\}, f_1\} + \{\{f_3, f_1\}, f_2\} = 0$.

2. A **Poisson map** from a Poisson manifold $(M, \{ , \}_M)$ to a Poisson manifold $(N, \{ , \}_N)$ is a smooth map $\phi : M \rightarrow N$ that satisfies

$$\{f_1 \circ \phi, f_2 \circ \phi\}_M = \{f_1, f_2\}_N \circ \phi \quad \forall f_1, f_2 \in C^\infty(N).$$

Example 1.

1. Every smooth manifold M is a Poisson manifold with the **trivial Poisson structure** $\{f_1, f_2\} = 0$ for all $f_1, f_2 \in C^\infty(M)$. The identity map $\text{id}_M : M \rightarrow M$ is a Poisson map for any Poisson manifold M .
2. If M and N are Poisson manifolds, then $M \times N$ becomes a Poisson manifold with the **product Poisson structure**

$$\{f_1, f_2\}_{M \times N}(m, n) = \{f_1(-, n), f_2(-, n)\}_M(m) + \{f_1(m, -), f_2(m, -)\}_N(n). \quad (1)$$

The **projections** $\pi_M : M \times N \rightarrow M, (m, n) \mapsto m$ and $\pi_N : M \times N \rightarrow N, (m, n) \mapsto n$ are Poisson maps. For all Poisson maps $\phi : M \rightarrow M'$ and $\psi : N \rightarrow N'$, the map $\phi \times \psi : M \times N \rightarrow M' \times N', (m, n) \mapsto (\phi(m), \psi(n))$ is a Poisson map.

3. A **symplectic manifold** (M, ω) is a smooth manifold M together with a non-degenerate closed 2-form ω on M . Every symplectic manifold (M, ω) is a Poisson manifold with the Poisson bracket

$$\{f_1, f_2\} = X_{f_1} \cdot f_2 \quad \text{with the vector field } X_f \text{ defined by } df = \omega(X_f, -). \quad (2)$$

The non-degeneracy of ω is required to define the vector field X_f for any function $f \in C^\infty(M)$. The antisymmetry of ω ensures the antisymmetry, and the closedness of ω ensures the Jacobi identity for the Poisson bracket.

4. A Poisson manifold with a Poisson bracket of the form (2) is called a **symplectic Poisson manifold**. A Poisson manifold M is symplectic if and only if $\{f_1, f_2\}(m) = 0$ for all $f_2 \in C^\infty(M)$ implies that $df_1(m) = 0$. In this case, (2) defines a 2-form ω on M , and the Jacobi identity ensures that ω is closed.

5. For every smooth manifold M , the **cotangent bundle** T^*M has a canonical symplectic structure. If we interpret functions on M and vector fields on M as functions on T^*M , their Poisson brackets are given by

$$\{f_1, f_2\} = 0 \quad \{X_1, f_1\} = X_1 \cdot f_1 \quad \{X_1, X_2\} = [X_1, X_2],$$

where $f_1, f_2 \in C^\infty(M)$, $X_1, X_2 \in \text{Vec}(M)$ and $[\cdot, \cdot]$ denotes the Lie bracket of vector fields

$$[X, Y] \cdot f = X \cdot (Y \cdot f) - Y \cdot (X \cdot f) \quad \forall X, Y \in \text{Vec}(M), f \in C^\infty(M). \quad (3)$$

We now consider Lie group actions on Poisson manifolds. In the following, we assume that all Lie groups are finite-dimensional.

Definition 2.

1. A **Lie group** G is a smooth manifold G with a group structure such that the multiplication map $\mu : G \times G \rightarrow G$, $(g, h) \rightarrow gh$ and the inversion $i : G \rightarrow G$, $g \mapsto g^{-1}$ are smooth.
2. A **Lie group action** of G on a smooth manifold M is a smooth map $\triangleright : G \times M \rightarrow M$, $(g, m) \mapsto g \triangleright m$ with

$$(gh) \triangleright m = g \triangleright (h \triangleright m) \quad 1 \triangleright m = m \quad \forall g, h \in G, m \in M.$$

- The set $G \triangleright m = \{g \triangleright m \mid g \in G\}$ for $m \in M$ is called the **orbit** of $m \in M$.
- The set $G \backslash M = \{G \triangleright m \mid m \in M\}$ of orbits is called the **orbit space**.
- The set of **invariant functions** on M is denoted

$$C^\infty(M)^G = \{f \in C^\infty(M) \mid f(g \triangleright m) = f(m) \forall m \in M, g \in G\}.$$

Example 2.

1. If G and H are Lie groups, then $G \times H$ is a Lie group with $(g, h) \cdot (g', h') = (gg', hh')$. It is called the **direct product** of G and H .
2. \mathbb{R}^n and \mathbb{C}^n with the usual addition are Lie groups.
3. Any closed subgroup of the group $GL(n, \mathbb{C})$ of invertible $n \times n$ -matrices is a Lie group. A Lie group of this form is called a **matrix Lie group**. This includes the
 - **special linear groups**

$$SL(n, \mathbb{C}) = \{M \in \text{Mat}(n \times n, \mathbb{C}) \mid \det M = 1\}$$

$$SL(n, \mathbb{R}) = \{M \in \text{Mat}(n \times n, \mathbb{R}) \mid \det M = 1\}$$

- **unitary groups**

$$U(n) = \{M \in \text{Mat}(n \times n, \mathbb{C}) \mid M^\dagger M = 1\}$$

- **special unitary groups**

$$SU(n) = U(n) \cap SL(n, \mathbb{C})$$

- **orthogonal groups**

$$O(p, q) = \{M \in \text{Mat}(n \times n, \mathbb{R}) \mid M^T I_{p,q} M = I_{p,q}\} \quad I_{p,q} = \begin{pmatrix} -1_p & 0 \\ 0 & 1_q \end{pmatrix}$$

$$O(n) = O(0, n) \quad \text{with } p, q \geq 0, p + q = n$$

- **special orthogonal groups**

$$SO(p, q) = O(p, q) \cap SL(n, \mathbb{R}) \quad SO(n) = SO(0, n)$$

- **symplectic groups**

$$\mathrm{Sp}(n, \mathbb{R}) = \{M \in \mathrm{Mat}(2n \times 2n, \mathbb{R}) \mid M^T I_n M = I_n\} \quad I_n = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$$

- Any Lie group acts on itself by left multiplication $\triangleright : G \times G \rightarrow G$, $g \triangleright h = gh$, by right multiplication with inverses $\triangleright : G \times G \rightarrow G$, $g \triangleright h = hg^{-1}$, and by conjugation $\triangleright : G \times G \rightarrow G$, $g \triangleright h = ghg^{-1}$.
- Any matrix Lie group $G \subset \mathrm{GL}(n, \mathbb{C})$ acts on \mathbb{C}^n by $\triangleright : G \times \mathbb{C}^n \rightarrow \mathbb{C}^n$, $M \triangleright v = M \cdot v$ and any matrix Lie group $G \subset \mathrm{GL}(n, \mathbb{R})$ acts on \mathbb{R}^n by $\triangleright : G \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $M \triangleright v = M \cdot v$.
- The orbits of the group action $\triangleright : \mathrm{SO}(n) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $M \triangleright v = M \cdot v$ are the origin and the $(n-1)$ -dimensional **spheres** of radius $r > 0$

$$S_r^{n-1} = \{v \in \mathbb{R}^n \mid v_1^2 + \dots + v_n^2 = r^2\}.$$

- The orbits of the group action $\triangleright : \mathrm{O}(1, n-1) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $M \triangleright v = M \cdot v$ are the origin, the timelike or two-sheeted hyperboloids

$$H_{r+}^{n-1} = \{v \in \mathbb{R}^n \mid v_1^2 - v_2^2 - \dots - v_n^2 = r^2\} \quad r > 0,$$

the spacelike or one-sheeted hyperboloids

$$H_{r-}^{n-1} = \{v \in \mathbb{R}^n \mid v_1^2 - v_2^2 - \dots - v_n^2 = -r^2\} \quad r > 0,$$

and the light cone

$$L^{n-1} = \{v \in \mathbb{R}^n \mid v_1^2 - v_2^2 - \dots - v_n^2 = 0, v \neq 0\}.$$

- If G and H are Lie groups and $\triangleright : G \times H \rightarrow H$ is a smooth group action such that $g \triangleright - : H \rightarrow H$ is a homomorphism of Lie groups for all $g \in G$, then $G \times H$ is a Lie group with the group multiplication

$$(g, h) \cdot (g', h') = (gg', h \cdot (g \triangleright h')).$$

It is called the **semidirect product** of G and H and denoted $G \ltimes H$.

- Examples are the n -dimensional **Euclidean group** $E_n = \mathrm{SO}(n) \ltimes \mathbb{R}^n$ and the n -dimensional **Poincaré group** $P_n = \mathrm{SO}(1, n-1) \ltimes \mathbb{R}^n$.

Leaving aside questions of smoothness, one sees that invariant functions on M correspond to functions on orbit space. Every invariant function on M is constant on each orbit and defines a function on the orbit space. Similarly, every function on the orbit space can be lifted to an invariant function on M . This raises the question:

Given a Poisson manifold M with a Lie group action $\triangleright : G \times M \rightarrow M$, what is a practical condition on the group action that ensures that the Poisson bracket of two invariant functions is again invariant?

It seems plausible to impose that the maps $g \triangleright - : M \rightarrow M$, $m \mapsto g \triangleright m$ are Poisson maps for all $g \in G$. Indeed, this condition implies $\{f_1, f_2\} \in C^\infty(M)^G$ for all $f_1, f_2 \in C^\infty(M)^G$. However, it turns out that this is too restrictive and excludes many interesting examples.

To relax this requirement, we impose that the Lie group G is equipped with a Poisson structure and that the action map $\triangleright : G \times M \rightarrow M$ is a Poisson map with respect to the Poisson structure on M and the *product Poisson structure* on $G \times M$ from Example 1, 2.

Indeed, if $\triangleright : G \times M \rightarrow M$ is a Poisson map, we obtain for all $f_1, f_2 \in C^\infty(M)^G$, $g \in G$ and $m \in M$

$$\begin{aligned} \{f_1, f_2\}_M(g \triangleright m) &= \{f_1, f_2\}_M \circ \triangleright(g, m) = \{f_1 \circ \triangleright, f_2 \circ \triangleright\}_{G \times M}(g, m) \\ &= \{f_1(g \triangleright -), f_2(g \triangleright -)\}_M(m) + \{f_1(- \triangleright m), f_2(- \triangleright m)\}_G(g) = \{f_1, f_2\}_M(m), \end{aligned} \quad (4)$$

since $f \in C^\infty(M)^G$ implies $f(g \triangleright -) = f : M \rightarrow \mathbb{R}$ and $f(- \triangleright m) : G \rightarrow \mathbb{R}$ constant. Thus, the Poisson bracket of two invariant functions is again invariant. The stricter condition that $g \triangleright - : M \rightarrow M$ is a Poisson map for all $g \in G$ arises as a special case, namely the one where G is equipped with the trivial Poisson bracket.

It remains to check if there are additional conditions that should be imposed on the Poisson structure on G . The condition that $\triangleright : G \times M \rightarrow M$ is a group action implies that the diagram

$$\begin{array}{ccc} G \times G \times M & \xrightarrow{\text{id}_G \times \triangleright} & G \times M \\ \mu \otimes \text{id}_M \downarrow & & \downarrow \triangleright \\ G \times M & \xrightarrow{\triangleright} & M \end{array} \quad (5)$$

commutes, where $\mu : G \times G \rightarrow G$ is the group multiplication. As $\triangleright : G \times M \rightarrow M$ is Poisson, three of the four arrows in the diagram are Poisson maps. It is thus natural to demand that this holds for the multiplication map $\mu : G \times G \rightarrow G$ as well. Another motivation to impose this is that every Lie group G acts on itself by left multiplication. This group action is used as a starting point in many constructions and hence should be required to be Poisson if one wants to carry over those constructions to a Poisson setting.

Definition 3 ([9]).

1. A **Poisson–Lie group** is a Lie group G that is also a Poisson manifold in such a way that the multiplication $\mu : G \times G \rightarrow G$, $(g, h) \mapsto gh$ is a Poisson map with respect to the Poisson structure on G and the product Poisson structure on $G \times G$:

$$\{f_1, f_2\}(gh) = \{f_1(g-), f_2(g-)\}(h) + \{f_1(-h), f_2(-h)\}(g) \quad \forall f_1, f_2 \in C^\infty(G). \quad (6)$$

2. A **homomorphism of Poisson–Lie groups** from G to H is a homomorphism of Lie groups $\phi : G \rightarrow H$ that is also a Poisson map.

Remark 1. A Poisson–Lie group is never symplectic, since (6) implies for $f_1, f_2 \in C^\infty(G)$

$$\begin{aligned} \{f_1, f_2\}(1) &= \{f_1(1-), f_2(1-)\}(1) + \{f_1(-1), f_2(-1)\}(1) = 2\{f_1, f_2\}(1) \\ &\Rightarrow \{f_1, f_2\}(1) = 0. \end{aligned}$$

Example 3.

1. Every Lie group G becomes a Poisson–Lie group with the trivial Poisson structure.
2. If $(G, \{ , \}_G)$ and $(H, \{ , \}_H)$ are Poisson–Lie groups then, $(G \times H, \{ , \}_{G \times H})$ is a Poisson–Lie group. This follows directly from (1) and (6).
3. The Lie group $G = \mathbb{R} \ltimes \mathbb{R}$ with group multiplication

$$(x, y) \cdot (x', y') = (x + x', y + e^x y'),$$

can be identified with the matrix Lie group

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ y & e^x \end{pmatrix} \mid x, y \in \mathbb{R} \right\} \subset GL(2, \mathbb{R}).$$

Due to the Leibniz identity and antisymmetry, any Poisson structure on G is of the form

$$\{f_1, f_2\}(x, y) = C(x, y)(\partial_x f_1 \partial_y f_2 - \partial_y f_1 \partial_x f_2)(x, y) \quad \text{with } C \in C^\infty(\mathbb{R}^2). \quad (7)$$

A short computation shows that the Jacobi identity is satisfied for all brackets of the form (7). The compatibility (6) between Poisson structure and group multiplication translates into the requirement

$$C(x + x', y + e^x y') = e^x C(x', y') + C(x, y),$$

which is satisfied if and only if $C(x, y) = a(e^x - 1) + by$ for some constants $a, b \in \mathbb{R}$. Hence, all Poisson–Lie structures on G are of the form

$$\{f_1, f_2\}(x, y) = (a(e^x - 1) + by)(\partial_x f_1 \partial_y f_2 - \partial_y f_1 \partial_x f_2)(x, y) \quad a, b \in \mathbb{R}.$$

4. We consider the Lie group $GL(2, \mathbb{R}) = \{M \in Mat(2 \times 2, \mathbb{R}) \mid \det M \neq 0\}$. One can show that up to isomorphisms of Poisson–Lie groups, there are three inequivalent Poisson–Lie structures $\{, \}_1, \{, \}_2, \{, \}_3$ on $GL(2, \mathbb{R})$. In terms of the matrix element functions

$$a_{ij} : GL(2, \mathbb{R}) \rightarrow \mathbb{R}, \quad M = (m_{kl}) \mapsto m_{ij}$$

they are given by the following formulas from [1], up to some typos that are corrected here:

$$\begin{aligned} \{a_{11}, a_{12}\}_1 &= a_{11}a_{12} & \{a_{11}, a_{12}\}_2 &= a_{11}a_{22} - a_{12}a_{21} + a_{11}^2 + a_{12}^2 & \{a_{11}, a_{12}\}_3 &= a_{11}^2 - a_{12}a_{21} + a_{11}a_{22} \\ \{a_{11}, a_{21}\}_1 &= a_{11}a_{21} & \{a_{11}, a_{21}\}_2 &= a_{11}a_{22} - a_{12}a_{21} + a_{11}^2 + a_{21}^2 & \{a_{11}, a_{21}\}_3 &= a_{21}^2 \\ \{a_{11}, a_{22}\}_1 &= 2a_{12}a_{21} & \{a_{11}, a_{22}\}_2 &= (a_{12} + a_{21})(a_{11} + a_{22}) & \{a_{11}, a_{22}\}_3 &= a_{21}(a_{11} + a_{22}) \\ \{a_{12}, a_{21}\}_1 &= 0 & \{a_{12}, a_{21}\}_2 &= (a_{12} - a_{21})(a_{11} - a_{22}) & \{a_{12}, a_{21}\}_3 &= a_{21}(a_{22} - a_{11}) \\ \{a_{12}, a_{22}\}_1 &= a_{12}a_{22} & \{a_{12}, a_{22}\}_2 &= a_{11}a_{22} - a_{12}a_{21} + a_{12}^2 + a_{22}^2 & \{a_{12}, a_{22}\}_3 &= a_{11}a_{22} - a_{12}a_{21} + a_{22}^2 \\ \{a_{21}, a_{22}\}_1 &= a_{21}a_{22} & \{a_{21}, a_{22}\}_2 &= a_{11}a_{22} - a_{12}a_{21} + a_{21}^2 + a_{22}^2 & \{a_{21}, a_{22}\}_3 &= a_{21}^2. \end{aligned}$$

As the determinant function $\det = a_{11}a_{22} - a_{12}a_{21}$ satisfies $\{\det, a_{ij}\}_k = 0$ for all $i, j \in \{1, 2\}$ and $k \in \{1, 2, 3\}$, they induce Poisson–Lie structures on the subgroup $SL(2, \mathbb{R})$.

We now consider smooth group actions of Poisson–Lie groups G on Poisson manifolds M , for which the action maps $\triangleright : G \times M \rightarrow M$ are Poisson.

Definition 4. Let G be a Poisson–Lie group.

1. A **Poisson- G space** is a Poisson manifold M with a smooth group action $\triangleright : G \times M \rightarrow M$ that is a Poisson map with respect to the Poisson structure on M and the product Poisson structure on $G \times M$:

$$\begin{aligned} \{f_1, f_2\}_M(g \triangleright m) &= \{f_1 \circ (g \triangleright -), f_2 \circ (g \triangleright -)\}_M(m) \\ &\quad + \{f_1 \circ (- \triangleright m), f_2 \circ (- \triangleright m)\}_G(g) \quad \forall g \in G, m \in M. \end{aligned} \quad (8)$$

2. A **homomorphism of Poisson- G spaces** from M to N is a Poisson map $\phi : M \rightarrow N$ that intertwines the G -actions on M and N :

$$\phi(g \triangleright_M m) = g \triangleright_N \phi(m) \quad \forall g \in G, m \in M.$$

Corollary 1. If $(M, \{, \}, \triangleright)$ is a Poisson G -space, then $C^\infty(M)^G \subset C^\infty(M)$ is a Poisson subalgebra: $\{f_1, f_2\} \in C^\infty(M)^G$ for all $f_1, f_2 \in C^\infty(M)^G$.

Example 4.

1. Let $(G, \{, \}_G)$ be a Poisson–Lie group and $\overline{G} = (G, -\{, \}_G)$. Then G is a Poisson space over itself and over \overline{G} with the group actions

$$\triangleright : G \times G \rightarrow G, (g, h) \mapsto gh \quad \triangleright : \overline{G} \times G \rightarrow G, (g, h) \mapsto hg^{-1}.$$

As these group actions commute, this gives G the structure of a Poisson $G \times \overline{G}$ -space.

2. A **Poisson–Lie subgroup** of a Poisson–Lie group G is a closed subgroup $H \subset G$ that is a Poisson–Lie group and for which the inclusion $\iota : H \rightarrow G, h \mapsto h$ is a Poisson map.

If $H \subset G$ is a Poisson–Lie subgroup, then there is a unique Poisson structure on the homogeneous space G/H for which the projection $\pi : G \rightarrow G/H, g \mapsto gH$ is a Poisson map. The homogeneous space G/H with this Poisson structure and the canonical G -action $\triangleright : G \times G/H \rightarrow G/H, g \triangleright (g'H) = (gg')H$ is a Poisson G -space:

$$\begin{aligned} \{f_1, f_2\}_{G/H}(g \triangleright g'H) &= \{f_1, f_2\}_{G/H} \circ \pi(gg') = \{f_1 \circ \pi, f_2 \circ \pi\}_G(gg') \\ &= \{f_1 \circ \pi(g-), f_2 \circ \pi(g-)\}_G(g') + \{f_1 \circ \pi(-g'), f_2 \circ \pi(-g')\}_G(g) \\ &= \{f_1(g \triangleright -), f_2(g \triangleright -)\}_{G/H}(g'H) + \{f_1(- \triangleright g'H), f_2(- \triangleright g'H)\}_G(g), \end{aligned}$$

where we used first that π is Poisson, then that G is a Poisson–Lie group to pass to the second line and then the identity $\pi(-g') = - \triangleright g'H$ and that π is Poisson to pass to the third line.

2. Poisson–Lie Groups and Lie Bialgebras

In general, it is not easy to construct or classify Poisson–Lie groups. To systematically investigate Poisson–Lie groups, one uses the same strategy as for Lie groups, namely to consider the associated infinitesimal structures. For every Lie group the tangent space at the unit element is a Lie algebra and every finite-dimensional Lie algebra exponentiates to a unique connected and simply connected Lie group. To extend these statements to Poisson–Lie groups, one determines the infinitesimal structures induced by their Poisson brackets.

2.1. Lie Bialgebras

We first introduce some background on Lie algebras. In the following, all Lie algebras will be real and finite-dimensional unless stated otherwise.

Definition 5.

1. A **Lie algebra** \mathfrak{g} is a real vector space \mathfrak{g} together with an antisymmetric linear map $[,] : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}, X \otimes Y \mapsto [X, Y]$, the **Lie bracket**, that satisfies the **Jacobi identity**

$$[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0 \quad \forall X, Y, Z \in \mathfrak{g}.$$

2. A **Lie algebra homomorphism** from \mathfrak{g} to \mathfrak{h} is a linear map $\phi : \mathfrak{g} \rightarrow \mathfrak{h}$ with

$$\phi \circ [,]_{\mathfrak{g}} = [,]_{\mathfrak{h}} \circ (\phi \otimes \phi).$$

Example 5.

1. For every Poisson manifold M , the smooth functions on M form a Lie algebra $C^\infty(M)$ with the Poisson bracket.
2. For every smooth manifold M , the vector fields on M form an infinite-dimensional Lie algebra $\text{Vec}(M)$ with the Lie bracket

$$[X, Y].f = X.(Y.f) - Y.(X.f) \quad \forall X, Y \in \text{Vec}(M), f \in C^\infty(M). \quad (9)$$

3. Every associative algebra A becomes a Lie algebra with the **commutator**

$$[A, B] := A \cdot B - B \cdot A$$

In particular, this applies to $\mathfrak{gl}(n, \mathbb{C}) = \text{Mat}(n \times n, \mathbb{C})$ and to every \mathbb{R} -linear subspace $V \subset \text{Mat}(n \times n, \mathbb{C})$ with $[V, V] \subset V$.

4. Examples of the latter are the following Lie algebras with the commutator brackets:

- $\mathfrak{sl}(n, \mathbb{C}) = \{M \in \text{Mat}(n \times n, \mathbb{C}) \mid \text{tr}(M) = 0\}$
- $\mathfrak{sl}(n, \mathbb{R}) = \{M \in \text{Mat}(n \times n, \mathbb{R}) \mid \text{tr}(M) = 0\}$
- $\mathfrak{u}(n) = \{M \in \text{Mat}(n \times n, \mathbb{C}) \mid M^\dagger = -M\}$
- $\mathfrak{su}(n) = \mathfrak{u}(n) \cap \mathfrak{sl}(n, \mathbb{C})$
- $\mathfrak{o}(p, q) = \{M \in \text{Mat}(n \times n, \mathbb{R}) \mid M^T I_{p,q} + I_{p,q} M = 0\}$, $\mathfrak{o}(n) = \mathfrak{o}(0, n)$
- $\mathfrak{so}(p, q) = \mathfrak{o}(p, q) \cap \mathfrak{sl}(n, \mathbb{R})$
- $\mathfrak{sp}(n, \mathbb{R}) = \{M \in \text{Mat}(n \times n, \mathbb{R}) \mid M^T I_n + I_n M = 0\}$.

5. If $\mathfrak{g}, \mathfrak{h}$ are Lie algebras then the vector space $\mathfrak{g} \oplus \mathfrak{h}$ becomes a Lie algebra with

$$[X, X'] = [X, X']_{\mathfrak{g}} \quad [X, Y] = 0 \quad [Y, Y'] = [Y, Y']_{\mathfrak{h}} \quad \forall X, X' \in \mathfrak{g}, Y, Y' \in \mathfrak{h}.$$

This is called the **direct sum** of the Lie algebras \mathfrak{g} and \mathfrak{h} .

6. If $\mathfrak{g}, \mathfrak{h}$ are Lie algebras and $\triangleright : \mathfrak{g} \times \mathfrak{h} \rightarrow \mathfrak{h}$, $(X, Y) \mapsto X \triangleright Y$ a map with

$$\begin{aligned} X \triangleright [Y, Y']_{\mathfrak{h}} &= [X \triangleright Y, Y']_{\mathfrak{h}} + [Y, X \triangleright Y']_{\mathfrak{h}} \\ [X, X']_{\mathfrak{g}} \triangleright Y &= X \triangleright (X' \triangleright Y) - X' \triangleright (X \triangleright Y), \end{aligned}$$

then the vector space $\mathfrak{g} \oplus \mathfrak{h}$ becomes a Lie algebra with the Lie bracket

$$[X, X'] = [X, X']_{\mathfrak{g}} \quad [X, Y] = X \triangleright Y \quad [Y, Y'] = [Y, Y']_{\mathfrak{h}} \quad \forall X, X' \in \mathfrak{g}, Y, Y' \in \mathfrak{h}.$$

It is called the **semidirect product** of the Lie algebras \mathfrak{g} and \mathfrak{h} and denoted $\mathfrak{g} \ltimes \mathfrak{h}$.

To introduce the additional structure on a Lie algebra that is the counterpart of the Poisson bracket on a Poisson–Lie group, we require the adjoint action of a Lie algebra \mathfrak{g} on itself and on its n -fold tensor products

$$\text{ad}_X : \mathfrak{g}^{\otimes n} \rightarrow \mathfrak{g}^{\otimes n} \quad (10)$$

$$\text{ad}_X(Y_1 \otimes \dots \otimes Y_n) = [X, Y_1] \otimes Y_2 \otimes \dots \otimes Y_n + Y_1 \otimes [X, Y_2] \otimes \dots \otimes Y_n + \dots + Y_1 \otimes \dots \otimes Y_{n-1} \otimes [X, Y_n].$$

For an element $X = X_1 \otimes \dots \otimes X_n \in \mathfrak{g}^{\otimes n}$, we set

$$\sum_{\text{cyc}} X := X_1 \otimes \dots \otimes X_n + X_2 \otimes \dots \otimes X_n \otimes X_1 + \dots + X_n \otimes X_1 \otimes \dots \otimes X_{n-1}$$

and extend this linearly to $\mathfrak{g}^{\otimes n}$. With these definitions, we can introduce the Lie algebra counterpart of a Poisson–Lie group.

Definition 6.

1. A **Lie bialgebra** is a Lie algebra $(\mathfrak{g}, [\ , \])$ together with an antisymmetric linear map $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$, the **cocommutator**, that satisfies the

$$(C1) \text{ cocycle condition: } \delta([X, Y]) = \text{ad}_X(\delta(Y)) - \text{ad}_Y(\delta(X)) \quad \forall X, Y \in \mathfrak{g}.$$

$$(C2) \text{ coJacobi identity: } \sum_{\text{cyc}} (\delta \otimes \text{id}) \circ \delta(X) = 0.$$

2. A **homomorphism of Lie bialgebras** from a Lie bialgebra \mathfrak{g} to a Lie bialgebra \mathfrak{h} is a linear map $\phi : \mathfrak{g} \rightarrow \mathfrak{h}$ that satisfies

$$\phi \circ [\ , \]_{\mathfrak{g}} = [\ , \]_{\mathfrak{h}} \circ (\phi \otimes \phi) \quad \delta_{\mathfrak{h}} \circ \phi = (\phi \otimes \phi) \circ \delta_{\mathfrak{g}}.$$

Example 6.

1. Every Lie algebra \mathfrak{g} becomes a Lie bialgebra with $\delta = 0 : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$. This corresponds to the trivial Poisson structure on a Lie group G with Lie algebra \mathfrak{g} .
2. For every Lie bialgebra $(\mathfrak{g}, [\cdot, \cdot], \delta)$ and $a, b \in \mathbb{R}$, $(\mathfrak{g}, a[\cdot, \cdot], b\delta)$ is also a Lie bialgebra.
3. ([10] (Example 1)) Consider the two-dimensional Lie algebra $\mathfrak{g} = \mathbb{R} \ltimes \mathbb{R}$ with basis $\{X, Y\}$ and Lie bracket

$$[X, X] = [Y, Y] = 0 \quad [X, Y] = Y.$$

Every antisymmetric map $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ defines a cocommutator on \mathfrak{g} . Up to isomorphisms of Lie bialgebras, there are exactly two non-trivial Lie bialgebra structures on \mathfrak{g} with

$$\begin{aligned} \delta_1(X) &= 0 & \delta_1(Y) &= X \wedge Y \\ \delta_2(X) &= X \wedge Y & \delta_2(Y) &= 0, \end{aligned}$$

where $X \wedge Y := X \otimes Y - Y \otimes X$.

4. We consider the Lie algebra $\mathfrak{sl}(2, \mathbb{R}) = \{M \in \text{Mat}(2 \times 2, \mathbb{R}) \mid \text{tr}(M) = 0\}$ with the basis

$$J_0 = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad J_1 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad J_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and Lie bracket

$$[J_0, J_1] = J_2 \quad [J_0, J_2] = -J_1 \quad [J_1, J_2] = -J_0.$$

Then, the following are non-isomorphic Lie bialgebra structures on $\mathfrak{sl}(2, \mathbb{R})$

$$\begin{aligned} \delta_1(J_0) &= J_0 \wedge J_1 & \delta_1(J_1) &= 0 & \delta_1(J_2) &= J_2 \wedge J_1 \\ \delta_2(J_0) &= 0 & \delta_2(J_1) &= J_0 \wedge J_1 & \delta_2(J_2) &= J_0 \wedge J_2 \\ \delta_3(J_0) &= J_0 \wedge J_2 & \delta_3(J_1) &= J_1 \wedge (J_2 - J_0) & \delta_3(J_2) &= J_0 \wedge J_2, \end{aligned} \quad (11)$$

where $X \wedge Y := X \otimes Y - Y \otimes X$.

Remark 2. Condition (C1) in Definition 6 is called **cocycle condition**, because it states that $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ is a 1-cocycle with values in $\mathfrak{g} \otimes \mathfrak{g}$ in the sense of Lie algebra cohomology.

If we equip $\mathfrak{g} \otimes \mathfrak{g}$ with the \mathfrak{g} -module structure $X \triangleright (Y \otimes Z) = \text{ad}_X(Y \otimes Z) = [X, Y] \otimes Z + Y \otimes [X, Z]$, then the cocycle condition takes the form

$$d\delta(X, Y) := \delta([X, Y]) - X \triangleright \delta(Y) + Y \triangleright \delta(X) = 0.$$

In Section 3, we investigate the case, where δ is not just a 1-cocycle, but a 1-coboundary.

The concept of a Lie bialgebra is symmetric with respect to vector space duals. For every Lie algebra \mathfrak{g} consider the dual vector space \mathfrak{g}^* and the **pairing** or **evaluation map**

$$\langle \cdot, \cdot \rangle : \mathfrak{g}^{*\otimes n} \times \mathfrak{g}^{\otimes n} \rightarrow \mathbb{R}, \quad \langle \alpha_1 \otimes \dots \otimes \alpha_n, X_1 \otimes \dots \otimes X_n \rangle = \alpha_1(X_1) \cdots \alpha_n(X_n). \quad (12)$$

The dual $f^* : \mathfrak{h}^{*\otimes k} \rightarrow \mathfrak{g}^{*\otimes n}$ of a linear map $f : \mathfrak{g}^{\otimes n} \rightarrow \mathfrak{h}^{\otimes k}$ is then characterised by the condition

$$\langle f^*(\alpha_1 \otimes \dots \otimes \alpha_k), x_1 \otimes \dots \otimes x_n \rangle = \langle \alpha_1 \otimes \dots \otimes \alpha_k, f(x_1 \otimes \dots \otimes x_n) \rangle \quad \forall x_i \in \mathfrak{g}, \alpha_i \in \mathfrak{h}^*. \quad (13)$$

In particular, the cocommutator $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ defines an antisymmetric map $\delta^* : \mathfrak{g}^* \otimes \mathfrak{g}^* \rightarrow \mathfrak{g}^*$, and the Lie bracket $[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ defines an antisymmetric linear map $[\cdot, \cdot]^* : \mathfrak{g}^* \rightarrow \mathfrak{g}^* \otimes \mathfrak{g}^*$. Together, these maps define a Lie bialgebra structure on \mathfrak{g}^* .

Lemma 1. If $(\mathfrak{g}, [\cdot, \cdot], \delta)$ is a Lie bialgebra, then $(\mathfrak{g}^*, \delta^*, [\cdot, \cdot]^*)$ is a Lie bialgebra. It is called the **dual Lie bialgebra** to $(\mathfrak{g}, [\cdot, \cdot], \delta)$.

Proof. With the notation $\delta' := [\cdot, \cdot]^*, [\cdot, \cdot]' := \delta^*$ and $\text{ad}'_\alpha(\beta \otimes \gamma) := [\alpha, \beta]' \otimes \gamma + \beta \otimes [\alpha, \gamma]'$ for all $\alpha, \beta, \gamma \in \mathfrak{g}^*$, we have by definition

$$\langle [\alpha, \beta]', X \rangle = \langle \alpha \otimes \beta, \delta(X) \rangle \quad \langle \delta'(\alpha), X \otimes Y \rangle = \langle \alpha, [X, Y] \rangle \quad \forall X, Y \in \mathfrak{g},$$

and this implies

$$\begin{aligned} \langle \alpha, [[X, Y], Z] \rangle &= \langle \delta'(\alpha), [X, Y] \otimes Z \rangle = \langle (\delta' \otimes \text{id}) \circ \delta'(\alpha), X \otimes Y \otimes Z \rangle \\ \langle \alpha \otimes \beta, \delta([X, Y]) \rangle &= \langle [\alpha, \beta]', [X, Y] \rangle = \langle \delta'([\alpha, \beta]'), X \otimes Y \rangle \\ \langle \alpha \otimes \beta, \text{ad}_X(\delta(Y)) - \text{ad}_Y(\delta(X)) \rangle &= \langle \text{ad}'_\alpha(\delta'(\beta)) - \text{ad}'_\beta(\delta'(\alpha)), X \otimes Y \rangle. \end{aligned}$$

The first identity and the Jacobi identity for the Lie bracket guarantees the coJacobi identity for δ' . The second identity and the coJacobi identity for δ' guarantee the Jacobi identity for $[\cdot, \cdot]'$. The last two identities show that the cocycle condition is self-dual. \square

Example 7.

1. If \mathfrak{g} is a Lie bialgebra with a trivial cocommutator, then \mathfrak{g}^* is abelian and vice versa.
2. We consider the real Lie algebra $\mathfrak{g} = \mathbb{R} \ltimes \mathbb{R}$ with basis $\{X, Y\}$ and Lie bracket

$$[X, X] = [Y, Y] = 0 \quad [X, Y] = Y$$

from Example 6, 3. with the two Lie bialgebra structures

$$\begin{aligned} \delta_1(X) &= 0 & \delta_1(Y) &= X \wedge Y \\ \delta_2(X) &= X \wedge Y & \delta_2(Y) &= 0. \end{aligned}$$

Denoting by $\{x, y\}$ the basis dual to $\{X, Y\}$, we find that the dual Lie bialgebra structures have the cocommutator and the Lie brackets

$$\delta'(x) = 0 \quad \delta'(y) = x \wedge y \quad [x, y]'_1 = y \quad [x, y]'_2 = x.$$

By exchanging the basis elements, we can transform the Lie bracket $[\cdot, \cdot]_2$ into $[\cdot, \cdot]_1$, and we find that the Lie bialgebras $(\mathfrak{g}, [\cdot, \cdot], \delta_1)$ and $(\mathfrak{g}, [\cdot, \cdot], \delta_2)$ are self-dual.

3. We consider the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ with the basis

$$J_0 = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad J_1 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad J_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (14)$$

and the three Lie bialgebra structures on $\mathfrak{sl}(2, \mathbb{R})$ from Example 6, given in (11). Denoting by $\{j^0, j^1, j^2\}$ the dual basis with $\langle j^i, J_k \rangle = \delta_k^i$, we find that the cocommutator of $\mathfrak{sl}(2, \mathbb{R})^*$ is given by

$$\delta'(j^0) = j^2 \wedge j^1 \quad \delta'(j^1) = j^2 \wedge j^0 \quad \delta'(j^2) = j^0 \wedge j^1$$

and the Lie brackets by

$$\begin{aligned} [j^0, j^1]'_1 &= j^0 & [j^0, j^2]'_1 &= 0 & [j^1, j^2]'_1 &= j^2 \\ [j^0, j^1]'_2 &= j^1 & [j^0, j^2]'_2 &= j^2 & [j^1, j^2]'_2 &= 0 \\ [j^0, j^1]'_3 &= j^1 & [j^0, j^2]'_3 &= j^0 + j^2 & [j^1, j^2]'_3 &= j^1. \end{aligned}$$

The Lie brackets $[\cdot, \cdot]_1'$ and $[\cdot, \cdot]_2'$ define non-isomorphic Lie algebras that are semidirect products $\mathbb{R} \ltimes \mathbb{R}^2$ of the abelian Lie algebras \mathbb{R} and \mathbb{R}^2 . By passing to a different basis, we see that the Lie algebra with bracket $[\cdot, \cdot]_3'$ is isomorphic to the one with bracket $[\cdot, \cdot]_2'$:

$$[\frac{1}{2}(j^0 - j^2), \frac{1}{2}(j^0 + j^2)]_3' = \frac{1}{2}(j^0 + j^2) \quad [\frac{1}{2}(j^0 + j^2), j^1]_3' = 0 \quad [\frac{1}{2}(j^0 - j^2), j^1]_3' = j^1.$$

2.2. Tangent Lie Bialgebras of a Poisson–Lie Group

We now show that Lie bialgebras are the infinitesimal structures associated with Poisson–Lie groups. This requires some preliminaries about the relation between Lie groups and Lie algebras. Every Lie group defines a Lie algebra, which is given as its tangent space at the unit element. Its Lie bracket can be characterised in terms of the exponential map $\exp : T_1G \rightarrow G$. If G is a matrix Lie group, this is just the usual matrix exponential. The relation between Lie groups and Lie algebras is then summarised by the following theorem.

Theorem 1. Let G be a Lie group. Then:

1. The tangent space $\mathfrak{g} = T_1G$ is a Lie algebra with the Lie bracket

$$[X, Y] = \left. \frac{d^2}{dsdt} \right|_{s=t=0} e^{tX} \cdot e^{sY} \cdot e^{-tX},$$

where $\exp : \mathfrak{g} \rightarrow G$, $X \mapsto e^X$ is the exponential map.

2. For every smooth group homomorphism $\phi : G \rightarrow H$, the tangent map $T_1\phi : \mathfrak{g} \rightarrow \mathfrak{h}$ is a homomorphism of Lie algebras.
3. For every finite-dimensional real Lie algebra \mathfrak{g} , there is a unique connected and simply connected Lie group G with $T_1G = \mathfrak{g}$.
4. If G and H are connected and simply connected with Lie algebras $T_1G = \mathfrak{g}$ and $T_1H = \mathfrak{h}$, then, for every Lie algebra homomorphism $\phi' : \mathfrak{g} \rightarrow \mathfrak{h}$, there is a unique smooth group homomorphism $\phi : G \rightarrow H$ with $T_1\phi = \phi'$.
5. There is a canonical group action of G on \mathfrak{g} , the **adjoint action**

$$\triangleright_{\text{Ad}} : G \times \mathfrak{g} \rightarrow \mathfrak{g}, \quad (g, x) \mapsto \text{Ad}(g)X = \left. \frac{d}{dt} \right|_{t=0} g \cdot e^{tX} g^{-1}. \quad (15)$$

For all $g \in G$, the maps $\text{Ad}(g) = g \triangleright_{\text{Ad}} - : \mathfrak{g} \rightarrow \mathfrak{g}$ are Lie algebra homomorphisms. Differentiating the adjoint action yields the Lie algebra homomorphisms

$$\text{ad}_X : \mathfrak{g} \rightarrow \mathfrak{g}, \quad Y \mapsto [X, Y] = \left. \frac{d}{dt} \right|_{t=0} \text{Ad}(e^{tX})Y. \quad (16)$$

Example 8.

1. The Lie algebras of the matrix Lie groups $GL(n, \mathbb{C})$ and $GL(n, \mathbb{R})$ are $\mathfrak{gl}(n, \mathbb{C}) = \text{Mat}(n \times n, \mathbb{C})$ and $\mathfrak{gl}(n, \mathbb{R}) = \text{Mat}(n \times n, \mathbb{R})$ with the matrix commutator as the Lie bracket.
2. For every matrix Lie group $G \subset GL(n, \mathbb{C})$, the associated Lie algebra is an \mathbb{R} -linear subspace $V \subset \text{Mat}(n \times n, \mathbb{C})$ that is closed under the matrix commutator. The Lie algebras for the Lie groups from Example 2, are the Lie algebras in Example 5:
 - $T_1\text{SL}(n, \mathbb{C}) = \mathfrak{sl}(n, \mathbb{C}) = \{M \in \text{Mat}(n \times n, \mathbb{C}) \mid \text{tr}(M) = 0\}$
 - $T_1\text{SL}(n, \mathbb{R}) = \mathfrak{sl}(n, \mathbb{R}) = \{M \in \text{Mat}(n \times n, \mathbb{R}) \mid \text{tr}(M) = 0\}$
 - $T_1\text{U}(n) = \mathfrak{u}(n) = \{M \in \text{Mat}(n \times n, \mathbb{C}) \mid M^\dagger = -M\}$
 - $T_1\text{SU}(n, \mathbb{C}) = \mathfrak{su}(n) = \mathfrak{u}(n) \cap \mathfrak{sl}(n, \mathbb{C})$
 - $T_1\text{O}(p, q) = \mathfrak{o}(p, q) = \{M \in \text{Mat}(n \times n, \mathbb{R}) \mid M^T I_{p,q} + I_{p,q} M = 0\}$
 - $T_1\text{O}(n) = \mathfrak{o}(n) = \mathfrak{o}(0, n)$
 - $T_1\text{SO}(p, q) = \mathfrak{so}(p, q) = \mathfrak{o}(p, q) \cap \mathfrak{sl}(n, \mathbb{R})$

- $T_1\mathrm{Sp}(n, \mathbb{R}) = \mathfrak{sp}(n, \mathbb{R}) = \{M \in \mathrm{Mat}(n \times n, \mathbb{R}) \mid M^T I_n + I_n M = 0\}$.
- 3. If G and H are Lie groups with Lie algebras \mathfrak{g} and \mathfrak{h} , then the Lie algebra of the direct product $G \times H$ is the direct sum $\mathfrak{g} \oplus \mathfrak{h}$.
- 4. If G, H are Lie groups and $\triangleright : G \times H \rightarrow H$ a smooth group action such that $g \triangleright - : H \rightarrow H$ is a group homomorphism for all $g \in G$, then the Lie algebra of the semidirect product $G \ltimes H$ from Example 2 is the semidirect product $\mathfrak{g} \ltimes \mathfrak{h}$ with

$$\begin{aligned} [X, X'] &= [X, X']_{\mathfrak{g}} \quad [Y, Y'] = [Y, Y']_{\mathfrak{h}} \\ [X, Y] &= \frac{d^2}{dsdt} \Big|_{t=s=0} e^{tX} \triangleright e^{sY} \quad \forall X, X' \in \mathfrak{g}, Y, Y' \in \mathfrak{h}. \end{aligned}$$

To describe the Poisson bracket on a Poisson–Lie group concretely, it is useful to work with distinguished vector fields, namely the vector fields associated with the group action of the Lie group on itself by left and right multiplication, the *right* and *left invariant* vector fields.

Definition 7. Let G be a Lie group and $\triangleright : G \times M \rightarrow M$ a smooth group action of G on a smooth manifold M .

1. The **action vector field** $X^{\triangleright} \in \mathrm{Vec}(M)$ for $X \in \mathfrak{g}$ is given by

$$(X^{\triangleright} \cdot f)(m) = \frac{d}{dt} \Big|_{t=0} f(e^{-tX} \triangleright m) \quad \forall m \in M, f \in C^{\infty}(M). \quad (17)$$

2. The **right invariant** and **left invariant** vector fields on G are the action vector fields for the action of G on itself by left and right multiplication

$$(X^L \cdot f)(g) = \frac{d}{dt} \Big|_{t=0} f(e^{-tX} g) \quad (X^R \cdot f)(g) = \frac{d}{dt} \Big|_{t=0} f(g e^{tX}) \quad \forall g \in G, f \in C^{\infty}(G). \quad (18)$$

Remark 3.

1. The vector fields X^L are called **right invariant** and the vector fields X^R **left invariant**, since they commute with the right and left multiplication maps $R_h : G \rightarrow G, g \mapsto gh$ and $L_h : G \rightarrow G, g \mapsto hg$ for $h \in G$:

$$(X^L \cdot f)(R_h(g)) = (X^L \cdot (f \circ R_h))(g) \quad (X^R \cdot f)(L_h(g)) = (X^R \cdot (f \circ L_h))(g).$$

This implies that they are determined uniquely by their value at the unit element $1 \in G$:

$$(X^L \cdot f)(g) = X^L \cdot (f \circ R_g)(1) \quad (X^R \cdot f)(g) = X^R \cdot (f \circ L_g)(1). \quad (19)$$

For any basis $\{T_a\}$ of \mathfrak{g} and any $g \in G$, the sets $\{T_a^L(g)\}$ and $\{T_a^R(g)\}$ are bases of $T_g G$.

2. The left- and right invariant vector fields are related by

$$X^L(g) = -(\mathrm{Ad}(g^{-1})X)^R(g) \quad X^R(g) = -(\mathrm{Ad}(g)X)^L(g) \quad \forall g \in G, X \in \mathfrak{g}. \quad (20)$$

3. The vector fields associated with a group action $\triangleright : G \times M \rightarrow M$ form a Lie subalgebra of the Lie algebra of vector fields on M , since we have

$$\begin{aligned} [X^{\triangleright}, Y^{\triangleright}] \cdot f(m) &= X^{\triangleright} \cdot (Y^{\triangleright} \cdot f)(m) - Y^{\triangleright} \cdot (X^{\triangleright} \cdot f)(m) \\ &= \frac{d^2}{dtds} \Big|_{s=t=0} f(e^{tX} e^{-sY} e^{-tX} \triangleright m) = \frac{d}{dt} \Big|_{t=0} (\mathrm{Ad}(e^{tX})Y)^{\triangleright} \cdot f(m) \\ &= [X, Y]^{\triangleright} \cdot f(m). \end{aligned} \quad (21)$$

4. In particular, the right and left invariant vector fields on a Lie group G form Lie subalgebras of $\text{Vec}(G)$ isomorphic to \mathfrak{g} .

To relate Poisson–Lie groups and Lie bialgebras, we describe the Poisson bracket of a Poisson–Lie group G in terms of a **Poisson bivector** $B_G \in \Lambda^2(TG)$ that assigns to every point $g \in G$ an antisymmetric element $B(g) \in T_g G \otimes T_g G$. The Poisson bracket is then given by

$$\{f_1, f_2\}_G = B.(f_1 \otimes f_2).$$

By Remark 3, 1. the right and left invariant vector fields associated with a basis $\{T_a\}$ of $\mathfrak{g} = T_1 G$ form bases of $T_g G$ at every point $g \in G$. We can therefore describe the Poisson bivector and the Poisson–Lie structure by its coefficient functions $B^{ab} \in C^\infty(G)$ with respect to one of these bases:

$$B = B^{ab} T_a^L \otimes T_b^L \quad \{f_1, f_2\}_G = B^{ab} (T_a^L \cdot f_1) (T_b^L \cdot f_2), \quad (22)$$

where we use Einstein summation conventions and sum over repeated upper and lower indices. As the Poisson bracket is antisymmetric, we have $B^{ab} = -B^{ba}$. The action of the left and right invariant vector fields on the Poisson bivector at the unit element then defines the cocommutator on the Lie algebra $\mathfrak{g} = T_1 G$.

Theorem 2 ([9]).

1. Let G be a Poisson–Lie group with Poisson bivector $B = B^{ab} T_a^L \otimes T_b^L$. Then, its Lie algebra \mathfrak{g} is a Lie bialgebra with cocommutator given by

$$\begin{aligned} \delta(X)^R.(f_1 \otimes f_2)(1) &= X^R.\{f_1, f_2\}(1) = -X^L.\{f_1, f_2\}(1) \quad \forall f_1, f_2 \in C^\infty(G) \\ \delta(X) &= (X^R.B^{ab})(1) T_a \otimes T_b \quad \forall X \in \mathfrak{g}. \end{aligned} \quad (23)$$

It is called the **tangent Lie bialgebra** of G .

2. Every homomorphism of Poisson–Lie groups $\phi : G \rightarrow H$ induces a Lie bialgebra homomorphism $T_1 \phi : \mathfrak{g} \rightarrow \mathfrak{h}$ between their tangent Lie bialgebras.
3. For every Lie bialgebra \mathfrak{g} , there is a unique connected and simply connected Poisson–Lie group G with tangent Lie bialgebra \mathfrak{g} .
4. Every homomorphism of Lie bialgebras $\phi' : \mathfrak{g} \rightarrow \mathfrak{h}$ lifts to a unique Lie group homomorphism $\phi : G \rightarrow H$ with $T_1 \phi = \phi'$ between the associated connected and simply connected Poisson–Lie groups G, H .

Proof. We prove the Statements 1. and 2. Statements 3. and 4. then follow by integrating the structures on the Lie bialgebras to the associated connected and simply connected Lie group.

1. As the Poisson bracket of a Lie group vanishes at the unit element by Remark 1, we have $B^{ab}(1) = 0$ for all coefficient functions B^{ab} of the Poisson bivector. This implies

$$\delta(X)^R.(f_1 \otimes f_2)(1) = X^R.\{f_1, f_2\}(1) = (X^R.B^{ab})(T_a^L \cdot f_1)(T_b^L \cdot f_2)(1),$$

and hence $\delta(X) = (X^R.B^{ab}) T_a \otimes T_b$. The antisymmetry of the Poisson bracket guarantees the antisymmetry of δ . The cocycle condition follows because the multiplication is a Poisson map. To see this, we express this condition in terms of the Poisson bivector. Using the identities

$$X^L.(f \circ L_g)(h) = (\text{Ad}(g)X)^L.f(gh) \quad X^L.(f \circ R_h)(g) = X^L.f(gh)$$

that follow directly from the definitions of the right and left invariant vector fields, we obtain

$$\begin{aligned}\{f_1, f_2\}(gh) &= B^{ab}(gh)(T_a^L \otimes T_b^L) \cdot (f_1 \otimes f_2)(gh) \\ &= \{f_1(g-), f_2(g-)\}(h) + \{f_1(-h), f_2(-h)\}(g) \\ &= B^{ab}(h) \cdot (\text{Ad}(g)T_a \otimes \text{Ad}(g)T_b)^L \cdot (f_1 \otimes f_2)(gh) + B^{ab}(g)(T_a^L \otimes T_b^L) \cdot (f_1 \otimes f_2)(gh).\end{aligned}$$

Hence, we have shown that

$$B^{ab}(gh) = B^{ab}(g) + (\text{Ad}(g) \otimes \text{Ad}(g))B(h) \quad \forall g, h \in G. \quad (24)$$

Together with the formulas for the left invariant vector fields and their Lie bracket, this implies

$$\begin{aligned}\delta([X, Y]) &= ([X, Y]^R \cdot B)(1) = (X^R \cdot (Y^R \cdot B^{ab})(1) - Y^R \cdot (X^R \cdot B^{ab})(1))T_a \otimes T_b \\ &= \text{ad}_X(\delta(Y)) - \text{ad}_Y(\delta(X)).\end{aligned}$$

The coJacobi identity follows from the Jacobi identity for the Poisson bracket on G . With formula (23) we obtain

$$(\delta \otimes \text{id}) \circ \delta(X) = -(X^L \cdot B^{ab})(1) \delta(T_a) \otimes T_b = (X^L \cdot B^{ab})(1)(T_a^L \cdot B^{cd})(1)T_c \otimes T_d \otimes T_b.$$

As the Poisson bivector vanishes at the unit element, this yields

$$\begin{aligned}X^L \cdot \{\{f_1, f_2\}, f_3\}(1) &= (X^L \cdot B^{ab})(1)T_a^L \cdot \{f_1, f_2\}(1)T_b^L \cdot f_3(1) \\ &= (X^L \cdot B^{ab})(1)(T_a^L \cdot B^{cd})(1)(T_c^L \otimes T_d^L \otimes T_b^L) \cdot (f_1 \otimes f_2 \otimes f_3)(1) = (\delta \otimes \text{id}) \circ \delta(X)^L \cdot (f_1 \otimes f_2 \otimes f_3)(1),\end{aligned}$$

and by applying the Jacobi identity for the Poisson bracket, we obtain

$$\begin{aligned}0 &= X^L \cdot (\{\{f_1, f_2\}, f_3\} + \{\{f_2, f_3\}, f_1\} + \{\{f_3, f_1\}, f_2\})(1) \\ &= (\Sigma_{\text{cycl}}(\delta \otimes \text{id}) \circ \delta(X)) \cdot (f_1 \otimes f_2 \otimes f_3)(1)\end{aligned}$$

for all $f_1, f_2, f_3 \in C^\infty(G)$. This proves the coJacobi identity.

2. It is a standard result from the Lie theory that for any homomorphism of Lie groups $\phi : G \rightarrow H$ the tangent map $T_1\phi : \mathfrak{g} \rightarrow \mathfrak{h}$ is a homomorphism of Lie algebras and satisfies

$$X^R \cdot (f \circ \phi) = T\phi(X^R) \cdot f = ((T_1\phi)(X)^R \cdot f) \circ \phi \quad \forall X \in \mathfrak{g}. \quad (25)$$

If $\phi : G \rightarrow H$ is a homomorphism of Poisson–Lie groups, it is also a Poisson map, and from formula (23), we obtain for all $X \in \mathfrak{g}$

$$\begin{aligned}\delta_G(X)^R \cdot ((f_1 \circ \phi) \otimes (f_2 \circ \phi))(1_G) &\stackrel{(25)}{=} ((T_1\phi \otimes T_1\phi) \circ \delta_G(X))^R \cdot (f_1 \otimes f_2)(1_H) \\ &\stackrel{(23)}{=} X^R \cdot \{f_1 \circ \phi, f_2 \circ \phi\}_G(1_G) = X^R \cdot (\{f_1, f_2\}_H \circ \phi)(1_G) \stackrel{(25)}{=} (T_1\phi(X)^R \cdot \{f_1, f_2\}_H)(1_H) \\ &= (\delta_H \circ T_1\phi(X))^R \cdot (f_1 \otimes f_2)(1_H).\end{aligned}$$

Hence, $\delta_H \circ T_1\phi = (T_1\phi \otimes T_1\phi) \circ \delta_G$, and $T_1\phi$ is a homomorphism of Lie bialgebras. \square

Example 9.

We consider the Poisson–Lie group $G = \mathbb{R} \times \mathbb{R}$ from Example 3, 3. with

$$\begin{aligned}(x, y) \cdot (x', y') &= (x + x', y + e^x y') \\ \{f_1, f_2\}(x, y) &= (a(e^x - 1) + by)((\partial_x f_1)(\partial_y f_2) - (\partial_x f_2)(\partial_y f_1)).\end{aligned}$$

By identifying G with the matrix Lie group

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ y & e^x \end{pmatrix} \middle| x, y \in \mathbb{R} \right\} \subset GL(2, \mathbb{C})$$

we obtain the following basis of the Lie algebra $T_1 G = \mathfrak{g}$

$$X = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{d}{dt} \bigg|_{t=0} \begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \frac{d}{dt} \bigg|_{t=0} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

in which the Lie bracket reads $[X, Y] = Y$. The left invariant vector fields on \mathfrak{g} are given by

$$\begin{aligned} X^R \cdot f(x, y) &= d_t|_{t=0} f(x + t, y) = \partial_x f(x, y) \\ Y^R \cdot f(x, y) &= d_t|_{t=0} f(x, y + e^x t) = e^x \partial_y f(x, y). \end{aligned}$$

This implies

$$\begin{aligned} X^R \cdot \{f_1, f_2\}(0, 0) &= a(X \wedge Y)^R \cdot (f_1 \otimes f_2)(0, 0) \\ Y^R \cdot \{f_1, f_2\}(0, 0) &= b(X \wedge Y)^R \cdot (f_1 \otimes f_2)(0, 0) \end{aligned}$$

and shows that the cocommutator is given by $\delta(X) = aX \wedge Y$, $\delta(Y) = bX \wedge Y$. This shows that the tangent Lie bialgebra of G is the Lie bialgebra $(\mathbb{R} \ltimes \mathbb{R}, a\delta_2 + b\delta_1)$ from Example 6.

Theorem 2 generalises the relation between Lie groups and Lie algebras. It shows that the additional structure of a Poisson–Lie group, the Poisson bracket, corresponds to the additional structure of the Lie bialgebra, the cocommutator. The ambiguity in exponentiation is the standard one between Lie groups and Lie algebras. The Lie (bi)algebras are determined uniquely by the (Poisson–)Lie groups, while uniqueness in passing from Lie (bi)algebras to (Poisson–)Lie groups requires connectedness and simply-connectedness.

Keeping this ambiguity in mind, one can straightforwardly extend concepts from Lie bialgebras to Poisson–Lie groups. In particular, this applies to the notion of a dual. The relation between Poisson–Lie groups and Lie bialgebras implies that the Poisson bracket and multiplication of a Poisson–Lie group define the multiplication and Poisson bracket of another Poisson–Lie group.

Definition 8. Two Poisson–Lie groups G, H are called **dual** to each other, if their tangent Lie bialgebras are dual Lie bialgebras.

Example 10.

1. If G is a Lie group with the trivial Poisson bracket, then \mathfrak{g} is a Lie bialgebra with the trivial cocommutator, and \mathfrak{g}^* is abelian. In this case, the connected and simply connected dual is the abelian Lie group $G^* = \mathfrak{g}^*$ with the vector addition as the group multiplication and the Poisson bracket

$$\{f_1, f_2\}(\alpha) = \langle \alpha, [d_\alpha f_1, d_\alpha f_2] \rangle \quad \forall f_1, f_2 \in C^\infty(\mathfrak{g}^*), \alpha \in \mathfrak{g}^*. \quad (26)$$

2. The Poisson–Lie group $(\mathbb{R} \ltimes \mathbb{R}, \delta)$ from Example 3 and Example 9 is self-dual, since by Example 7, its Lie bialgebra is self-dual.

3. Coboundary and Quasitriangular Poisson–Lie Groups

3.1. Coboundary and Quasitriangular Lie Bialgebras

Theorem 2 largely reduces the task of finding Poisson–Lie structures on a Lie group G to finding Lie bialgebra structures on its Lie algebra \mathfrak{g} . However, this is still difficult, since a cocommutator has to obey two nontrivial equations simultaneously, the cocycle condition and the coJacobi identity. The latter corresponds to the Jacobi identity of a Lie

bracket on the dual vector space \mathfrak{g}^* . The former is a compatibility condition between the Lie algebra structures on \mathfrak{g} and \mathfrak{g}^* .

The name *cocycle condition* and Remark 2 suggest a way of replacing this condition by a simpler one. This is to consider *coboundaries*, linear maps $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$, $X \mapsto \text{ad}_X(u)$, with a fixed element $u \in \mathfrak{g} \otimes \mathfrak{g}$. As every cocycle is a coboundary, the cocycle condition is then satisfied automatically. The antisymmetry of δ follows if we require u to be antisymmetric and the coJacobi identity translates into the following condition.

Proposition 1 ([9]). *Let \mathfrak{g} be a Lie algebra and $u \in \mathfrak{g} \otimes \mathfrak{g}$ antisymmetric. Then $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$, $X \mapsto \text{ad}_X(u)$ is a cocommutator if and only if the **Schouten bracket** of u*

$$\begin{aligned} [[u, u]] &:= [u_{12}, u_{13}] + [u_{12}, u_{23}] + [u_{13}, u_{23}] \\ &= u^{ab} u^{cd} ([T_a, T_c] \otimes T_b \otimes T_d + T_a \otimes [T_b, T_c] \otimes T_d + T_a \otimes T_c \otimes [T_b, T_d]) \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \end{aligned} \quad (27)$$

is ad-invariant: $\text{ad}_X([u, u]) = 0$ for all $X \in \mathfrak{g}$.

Proof. A direct computation shows that the map $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$, $X \mapsto \text{ad}_X(u)$ satisfies the cocycle condition for any element $u \in \mathfrak{g} \otimes \mathfrak{g}$. To show that the coJacobi identity is satisfied if and only if $[[u, u]]$ is ad-invariant, we choose a basis $\{T_a\}$ of \mathfrak{g} . Then, u takes the form $u = u^{ab} T_a \otimes T_b$ with $u^{ab} = -u^{ba}$ and

$$\begin{aligned} (\delta \otimes \text{id}) \circ \delta(x) &= u^{ab} (\delta([x, T_a]) \otimes T_b + \delta(T_a) \otimes [x, T_b]) \\ &= u^{ab} u^{cd} ([x, T_a], T_c] \otimes T_d \otimes T_b + T_c \otimes [x, T_a], T_d] \otimes T_b \\ &\quad + [T_a, T_c] \otimes T_d \otimes [x, T_b] + T_c \otimes [T_a, T_d] \otimes [x, T_b]). \end{aligned} \quad (28)$$

Using the antisymmetry of u together with the antisymmetry and the Jacobi identity of the Lie bracket, one obtains after some computations

$$\Sigma_{\text{cyc}}(\delta \otimes \text{id}) \circ \delta(X) = -\text{ad}_X([u, u]).$$

□

Given an element $u \in \mathfrak{g} \otimes \mathfrak{g}$ that defines a cocommutator on \mathfrak{g} , we can modify u by adding an ad-invariant element of $t \in \mathfrak{g} \otimes \mathfrak{g}$ without changing its cocommutator. If additionally t is *symmetric*, this does not affect the ad-invariance of the Schouten bracket.

Lemma 2. *Let \mathfrak{g} be a Lie algebra.*

1. *If $r = t + u \in \mathfrak{g} \otimes \mathfrak{g}$ with $u \in \mathfrak{g} \otimes \mathfrak{g}$ antisymmetric and $t \in \mathfrak{g} \otimes \mathfrak{g}$ ad-invariant, one has*

$$[[r, r]] = [[t + u, t + u]] = [[t, t]] + [[u, u]].$$

2. *If $t \in \mathfrak{g} \otimes \mathfrak{g}$ is ad-invariant, then $[[t, t]]$ is ad-invariant as well.*

Proof. We choose a basis $\{T_a\}$ of \mathfrak{g} and express u and t as linear combinations of basis elements: $u = u^{ab} T_a \otimes T_b$ and $t = t^{ab} T_a \otimes T_b$. Then, the antisymmetry of u implies $u^{ab} = -u^{ba}$, and the Ad-invariance of t implies $\text{ad}_X(t) = t^{ab} ([X, T_a] \otimes T_b + T_a \otimes [X, T_b]) = 0$ for all $X \in \mathfrak{g}$. The claims then follow by a direct computation using these identities together with the antisymmetry and the Jacobi identity for the Lie-bracket. □

Lemma 2 shows that an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ defines a cocommutator on \mathfrak{g} if and only if its symmetric component is ad-invariant and its antisymmetric component has an ad-invariant Schouten bracket. One possibility to satisfy the second condition is to require that the Schouten bracket $[[r, r]]$ vanishes. Such an element is called a *classical r -matrix* for \mathfrak{g} .

Definition 9. *Let \mathfrak{g} be a Lie algebra with basis $\{T_a\}$. A **classical r -matrix** for \mathfrak{g} is an element $r = r^{ab} T_a \otimes T_b \in \mathfrak{g} \otimes \mathfrak{g}$ that satisfies the following conditions*

1. its symmetric component $r_{(s)} = \frac{1}{2}(r^{ab} + r^{ba})T_a \otimes T_b$ is **ad-invariant**;
2. the **classical Yang–Baxter equation (CYBE)**: $[[r, r]] = 0$.

Corollary 2. If $r \in \mathfrak{g} \otimes \mathfrak{g}$ is a classical r -matrix for a Lie algebra \mathfrak{g} , then $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}, X \mapsto \text{ad}_X(r)$ is a cocommutator for \mathfrak{g} .

Proof. As any classical r -matrix satisfies $\text{ad}_X(r_{(s)}) = 0$, the map $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}, X \mapsto \text{ad}_X(r)$ depends only on the antisymmetric component of r . By Lemma 2, we have $[[r_{(a)}, r_{(a)}]] = -[[r_{(s)}, r_{(s)}]]$ and $[[r_{(s)}, r_{(s)}]]$ ad-invariant. Hence $r_{(a)}$ satisfies the conditions in Proposition 1 and defines a cocommutator $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}, X \mapsto \text{ad}_X(r)$ on \mathfrak{g} . \square

Remark 4.

1. As the CYBE is quadratic in r and invariant under the reversal of the factors in the tensor product, for any solution $r = r^{ab}T_a \otimes T_b \in \mathfrak{g} \otimes \mathfrak{g}$, the elements $r_{21} = r^{ab}T_b \otimes T_a$ and λr for $\lambda \in \mathbb{R}$ are solutions as well.
2. Every non-degenerate ad-invariant symmetric bilinear form κ on \mathfrak{g} determines an ad-invariant symmetric element $t \in \mathfrak{g} \otimes \mathfrak{g}$, the **Casimir element** associated to κ . It is given by $t = \kappa^{ab}T_a \otimes T_b$ for any basis $\{T_a\}$ of \mathfrak{g} , where $\kappa_{ab} = \kappa(T_a, T_b)$ are the entries of the coefficient matrix of κ with respect to this basis and κ^{ab} the entries of the inverse matrix with $\kappa^{ab}\kappa_{bc} = \delta_c^a$.
By Lemma 2, the element $r = t + u$ with antisymmetric u is a classical r -matrix if and only if u satisfies the **modified classical Yang–Baxter equation (MCYBE)** for κ

$$[[u, u]] = -[[t, t]].$$

Lie bialgebras with coboundary cocommutators $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}, X \mapsto \text{ad}_X(r)$ are particularly easy to construct and can be classified with methods from Lie algebra cohomology. Among these Lie bialgebras, the ones whose cocommutators are given by a classical r -matrix play a special role. We will see in the following sections that their Poisson–Lie structure is particularly simple and that they define integrable systems with Lax pairs.

Definition 10. A Lie bialgebra $(\mathfrak{g}, [\cdot, \cdot], \delta)$ is called:

1. **coboundary** if its cocommutator is of the form $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}, X \mapsto \text{ad}_X(r)$ with an antisymmetric element $r \in \mathfrak{g} \otimes \mathfrak{g}$;
2. **quasitriangular** if its cocommutator is of the form $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}, X \mapsto \text{ad}_X(r)$ with a classical r -matrix $r \in \mathfrak{g} \otimes \mathfrak{g}$;
3. **triangular** if its cocommutator is of the form $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}, X \mapsto \text{ad}_X(r)$ with an antisymmetric classical r -matrix $r \in \mathfrak{g} \otimes \mathfrak{g}$.

Example 11.

1. Consider the real Lie algebra $\mathfrak{g} = \mathbb{R} \ltimes \mathbb{R}$ with basis $\{X, Y\}$ and Lie bracket

$$[X, X] = [Y, Y] = 0 \quad [X, Y] = Y$$

from Example 6 with the Lie bialgebra structures

$$\begin{aligned} \delta_1(X) &= 0 & \delta_1(Y) &= X \wedge Y \\ \delta_2(X) &= X \wedge Y & \delta_2(Y) &= 0. \end{aligned}$$

Any antisymmetric element of $\mathfrak{g} \otimes \mathfrak{g}$ is of the form $r_{(a)} = \lambda X \wedge Y$ with $\lambda \in \mathbb{R}$ and satisfies

$$\text{ad}_X(r_{(a)}) = \lambda X \wedge Y, \quad \text{ad}_Y(r_{(a)}) = 0, \quad [[r_{(a)}, r_{(a)}]] = 0.$$

This shows that the Lie bialgebra structure on \mathfrak{g} with cocommutator δ_2 is triangular, while the Lie bialgebra structure on \mathfrak{g} with cocommutator δ_1 is not even coboundary.

2. Consider the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ with the basis (14) and the three Lie bialgebra structures on $\mathfrak{sl}(2, \mathbb{R})$ from Example 6 in (11). From the Lie bracket of $\mathfrak{sl}(2, \mathbb{R})$, it follows directly that the three cocommutators are all given by antisymmetric elements of $\mathfrak{sl}(2, \mathbb{R}) \otimes \mathfrak{sl}(2, \mathbb{R})$

$$\delta_1(X) = \text{ad}_X(J_2 \wedge J_0) \quad \delta_2(X) = \text{ad}_X(J_1 \wedge J_2) \quad \delta_3(X) = \text{ad}_X(J_1 \wedge (J_2 - J_0))$$

and hence all three Lie bialgebra structures are coboundary. To determine if they are quasitriangular, we look for an ad -invariant symmetric element of $\mathfrak{sl}(2, \mathbb{R}) \otimes \mathfrak{sl}(2, \mathbb{R})$ whose Schouten bracket is minus the Schouten bracket of the elements in the last equation.

Up to multiplication by scalars, there is a unique ad -invariant symmetric bilinear form on $\mathfrak{sl}(2, \mathbb{R})$, the Killing form given by $\langle J_0, J_0 \rangle = -1$ and $\langle J_1, J_1 \rangle = \langle J_2, J_2 \rangle = 1$. It follows that every ad -invariant symmetric element of $\mathfrak{sl}(2, \mathbb{R}) \otimes \mathfrak{sl}(2, \mathbb{R})$ is of the form

$$\lambda t = \lambda(J_0 \otimes J_0 - J_1 \otimes J_1 - J_2 \otimes J_2) \quad \lambda \in \mathbb{R}.$$

This is the only candidate for the symmetric component of the classical r -matrix. A straightforward but lengthy computation shows that it satisfies

$$[[t, t]] = -J_0 \otimes J_1 \otimes J_2 - J_2 \otimes J_0 \otimes J_1 - J_1 \otimes J_2 \otimes J_0 + J_0 \otimes J_2 \otimes J_1 + J_1 \otimes J_0 \otimes J_2 + J_2 \otimes J_1 \otimes J_0.$$

Similarly, we compute the Schouten brackets for $J_2 \wedge J_0$, $J_1 \wedge J_2$ and $J_1 \wedge (J_2 - J_0)$:

$$\begin{aligned} [[J_2 \wedge J_0, J_2 \wedge J_0]] &= -[[t, t]] \\ [[J_1 \wedge J_2, J_1 \wedge J_2]] &= [[t, t]] \\ [[J_1 \wedge (J_2 - J_0), J_1 \wedge (J_2 - J_0)]] &= 0. \end{aligned}$$

This shows that the first Lie bialgebra structure is quasitriangular with r -matrix

$$r = J_0 \otimes J_0 - J_1 \otimes J_1 - J_2 \otimes J_2 + J_2 \otimes J_0 - J_0 \otimes J_2 \quad (29)$$

the second is coboundary, but not quasitriangular, and the third is triangular.

3. For complex semisimple Lie algebras and their compact and normal real forms, one can show that every Lie bialgebra structure is coboundary. This follows from an argument based on Lie algebra cohomology and allows one to classify their Lie bialgebra structures.

At this stage, it is not apparent what is the advantage of a quasitriangular Lie bialgebra compared to a coboundary one. As the cocommutator depends only on the antisymmetric component of r , the symmetric part of a classical r -matrix is irrelevant for the Lie bialgebra structure. Its advantage is that it defines Lie bialgebra homomorphisms between \mathfrak{g} and its dual \mathfrak{g}^* . If the symmetric part is non-degenerate, this gives rise to a decomposition of the dual Lie bialgebra into Lie subbialgebras.

Lemma 3. Let \mathfrak{g} be a Lie algebra with basis $\{T_a\}$ and $r = r^{ab}T_a \otimes T_b$ a classical r -matrix for \mathfrak{g} . Denote by \mathfrak{g}^{*cop} the Lie bialgebra with the same Lie algebra bracket as \mathfrak{g}^* but the opposite cocommutator. Then, the maps

$$\begin{aligned} \sigma_+ : \mathfrak{g}^{*cop} &\rightarrow \mathfrak{g}, & \alpha &\mapsto (\langle \alpha, - \rangle \otimes \text{id}_{\mathfrak{g}})(r) = r^{ab} \langle \alpha, T_a \rangle T_b \\ \sigma_- : \mathfrak{g}^{*cop} &\rightarrow \mathfrak{g}, & \alpha &\mapsto -(\text{id}_{\mathfrak{g}} \otimes \langle \alpha, - \rangle)(r) = -r^{ba} \langle \alpha, T_a \rangle T_b \end{aligned}$$

are Lie bialgebra homomorphisms:

$$[\ , \]_{\mathfrak{g}} \circ (\sigma_{\pm} \otimes \sigma_{\pm}) = \sigma_{\pm} \circ [\ , \]_{\mathfrak{g}^*} \quad \delta_{\mathfrak{g}} \circ \sigma_{\pm} = -(\sigma_{\pm} \otimes \sigma_{\pm}) \circ \delta_{\mathfrak{g}^*}.$$

If the symmetric component of r is non-degenerate, one has $\mathfrak{g} = \sigma_+(\mathfrak{g}^*) \oplus \sigma_-(\mathfrak{g}^*)$ as a vector space with Lie subalgebras $\sigma_{\pm}(\mathfrak{g}^*) \subset \mathfrak{g}$.

Proof. This follows by a direct computation. The cocommutator of \mathfrak{g} is given by

$$\delta(T_a) = \text{ad}_{T_a}(r) = r^{cd}([T_a, T_c] \otimes T_d + T_c \otimes [T_a, T_d]), \quad (30)$$

and this implies for the Lie bracket and the cocommutator

$$\begin{aligned} [\sigma_+(\alpha), \sigma_+(\beta)]_{\mathfrak{g}} &\stackrel{\text{Def } \sigma_+}{=} r^{ab} r^{cd} \langle \alpha, T_a \rangle \langle \beta, T_c \rangle [T_b, T_d] \\ &\stackrel{\text{CYBE}}{=} -r^{ab} r^{cd} \langle \alpha, [T_a, T_c] \rangle \langle \beta, T_b \rangle T_d - r^{ab} r^{cd} \langle \alpha, T_a \rangle \langle \beta, [T_b, T_c] \rangle T_d \\ &\stackrel{(30)}{=} r^{cd} \langle \alpha \otimes \beta, \delta(T_c) \rangle T_d \stackrel{\text{Def } [\cdot]_{\mathfrak{g}^*}}{=} r^{cd} \langle [\alpha, \beta]_{\mathfrak{g}^*}, T_c \rangle T_d \\ &\stackrel{\text{Def } \sigma_+}{=} \sigma_+([\alpha, \beta]_{\mathfrak{g}^*}) \\ \delta_{\mathfrak{g}}(\sigma_+(\alpha)) &\stackrel{\text{Def } \sigma_+}{=} r^{ab} \langle \alpha, T_a \rangle \delta_{\mathfrak{g}}(T_b) \stackrel{(30)}{=} r^{ab} r^{cd} \langle \alpha, T_a \rangle ([T_b, T_c] \otimes T_d + T_c \otimes [T_b, T_d]) \\ &\stackrel{\text{CYBE}}{=} -r^{ab} r^{cd} \langle \alpha, [T_a, T_c] \rangle T_b \otimes T_d = -r^{ab} r^{cd} \langle \delta_{\mathfrak{g}^*}(\alpha), T_a \otimes T_c \rangle T_b \otimes T_d \\ &\stackrel{\text{Def } \sigma_+}{=} -(\sigma_+ \otimes \sigma_+)(\delta_{\mathfrak{g}^*}(\alpha)). \end{aligned}$$

The computations for σ_- are analogous. \square

Quasitriangular Lie bialgebras can be constructed systematically in a very simple way. In fact, every Lie bialgebra \mathfrak{g} defines a canonical quasitriangular Lie bialgebra structure on the vector space $\mathfrak{g} \oplus \mathfrak{g}^*$. This construction is due to Drinfeld [9] and can be viewed as the Lie bialgebra counterpart of a corresponding construction for quantum groups and monoidal categories.

Theorem 3 ([9]). Let \mathfrak{g} be a Lie bialgebra with dual \mathfrak{g}^* and \mathfrak{g}^{*cop} the Lie bialgebra with opposite co-commutator.

1. There is a unique quasitriangular Lie bialgebra structure on the vector space $\mathfrak{g} \oplus \mathfrak{g}^*$ such that the inclusions $\iota : \mathfrak{g} \rightarrow D(\mathfrak{g})$ and $\iota : \mathfrak{g}^{*cop} \rightarrow D(\mathfrak{g})$ are Lie bialgebra homomorphisms. This Lie bialgebra is called the **classical double** $D(\mathfrak{g})$.
2. If $\{T_a\}$ is a basis of \mathfrak{g} with dual basis $\{t^a\}$, then the classical r -matrix of $D(\mathfrak{g})$ is $r = T_a \otimes t^a$ and the Lie algebra structure of $D(\mathfrak{g})$ reads

$$[T_a, T_b] = f_{ab}^c T_c \quad [T_a, t^b] = C_a^{bc} T_c - f_{ac}^b t^c \quad [t^a, t^b] = C_c^{ab} t^c, \quad (31)$$

where f_{ab}^c and C_a^{bc} are the structure constants of \mathfrak{g} and \mathfrak{g}^* with respect to $\{T_a\}$ and $\{t^a\}$.

Proof. Suppose that the Lie bracket and the cocommutator of \mathfrak{g} take the form

$$[T_a, T_b] = f_{ab}^c T_c \quad \delta(T_a) = C_a^{bc} T_b \otimes T_c$$

with structure constants f_{ab}^c, C_a^{bc} . The antisymmetry of the commutator and cocommutator reads $f_{ab}^c = -f_{ba}^c$ and $C_a^{bc} = -C_a^{cb}$. The Jacobi identity, the coJacobi identity and the cocycle condition for δ take the form

$$\begin{aligned} f_{ab}^c f_{cd}^e + f_{da}^c f_{cb}^e + f_{bd}^c f_{ca}^e &= 0 \\ C_a^{bc} C_b^{de} + C_a^{bf} C_b^{cd} + C_a^{bd} C_b^{fc} &= 0 \\ f_{ab}^c C_c^{dg} &= C_b^{cg} f_{ac}^d + C_b^{dc} f_{ac}^g - C_a^{cg} f_{bc}^d - C_a^{dc} f_{bc}^g. \end{aligned} \quad (32)$$

The condition that the inclusion maps are homomorphisms of Lie bialgebras then implies that the Lie bracket and cocommutator of $D(\mathfrak{g})$ satisfy

$$\begin{aligned} [T_a, T_b] &= f_{ab}^c T_c & [t^a, t^b] &= C_c^{ab} t^c \\ \delta(T_a) &= C_a^{bc} T_b \otimes T_c & \delta(t^a) &= -f_{bc}^a t^b \otimes t^c. \end{aligned}$$

Hence, the cocommutator of $D(\mathfrak{g})$ is determined uniquely by the cocommutator and Lie bracket of \mathfrak{g} and satisfies the coJacobi identity. If $D(\mathfrak{g})$ is quasitriangular with r -matrix $r = T_a \otimes t^a$, then

$$\begin{aligned} \delta(T_a) &= C_a^{bc} T_b \otimes T_c = \text{ad}_{T_a}(r) = [T_a, T_b] \otimes t^b + T_b \otimes [T_a, t^b] = f_{ab}^c T_c \otimes t^b + T_b \otimes [T_a, t^b] \\ \delta(t^a) &= -f_{bc}^a t^b \otimes t^c = \text{ad}_{t^a}(r) = [t^a, T_b] \otimes t^b + T_b \otimes [t^a, t^b] = [t^a, T_b] \otimes t^b + C_c^{ab} T_b \otimes t^c. \end{aligned}$$

From these conditions, one finds that the Lie bracket of $D(\mathfrak{g})$ must take the form (31).

By definition of $D(\mathfrak{g})$, the Jacobi identity holds for Lie brackets involving only elements of \mathfrak{g} or of \mathfrak{g}^* . For brackets involving elements of \mathfrak{g} and of \mathfrak{g}^* , it can be established by a direct computation using expressions (31) and the identities (32). A similar computation proves the ad-invariance of the symmetric component of r and for the CYBE. \square

Example 12.

1. If \mathfrak{g} is a Lie algebra with the trivial cocommutator $\delta = 0$, then the classical double $D(\mathfrak{g})$ is a semidirect product $\mathfrak{g} \ltimes \mathfrak{g}^*$ with Lie-bracket and cocommutator given by

$$\begin{aligned} [T_a, T_b] &= f_{ab}^c T_c & [T_a, t^b] &= -f_{ac}^b t^c & [t^a, t^b] &= 0 \\ \delta(T_a) &= 0 & \delta(t^a) &= -f_{bc}^a t^b \otimes t^c. \end{aligned}$$

2. We consider the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ with the basis

$$J_0 = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad J_1 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad J_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

Lie bracket

$$[J_0, J_1] = J_2 \quad [J_0, J_2] = -J_1 \quad [J_1, J_2] = -J_0 \quad (33)$$

and the second cocommutator from Example 6,

$$\delta(J_0) = 0 \quad \delta(J_1) = J_0 \wedge J_1 \quad \delta(J_2) = J_0 \wedge J_2.$$

Then, by Example 7, the dual Lie algebra has the Lie bracket

$$[j^0, j^1] = j^1 \quad [j^0, j^2] = j^2 \quad [j^1, j^2] = 0. \quad (34)$$

The associated classical double $D(\mathfrak{sl}(2, \mathbb{R}))$ has the Lie brackets (33), (34) and

$$\begin{aligned} [J_0, j^0] &= 0 & [J_0, j^1] &= j^2 & [J_0, j^2] &= -j^1 \\ [J_1, j^0] &= J_1 + j^2 & [J_1, j^1] &= -J_0 & [J_1, j^2] &= j^0 \\ [J_2, j^0] &= J_2 - j^1 & [J_2, j^1] &= -j^0 & [J_2, j^2] &= -J_0. \end{aligned}$$

Using these expressions for the Lie bracket, one can show that $D(\mathfrak{sl}(2, \mathbb{R})) \cong \mathfrak{sl}(2, \mathbb{C})$.

3.2. Application: Integrable Systems from Quasitriangular Lie Bialgebras

Quasitriangular Lie bialgebras have important applications in integrable systems. A quasitriangular Lie bialgebra \mathfrak{g} with a non-degenerate symmetric component of the classical r -matrix allows one to construct a Hamiltonian system with a Lax pair and conserved

quantities in involution on the dual vector space \mathfrak{g}^* . In fact, the role of Poisson–Lie and Lie bialgebra symmetries in integrable systems was one the origins of these structures.

Definition 11. A **Hamiltonian system** is a Poisson manifold $(M, \{, \})$ together with the choice of a function $H \in C^\infty(M)$, the **Hamiltonian**.

- The **time evolution equation** for a function $f \in C^\infty(M)$ is $\dot{f} = \{f, H\}$.
- A function $f \in C^\infty(M)$ is called a **conserved quantity** if $\{f, H\} = 0$.
- Two functions $f_1, f_2 \in C^\infty(M)$ are called **in involution** if $\{f_1, f_2\} = 0$.

The Poisson manifold M stands for the phase space of the system and the Hamiltonian H for its total energy. The time evolution equation is then the usual time evolution equation in the Hamilton–Jacobi formalism, and the conserved quantities are the constants of motion. To solve the equations of motion, one is interested in having as many independent conserved quantities in involution as possible. By taking such functions as coordinates, one can reduce the number of degrees of freedom in the equations of motion and solve these equations efficiently.

The condition that conserved quantities $f_1, f_2, \dots, f_n \in C^\infty(M)$ are independent is usually stated as the requirement that they combine into the function $f = (f_1, \dots, f_n) : M \rightarrow \mathbb{R}^n$ such that $\text{rank}(df) = n$ almost everywhere.

It is often also required that the Poisson bracket of a Hamiltonian system is symplectic, which requires M to be even-dimensional. On a $2n$ -dimensional symplectic manifold M , there can be at most n independent conserved quantities in involution. If there are n independent conserved quantities in involution, one calls the system *integrable*.

Definition 12. A Hamiltonian system $(M, \{, \}, H)$ is called **integrable** if $(M, \{, \})$ is a $2n$ -dimensional symplectic manifold and there are n independent conserved quantities in involution.

The name *integrable* is motivated as follows. If f_1, \dots, f_n are independent conserved quantities in involution, then the Hamiltonian is a conserved quantity as well, it can be expressed as a function of the variables f_1, \dots, f_n . One usually takes it as one of the conserved quantities and sets $H = f_1$. One may then introduce n additional phase space coordinates q_1, \dots, q_n , that are conjugate to the variables f_1, \dots, f_n with respect to the symplectic structure, and their equations of motion read $\dot{q}_i = \{H, q_i\} = \partial H / \partial f_i$. The right-hand side is a function of the conserved quantities f_i , and thus, the equations of motion may be solved explicitly by integration.

It is often not easy to find conserved quantities. If the Hamiltonian system has obvious symmetries, then by Noether’s theorem, these symmetries usually give rise to conserved quantities, and one may determine conserved quantities by studying the symmetries of the system. An important method to find conserved quantities of a Hamiltonian system are Lax pairs. This concept goes back to Lax [11], for the construction of Lax pairs from classical r -matrices, see [12–15]. For an accessible and broad treatment of Lax pairs and classical r -matrices in different integrable systems, see the textbook [7].

Definition 13. A **Lax Pair** for a Hamiltonian system $(M, \{, \}, H)$ is a pair of smooth functions $L, P : M \rightarrow \mathfrak{g}$ into a Lie algebra \mathfrak{g} such that

$$\dot{L}(m) = \{H, L\}(m) = [L(m), P(m)] \quad \forall m \in M. \quad (35)$$

The left-hand side and the right-hand-side of equation (35) are understood as equations for the coefficient functions of L, P with respect to a basis. If $\{T_a\}$ is a basis of \mathfrak{g} with associated structure constants $[T_a, T_b] = f_{ab}^c T_c$, then one can write $L = L^a T_a$, $P = P^a T_a$ with functions $L^a, P^a \in C^\infty(M)$, and equation (35) reads

$$\dot{L}_a = \{H, L_a\} = f_{ab}^c L^a P^b.$$

Remark 5. It follows directly from the Lax equations that Lax pairs are not unique. If G is a Lie group with Lie algebra \mathfrak{g} , then for any Lax pair (L, P) for a Hamiltonian system $(M, \{, \}, H)$ and any function $\gamma : M \rightarrow G$, one obtains another Lax pair (L', P') with

$$L' = \text{Ad}(\gamma)L \quad P' = \text{Ad}(\gamma)P - \dot{\gamma}\gamma^{-1}.$$

The advantage of a Lax pair is that it gives rise to many conserved quantities for the underlying Hamiltonian system. The Lax equations guarantee that for any ad-invariant function f on \mathfrak{g} , the function $f \circ L \in C^\infty(M)$ is a conserved quantity. The simplest ad-invariant functions on \mathfrak{g} are the trace polynomials in representations of \mathfrak{g} .

Lemma 4. Let $(M, \{, \}, H)$ be a Hamiltonian system with a Lax pair $L, P : M \rightarrow \mathfrak{g}$. Then, the trace polynomials and eigenvalues of L in any representation of \mathfrak{g} are conserved quantities.

Proof. Let $\rho : \mathfrak{g} \rightarrow \text{End}_{\mathbb{K}}(V)$ be a representation of \mathfrak{g} on V . Then, the associated trace polynomials are the functions $f_k^\rho : M \rightarrow \mathbb{R}, m \mapsto \text{tr}_V(\rho(L(m))^k)$. Their equations of motion read

$$\begin{aligned} \dot{f}_k^\rho &= \sum_{i=0}^{k-1} \text{tr}_V(\rho(L)^i \cdot \rho(\dot{L}) \cdot \rho(L)^{k-i-1}) \\ &= k \text{tr}_V(\rho(L)^{k-1} [\rho(L), \rho(P)]) = \text{tr}_V([\rho(L)^k, \rho(P)]) = 0, \end{aligned}$$

where we used the cyclic invariance of the trace. As the eigenvalues of $\rho(L)$ are functions of the trace polynomials, they are conserved as well. \square

Lemma 4 allows one to construct conserved quantities from a Lax pair. However, in general, it is not guaranteed that they are conserved quantities *in involution*. Their Poisson brackets are given by the Poisson brackets of the matrix elements of L in different representations, and to ensure that these matrix elements Poisson commute, one needs additional conditions on L .

Quasitriangular Lie bialgebras give a systematic way of constructing Hamiltonian systems with Lax pairs whose conserved quantities are *in involution*. This construction makes use of the symmetric part of the r -matrix and requires that its symmetric part is non-degenerate. The key observation is that in that situation, we can use the symmetric part of r to pull back the Lie bracket on \mathfrak{g}^* defined by its antisymmetric part to a Lie bracket on \mathfrak{g} .

Lemma 5. Let \mathfrak{g} be a Lie algebra, κ a non-degenerate ad-invariant symmetric bilinear form on \mathfrak{g} and $K : \mathfrak{g} \rightarrow \mathfrak{g}^*, X \mapsto \kappa(X, -)$ the associated linear isomorphism.

Then, for any antisymmetric element $u = u^{ab} T_a \otimes T_b \in \mathfrak{g} \otimes \mathfrak{g}$ with $[[u, u]]$ ad-invariant, the associated Lie bracket on \mathfrak{g}^* is given by $[K(X), K(Y)]_{\mathfrak{g}^*} = K([X, Y]_u)$ with

$$[X, Y]_u = [\phi(X), Y] + [Y, \phi(Y)] \quad \text{with} \quad \phi : \mathfrak{g} \rightarrow \mathfrak{g}, X \mapsto u^{ab} \kappa(X, T_a) T_b. \quad (36)$$

Proof. By Proposition 1, the map $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}, X \mapsto \text{ad}_X(u)$ is a cocommutator on \mathfrak{g} and hence defines a Lie bracket on \mathfrak{g}^* . In terms of a basis $\{T_a\}$ of \mathfrak{g} , this Lie bracket reads

$$\langle [\alpha, \beta]_{\mathfrak{g}^*}, T_a \rangle = \langle \alpha \otimes \beta, \delta(T_a) \rangle = u^{bc} \langle \alpha \otimes \beta, [T_a, T_b] \otimes T_c + T_b \otimes [T_a, T_c] \rangle \quad (37)$$

for all $\alpha, \beta \in \mathfrak{g}^*$ and basis elements T_a . The ad-invariance of κ reads

$$\kappa([X, Y], Z) + \kappa(Y, [X, Z]) = 0 \quad \forall X, Y, Z \in \mathfrak{g}. \quad (38)$$

Using the definition of K , the antisymmetry of u and the Lie bracket and the ad-invariance of κ , we then obtain for all basis elements T_a

$$\begin{aligned}
\langle [K(X), K(Y)]_{\mathfrak{g}^*}, T_a \rangle &\stackrel{(37)}{=} u^{bc} \langle K(X) \otimes K(Y), [T_a, T_b] \otimes T_c + T_b \otimes [T_a, T_c] \rangle \\
&\stackrel{\text{Def } K}{=} u^{bc} \kappa(X, [T_a, T_b]) \kappa(Y, T_c) + \kappa(X, T_b) \kappa(Y, [T_a, T_c]) \\
&\stackrel{(38)}{=} -u^{bc} \kappa([X, T_b], T_a) \kappa(Y, T_c) - u^{bc} \kappa(X, T_b) \kappa([Y, T_c], T_a) \\
&\stackrel{(36)}{=} \kappa([X, \phi(Y)], T_a) + \kappa([\phi(X), Y], T_a) \stackrel{\text{Def } K}{=} \langle K([X, \phi(Y)] + [\phi(X), Y]), T_a \rangle.
\end{aligned}$$

□

We can now consider the Lie algebra \mathfrak{g} with either its original Lie bracket $[\cdot, \cdot]$ or with the Lie bracket $[\cdot, \cdot]_u$ from (36) and with the trivial cocommutator. Then, its dual Lie bialgebra is the vector space \mathfrak{g}^* with the trivial Lie bracket and the cocommutator $\delta = [\cdot, \cdot]^* : \mathfrak{g}^* \rightarrow \mathfrak{g}^* \otimes \mathfrak{g}^*$ or $\delta_u = [\cdot, \cdot]_u^* : \mathfrak{g}^* \rightarrow \mathfrak{g}^* \otimes \mathfrak{g}^*$. By Example 10, the associated Poisson–Lie group is the vector space \mathfrak{g}^* with the usual vector space addition and with the Poisson bracket from (26)

$$\{f_1, f_2\}(\alpha) = \langle \alpha, [d_\alpha f_1, d_\alpha f_2] \rangle \quad \{f_1, f_2\}_u(\alpha) = \langle \alpha, [d_\alpha f_1, d_\alpha f_2]_u \rangle. \quad (39)$$

For these Poisson brackets, it is straightforward to construct quantities in involution. It turns out that any function $f \in C^\infty(\mathfrak{g}^*)$ that is invariant under the **coadjoint action** of G on \mathfrak{g}^* is a conserved quantity. The coadjoint action $\triangleright^* : G \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ and its infinitesimal counterpart $\text{ad}^* : \mathfrak{g} \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ are given in terms of the adjoint action by

$$\langle g \triangleright^* \alpha, X \rangle = \langle \alpha, \text{Ad}(g^{-1})X \rangle \quad \langle \text{ad}_X^*(\alpha), Y \rangle = -\langle \alpha, [X, Y] \rangle \quad \forall \alpha \in \mathfrak{g}^*, X, Y \in \mathfrak{g}, g \in G. \quad (40)$$

Proposition 2. Let \mathfrak{g} be a Lie algebra, κ a non-degenerate ad-invariant symmetric bilinear form on \mathfrak{g} and $u \in \mathfrak{g} \otimes \mathfrak{g}$ antisymmetric with $[[u, u]]$ ad-invariant. Then, the functions $f \in C^\infty(\mathfrak{g}^*)$ that are invariant under the coadjoint action $\triangleright^* : G \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ are in involution for the Poisson-brackets $\{\cdot, \cdot\}$ and $\{\cdot, \cdot\}_u$ on \mathfrak{g}^* from (39).

Proof. If $f \in C^\infty(\mathfrak{g}^*)$ is invariant under the coadjoint action of G on \mathfrak{g}^* , then we have $f(g \triangleright^* \alpha) = f(\alpha)$ for all $g \in G$ and $\alpha \in \mathfrak{g}^*$, and this implies for all $X \in \mathfrak{g}$

$$0 = \frac{d}{dt} \Big|_{t=0} f(e^{-tX} \triangleright^* \alpha) = -\langle \text{ad}_X^*(\alpha), d_\alpha f \rangle = \langle \alpha, [X, d_\alpha f] \rangle. \quad (41)$$

For the Poisson brackets of two functions $f_1, f_2 \in C^\infty(\mathfrak{g}^*)$ that are invariant under the coadjoint action, this implies $\{f_1, f_2\} = \{f_1, f_2\}_u = 0$ by (39). □

So far, we considered the Poisson brackets $\{\cdot, \cdot\}_u$ for any non-degenerate ad-invariant symmetric bilinear form κ on \mathfrak{g} and any antisymmetric element $u \in \mathfrak{g} \otimes \mathfrak{g}$ with $[[u, u]]$ ad-invariant. The ad-invariance of $[[u, u]]$ was needed to obtain a Lie bracket on \mathfrak{g}^* and the non-degenerate ad-invariant symmetric bilinear form κ to pull back this Lie bracket to \mathfrak{g} .

Given a classical r -matrix $r \in \mathfrak{g} \otimes \mathfrak{g}$ whose symmetric component is non-degenerate, we can take its antisymmetric component $u = r_{(a)}$ for u and use its symmetric component to define a non-degenerate ad-invariant symmetric bilinear form κ on \mathfrak{g} . If $\{T_a\}$ is a basis of \mathfrak{g} , in which $r_{(s)} = r_{(s)}^{ab} T_a \otimes T_b$, we define the coefficient matrix of κ by

$$\kappa(T_a, T_b) = \kappa_{ab} \quad \kappa_{ab} r_{(s)}^{bc} = \delta_a^c. \quad (42)$$

By considering the Poisson bracket $\{\cdot, \cdot\}_u$ for the antisymmetric component of r and any smooth function that is invariant under the coadjoint action as a Hamiltonian, we then obtain a Hamiltonian system on \mathfrak{g}^* that is equipped with a Lax pair whose trace polynomials are conserved quantities in involution.

Theorem 4. Let $r \in \mathfrak{g} \otimes \mathfrak{g}$ be a classical r -matrix for \mathfrak{g} with a non-degenerate symmetric component $r_{(s)}$ and antisymmetric component u .

1. For any function $H \in C^\infty(\mathfrak{g}^*)$ that is invariant under the coadjoint action, the Hamiltonian system $(\mathfrak{g}^*, \{, \}_u, H)$ admits a Lax pair $L, P : \mathfrak{g}^* \rightarrow \mathfrak{g}$ satisfying

$$\{L, L\}_u(\alpha) = [L(\alpha) \otimes 1 + 1 \otimes L(\alpha), u].$$

2. For any representation $\rho : \mathfrak{g} \rightarrow \text{End}_{\mathbb{R}}(V)$, the trace polynomials f_k^ρ are conserved quantities in involution.

Proof. 1. We consider the maps

$$P : \mathfrak{g}^* \rightarrow \mathfrak{g}, \quad \alpha \mapsto \phi(d_\alpha H) \quad L : \mathfrak{g}^* \rightarrow \mathfrak{g}, \quad \alpha \mapsto (\langle \alpha, - \rangle \otimes \text{id}) r_{(s)} = r_{(s)}^{ab} \langle \alpha, T_a \rangle T_b,$$

where $\phi : \mathfrak{g} \rightarrow \mathfrak{g}$ is defined as in (36). Then, we have $L = L^a T_a$ with $L^a(\alpha) = r_{(s)}^{ab} \langle \alpha, T_b \rangle$, which implies $d_\alpha L^a = r_{(s)}^{ab} T_b$. By (41) the invariance of H under the coadjoint action implies $\langle \alpha, [X, d_\alpha H] \rangle = 0$ for all $X \in \mathfrak{g}$. Inserting this in the equations of motion for L , we obtain

$$\begin{aligned} \dot{L}(\alpha) &= \{H, L^b\}_u(\alpha) T_b \stackrel{(39)}{=} \langle \alpha, [d_\alpha H, d_\alpha L^b]_u \rangle T_b \stackrel{(36)}{=} \langle \alpha, [\phi(d_\alpha H), d_\alpha L^b] + [d_\alpha H, \phi(d_\alpha L^b)] \rangle T_b \\ &= \langle \alpha, [\phi(d_\alpha H), d_\alpha L^b] \rangle T_b = r_{(s)}^{ab} \langle \alpha, [\phi(d_\alpha H), T_a] \rangle T_b \stackrel{(iii)}{=} -r_{(s)}^{ab} \langle \alpha, T_a \rangle [\phi(d_\alpha H), T_b] \\ &= [L(\alpha), P(\alpha)], \end{aligned}$$

where we used the ad-invariance of $r_{(s)}$ in (iii). This shows that L and P form a Lax pair. To prove the formula for the Poisson brackets of L , we compute

$$\begin{aligned} \{L, L\}_u(\alpha) &= \{L^i, L^j\}_u(\alpha) T_i \otimes T_j \stackrel{(39)}{=} \langle \alpha, [d_\alpha L^i, d_\alpha L^j]_u \rangle T_i \otimes T_j \\ &\stackrel{(36)}{=} \langle \alpha, [\phi(d_\alpha L^i), d_\alpha L^j] + [d_\alpha L^i, \phi(d_\alpha L^j)] \rangle T_i \otimes T_j \\ &= r_{(s)}^{ai} r_{(s)}^{bj} \langle \alpha, [\phi(T_a), T_b] + [T_a, \phi(T_b)] \rangle T_i \otimes T_j \\ &\stackrel{(36)}{=} r_{(s)}^{ai} r_{(s)}^{bj} (\kappa_{af} u^{fg} \langle \alpha, [T_g, T_b] \rangle + \kappa_{bf} u^{fg} \langle \alpha, [T_a, T_g] \rangle) T_i \otimes T_j \\ &\stackrel{(42)}{=} (r_{(s)}^{bj} u^{ig} \langle \alpha, [T_g, T_b] \rangle + r_{(s)}^{ai} u^{ig} \langle \alpha, [T_a, T_g] \rangle) T_i \otimes T_j \\ &= -r_{(s)}^{bj} u^{ig} \langle \alpha, T_b \rangle T_i \otimes [T_g, T_j] - r_{(s)}^{ai} u^{ig} \langle \alpha, T_a \rangle [T_i, T_g] \otimes T_j \\ &= -u^{ig} T_i \otimes [T_g, L(\alpha)] - u^{ig} [L(\alpha), T_g] \otimes T_j = -[u, L(\alpha) \otimes 1 + 1 \otimes L(\alpha)]. \end{aligned}$$

2. Let now $\rho : \mathfrak{g} \rightarrow \text{End}_{\mathbb{R}}(V)$ and $\tau : \mathfrak{g} \rightarrow \text{End}_{\mathbb{R}}(W)$ be representations of \mathfrak{g} . Their trace polynomials are conserved quantities by Lemma 4. Using the cyclic invariance of the trace, one obtains for the associated trace polynomials

$$\begin{aligned} \{f_k^\rho, f_l^\tau\}_u(\alpha) &= \{\text{tr}_V(\rho(L)^k), \text{tr}_W(\tau(L)^l)\}_u(\alpha) \\ &= \sum_{i=0}^{k-1} \sum_{j=0}^{l-1} (\text{tr}_V \otimes \text{tr}_W)(\rho \otimes \tau)((L^i \otimes L^j) \cdot \{L, L\}_u \cdot (L^{k-i-1} \otimes L^{l-j-1})) \\ &= kl(\text{tr}_V \otimes \text{tr}_W)(\rho \otimes \tau)((L^{k-1} \otimes L^{l-1}) \cdot [L \otimes 1 + 1 \otimes L, u]) \\ &= kl u^{ab} (\text{tr}_V \otimes \text{tr}_W)(\rho \otimes \tau)((L^{k-1} \otimes L^{l-1}) \cdot ([L, T_a] \otimes T_b + T_a \otimes [L, T_b])) \\ &= u^{ab} (\text{tr}_V \otimes \text{tr}_W)(\rho \otimes \tau)(([L^k, T_a] \otimes (l L^{l-1} T_b) + (k L^{k-1} T_a) \otimes [L^l, T_b])) = 0. \end{aligned}$$

□

Example 13. We consider the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ with the basis

$$H = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

in which the Lie bracket takes the form

$$[H, J_{\pm}] = \pm J_{\pm} \quad [J_+, J_-] = 2H$$

and with the classical r -matrix from (29)

$$r = -H \otimes H - J_+ \otimes J_-.$$

The ad-invariant symmetric bilinear form defined by the symmetric part of the r takes the form

$$\kappa(H, H) = -1 \quad \kappa(J_+, J_-) = \kappa(J_-, J_+) = -2 \quad \kappa(H, J_{\pm}) = \kappa(J_{\pm}, J_{\pm}) = 0.$$

The map $\phi : \mathfrak{sl}(2, \mathbb{R}) \rightarrow \mathfrak{sl}(2, \mathbb{R})$ and the Lie bracket $[\cdot, \cdot]_u$ on $\mathfrak{sl}(2, \mathbb{R})$ from (36) are given by

$$\phi(H) = 0 \quad \phi(J_{\pm}) = \pm J_{\pm} \quad [H, J_{\pm}]_u = J_{\pm} \quad [J_+, J_-]_u = 0.$$

If we interpret the elements H, J_{\pm} as functions on $\mathfrak{sl}(2, \mathbb{R})^*$, the associated Poisson structure on $\mathfrak{sl}(2, \mathbb{R})^*$ reads

$$\{H, J_{\pm}\}_u = J_{\pm} \quad \{J_+, J_-\}_u = 0.$$

This allows us to restrict this Poisson bracket to the linear subspace $\ker(J_+ - J_-) \subset \mathfrak{sl}(2, \mathbb{R})$, which is invariant under the coadjoint action of $\mathfrak{sl}(2, \mathbb{R})$ on $\mathfrak{sl}(2, \mathbb{R})^*$. As a Hamiltonian, it is natural to choose the function $H = H^2 + J_+^2$, which is invariant under the coadjoint action.

If we set $p = H$ and $J_+ = J_- = e^{-q}$, we obtain the usual symplectic structure on \mathbb{R}^2 and the integrable Hamiltonian system with Hamiltonian $H = p^2 + e^{-2q}$.

Although this example is very simple and can be solved explicitly, it illustrates the general method. It can be generalised to normal real forms of complex semisimple Lie algebras, for details see [5] [2.3D]. The associated integrable systems are the Toda lattice systems.

3.3. Coboundary and Quasitriangular Poisson–Lie Groups

If a Poisson–Lie group has a Lie bialgebra that is coboundary or even quasitriangular, its Poisson structure becomes much simpler, and the same holds for its dual and its classical double. In the quasitriangular case, these simplifications also extend to Poisson G -spaces. They allow one to form products of Poisson G -spaces and to write down simple expressions for Poisson structures on G that give it the structure of a Poisson G -space.

Definition 14. A Poisson–Lie group is called **coboundary** or **(quasi)triangular** if its tangent Lie bialgebra is coboundary or (quasi)triangular.

Example 14.

1. To every connected and simply connected Poisson–Lie group G , one can associate a quasitriangular Poisson–Lie group, its **classical double** $D(G)$. This is the unique connected and simply connected Poisson–Lie group $D(G)$ with tangent Lie bialgebra $D(\mathfrak{g})$.
2. If G is a complex semisimple Lie group or its compact or normal real form, then G is coboundary. This follows from the corresponding statement for complex semisimple Lie algebras and their real forms in Example 11.

Just as the cocommutator of a coboundary Lie bialgebra, the Poisson bracket of a coboundary Poisson–Lie group has a particularly simple description. In this case, one can

express the Poisson bracket in terms of the left and right invariant vector fields on G with structure constants that are the coefficients of the coboundary with respect to a basis.

Theorem 5 ([14,15]). *Let G be a coboundary Poisson–Lie group with cocommutator $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$, $X \mapsto \text{ad}_X(u)$ for an antisymmetric element $u = u^{ab}T_a \otimes T_b$. Then, its Poisson bracket is given by the Sklyanin bracket on G*

$$\{f_1, f_2\} = u^{ab}((T_a^R \cdot f_1)(T_b^R \cdot f_2) - (T_a^L \cdot f_1)(T_b^L \cdot f_2)). \quad (43)$$

Proof. As $u \in \mathfrak{g} \otimes \mathfrak{g}$ defines a cocommutator on the Lie algebra \mathfrak{g} , its Schouten bracket $[[u, u]]$ is ad-invariant by Proposition 1. To show that the Sklyanin bracket satisfies the Jacobi identity, one computes

$$\begin{aligned} & \{\{f_1, f_2\}, f_3\} + \{\{f_3, f_1\}, f_2\} + \{\{f_2, f_3\}, f_1\} \\ &= -[[u, u]]^L \cdot (f_1 \otimes f_2 \otimes f_3) - [[u, u]]^R \cdot (f_1 \otimes f_2 \otimes f_3) = ([u, u] - [[u, u]])^L \cdot (f_1 \otimes f_2 \otimes f_3) = 0, \end{aligned}$$

where one uses the antisymmetry of u , the fact that left and right invariant vector fields on a Lie group G commute and formula (21) for their Lie bracket to pass to the second line. The last identity then follows from the relation (20) between left and right invariant vector fields on G and the ad-invariance of $[[u, u]]$.

To show that this is a Poisson–Lie structure on G , one uses the maps $R_g : G \rightarrow G$, $h \mapsto hg$ and $L_g : G \rightarrow G$, $h \mapsto gh$ and the relation (20) between left and right invariant vector fields on G to compute

$$\begin{aligned} \{f_1, f_2\}(gh) &= u^{ab}(T_a^R \cdot f_1)(gh)(T_b^R \cdot f_2)(gh) - u^{ab}(T_a^L \cdot f_1)(gh)(T_b^L \cdot f_2)(gh) \\ &= \{f_1 \circ R_h, f_2 \circ R_h\}(g) + \{f_1 \circ L_g, f_2 \circ L_g\}(h). \end{aligned}$$

To show that this coincides with the Poisson–Lie structure on G defined by u , we compute the cocommutator of the associated Lie bialgebra by differentiating the Poisson bivector. Using formulas (20) that relate the left and right invariant vector fields on G , we find that the Poisson bivector of the Sklyanin bracket is given by

$$B(g) = B^{ab}(g)T_a^L(g) \otimes T_b^L(g) = -u^{ab}T_a^L(g) \otimes T_b^L(g) + u^{ab}(\text{Ad}(g)T_a)^L(g) \otimes (\text{Ad}(g)T_b)^L(g),$$

and formula (23) for the cocommutator yields

$$\delta(X) = X^R \cdot B(1) = u^{ab}([X, T_a] \otimes T_b + T_a \otimes [X, T_b]) = \text{ad}_X(u).$$

□

The description in Theorem 5 has the advantages that it depends only on the choice of a Lie algebra basis and exhibits clearly the relation with the associated Lie bialgebra structures. Note in particular that the conditions in Theorem 5 are satisfied for any quasitriangular Poisson–Lie group, since in that case the Schouten bracket of the antisymmetric part of the r -matrix is ad-invariant by Lemma 2.

Just as in the case of a quasitriangular Lie bialgebra, one might wonder what is the additional benefit of having a quasitriangular Poisson–Lie group and not just a coboundary one. The answer is that it allows one to describe the Poisson–Lie structure of the dual Poisson–Lie group G^* and the classical double $D(G)$ in terms of a Poisson bracket on G . This result is due to Semenov–Tian–Shansky [15] and Lu and Weinstein [16].

Recall from Lemma 3 that a classical r -matrix $r = r^{ab}T_a \otimes T_b \in \mathfrak{g} \otimes \mathfrak{g}$ defines Lie bialgebra homomorphisms

$$\sigma_+ : \mathfrak{g}^{*cop} \rightarrow \mathfrak{g}, \quad t^a \mapsto r^{ab}T_b \quad \sigma_- : \mathfrak{g}^{*cop} \rightarrow \mathfrak{g}, \quad t^a \mapsto -r^{ba}T_b, \quad (44)$$

where \mathfrak{g}^{*cop} denotes the Lie bialgebra \mathfrak{g}^* with the opposite cocommutator, $\{T_a\}$ is a basis of \mathfrak{g} and $\{t^a\}$ the associated dual basis of \mathfrak{g}^* . By Theorem 2, there are unique connected and simply connected Poisson–Lie groups G and G^* with tangent Lie bialgebras \mathfrak{g} and \mathfrak{g}^* . The Lie bialgebra \mathfrak{g}^{*cop} is the tangent Lie bialgebra of the Poisson–Lie group G^{*cop} with the group multiplication and minus the Poisson bracket of G^* . Again by Theorem 2, the Lie bialgebra homomorphisms $\sigma_{\pm} : \mathfrak{g}^{*cop} \rightarrow \mathfrak{g}$ lift to unique homomorphisms of Poisson–Lie groups $S_{\pm} : G^{*cop} \rightarrow G$ with tangent maps $T_1 S_{\pm} = \sigma_{\pm}$. We can use these maps to describe the Poisson–Lie structures of the classical double $D(G)$ in terms of a Poisson bracket on $G \times G$.

Theorem 6 ([15,16]). *Let G be a connected and simply connected quasitriangular Poisson–Lie group, G^* its connected and simply connected dual and $D(G)$ its connected and simply connected double. Denote by $r = r^{ab} T_a \otimes T_b \in \mathfrak{g} \otimes \mathfrak{g}$ the classical r -matrix of G .*

1. *The map $\psi : G \times G^* \rightarrow G \times G$, $(g, \alpha) \mapsto (S_+(\alpha)g, S_-(\alpha)g)$ is a homomorphism of Poisson–Lie groups from the classical double $D(G)$ to $G \times G$ with the Poisson bracket*

$$\begin{aligned} \{f_1, f_2\}_{D(G)} = & r^{ab}_{(a)} ((T_a^{R+} \cdot f_1)(T_b^{R+} \cdot f_2) - (T_a^{L+} \cdot f_1)(T_b^{L+} \cdot f_2) \\ & + (T_a^{R-} \cdot f_1)(T_b^{R-} \cdot f_2) - (T_a^{L-} \cdot f_1)(T_b^{L-} \cdot f_2)) \\ & + r^{ba} ((T_a^{L+} \cdot f_1)(T_b^{L-} \cdot f_2) - (T_a^{R+} \cdot f_1)(T_b^{R-} \cdot f_2) \\ & - (T_b^{L-} \cdot f_1)(T_a^{L+} \cdot f_2) + (T_b^{R-} \cdot f_1)(T_a^{R+} \cdot f_2)), \end{aligned}$$

where T_a^{L+} , T_a^{R+} and T_a^{L-} , T_a^{R-} denote the right and left invariant vector fields associated with the first and second component in $G \times G$, respectively.

2. *The map $S : G^* \rightarrow G$, $\alpha \mapsto S_+(\alpha)S_-(\alpha)^{-1}$ is a Poisson map from the dual Poisson–Lie group G^* to the Poisson manifold G with the Poisson bracket*

$$\begin{aligned} \{f_1, f_2\}_* = & r^{ba}_{(a)} ((T_a^L \cdot f_1)(T_b^L \cdot f_2) + (T_a^R \cdot f_1)(T_b^R \cdot f_2)) \\ & - r^{ab} ((T_a^L \cdot f_1)(T_b^R \cdot f_2) - (T_b^R \cdot f_1)(T_a^L \cdot f_2)). \end{aligned} \quad (45)$$

Proof. The proof proceeds by noting that both brackets define a Poisson–Lie structure on the manifold G . To verify that ψ and S are homomorphisms of Poisson–Lie groups, it is then sufficient to consider their tangent maps at the unit element and to show that these are homomorphisms of Lie bialgebras. Details of the proof are given in [15,16], see also ([7] (Section 14.7)). \square

Of special interest is the case where the map $\sigma = \sigma_+ - \sigma_- : \mathfrak{g}^{*cop} \rightarrow \mathfrak{g}$ is a linear isomorphism. This allows one to write every element $X \in \mathfrak{g}$ as $X = \sigma_+(\alpha) - \sigma_-(\alpha)$ with a unique element $\alpha = \sigma^{-1}(X) \in \mathfrak{g}^*$. It also implies that the map $S : G^* \rightarrow G$, $\alpha \mapsto S_+(\alpha)S_-(\alpha)^{-1}$ with tangent map $T_1 S = \sigma$ is a diffeomorphism between neighbourhoods of the unit elements of G^* and G . This allows one to factorise elements of G near the unit element uniquely as $g = g_+ g_-^{-1}$ with $g_{\pm} = S_{\pm}(S^{-1}(g))$. If the map S is a global diffeomorphism, this applies not only near the unit element but on the entire group G .

Definition 15.

1. *A quasitriangular Lie bialgebra \mathfrak{g} is called **factorisable** if the map $\sigma_+ - \sigma_- : \mathfrak{g}^{*cop} \rightarrow \mathfrak{g}$ is a linear isomorphism.*
2. *A quasitriangular Poisson–Lie group G is called **factorisable** if the map $S : G^* \rightarrow G$, $\alpha \mapsto S_+(\alpha)S_-(\alpha)^{-1}$ is a diffeomorphism.*

Example 15.

1. *For every Lie bialgebra \mathfrak{g} , the classical double $D(\mathfrak{g})$ is a factorisable Lie bialgebra.*

2. We consider the Lie bialgebra $\mathfrak{sl}(2, \mathbb{R})$ with the basis (14), the Lie bracket

$$[J_0, J_1] = J_2 \quad [J_0, J_2] = -J_1 \quad [J_1, J_2] = -J_0$$

and the classical r -matrix from (29)

$$r = J_0 \otimes J_0 - J_1 \otimes J_1 - J_2 \otimes J_2 + J_2 \otimes J_0 - J_0 \otimes J_2.$$

In terms of the dual basis $\{j^0, j^1, j^2\}$ of \mathfrak{g}^* with $\langle j^a, J_b \rangle = \delta_b^a$, the maps $\sigma_{\pm} : \mathfrak{g}^{*cop} \rightarrow \mathfrak{g}$ are given by

$$\begin{aligned} \sigma_+(j^0) &= J_0 - J_2 & \sigma_+(j^1) &= -J_1 & \sigma_+(j^2) &= J_0 - J_2 \\ \sigma_-(j^0) &= -(J_2 + J_0) & \sigma_-(j^1) &= J_1 & \sigma_-(j^2) &= J_2 + J_0. \end{aligned}$$

The map $\sigma_+ - \sigma_-$ is a linear isomorphism, and we have $\text{im}(\sigma_+) = \text{span}\{J_1, J_0 - J_2\}$ and $\text{im}(\sigma_-) = \text{span}\{J_1, J_0 + J_2\}$. The factorisation of an element $X = x_0 J_0 + x_1 J_1 + x_2 J_2$ with $\sigma^{-1}(X) = \frac{1}{2}x_0 j^0 - \frac{1}{2}x_1 j^1 - \frac{1}{2}x_2 j^2$ is given by

$$\begin{aligned} X_+ &= \frac{1}{2}(x_0 - x_2)J_0 + \frac{1}{2}x_1 J_1 + \frac{1}{2}(x_2 - x_0)J_2 \\ X_- &= -\frac{1}{2}(x_0 + x_2)J_0 - \frac{1}{2}x_1 J_1 - \frac{1}{2}(x_0 + x_2)J_2. \end{aligned}$$

Using the matrix expressions (14) for the basis elements, one obtains

$$X_+ = \frac{1}{2} \begin{pmatrix} \frac{x_1}{2} & x_2 - x_0 \\ 0 & -\frac{x_1}{2} \end{pmatrix} \quad X_- = \frac{1}{2} \begin{pmatrix} -\frac{x_1}{2} & 0 \\ -(x_0 + x_2) & \frac{x_1}{2} \end{pmatrix}.$$

3. If G is a connected and simply connected Lie group with the trivial Poisson–Lie structure, then the classical double is the semidirect product $G \ltimes \mathfrak{g}^*$ with group multiplication

$$(g, \alpha) \cdot (h, \beta) = (gh, \alpha + \text{Ad}^*(g)\beta).$$

The factorisation is then given by $(g, \alpha) = (g, \alpha)_+ \cdot (g, \alpha)_-^{-1}$ with $(g, \alpha)_+ = (0, \alpha)$ and $(g, \alpha)_- = (g^{-1}, 0)$.

Factorisability of a Poisson–Lie group G leads to many simplifications. In particular, the expression for the Poisson bracket $\{, \}_*$ in Theorem 6 allows one to determine the symplectic leaves of the dual Poisson–Lie group G^* . A direct computation shows that any conjugation invariant function on G Poisson commutes with all functions on G with respect to the Poisson bracket $\{, \}_*$ on G . This shows that its symplectic leaves must be contained in the conjugacy classes of G . In fact, for a factorisable Poisson–Lie group G , the diffeomorphism S from Theorem 6 identifies the symplectic leaves of G^* with the conjugacy classes of G . This simplifies a more general description of the symplectic leaves of a Poisson–Lie group in terms of dressing transformations. For more background on factorisable Poisson–Lie groups, symplectic leaves and dressing transformations, see the original articles [16–18], the textbook ([5] (Chapter 1.5)) and the references given therein.

4. Poisson Spaces from (Dynamical) Classical r -Matrices

4.1. Poisson Spaces over Quasitriangular Poisson–Lie Groups

These simplifications of the Poisson structure for quasitriangular Poisson–Lie groups G also extend to the associated Poisson G -spaces. The (antisymmetric part of) the classical r -matrix defines several Poisson G -space structures on G . Examples are the dual Poisson–Lie structure from Theorem 6, which is a Poisson G -space with respect to the conjugation action, and the Heisenberg double Poisson structure on G that gives G the structure of a Poisson $G \times G$ -space. The Heisenberg double Poisson structure is also called Semenov–Tian–Shansky bracket and was first introduced in [15,19]. The term Heisenberg double is

from [20], where it was related to a symplectic form and considered as a generalisation of a cotangent bundle symplectic structure.

Proposition 3. Let G be a Poisson–Lie group.

1. If G is a coboundary with $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$, $X \mapsto \text{ad}_X(u)$ for an antisymmetric element $u = u^{ab}T_a \otimes T_b$, then the **Heisenberg double Poisson structure**

$$\{f_1, f_2\} = u^{ba}(T_a^L \cdot f_1)(T_b^L \cdot f_2) + u^{ba}(T_a^R \cdot f_1)(T_b^R \cdot f_2)$$

gives G the structure of a Poisson $G \times G$ -space with respect to $\triangleright : G \times G \times G \rightarrow G$, $(g_1, g_2) \triangleright h = g_1 h g_2^{-1}$.

2. If G is quasitriangular with classical r -matrix $r = r^{ab}T_a \otimes T_b \in \mathfrak{g} \otimes \mathfrak{g}$ then G becomes a Poisson G -space with $\triangleright : G \times G \rightarrow G$, $g \triangleright h = ghg^{-1}$ and the **dual Poisson structure** from Theorems 2 and 6.

Proof. That the Poisson brackets indeed satisfy the Jacobi identity follows by direct computations similar to the ones in the proof of Theorem 5. For the bracket in 1. one obtains from the Ad-invariance of $[[u, u]]$ and the relation between left and right invariant vector fields

$$\begin{aligned} & \{\{f_1, f_2\}, f_3\} + \{\{f_3, f_1\}, f_2\} + \{\{f_2, f_3\}, f_1\} \\ &= -[[u, u]]^L \cdot (f_1 \otimes f_2 \otimes f_3) - [[u, u]]^R \cdot (f_1 \otimes f_2 \otimes f_3) = 0. \end{aligned}$$

The computations for the bracket in 2. are similar. The compatibility condition from Definition 4 between the Poisson structure and the group actions follows by a direct computation from the expressions for the Poisson brackets and formula (43) for the Sklyanin bracket. \square

Besides giving simple expressions for the Poisson brackets of certain Poisson G -spaces, the classical r -matrix of a quasitriangular Poisson–Lie group also has more conceptual implications. It allows one to form *products* of Poisson G -spaces that can be viewed as the Poisson–Lie counterpart of the braided tensor product of module algebras over a quasitriangular Hopf algebra.

For this, note that if M and N are Poisson G -spaces over a Poisson–Lie group G , then $M \times N$ has a canonical Poisson structure and a canonical G -action, namely the *product Poisson structure* $\{, \}_{M \times N}$ and the *diagonal action* $\triangleright : G \times M \times N \rightarrow M \times N$, $(g, m, n) \mapsto (g \triangleright m, g \triangleright n)$. However, together they do not equip $M \times N$ with the structure of a Poisson G -space. This becomes obvious when one considers functions $f_1 \in C^\infty(M)$, $f_2 \in C^\infty(N)$ together with the projection maps $\pi_M : M \times N \rightarrow M$, $(m, n) \mapsto m$ and $\pi_N : M \times N \rightarrow N$, $(m, n) \mapsto n$. Then, the definition of the product Poisson structure implies

$$\{f_1 \circ \pi_M, f_2 \circ \pi_N\}_{M \times N}(m, n) = \{f_1, f_2(n)\}_M(m) + \{f_1(m), f_2\}_N(n) = 0$$

for all $m \in M$ and $n \in N$. However, the condition that $M \times N$ is a Poisson G -space requires

$$\begin{aligned} 0 &= \{f_1 \circ \pi_M, f_2 \circ \pi_N\}_{M \times N}(g \triangleright m, g \triangleright n) \\ &= \{f_1 \circ \pi_M(g \triangleright -), f_2 \circ \pi_N(g \triangleright -)\}_{M \times N}(m, n) + \{f_1(- \triangleright m), f_2(- \triangleright n)\}_G(g) \\ &= \{f_1(- \triangleright m), f_2(- \triangleright n)\}_G(g). \end{aligned}$$

Hence, $M \times N$ with the product Poisson structure and the diagonal G -action is in general *not* a Poisson G -space, unless the Poisson–Lie structure on G is trivial.

It turns out that if G is quasitriangular, one can modify the Poisson structure on $M \times N$ with the classical r -matrix to obtain a Poisson G -space structure. This result goes back to the work of Fock and Rosly [21–23], but was not stated explicitly there. The following theorem was given in [24,25] and independently in [26]. It was shown in [24,26] that this product of Poisson G -spaces gives the category of Poisson G -spaces over a quasitriangular

Poisson–Lie group G the structure of a monoidal category and in [24] that this monoidal category is monoidally equivalent to the category of quasi-Hamiltonian G -spaces.

Theorem 7. *Let G be a quasitriangular Poisson–Lie group with classical r -matrix $r = r^{ab}T_a \otimes T_b$ and $(M, \{, \}_M, \triangleright_M)$ and $(N, \{, \}_N, \triangleright_N)$ Poisson G -spaces. Then:*

1. $M \times N$ becomes a Poisson G -space with the diagonal G -action and the Poisson bracket

$$\{f_1, f_2\}_r = \{f_1, f_2\}_{M \times N} - r^{ab}(T_a^{\triangleright M} \cdot f_1)(T_b^{\triangleright N} \cdot f_2) + r^{ab}(T_b^{\triangleright N} \cdot f_1)(T_a^{\triangleright M} \cdot f_2).$$

2. Every manifold M with a G -action $\triangleright : G \times M \rightarrow M$ becomes a Poisson G -space with

$$\{f_1, f_2\}_M = -r_{(a)}^{ab}(T_a^{\triangleright} \cdot f_1)(T_b^{\triangleright} \cdot f_2).$$

Proof. Clearly, the brackets $\{, \}_r$ and $\{, \}_M$ are antisymmetric and satisfy the Leibniz identity. That they satisfy the Jacobi identity follows by direct computations that use the CYBE for r , and for the bracket $\{, \}_r$ additionally the fact that M and N are Poisson G -spaces. The compatibility conditions from Definition 4 between the brackets and the G -actions follow by a direct computation that makes use of expression (43) for the Sklyanin bracket, and for the bracket $\{, \}_r$ also of the definition of the product Poisson structure and the Ad-invariance of r . \square

4.2. Application: Moduli Spaces of Flat Connections

An important application of the product of Poisson G -spaces is the description of the canonical symplectic structure on the moduli space of flat G -connections on an orientable surface Σ . The moduli space of flat G -connections is given as the quotient $\text{Hom}(\pi_1(\Sigma), G)/G$ of the set $\text{Hom}(\pi_1(\Sigma), G)$ of group homomorphisms $\rho : \pi_1(\Sigma) \rightarrow G$ modulo the conjugation action $\triangleright : G \times \text{Hom}(\pi_1(\Sigma), G) \rightarrow \text{Hom}(\pi_1(\Sigma), G)$, $(g, \rho) \mapsto g \cdot \rho \cdot g^{-1}$. The moduli space of flat connections has a canonical symplectic structure described by Atiyah and Bott [27] and Goldman [28,29] that depends only on the choice of a non-degenerate Ad-invariant symmetric bilinear form on $\mathfrak{g} = \text{Lie } G$.

From the physics perspective, the moduli space of flat G -connections is the gauge invariant or reduced phase space of a Chern–Simons gauge theory with gauge group G on $\mathbb{R} \times \Sigma$. The choice of a non-degenerate Ad-invariant symmetric bilinear form on \mathfrak{g} defines the Chern–Simons gauge functional, and its canonical symplectic structure arises from symplectic reduction of the induced symplectic structure on the space of gauge fields [27].

A choice of a quasitriangular Poisson–Lie group structure on the gauge group G allows one to give a simple description of the canonical symplectic structure on $\text{Hom}(\pi_1(\Sigma), G)/G$ due to Fock and Rosly [21–23] that resembles a lattice gauge theory. In this description, the orientable surface Σ is modelled by a directed graph Γ embedded in Σ . The graph needs to be sufficiently fine to resolve the topology of the surface, so one requires that $\Sigma \setminus \Gamma$ is a disjoint union of discs. The connected components of $\Sigma \setminus \Gamma$ are called *faces* of Γ .

We denote by V, E, F , respectively, the sets of vertices, edges and faces of Γ . For each edge e , we write $s(e)$ for the starting and $t(e)$ for the target vertex of e . The orientation of the surface induces a cyclic ordering of the edge ends incident at each vertex $v \in V$. We equip each vertex with a marking, the *cilium*, that transforms this cyclic ordering into a linear ordering. The order of the edges is taken counterclockwise from the cilium, and we write $e < f$ if the edge end e is of lower order than the edge end f . The faces of Γ then correspond bijectively to closed paths in Γ , up to cyclic permutations, that follow the face counterclockwise, turn maximally left at each vertex in the path and traverse each edge at most once in each direction. Cyclic permutations of paths correspond to different choices of their starting and endpoints.

Given a Lie group G , one can define a **graph connection** as a map $A : E \rightarrow G, e \mapsto g_e$ that assigns an element of G to each oriented edge of Γ . Reversing the orientation of an edge e corresponds to replacing g_e by g_e^{-1} . The set $\mathcal{A}(\Gamma)$ of graph connections on Γ is thus

given by the E -fold product $G^{\times E}$. We denote by $\pi_e : G^{\times E} \rightarrow G, (g_1, \dots, g_n) \rightarrow g_e$ the maps that project on the factor associated with the edge $e \in E$.

The appropriate notion of curvature in this context is obtained by considering the ordered product of the group elements taken along a path that borders a face of Γ . For a **flat graph connection**, this product must vanish for the paths around each face of Γ . One thus chooses for each face of Γ an associated path f in Γ that starts and ends at a vertex and defines a moment map

$$\mu_f : G^{\times E} \rightarrow G, (g_1, \dots, g_n) \mapsto \prod_{e \in f} g_e^{\epsilon_e}, \quad (46)$$

that encodes the curvature of a graph connection at the face f . Here, the product is taken over all edges traversed by f , in the order in which they occur in f and $\epsilon_e = 1$ if e is traversed in its orientation and $\epsilon_e = -1$ if it is traversed against its orientation. Group elements of edges that are traversed twice by f occur twice in the product, but with opposite signs. Although μ_f depends on the choice of the starting point of the path f , the preimage $\mu_f^{-1}(1)$ does not, since different choices of paths f for a given face are related by conjugation. The space of flat graph connections is thus independent of the choices of paths

$$\mathcal{F}(\Gamma) = \cap_{f \in F} \mu_f^{-1}(1) \subseteq G^{\times E}.$$

Graph gauge transformations are modelled by group actions of G assigned to the vertices of Γ . The group action $\triangleright_v : G \times G^{\times E} \rightarrow G^{\times E}$ for a vertex $v \in V$ acts only on those copies of G that correspond to edges incident at v . The action is by left multiplication for edges that are incoming at v , by right multiplication with the inverse for edges that are outgoing at v and by conjugation for loops at v :

$$\pi_e(g \triangleright_v (g_1, \dots, g_n)) = \begin{cases} g \cdot g_e & v = t(e) \neq s(e) \\ g_e \cdot g^{-1} & v = s(e) \neq t(e) \\ g \cdot g_e \cdot g^{-1} & v = t(e) = s(e) \\ g_e & v \notin \{s(e), t(e)\}. \end{cases} \quad (47)$$

As the group actions \triangleright_v and \triangleright_w for distinct vertices $v, w \in V$ commute, combining them yields a group action of $G^{\times V}$ on $G^{\times E}$. One thus defines the group of **graph gauge transformations** as $\mathcal{G}(\Gamma) = G^{\times V}$ and obtains a group action of $\mathcal{G}(\Gamma)$ on the set $\mathcal{A}(\Gamma)$ of graph connections. As this group action sends flat connections to flat connections, it restricts to an action $\triangleright : \mathcal{G}(\Gamma) \times \mathcal{F}(\Gamma) \rightarrow \mathcal{F}(\Gamma)$. The moduli space of flat G -connections on Σ is then given as the orbit space of this group action

$$\text{Hom}(\pi_1(\Sigma), G)/G = \mathcal{F}(\Gamma)/\mathcal{G}(\Gamma).$$

In [21–23], Fock and Rosly showed that if G has a quasitriangular Poisson–Lie group structure, the symplectic structure on $\text{Hom}(\pi_1(\Sigma), G)/G$ can be described in terms of products of Poisson G -spaces. More specifically, this requires one or more classical r -matrices $r \in \mathfrak{g} \otimes \mathfrak{g}$ whose symmetric components are dual to a fixed non-degenerate Ad-invariant symmetric bilinear form on \mathfrak{g} . They define a Poisson $G^{\times V}$ -space structure on $G^{\times E}$ that induces Goldman’s and Atiyah–Bott’s symplectic structure on $\text{Hom}(\pi_1(\Sigma), G)/G$.

This Poisson structure on $G^{\times E}$ is obtained by assigning classical r -matrices $r(v)$ to the vertices of Γ whose symmetric components are dual to a fixed non-degenerate Ad-invariant symmetric bilinear form on \mathfrak{g} . This amounts to assigning a quasitriangular Poisson–Lie group to each vertex of Γ . Each edge of Γ is assigned a generalisation of the Heisenberg double Poisson structure from Proposition 3, which is given by the r -matrices at its starting and target vertex and an example of the mixed product Poisson structures in [25].

Taking the product of the Poisson G -spaces from Theorem 7 at each vertex of Γ then yields Fock and Rosly’s Poisson structure on $G^{\times E}$. To describe it explicitly, we choose a

basis $\{T_a\}$ of \mathfrak{g} . We denote by T_a^{Le} and T_a^{Re} the right and left invariant vector fields on the copy of G for each edge $e \in E$, whose action on functions $F \in C^\infty(G^{\times E})$ is given by

$$\begin{aligned} T_a^{Le} \cdot F(g_1, \dots, g_n) &= \frac{d}{dt} \Big|_{t=0} F(g_1, \dots, e^{-tT_a} \cdot g_e, \dots, g_n) \\ T_a^{Re} \cdot F(g_1, \dots, g_n) &= \frac{d}{dt} \Big|_{t=0} F(g_1, \dots, g_e \cdot e^{tT_a}, \dots, g_n). \end{aligned}$$

Fock and Rosly's Poisson structure can then be summarised as follows.

Theorem 8 ([21–23]). *Let G be a quasitriangular Poisson–Lie group and Γ a directed graph embedded into an orientable surface Σ such that $\Sigma \setminus \Gamma$ is a disjoint union of discs. Assign to each vertex $v \in V$ a classical r -matrix $r(v) = r^{ab}(v)T_a \otimes T_b$, such that their symmetric components are non-degenerate and agree for all vertices $v \in V$.*

1. *The Poisson bivector*

$$\begin{aligned} B &= \sum_{v \in V} r_{(a)}^{ab}(v) \left(\sum_{s(e)=v} T_a^{Re} \wedge T_b^{Re} + \sum_{t(e)=v} T_a^{Le} \wedge T_b^{Le} \right) \\ &+ \sum_{v \in V} r^{ab}(v) \left(\sum_{\substack{v=t(e)=t(f) \\ e < f}} T_a^{Le} \wedge T_b^{Lf} + \sum_{\substack{v=t(e)=s(f) \\ e < f}} T_a^{Le} \wedge T_b^{Rf} + \sum_{\substack{v=s(e)=t(f) \\ e < f}} T_a^{Re} \wedge T_b^{Lf} \right. \\ &\left. + \sum_{\substack{v=s(e)=s(f) \\ e < f}} T_a^{Re} \wedge T_b^{Rf} \right) \end{aligned} \quad (48)$$

defines a Poisson $G^{\times V}$ -space structure on $G^{\times E}$ for the $G^{\times V}$ -action from (47).

2. *The Poisson brackets of $G^{\times V}$ -invariant functions on $G^{\times E}$ are independent of the choice of the cilia and depend only on the common symmetric component of all r -matrices $r^{ab}(v)$.*
3. *For all class functions $F \in C^\infty(G)^G$ and faces $f \in F$, the function $F \circ \mu_f \in C^\infty(G^{\times E})$ Poisson commutes with all functions in $C^\infty(G^{\times E})$.*

Proof. 1. The terms in the first line of (48) are the product Poisson structure on $G^{\times E}$ defined by the Poisson structures on G with bivectors

$$r^{ab}(s(e))T_a^R \wedge T_b^R + r^{ab}(t(e))T_a^L \wedge T_b^L$$

for each edge $e \in E$. This generalises the Heisenberg double Poisson structure from Proposition 3. As in the proof of Proposition 3, one can show that this defines a Poisson $G \times G$ -space structure on G with respect to the G -actions by left multiplication and by right multiplication with the inverse. As each G -action \triangleright_v from (47) is trivial on edges that are not incident at v , it follows that G with this Poisson structure is a Poisson $G^{\times V}$ space with $G^{\times V}$ -action from (47) if none of the edges is a loop.

For an edge e that is a loop based at $v \in V$, the second line of (48) yields an additional term of the form $r^{ab}(v)T_a^{Le} \wedge T_b^{Re}$ or $-r^{ba}(v)T_a^{Le} \wedge T_b^{Re}$ that combines with the terms for e in the first line of (48) into the dual Poisson–Lie structure from (45). As (45) defines a Poisson G -space structure on G for the conjugation action, it is a Poisson $G^{\times V}$ -space for the $G^{\times V}$ -action from (47). The Poisson bivector (48) is then obtained by taking the product of these Poisson $G^{\times V}$ -spaces from Theorem 7 and hence becomes a Poisson $G^{\times V}$ -space with respect to this action by Theorem 7.

2. That the Poisson brackets of $G^{\times V}$ -invariant functions depend only on the symmetric components of the classical r -matrices follows by expressing the Poisson bivector B from

(48) as the sum of a bivector B_a that depends only on their antisymmetric and a bivector B_s that depends only on their symmetric components. One finds that the former is given by

$$B_a = \sum_{v \in V} r_{(a)}^{ab}(v) \left(\sum_{v=s(e)} T_a^{Re} + \sum_{v=t(e)} T_a^{Le} \right) \otimes \left(\sum_{v=s(e)} T_b^{Re} + \sum_{v=t(e)} T_b^{Le} \right).$$

As any $G^{\times V}$ -invariant function $F \in C^\infty(G^{\times E})$ satisfies

$$\left(\sum_{v=s(e)} T_a^{Re} + \sum_{v=t(e)} T_a^{Le} \right) \cdot F = 0,$$

the contribution of B_a to Poisson brackets of such an invariant function with any function in $C^\infty(G^{\times E})$ vanishes. The symmetric component of the bivector is given by

$$\begin{aligned} B_s = \sum_{v \in V} r_{(s)}^{ab} \left(\sum_{\substack{v=t(e)=t(f) \\ e < f}} T_a^{Le} \wedge T_b^{Lf} + \sum_{\substack{v=t(e)=s(f) \\ e < f}} T_a^{Le} \wedge T_b^{Rf} + \sum_{\substack{v=s(e)=t(f) \\ e < f}} T_a^{Re} \wedge T_b^{Lf} \right. \\ \left. + \sum_{\substack{v=s(e)=s(f) \\ e < f}} T_a^{Re} \wedge T_b^{Rf} \right) \end{aligned}$$

and independent of the choice of the cilia, which differ by cyclic permutations of the orderings at each vertex.

3. For each function $F \in C^\infty(G)^G$ and any face f , the function $F \circ \mu_f \in C^\infty(G^{\times E})$ is $G^{\times V}$ -invariant and hence its Poisson bracket with any function in $C^\infty(G^{\times E})$ is independent of the choice of the cilia. One may thus orient the cilia at the vertices in such a way that the associated path f does not traverse any cilia. This implies that there are no edges that are between consecutive edges in f with respect to the ordering at their common vertex. A direct computation then shows that the Poisson bracket of $F \circ \mu_f$ with any function in $C^\infty(G^{\times E})$ vanishes. \square

The description in Theorem 8 involves the graph gauge transformations acting on the graph connections but not the moment maps $\mu_f : G^{\times E} \rightarrow G$ from (46) that encode the flatness conditions. From the viewpoint of constrained mechanical systems, the conditions $\mu_f(g_1, \dots, g_n) = 1$ for the moment maps $\mu_f : G^{\times E} \rightarrow G$ from (46) can be viewed as group valued first-class constraints that generate the graph gauge transformations. Reducing this gauge freedom corresponds to a Poisson reduction of the Poisson $G^{\times V}$ -space structure from Theorem 8. It was shown by Fock and Rosly [21–23] that this yields the canonical symplectic structure on the moduli space of flat G connections.

Theorem 9 ([21–23]). *Poisson reduction of the Poisson structure from Theorem 8 with respect to the moment maps $\mu_f : G^{\times E} \rightarrow G$ for the faces $f \in F$ and the group action $\triangleright : G^{\times V} \times G^{\times E} \rightarrow G^{\times E}$ induces Goldman's and Atiyah–Bott's symplectic structure on $\text{Hom}(\pi_1(\Sigma), G)/G$ for the non-degenerate symmetric bilinear form defined by the symmetric component of r .*

Remark 6. *This description of the moduli space of flat G -connections is closely related to its description via quasi Hamiltonian systems and Lie group value moment maps [30], for an accessible review see also the lecture notes [31]. In fact, it was shown in [24] that the monoidal category of Poisson G -spaces over a quasitriangular Poisson–Lie group G is monoidally equivalent to the monoidal category of quasi Hamiltonian G -spaces. The monoidal structures are the product of Poisson G -spaces from Theorem 7 and the fusion product of Poisson G -spaces, respectively.*

Remark 7. *In the case where the quasitriangular Poisson–Lie group G in Theorem 8 is a classical double $D(H)$ of a Poisson–Lie group H , the description in Theorem 8 can be factorised such that group elements of H are associated with Γ and elements of the dual Poisson–Lie group H^* with the*

dual graph of Γ . This yields a factorisation of the Poisson structure from Theorem 8 as a product of Heisenberg double Poisson structures on $D(G)$ associated with each edge of Γ . For a graph with one vertex, this was first shown in the symplectic framework in [32]. For a general graph, this was established in [33], where it was shown that the resulting description of the moduli space can be viewed as a Poisson analogue of Kitaev lattice models [34] considered in condensed matter physics and topological quantum computing.

4.3. Poisson Structures from Dynamical r -Matrices

In this section, we generalise some of the results from Section 4.1 to dynamical classical r -matrices. While a classical r -matrix is an element of $\mathfrak{g} \otimes \mathfrak{g}$ for a Lie algebra \mathfrak{g} , a dynamical r -matrix is a function with values in $\mathfrak{g} \otimes \mathfrak{g}$. Thus, a classical r -matrix can be viewed as a special case of a dynamical r -matrix, namely a constant one.

Dynamical r -matrices were first considered in [35,36] in the context of integrable systems. They also arise in the Poisson reduction and symplectic reduction of systems with Poisson–Lie symmetries [37], for instance, in the description of moduli spaces of flat connections and the Chern–Simons gauge theory [38,39]. For an accessible introduction to classical dynamical r -matrices, see [8], for classification results [40–43].

To keep the discussion simple, we restrict attention to dynamical r -matrices associated with Cartan subalgebras. Most of the results hold in more generality but their description becomes more complicated.

Definition 16 ([41]). Let \mathfrak{g} be a Lie algebra with basis $\{T_a\}$ and $\mathfrak{h} \subseteq \mathfrak{g}$ a Cartan subalgebra. A **dynamical r -matrix** for $(\mathfrak{g}, \mathfrak{h})$ is a meromorphic function $r = r^{ab} T_a \otimes T_b : U \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ on an open subset $\emptyset \neq U \subset \mathfrak{h}^*$ that is invariant under the adjoint action of \mathfrak{h} on $\mathfrak{g} \otimes \mathfrak{g}$ and satisfies

1. the **dynamical classical Yang–Baxter equation**:

$$[[r, r]] + \text{Alt}(dr) := [[r, r]] + \frac{\partial r^{ab}}{\partial x^i} (x_i \otimes T_a \otimes T_b - T_a \otimes x_i \otimes T_b + T_a \otimes T_b \otimes x_i) = 0,$$

where $[[r, r]]$ is the Schouten bracket (27) and one sums over a basis $\{x_i\}$ of \mathfrak{h} and the dual basis $\{x^i\}$ of \mathfrak{h}^* ;

2. the **unitarity condition**:

$r_{(s)} = \frac{1}{2}(r^{ab} + r^{ba})T_a \otimes T_b$ is a constant element of $\mathfrak{g} \otimes \mathfrak{g}$ that is invariant under the adjoint action of \mathfrak{g} .

The concept of a dynamical classical r -matrix generalises the concept of a classical r -matrix. The latter is simply a dynamical classical r -matrix for the *trivial* Lie subalgebra $\mathfrak{h} = \{0\} \subset \mathfrak{g}$. However, dynamical r -matrices have some important advantages over the classical r -matrices from Definition 9. The first is that classical r -matrices whose symmetric component coincides with a fixed ad-invariant symmetric element $t \in \mathfrak{g} \otimes \mathfrak{g}$ for a real Lie algebra \mathfrak{g} may not exist unless one complexifies \mathfrak{g} , while the differential equation that defines dynamical classical r -matrices still has a solution. If a classical r -matrix for a fixed ad-invariant element of $\mathfrak{g} \otimes \mathfrak{g}$ exists, it often arises as a limit of a dynamical one. This is illustrated by the following examples.

Example 16.

1. Consider the Lie algebra $\mathfrak{su}(2)$ with a basis $\{J_0, J_1, J_2\}$, in which the Lie bracket reads

$$[J_0, J_1] = J_2, \quad [J_1, J_2] = J_0, \quad [J_2, J_0] = J_1.$$

Then, a dynamical classical r -matrix for the Lie subalgebra $\mathfrak{h} = \mathbb{R}J_0 \cong \mathbb{R}$ is given by

$$r : \mathbb{R} \rightarrow \mathfrak{su}(2) \otimes \mathfrak{su}(2), \quad t \mapsto J_0 \otimes J_0 + J_1 \otimes J_1 + J_2 \otimes J_2 - \tan(t) J_1 \wedge J_2.$$

The Lie algebra $\mathfrak{su}(2)$ has no real classical r -matrix in the sense of Definition 9.

2. Consider the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ with the basis $\{J_0, J_1, J_2\}$ from Example 6. Then, a dynamical classical r -matrix for $\mathfrak{h} = \mathbb{R}J_0$ is given by

$$r : \mathbb{R} \rightarrow \mathfrak{sl}(2, \mathbb{R}) \otimes \mathfrak{sl}(2, \mathbb{R}), \quad t \mapsto J_0 \otimes J_0 - J_1 \otimes J_1 - J_2 \otimes J_2 - \tan(t) J_1 \wedge J_2$$

and a dynamical classical r -matrix for $\mathfrak{h} = \mathbb{R}J_1$ by

$$r' : \mathbb{R} \rightarrow \mathfrak{sl}(2, \mathbb{R}) \otimes \mathfrak{sl}(2, \mathbb{R}), \quad t \mapsto J_0 \otimes J_0 - J_1 \otimes J_1 - J_2 \otimes J_2 - \tanh(t) J_2 \wedge J_0.$$

The Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ has no real classical r -matrix whose antisymmetric component is a real multiple of $J_1 \wedge J_2$. A classical r -matrix whose antisymmetric component is a real multiple of $J_2 \wedge J_0$ is given in (29). This classical r -matrix is obtained from r' in the limit $t \rightarrow -\infty$.

The second major advantage of dynamical over classical r -matrices is that the latter can be related by dynamical gauge transformations. This was used in [40–43] to classify them. Dynamical gauge transformations are especially simple for the case where $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra, and we restrict attention to this case.

Definition 17 ([41]). Let $\mathfrak{h} \subseteq \mathfrak{g}$ a Cartan subalgebra and $H \subset G$ the associated connected subgroup. A **dynamical gauge transformation** for \mathfrak{h} is a holomorphic map $\gamma : U \rightarrow H$ with $U \subseteq \mathfrak{h}^*$. Its action on a classical dynamical r -matrix $r : U \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ is given by

$$\gamma \triangleright r : U \rightarrow \mathfrak{g} \otimes \mathfrak{g}, \quad x \mapsto \text{Ad}(\gamma(x))(r(x) - \eta(x) + \eta^{21}(x)), \quad (49)$$

where $\eta : U \rightarrow \mathfrak{h} \otimes \mathfrak{h}$ is the map defined by the left invariant one-form $\omega = \gamma^{-1} d\gamma$ and η^{21} is obtained from η by flipping the two factors in the tensor product.

Proposition 4 ([41]). If $r : U \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ is a classical dynamical r -matrix for a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, then for all dynamical gauge transformations $\gamma : U \rightarrow H$ the map $\gamma \triangleright r : U \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ is a dynamical classical r -matrix for \mathfrak{h} .

This result was used in [40,42] to classify all dynamical r -matrices for complex simple Lie algebras, for a more general classification under weaker assumptions, see also [43]. These classification results uses dynamical gauge transformations to bring the dynamical r -matrix into a standard form, in which the dynamical classical Yang–Baxter equation becomes particularly simple.

Just as a classical r -matrix for a Lie algebra \mathfrak{g} defines a quasitriangular Poisson–Lie group structure on the associated Lie group G , dynamical classical r -matrices define Poisson groupoid structures. The Poisson groupoid structure defined by classical dynamical r -matrices $r : U \subset \mathfrak{h}^* \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ and $s : V \subset \mathfrak{k}^* \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ is a Poisson structure on $U \times G \times V$, where linear functions on U and V are identified with elements of \mathfrak{h} and \mathfrak{k} . Dynamical Poisson groupoids were introduced by Etingof and Varchenko in [42]. We summarise the more accessible formulation from [37], with some changes in sign conventions and a restriction to Cartan subalgebras not present in [37].

Proposition 5 ([37]). Let \mathfrak{g} be a Lie algebra with basis $\{T_a\}$ and $\mathfrak{h}, \mathfrak{k} \subset \mathfrak{g}$ Cartan subalgebras with bases $\{H_a = h_a^b T_b\}$ and $\{K_a = k_a^b T_b\}$, respectively, and dual bases $\{H^a\}$ and $\{K^a\}$. Let

$$r = r^{ab} T_a \otimes T_b : U \subset \mathfrak{h}^* \rightarrow \mathfrak{g} \otimes \mathfrak{g} \quad s = s^{ab} T_a \otimes T_b : V \subset \mathfrak{k}^* \rightarrow \mathfrak{g} \otimes \mathfrak{g}$$

be classical dynamical r -matrices with symmetric components $r_{(s)} = s_{(s)} = t$ for some \mathfrak{g} -invariant symmetric element $t \in \mathfrak{g} \otimes \mathfrak{g}$. Then, the following is a Poisson bracket on $U \times G \times V$

$$\begin{aligned}
\{f, g\} &= r_{(a)}^{ab}(T_a^L \cdot f)(T_b^L \cdot g) + s_{(b)}^{ab}(T_a^R \cdot f)(T_b^R \cdot g), \\
\{f, H_a\} &= h_a^b(T_b^L \cdot f), \quad \{f, K_a\} = k_a^b(T_b^R \cdot f), \\
\{H_a, H_b\} &= \{H_a, K_b\} = \{K_a, K_b\} = 0,
\end{aligned} \tag{50}$$

where $f, g \in C^\infty(G)$ and $H_a \in \mathfrak{h}$, $K_a \in \mathfrak{k}$ are viewed as functions on $U \subset \mathfrak{h}^*$ and $V \subset \mathfrak{k}^*$.

Proof. We check the Jacobi identity for the Poisson bracket (50). For functions $f, g, h \in C^\infty(G)$ one has

$$\begin{aligned}
&\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} \\
&= -([r_{(a)}, r_{(a)}]) + \text{Alt}(dr)^{abc}(T_a^L \cdot f)(T_b^L \cdot g)(T_c^L \cdot h) \\
&\quad - ([s_{(a)}, s_{(a)}]) + \text{Alt}(ds)^{abc}(T_a^R \cdot f)(T_b^R \cdot g)(T_c^R \cdot h) \\
&= [[r_{(s)}, r_{(s)}]]^{abc}(T_a^L \cdot f)(T_b^L \cdot g)(T_c^L \cdot h) + [[s_{(a)}, s_{(s)}]]^{abc}(T_a^R \cdot f)(T_b^R \cdot g)(T_c^R \cdot h) = 0,
\end{aligned} \tag{51}$$

where we used the dynamical classical Yang–Baxter equation for r and s , and in the last step the relation (20) between right- and left invariant vector fields and the Ad-invariance of $[[r_{(s)}, r_{(s)}]] = [[s_{(s)}, s_{(s)}]]$ that follows from the \mathfrak{g} -invariance of $r_{(s)} = s_{(s)}$. For $f, g \in C^\infty(G)$ one has

$$\{H_a, \{f, g\}\} + \{g, \{H_a, f\}\} + \{f, \{g, H_a\}\} = (f_{ac}^a r^{bd} + f_{dc}^b r^{ad})(T_a^L \cdot f)(T_b^L \cdot g)h_c^d,$$

where f_{ab}^c are the structure constants of \mathfrak{g} given by $[T_a, T_b] = f_{ab}^c T_c$. This vanishes due to the \mathfrak{h} -invariance of r . An analogous identity holds for brackets of the form $\{K_a, \{f, g\}\}$. That the remaining brackets vanish follows directly from the identities for the Lie brackets of the invariant vector fields on G in Remarks 3 and 4. \square

Instead of a Poisson structure on the group $U \times G \times V$, one can also use the dynamical classical r -matrices in Proposition 5 to define a Poisson structure on the double coset $H \backslash G / K$. Functions on this double coset space correspond to functions on G that are invariant under left multiplication with H and right multiplication with K . As the basis elements H_a and K_a act trivially on such functions, one obtains the following corollary.

Corollary 3. Under the assumptions of Proposition 5,

$$\{f, g\} = r_{(a)}^{ab}(u)(T_a^L \cdot f)(T_b^L \cdot g) + s_{(a)}^{ab}(v)(T_a^R \cdot f)(T_b^R \cdot g) \tag{52}$$

defines a Poisson bracket on the subalgebra of functions in $C^\infty(G)$ that are invariant under left multiplication with H and right multiplication with K .

Proof. The invariance of $f \in C^\infty(G)$ under left multiplication with H and right multiplication with K implies $(h_a^b T_b^L) \cdot f = (k_a^b T_b^R) \cdot f = 0$. It follows that the term proportional to $\text{Alt}(dr)$ in (51) vanishes if f, g, h all satisfy this invariance requirement. Thus, the terms involving derivatives of r in (51) vanish and the bracket (52) satisfies the Jacobi identity. \square

We now generalise Fock and Rosly's Poisson structure from Theorem 8 to dynamical r -matrices. For this, recall that the Poisson structure in Theorem 8 is obtained by assigning a classical r -matrix to each vertex of an embedded graph Γ , assigning a generalised Heisenberg double Poisson structure to each edge and then coupling them via the product Poisson G -space structure at each vertex.

This can be generalised by assigning a dynamical classical r -matrix $r(v) : U_v \subset \mathfrak{h}_v^* \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ to each vertex v of Γ instead of a classical r -matrix. The generalised Heisenberg double Poisson structure assigned to an edge of Γ in Theorem 8 is then replaced by the Poisson structure from Proposition 5. Coupling the contributions of the incident edges at a vertex

as in Theorem 8, but with a dynamical Poisson structure then yields a Poisson structure on the group $\Pi_{v \in V} U_v \times G^{\times E}$ that generalises the Poisson structure from Theorem 8. This is a special case of the results obtained in [44], in which this Poisson structure was considered for more general dynamical Poisson G -spaces.

Proposition 6. Let Γ be a graph as in Theorem 8 and \mathfrak{g} a Lie algebra with basis $\{T_a\}$. Assign to each vertex $v \in V$ a Cartan subalgebra $\mathfrak{h}_v \subset \mathfrak{g}$ with basis $\{H_a^v = h_a^b(v)T_b\}$ and a classical dynamical r -matrix $r(v) = r^{ab}(v)T_a \otimes T_b : U_v \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ for \mathfrak{h}_v such that their symmetric components are non-degenerate and agree.

1. The following defines a Poisson bracket on $\Pi_{v \in V} U_v \times G^{\times E}$

$$\begin{aligned} \{f, H_a^v\} &= \sum_{v=t(e)} h_a^b(v) T_b^L \cdot f + \sum_{v=s(e)} h_a^b(v) T_b^R \cdot f, \\ \{H_a^v, H_b^w\} &= 0, \quad \{f, g\} = B \cdot (f \otimes g), \end{aligned} \quad (53)$$

where $f, g \in C^\infty(G^{\times E})$, elements $H_a^v \in \mathfrak{h}_v$ are viewed as functions on $U_v \subset \mathfrak{h}_v^*$ and B is the Poisson bivector from (48).

2. The third bracket in (53) defines a Poisson structure on the subalgebra $C^\infty(G^{\times E})^{\Pi_{v \in V} H_v}$ of functions on $G^{\times E}$ that are invariant under the group actions $\triangleright_v : H_v \times G^{\times E} \rightarrow G^{\times E}$ from (47). It induces Goldman's and Atiyah–Bott's symplectic structure on $\text{Hom}(\pi_1(\Sigma), G)/G$ for the non-degenerate symmetric bilinear form defined by the symmetric component of r .

Proof. 1. That this defines a Poisson structure on $\Pi_{v \in V} U_v \times G^{\times E}$ follows by combining the proof of Theorem 8 and Proposition 5. The contribution to the Poisson bivector from the first line of (48) for an edge e together with contributions for e from the first term in (53) define the Poisson groupoid from Proposition 5. That the remaining terms combine into a Poisson structure on $\Pi_{v \in V} U_v \times G^{\times E}$ follows by computations analogous to the proof of Theorem 7, just that the CYBE is replaced by the dynamical CYBE in the proof of the Jacobi identity.

2. That the bracket induces a Poisson bracket on the subalgebra $C^\infty(G^{\times E})^{\Pi_{v \in V} H_v}$ of functions that are invariant under the action of the groups H_v associated with the vertices follows as in the proof of Proposition 5. All terms involving the action of $\text{Alt}(dr(v))$ vanish if they are applied to such functions. In particular, this holds for functions $f \in C^\infty(G^{\times E})^{G^{\times V}}$ that are invariant under graph gauge transformations acting at the vertices. As in the proof of Theorem 8, one finds that the resulting Poisson bracket on $C^\infty(G^{\times E})^{G^{\times V}}$ depends only on the symmetric parts of the dynamical r -matrices and hence coincides with the bracket induced by the Poisson structure defined in Theorem 8. The last claim then follows from Theorem 9. \square

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