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Lie Symmetries and Solutions of Reaction Diffusion Systems Arising in Biomathematics

Mariano Torrisi † and **Rita Traciná *,†**

Dipartimento di Matematica e Informatica, Universitá di Catania, 95125 Catania, Italy; torrisi@dmi.unict.it
* Correspondence: rita.tracina@unict.it
† These authors contributed equally to this work.

Abstract: In this paper, a special subclass of reaction diffusion systems with two arbitrary constitutive functions $\Gamma(v)$ and $H(u,v)$ is considered in the framework of transformation groups. These systems arise, quite often, as mathematical models, in several biological problems and in population dynamics. By using weak equivalence transformation the principal Lie algebra, $\mathcal{L}_{\mathcal{P}}$, is written and the classifying equations obtained. Then the extensions of $\mathcal{L}_{\mathcal{P}}$ are derived and classified with respect to $\Gamma(v)$ and $H(u,v)$. Some wide special classes of special solutions are carried out.

Keywords: weak equivalence transformations; classical symmetries; biomathematical models; exact solutions

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Citation: Torrisi, M.; Traciná, R. Lie Symmetries and Solutions of Reaction Diffusion Systems Arising in Biomathematics. *Symmetry* **2021**, *13*, 1530. https://doi.org/10.3390/sym13081530

Academic Editor: Alexander Shapovalov

Received: 23 July 2021
Accepted: 17 August 2021
Published: 20 August 2021

1. Introduction

Motivated by several papers concerned with the mathematical models describing the dispersal dynamics of the Aedes Aegypt mosquitoes (main vector of dengue, xicungunia, zika, and other similar diseases) [1–3] as well as *Anopheles* [4], in two previous papers [5,6], the following parabolic reaction-diffusion-advection system has been considered

$$\begin{cases} u_t = (f(u)u_x)_x + g(u,v,u_x), \\ v_t = h(u,v), \end{cases} \tag{1}$$

where the diffusion coefficient $f(u)$, the reaction-advection term $g(u,v,u_x)$ and the reaction term $h(u,v)$ are assumed to be analytic functions of their arguments. This system, apart from its mathematical interest, could be considered a quite general model of two interacting species. In this model the species u can be subjected to advection phenomena, such as wind effects or water currents while the species v does not feel advection effects and, moreover, does not show diffusive phenomena [7].

When the advection effects are negligible or absent it is possible to assume the constitutive function g only depending on u and v. Concerned with mosquito models, it might correspond to an infestation in a small region where the wind currents are very weak. It is worthwhile stressing that a system class (1) with $g(u,v)$ can model not only the dispersal dynamics of some mosquito species with negligible advection, but it can model the interaction between swimming and swarming populations in the colonies of Proteus Mirabilis [8–10] or analogous bacterial colonies.

Having in mind to look for symmetries of systems belonging to the class (1) and taking into account that a classification with respect to all the constitutive parameters usually brings a big number of cases, several of them without biological meaning, here we focus our attention in the following form of the functions f and g [3,11]

$$f = D_0, \qquad g = \gamma_1 u(\gamma_2 - u) + \Gamma(v),$$

with D_0, γ_1, γ_2 non zero constants, so that the system (1) reads

$$\begin{cases} u_t = D_0 u_{xx} + \gamma_1 u(\gamma_2 - u) + \Gamma(v), \\ \\ v_t = h(u,v). \end{cases} \tag{2}$$

In this way, it is assumed that there is a "weak interaction" of the species v over the species u [12,13]. It is useful stressing that even though the symmetry approach provides a methodological way to derive exact solutions of non linear system, the symmetry classification with respect to arbitrary constitutive functions, that appear in (2), could suggest special forms of them of a certain interest for the phenomena under consideration. Even though in the last decades several studies have been devoted to reaction diffusion equations only few papers have been devoted to the symmetry classification of *non-linear* systems like (1) or (2) with advection (convection) terms. It is possible instead to find, as in [14], a complete description of Lie symmetries for a class of diffusion systems with convection terms in both equations. Moreover, the paper [15] shows Lie symmetry derivation for a class of systems, which includes cases having a structure similar to system (1).

Here we do not follow the classical Lie criterion approach in order to get the infinitesimal coordinates of the Lie symmetry generators [16,17]. We apply a projection theorem introduced in [18] (see for some applications, e.g., [19–21]), that allows us to reduce the plethora of calculations, by using a known equivalence generator. In our case we use a *weak equivalence generator* [22] for class (1) derived in [5].

This paper is organized as follow. In the next section by applying a projection theorem we write the classifying equations (for the system (2)) and the principal Lie algebra is shown. In Section 3 the classifying equations are discussed and the extensions of $\mathcal{L}_{\mathcal{P}}$ are derived. In Section 4, after having specialized the form of the constitutive function H in a suitable way, we reduced the system by using the corresponding admitted generator. Then some wide classes of special exact solutions are obtained. The conclusions are given in Section 5.

2. Symmetries. A Projection Theorem

A *projection theorem* [18], reconsidered in [22] affirms:

Theorem 1. *Let*

$$\begin{aligned} Y = \alpha(x)\partial_x + \beta(t)\partial_t + \delta(t,u)\partial_u + \lambda(x,t,v)\partial_v + (2\alpha' - \beta')f\partial_f + \\ + \left(\delta_t + (\delta_u - \beta_t)g + (\alpha'' - \delta_{uu}u_x^2)f\right)\partial_g + ((\lambda_v - \beta')h + \lambda_t)\partial_h \end{aligned} \tag{3}$$

be an infinitesimal weak equivalence generator for the systems (1)*, then the operator*

$$X = \alpha(x)\partial_x + \beta(t)\partial_t + \delta(t,u)\partial_u + \lambda(x,t,v)\partial_v, \tag{4}$$

which corresponds to the projection of Y on the space (x,t,u,v), is an infinitesimal symmetry generator of the system (2) *if, and only if, the constitutive equations, specifying the forms of f, g and h, are invariant with respect to Y.*

For the system (2) the constitutive equations are given by

$$\begin{cases} f = D_0, \\ g = \gamma_1 u(\gamma_2 - u) + \Gamma(v), \\ h = H(u,v). \end{cases} \tag{5}$$

Applying Theorem 1 we need to require the invariance of (5) with respect to the generator (3) imposing that the functions f, g and h are given by (5). Specifically from

$$Y(f - D_0)|_{(5)} = 0, \tag{6}$$

we get

$$(2\alpha' - \beta')f|_{(5)} = 0 \tag{7}$$

that is

$$(2\alpha' - \beta')D_0 = 0. \tag{8}$$

Similarly, from

$$Y(g - \gamma_1\gamma_2 u + \gamma_1 u^2 - \Gamma(v))\Big|_{(5)} = 0, \tag{9}$$

we get

$$\delta_t + (\delta_u - \beta_t)(\gamma_1 u(\gamma_2 - u) + \Gamma(v)) + (\alpha'' - \delta_{uu}u_x^2)D_0 - \gamma_1\gamma_2\delta + 2\gamma_1 u\delta - \Gamma_2'\lambda = 0. \tag{10}$$

Finally, from

$$Y(h - H(u,v))|_{(5)} = 0 \tag{11}$$

we get

$$\big((\lambda_v - \beta')H + \lambda_t\big) - H_u\delta - H_v\lambda = 0. \tag{12}$$

Then the operator

$$X = \alpha(x)\partial_x + \beta(t)\partial_t + \delta(t,u)\partial_u + \lambda(x,t,v)\partial_v, \tag{13}$$

is an infinitesimal symmetry generator of the system (2) if the functions α, β, δ, and λ satisfy the conditions (8), (10), and (12). Taking into account that $D_0 \neq 0$, from (8) we get immediately

$$\alpha(x) = \alpha_1 x + \alpha_0, \quad \beta(t) = 2\alpha_1 t + \beta_0, \tag{14}$$

with α_0, α_1, and β_0 arbitrary constants. Substituting these forms of $\alpha(x)$ and $\beta(t)$ in the remaining conditions (10), and (12), we get

$$\delta_t + (\delta_u - 2\alpha_1)(\gamma_1 u(\gamma_2 - u) + \Gamma) - \delta_{uu}u_x^2 D_0 - \gamma_1(\gamma_2 - 2u)\delta - \Gamma'\lambda = 0, \tag{15}$$

$$((\lambda_v - 2\alpha_1)H + \lambda_t) - H_u\delta - H_v\lambda = 0. \tag{16}$$

We observe that the constants α_0 and β_0 do not appear in these conditions. Moreover, for arbitrary forms of the functions $\Gamma(v)$ and $H(u,v)$ these conditions are satisfied only if

$$\alpha_1 = \delta = \lambda = 0, \tag{17}$$

then the Principal Lie Algebra $\mathcal{L}_{\mathcal{P}}$ (see, e.g., [18]) (the algebra of all the Lie symmetries that leave the system (2) invariant for any form of the functions $\Gamma(v)$, and $H(u,v)$) is spanned by the following translation generators:

$$X_1 = \partial_t, \qquad X_2 = \partial_x. \tag{18}$$

3. Extensions of $\mathcal{L}_{\mathcal{P}}$

In this section, we are interested in getting extensions of the Principal Lie Algebra for the system (2). Then our goal is to find special forms of the functions $\Gamma(v)$, and $H(u,v)$, such that the conditions (15) and (16) are satisfied for α_1, δ, and λ not all zero. The discussion of (15) and (16) leads to a classification with respect to the functions $\Gamma(v)$ and $H(u,v)$.

From (15) we get

$$\delta_{uu} = 0 \;\Rightarrow\; \delta(t,u) = \delta_1(t)u + \delta_0(t) \tag{19}$$

so we can rewrite (15) in the form

$$\delta_{0t} + \gamma_1(\delta_1 + 2\alpha_1)u^2 + [\delta_{1t} + 2\gamma_1(\delta_0 - \alpha_1\gamma_2)]u - \gamma_1\gamma_2\delta_0 + (\delta_1 - 2\alpha_1)\Gamma - \lambda\Gamma' = 0, \tag{20}$$

from where we are able to derive

$$\gamma_1(\delta_1 + 2\alpha_1) = 0, \tag{21}$$
$$\delta_{1t} + 2\gamma_1(\delta_0 - \alpha_1\gamma_2) = 0, \tag{22}$$
$$\delta_{0t} - \gamma_1\gamma_2\delta_0 + (\delta_1 - 2\alpha_1)\Gamma - \lambda\Gamma' = 0. \tag{23}$$

By solving (21) and (22) we get

$$\delta_1 = -2\alpha_1,\; \delta_0 = \alpha_1\gamma_2. \tag{24}$$

By substituting these results in (23) and (16) we get the following coupled classifying equations for the functions $\Gamma(v)$ and $H(u,v)$

$$\alpha_1(\gamma_1\gamma_2^2 + 4\Gamma) + \lambda\Gamma' = 0, \tag{25}$$
$$(\lambda_v - 2\alpha_1)H + \lambda_t + \alpha_1(2u - \gamma_2)H_u - \lambda H_v = 0. \tag{26}$$

It is possible to ascertain that Equation (25) brings to the discussion of the following cases

3.1. $\Gamma' \neq 0$;
3.2. $\Gamma' = 0, \Gamma \neq -\frac{\gamma_1\gamma_2^2}{4}$ that implies $\alpha_1 = 0$;
3.3. $\Gamma' = 0, \Gamma = -\frac{\gamma_1\gamma_2^2}{4}$.

3.1. $\Gamma' \neq 0$

From (25) we get, for arbitrary $\Gamma(v)$ with $\Gamma' \neq 0$,

$$\lambda(t,x,v) = -\frac{\alpha_1(\gamma_1\gamma_2^2 + 4\Gamma)}{\Gamma'} \tag{27}$$

so we can write (26) as

$$\alpha_1\left(\frac{\gamma_1\gamma_2^2 + 4\Gamma}{\Gamma'}H_v + (2u - \gamma_2)H_u + \frac{(\gamma_1\gamma_2^2 + 4\Gamma)\Gamma'' - 6\Gamma'^2}{\Gamma'^2}H\right) = 0. \tag{28}$$

If $\alpha_1 = 0$ we do not obtain any extension of the principal Lie algebra. Then, for $\alpha_1 \neq 0$, (28) is satisfied only when

$$H(u,v) = \frac{(2u - \gamma_2)^3}{\Gamma'}\phi(\omega) \tag{29}$$

with $\phi(\omega)$ arbitrary function of $\omega = \dfrac{(2u - \gamma_2)^2}{\gamma_1\gamma_2^2 + 4\Gamma}$.

So the system

$$\begin{cases} u_t = D_0 u_{xx} + \gamma_1 u(\gamma_2 - u) + \Gamma(v), \\[2ex] v_t = \dfrac{(2u - \gamma_2)^3}{\Gamma'}\phi(\omega) \end{cases} \tag{30}$$

admits the following additional generator

$$X_3 = x\partial_x + 2t\partial_t + (\gamma_2 - 2u)\partial_u - \frac{(\gamma_1\gamma_2^2 + 4\Gamma)}{\Gamma'}\partial_v. \tag{31}$$

Remark 1. *If $\Gamma'(v) \neq 0$, it is possible to verify that the change of variable*

$$w = \Gamma(v) \tag{32}$$

maps the system (2) in the following equivalent form

$$\begin{cases} u_t = D_0 u_{xx} + \gamma_1 u(\gamma_2 - u) + w, \\ w_t = \Phi(u, w). \end{cases} \tag{33}$$

Remark 2. *Of course the system (33), when $\Phi(u,w)$ assumes the form*

$$\Phi(u,w) = (2u - \gamma_2)^3 \phi(\omega)$$

with $\omega = \dfrac{(2u-\gamma_2)^2}{\gamma_1\gamma_2^2 + 4w}$, admits the following extension with respect to (18)

$$X_3 = x\partial_x + 2t\partial_t + (\gamma_2 - 2u)\partial_u - (\gamma_1\gamma_2^2 + 4w)\partial_w. \tag{34}$$

Remark 3. *From Remarks 1 and 2, without loss of generality, we can assume $\Gamma(v) = v$ in the system (30), as well as in the generator (31).*

3.2. $\Gamma' = 0, \Gamma \neq -\frac{\gamma_1\gamma_2^2}{4}$

In this case, the system (2) reads,

$$\begin{cases} u_t = D_0 u_{xx} + \gamma_1 u(\gamma_2 - u) + \gamma_3, \\ v_t = H(u, v) \end{cases} \tag{35}$$

with γ_3 constant. As $\gamma_3 \neq -\frac{\gamma_1\gamma_2^2}{4}$ and $\alpha_1 = 0$, from (26) we have two possibilities

3.2.1. $H_u = 0$;
3.2.2. $H_u \neq 0$.

We analyze them separately.

3.2.1. $\Gamma(v) = \gamma_3 \neq -\frac{\gamma_1\gamma_2^2}{4}$, $H_u = 0$

From (26), by solving with respect to λ, it follows

$$\lambda(t, x, v) = \lambda_1(x, \omega) H(v),$$

where $\lambda_1(x,\omega)$ is an arbitrary function of x and ω, with $\omega = t - \displaystyle\int \frac{1}{H(v)}\,dv$.

Then, (35) becomes

$$\begin{cases} u_t = D_0 u_{xx} + \gamma_1 u(\gamma_2 - u) + \gamma_3, \\ v_t = H(v) \end{cases} \tag{36}$$

admitting the additional generator

$$X_{\lambda_1} = \lambda_1(x,\omega) H(v) \partial_v, \tag{37}$$

so the extended algebra is infinite dimensional.

3.2.2. $\Gamma(v) = \gamma_3 \neq -\frac{\gamma_1\gamma_2^2}{4}$, $H_u \neq 0$

In this case, we recall that $\alpha_1 = 0$, and, in order to have extensions of the principal Lie algebra, it must be $\lambda(t,x,v) \neq 0$. Then by differentiating the condition (26) with respect to u we can get

$$\frac{\lambda_v}{\lambda} = \frac{H_{uv}}{H_u}; \tag{38}$$

then

$$H(u,v) = \psi(u)e^{\phi_1(v)} + \phi_2(v), \tag{39}$$

and

$$\lambda = \lambda_1(t,x)e^{\phi_1(v)}. \tag{40}$$

Going back to (26) we can write

$$\phi'_2 - \phi'_1\phi_2 = \frac{\lambda_{1t}}{\lambda_1} \tag{41}$$

that implies

$$\phi_2(v) = e^{\phi_1(v)}\left(\phi_0 \int e^{-\phi_1(v)}\, dv + \phi_{01}\right), \tag{42}$$

$$\lambda_1(t,x) = \lambda_3(x)e^{\phi_0 t}. \tag{43}$$

In conclusion, when the function H assumes the form

$$H(u,v) = e^{\phi_1(v)}\left(\psi(u) + \phi_0 \int e^{-\phi_1(v)}\, dv\right),$$

with $\phi_1(v)$, $\psi(u)$ constitutive functions and ϕ_0 constitutive constant, the system admits the additional generator

$$X_{\lambda_3} = \lambda_3(x)e^{\phi_0 t + \phi_1(v)}\partial_v, \tag{44}$$

with $\lambda_3(x)$ arbitrary function. Even in this case the extended algebra is infinite dimensional.

3.3. $\Gamma(v) = -\frac{\gamma_1\gamma_2^2}{4}$

In this case, the system (2) assumes the form

$$\begin{cases} u_t = D_0 u_{xx} + \gamma_1 u(\gamma_2 - u) - \frac{\gamma_1\gamma_2^2}{4}, \\ v_t = H(u,v). \end{cases} \tag{45}$$

From (26) we have still two cases

3.3.1 $H_u = 0$;
3.3.2 $H_u \neq 0$.

We analyze them separately.

3.3.1. $\Gamma(v) = -\frac{\gamma_1\gamma_2^2}{4}$, $H_u = 0$

In this case, from (26), we get

$$\lambda(t,x,v) = \left(\lambda_1(x,\omega) + 2\alpha_1 \int \frac{1}{H(v)}\, dv\right)H(v)$$

with $\omega = t - \int \frac{1}{H(v)}\,dv$, then the system

$$\begin{cases} u_t = D_0 u_{xx} + \gamma_1 u(\gamma_2 - u) - \frac{\gamma_1\gamma_2^2}{4}, \\ v_t = H(v) \end{cases} \tag{46}$$

admits two additional generators:

$$X_3 = x\partial_x + 2t\partial_t + (\gamma_2 - 2u)\partial_u + \left(2\int \frac{1}{H(v)}\,dv\right)H(v)\partial_v, \tag{47}$$

$$X_{\lambda_1} = \lambda_1(x,\omega)H(v)\partial_v. \tag{48}$$

X_{λ_1} already obtained in the Section 3.2.1.

3.3.2. $\Gamma(v) = -\frac{\gamma_1\gamma_2^2}{4}$, $H_u \neq 0$

In this last case, deriving appropriately the condition (26), after some calculations, we are able to carry out the following subcases.

1. $H(u,v) = e^{\phi(v)}\left(\psi_0 + \psi_1(2u-\gamma_2)^k\right)$, with $\phi(v)$ arbitrary constitutive function and ψ_0, ψ_1, k constitutive constants.
   The system admits two additional generators, the generator

   $$X_3 = x\partial_x + 2t\partial_t + (\gamma_2 - 2u)\partial_u + \left(2k\psi_0 t - 2(k-1)\int e^{-\phi(v)}dv\right)e^{\phi(v)}\partial_v, \tag{49}$$

   and the generator

   $$X_{\lambda_2} = \lambda_2(x)e^{\phi(v)}\partial_v \tag{50}$$

   where $\lambda_2(x)$ is an arbitrary function. Even in this case, the extended algebra is infinite dimensional.
2. $H(u,v) = \frac{e^{-2\phi(v)}}{\phi'}\psi(\omega)$ where $\phi(v)$ and $\psi(\omega)$ are arbitrary constitutive functions with $\omega = (2u-\gamma_2)e^{2\phi(v)}$. It is possible to ascertain that in this case the system admits the following additional generator

   $$X_3 = x\partial_x + 2t\partial_t + (\gamma_2 - 2u)\partial_u + \frac{1}{\phi'}\partial_v. \tag{51}$$

The results of this section are summarized in the Table 1.

Table 1. The generators that appear in this table are the extensions of $\mathcal{L}_{\mathcal{P}}$ concerned with each couple of functions $\Gamma(v)$ and $H(u,v)$.

1	$\Gamma(v)$ arbitrary with $\Gamma' \neq 0$	
	$H(u,v) = \frac{(2u-\gamma_2)^3}{\Gamma'}\phi(\omega)$ with $\omega = \frac{(2u-\gamma_2)^2}{\gamma_1\gamma_2^2 + 4\Gamma}$	$X_3 = x\partial_x + 2t\partial_t + (\gamma_2 - 2u)\partial_u + -\frac{(\gamma_1\gamma_2^2+4\Gamma)}{\Gamma'}\partial_v$
2	$\Gamma(v) = \gamma_3 \neq -\frac{\gamma_1\gamma_2^2}{4}$	
	$H(u,v) = H(v)$	$X_{\lambda_1} = \lambda_1(x,\omega)H(v)\partial_v$ with $\omega = t - \int \frac{1}{H(v)}\,dv$
	$H(u,v) = e^{\phi_1(v)}\left(\psi(u) + \phi_0\int e^{-\phi_1(v)}\,dv\right)$	$X_{\lambda_3} = \lambda_3(x)e^{\phi_0 t + \phi_1(v)}\partial_v$

Table 1. *Cont.*

3	$\Gamma(v) = -\frac{\gamma_1\gamma_2^2}{4}$	
	$H(u,v) = H(v)$	$X_3 = x\partial_x + 2t\partial_t + (\gamma_2 - 2u)\partial_u + \left(2\int \frac{1}{H(v)}\, dv\right) H(v)\partial_v$ $X_{\lambda_1} = \lambda_1(x,\omega)H(v)\partial_v$ with $\omega = t - \int \frac{1}{H(v)}\, dv$
	$H(u,v) = e^{\phi(v)}\left(\psi_0 + \psi_1(2u-\gamma_2)^k\right)$	$X_3 = x\partial_x + 2t\partial_t + (\gamma_2 - 2u)\partial_u + 2\left(k\psi_0 t - (k-1)\int e^{-\phi} dv\right)e^{\phi}\partial_v$ $X_{\lambda_2} = \lambda_2(x)e^{\phi(v)}\partial_v$
	$H(u,v) = \dfrac{e^{-2\phi(v)}}{\phi'}\psi(\omega)$ with $\omega = (2u-\gamma_2)e^{2\phi(v)}$	$X_3 = x\partial_x + 2t\partial_t + (\gamma_2 - 2u)\partial_u + \frac{1}{\phi'}\partial_v$

4. Reduced Systems and Invariant Solutions

In this section we focus our attention in the subclass of systems (2) of the form

$$\begin{cases} u_t = D_0 u_{xx} + \gamma_1 u(\gamma_2 - u) + \Gamma(v), \\ v_t = \dfrac{(2u-\gamma_2)^3}{\Gamma'}\phi(\omega) \quad \text{where} \quad \omega = \dfrac{(2u-\gamma_2)^2}{\gamma_1\gamma_2^2 + 4\Gamma}, \end{cases} \tag{52}$$

with ϕ arbitrary function of ω, and $\Gamma(v)$ arbitrary function with $\Gamma'(v) \neq 0$.

Previously in the Section 3.1 we have verified that the system (52) admits the additional generator

$$X_3 = x\partial_x + 2t\partial_t - (2u-\gamma_2)\partial_u - \frac{(\gamma_1\gamma_2^2 + 4\Gamma)}{\Gamma'}\partial_v. \tag{53}$$

After having taken into account the remarks of Section 3.1, without loss of generality, we can assume $\Gamma(v) = v$.

However, for sake of readability we prefer to use the new variable $w = \Gamma(v)$. Then the system (52) reads

$$\begin{cases} u_t = D_0 u_{xx} + \gamma_1 u(\gamma_2 - u) + w, \\ w_t = (2u-\gamma_2)^3\phi(\omega) \quad \text{where} \quad \omega = \dfrac{(2u-\gamma_2)^2}{\gamma_1\gamma_2^2 + 4w}. \end{cases} \tag{54}$$

In this way it is clear that the results hold for any function $\Gamma(v)$ with $\Gamma'(v) \neq 0$.

By using the invariant surface conditions corresponding to the generator (34) we derive

$$\sigma = \frac{x^2}{t}, \quad u = \frac{1}{2}\gamma_2 + \frac{1}{t}U(\sigma), \quad w = \frac{1}{x^4}W(\sigma) - \frac{\gamma_1\gamma_2^2}{4},$$

then (54) is reduced to the following ODE system in the new dependent variables $U(\sigma)$ and $W(\sigma)$

$$\begin{cases} -U - \sigma U' = D_0(2U' + 4\sigma U'') - \gamma_1 U^2 + \dfrac{W}{\sigma^2}, \\ -W' = 8\sigma U^3\phi(\tilde{\omega}), \quad \text{where} \quad \tilde{\omega} = \dfrac{\sigma^2 U^2}{W}. \end{cases} \tag{55}$$

Remark 4. *By identifying $W(\sigma)$ with $V(\sigma)$, we observe that the reduced system (55) is also the reduced system of the system (52) for which the similarity variables corresponding to the generator (53) are*

$$\sigma = \frac{x^2}{t}, \quad u = \frac{1}{2}\gamma_2 + \frac{1}{t}U(\sigma), \tag{56}$$

while v is implicitly defined from

$$\Gamma(v) = \frac{1}{x^4}V(\sigma) - \frac{\gamma_1\gamma_2^2}{4}.$$

In order to look for exact solutions of the reduced system (55), we need to specialize the form of the constitutive function ϕ. In the following we assume ϕ of the form

$$\phi(\omega) = \frac{\gamma_4}{\omega} + \gamma_5 \tag{57}$$

with γ_i $(i = 4, 5)$ real constitutive constants.

So the system (52) becomes

$$\begin{cases} u_t = D_0 u_{xx} + \gamma_1 u(\gamma_2 - u) + w, \\ w_t = (2u - \gamma_2)\left[\gamma_4(\gamma_1\gamma_2^2 + 4w) + \gamma_5(2u - \gamma_2)^2\right] \end{cases} \tag{58}$$

while the reduced system reads

$$\begin{cases} -U - \sigma U' = D_0(2U' + 4\sigma U'') - \gamma_1 U^2 + \dfrac{W}{\sigma^2}, \\ -\sigma W' = 8U\left(\gamma_4 W + \gamma_5 \sigma^2 U^2\right). \end{cases} \tag{59}$$

It is a simple matter to ascertain that this system admits the following particular solutions:

1. If $\gamma_4 = \dfrac{\gamma_1}{4}$ and $\gamma_5 = -\dfrac{\gamma_1^2}{4}$,

$$U(\sigma) = -\frac{3(D_0 + \sigma)}{2D_0\gamma_1}, \qquad W(\sigma) = \frac{3\sigma^2(9D_0^2 + 10\sigma D_0 + 3\sigma^2)}{4D_0^2\gamma_1}, \tag{60}$$

that imply the following solution

$$u(t, x) = \frac{1}{2}\gamma_2 - \frac{3(D_0 t + x^2)}{2D_0\gamma_1 t^2}, \qquad w(t, x) = \frac{3(9D_0^2 t^2 + 10D_0 t x^2 + 3x^4)}{4D_0^2\gamma_1 t^4} - \frac{\gamma_1\gamma_2^2}{4}. \tag{61}$$

Taking into account (32) we are able to write

$$v(t, x) = \Gamma^{-1}(w) \tag{62}$$

where Γ^{-1} denotes the inverse function of Γ, so

$$v(t, x) = \Gamma^{-1}\left(\frac{3(9D_0^2 t^2 + 10D_0 t x^2 + 3x^4)}{4D_0^2\gamma_1 t^4} - \frac{\gamma_1\gamma_2^2}{4}\right). \tag{63}$$

If, for instance, $\Gamma(v) = \gamma_6 v^3$, then the solution (61) becomes

$$v(t, x) = \frac{1}{\gamma_6}\left(\frac{3(9D_0^2 t^2 + 10D_0 t x^2 + 3x^4)}{4D_0^2\gamma_1 t^4} - \frac{\gamma_1\gamma_2^2}{4}\right)^{\frac{1}{3}}. \tag{64}$$

In another way it is possible to derive v by taking into account (32) where v is implicitly definite by

$$\Gamma(v) - w = 0. \tag{65}$$

2. If $\gamma_5 = -\dfrac{1}{18}(\gamma_1 + 8\gamma_4)(\gamma_1 + 2\gamma_4)$,

$$U(\sigma) = \frac{3}{2(\gamma_1 - 4\gamma_4)}, \qquad W(\sigma) = \frac{3\sigma^2(\gamma_1 + 8\gamma_4)}{4(\gamma_1 - 4\gamma_4^2)}, \tag{66}$$

that imply the following spatially homogeneous solution

$$u(t,x) = \frac{1}{2}\gamma_2 + \frac{3}{2t(\gamma_1 - 4\gamma_4)}, \qquad w(t,x) = \frac{3(\gamma_1 + 8\gamma_4)}{4t^2(\gamma_1 - 4\gamma_4^2)} - \frac{\gamma_1\gamma_2^2}{4}. \tag{67}$$

3. If $\gamma_5 = 0$,

$$U(\sigma) = -\frac{1}{4\gamma_4}, \qquad W(\sigma) = \frac{1}{16\gamma_4^2}(\gamma_1 + 4\gamma_4)\sigma^2, \tag{68}$$

that imply the following spatially homogeneous solution

$$u(t,x) = \frac{1}{2}\gamma_2 - \frac{1}{4\gamma_4 t}, \qquad w(t,x) = \frac{1}{16\gamma_4^2 t^2}(\gamma_1 + 4\gamma_4) - \frac{\gamma_1\gamma_2^2}{4}. \tag{69}$$

4. For γ_4 and γ_5 arbitrary, we get

$$U(\sigma) = \frac{6D_0\gamma_4}{(\gamma_1\gamma_4 + \gamma_5)\sigma}, \qquad W(\sigma) = -\frac{36D_0^2\gamma_4\gamma_5}{(\gamma_1\gamma_4 + \gamma_5)^2}, \tag{70}$$

that imply the following temporally homogeneous solution

$$u(t,x) = \frac{1}{2}\gamma_2 + \frac{6D_0\gamma_4}{(\gamma_1\gamma_4 + \gamma_5)x}, \qquad w(t,x) = \frac{36D_0^2\gamma_4\gamma_5}{(\gamma_1\gamma_4 + \gamma_5)^2 x^4} - \frac{\gamma_1\gamma_2^2}{4}. \tag{71}$$

We wish to recall that, being our system invariant with respect to translations in t and x, it is possible to put in all solutions

$$t := t + t_0, \quad x := x + x_0.$$

5. Conclusions

This paper deals with the class of reaction-diffusion systems of PDEs (2). These systems are studied in the framework of symmetry methods in order to perform a classification of the different forms of the constitutive parameter functions $\Gamma(v)$ and $H(u,v)$ that allow to get some extensions of the principal Lie algebra. We have discussed the classifying Equations (8), (15) and (16) for the constitutive functions $\Gamma(v)$ and $H(u,v)$ obtained by applying a projection theorem where we used the *weak equivalence generator* (3) of the class (1).

It is useful stressing that in our classification the function Γ is arbitrary or constant, while the form of the function $H(u,v)$, even if assigned, is depending on arbitrary constitutive functions, that give us more degrees of freedom in the selection of cases of interest.

Between the cases of extensions of $\mathcal{L}_{\mathcal{P}}$ carried out, we considered the system (52) admitting the additional generator

$$X_3 = x\partial_x + 2t\partial_t - (2u - \Gamma)\partial_u - \frac{(\gamma_1\Gamma^2 + 4\Gamma)}{\Gamma'}\partial_v.$$

We remarked that as $\Gamma(v)$ is an arbitrary invertible function by a suitable change of variable it is possible to write (52) and its reduced system in a more simple form.

Several and wide classes of solutions have been obtained by specializing $\phi(\omega)$.

Author Contributions: M.T. and R.T. contributed equally to this work. Both authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: M.T. performed this paper in the framework of G.N.F.M. of INdAM.

Conflicts of Interest: The authors declare no conflict of interest.

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