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Distance Fibonacci Polynomials—Part II

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Abstract: In this paper we use a graph interpretation of distance Fibonacci polynomials to get a new generalization of Lucas polynomials in the distance sense. We give a direct formula, a generating function and we prove some identities for generalized Lucas polynomials. We present Pascal-like triangles with left-justified rows filled with coefficients of these polynomials, in which one can observe some symmetric patterns. Using a general Q-matrix and a symmetric matrix of initial conditions we also define matrix generators for generalized Lucas polynomials.

Keywords: generalized Fibonacci polynomials; generalized Lucas polynomials; Pascal's triangle; generating function; matrix generator

MSC: 11B37; 11B39

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1. Introduction

Fibonacci sequence $\{F_n\}$ defined by $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, where $n \geq 2$, and Lucas sequence $\{L_n\}$ defined by $L_0 = 2, L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$, where $n \geq 2$, in view of their connections with the golden ratio, are in the center of interest of many researchers and mathematics enthusiasts. These sequences have many interesting interpretations, applications and generalizations. Among numerous generalizations of Fibonacci and Lucas numbers there are generalizations in the distance sens, such as, for example, generalized Fibonacci numbers $F(k,n)$ introduced by M. Kwaśnik and I. Włoch [1] given by the formula $F(k,n) = F(k,n-1) + F(k,n-k)$, for $n \geq k$ and $k \geq 2$, with initial conditions $F(k,n) = n+1$ for $n = 0,1,\ldots k-1$ and generalized Lucas numbers defined by A. Włoch [2] as follows: $L(k,n) = L(k,n-1) + L(k,n-k)$ for $n \geq 2k$, with initial values $L(n,k) = n+1$ for $n = 0,1,2,\ldots,2k-1$. It is worth mentioning that generalizations in the distance sense are usually related to different graph parameters. Applications of Fibonacci-like numbers in graphs was initiated by the paper of H. Prodinger and R. F. Tichy [3], in which the relationship between Fibonacci numbers and independent sets (i.e., subsets of vertices of a graph being pairwise nonadjacent) was described. Independent sets, and consequently Fibonacci-like numbers play an important role in chemical combinatorics and many types of localization problems. Fibonacci and Lucas sequences, like other recursively defined sequences, are naturally generalized to polynomials. Fibonacci polynomials $f_n(x)$ are given by the recurrence relation $f_n(x) = xf_{n-1}(x) + f_{n-2}(x)$, for $n \geq 2$, with initial conditions $f_0(x) = 0, f_1(x) = 1$, Lucas polynomials are defined by the recursion $l_n(x) = xl_{n-1}(x) + l_{n-2}(x)$, for $n \geq 2$, with initial values $l_0(x) = 2, l_1(x) = x$. Obviously $f_n(1) = F_n$, and $l_n(1) = L_n$. Significant contributions to investigation on properties of Fibonacci and Lucas polynomials have been made by V. E. Hoggatt Jr. and M. Bicknell [4–7]. A few newer results on the classical Fibonacci and Lucas polynomials and their applications can be found in [8–10]. It is worth noting that Fibonacci and Lucas polynomials are used for determining approximate solutions of many types of integral equations such as for example Cauchy integral equations, Abel integral equations, Volterra-Fredholm integral equations and others (for details see [11–14]). Fibonacci numbers and polynomials by their connections with diophantine equations and Hilbert's 10th problem are also closely related with the so-called Pell surfaces studied by the Shaw prize winner J. Kollar [15] due to an important algorithmic embeddability problem for algebraic varieties [16].

The interest in Fibonacci-like polynomials has contributed to the emergence of many generalizations. Most of them are obtained by changing initial terms while preserving the recurrence relation (see References [17,18]) or by slight modifying the basic recursion (see References [19–21]). Some are obtained in the distance sense i.e., by changing distance between terms of a sequence. In the paper [22] we have introduced the distance Fibonacci polynomials $f_n(k,x)$ given by the following recurrence relation

$$f_n(k,x) = x f_{n-1}(k,x) + f_{n-k}(k,x) \quad \text{for} \quad n \geq k \tag{1}$$

with initial conditions $f_n(k,x) = x^n$ for $n = 0,1,\ldots,k-1$ for integers $k \geq 2$, $n \geq 0$.

In Table 1 we present some distance Fibonacci polynomials $f_n(k,x)$ for special values of k and n.

Table 1. Distance Fibonacci polynomials $f_n(k,x)$.

k \ n	**0**	**1**	**2**	**3**	**4**	**5**	**6**
$f_n(2,x)$	1	x	x^2+1	x^3+2x	x^4+3x^2+1	x^5+4x^3+3x	$x^6+5x^4+6x^2+1$
$f_n(3,x)$	1	x	x^2	x^3+1	x^4+2x	x^5+3x^2	x^6+4x^3+1
$f_n(4,x)$	1	x	x^2	x^3	x^4+1	x^5+2x	x^6+3x^2
$f_n(5,x)$	1	x	x^2	x^3	x^4	x^5+1	x^6+2x

We have found a direct formula, a generating function, matrix generators and some identities for generalized Fibonacci polynomials $f_n(k,x)$. We have also extended the distance Fibonacci polynomials $f_n(k,x)$ to negative integers n, namely

$$f_{-n}(k,x) = f_{-n+k}(k,x) - x f_{-n+k-1}(k,x) \text{ for } n \geq 0 \tag{2}$$

with initial conditions $f_n(k,x) = x^n$, for $n = 0,1,\ldots,k-1$.

In Table 2 we present the first few elements of $f_{-n}(k,x)$ polynomials for special k and negative n.

Table 2. Distance Fibonacci polynomials $f_{-n}(k,x)$.

k \ n	**−7**	**−6**	**−5**	**−4**	**−3**	**−2**	**−1**	**0**
$f_n(2,x)$	$-x^5-4x^3-3x$	x^4+3x^2+1	$-x^3-2x$	x^2+1	$-x$	1	0	1
$f_n(3,x)$	x^2	1	$-x$	0	1	0	0	1
$f_n(4,x)$	$-x$	0	0	1	0	0	0	1
$f_n(5,x)$	0	0	1	0	0	0	0	1

In this paper, which is a continuation of [22], based on a graph interpretation of the distance Fibonacci polynomials $f_n(k,x)$ we introduce a new generalization of Lucas polynomials in the distance sense. We derive a direct formula, a generating function and matrix generators for these polynomials. We also prove some identities that generalize the classical identities for Lucas polynomials and reveal some Pascal-like relations between coefficients of these polynomials.

2. From the Distance Fibonacci to the Distance Lucas Polynomials

In the paper [22] we have used a special kind of covering of a graph to obtain a graph interpretation of the distance Fibonacci polynomials $f_n(k,x)$. Let us recall the idea of that covering. Let G be an undirected, finite graph with the vertex set $V(G)$ and the edge set $E(G)$, $\mathbb{P}_n$ be an n-vetrex path (a sequence of distinct vertices $v_1, v_2, \ldots, v_n$ such that $\{v_i, v_{i+1}\} \in E(G)$ for $i = 1, 2, \ldots, n-1$) and C be a set of x colors, where $x \geq 1$. We cover the vertex set $V(G)$ by the subgraphs $\mathbb{P}_k$ and $\mathbb{P}_1$, with the vertex of a graph $\mathbb{P}_1$ additionally colored with one of x colors from the set C. This operation is called $(\mathbb{P}_k, \mathbb{P}_1)$-covering with $x\mathbb{P}_1$-coloring. By $\sigma(G,k)$ we denote the number of all $(\mathbb{P}_k, \mathbb{P}_1)$-covering with $x\mathbb{P}_1$-coloring of the graph G.

To illustrate $(\mathbb{P}_k, \mathbb{P}_1)$-covering with $x\mathbb{P}_1$-coloring of a graph G in Figure 1 we present $(\mathbb{P}_2, \mathbb{P}_1)$-covering with $x\mathbb{P}_1$-coloring of a graph $\mathbb{P}_3$ (we have surrounded paths $\mathbb{P}_2$ and $\mathbb{P}_1$ by dashed lines, the letter x below a path $\mathbb{P}_1$ denotes a number of colors we can use for coloring of this path). One can easy check that $\sigma(\mathbb{P}_3, 2) = x^3 + 2x$.

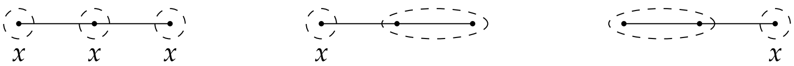

Figure 1. $(\mathbb{P}_2, \mathbb{P}_1)$-covering with $x\mathbb{P}_1$-coloring of a graph $\mathbb{P}_3$.

For a graph $\mathbb{P}_n$ we have proved the following theorem.

Theorem 1 ([22])**.** *Let $k \geq 2, n \geq 1, x \geq 1$ be integers. Then $\sigma(\mathbb{P}_n, k) = f_n(k,x)$.*

Now let us consider $(\mathbb{P}_k, \mathbb{P}_1)$-covering with $x\mathbb{P}_1$-coloring of an n-vertex cycle C_n (a closed path), where $n \geq 3$.

Theorem 2. *Let $k \geq 2, n \geq 3, x \geq 1$ be integers. Then $\sigma(\mathbb{C}_n, k) = x f_{n-1}(k,x) + k f_{n-k}(k,x)$.*

Proof (by induction on n). Let $k \geq 2, n \geq 3, x \geq 1$ be integers and $\mathbb{C}_n$ be a cycle with the vertex set $V(\mathbb{C}_n) = \{t_1, t_2, \ldots, t_n\}$.

If $n = 3, \ldots, k-1$, for $k \geq 4$, then we cover the vertices only by subgraphs $\mathbb{P}_1$ with coloring by one of x colors. Hence $\sigma(\mathbb{C}_n, k) = x^n$ for $n = 3, \ldots, k-1,\ k \geq 4$. If $n = k$ and $k \geq 3$, then we can cover the vertices of a cycle $\mathbb{C}_k$ by k subgraphs $\mathbb{P}_1$ which are colored with one of x colors or we can cover such a graph by one path $\mathbb{P}_k$, which can be chosen on k ways. Hence $\sigma(\mathbb{C}_k, k) = x^k + k$.

Assume that $n \geq k+1$, for $k \geq 2$, and the theorem is valid for all integers less then n. We will prove that it is true for n. We have to consider two possibilities:

1. $t_n \in V(\mathbb{P}_1)$.

Then a vertex t_n can be colored by one of x colors. By $\sigma_1(\mathbb{C}_n, k)$ let us denote the number of all $(\mathbb{P}_k, \mathbb{P}_1)$-covering with $x\mathbb{P}_1$-coloring of a graph $\mathbb{C}_n$ with t_n belonging to $\mathbb{P}_1$. Thus, $\sigma_1(\mathbb{C}_n, k) = x\sigma(\mathbb{P}_{n-1}, k)$.

2. $t_n \in V(\mathbb{P}_1)$.

Let $\sigma_k(\mathbb{C}_n, k)$ denote the number of all $(\mathbb{P}_k, \mathbb{P}_1)$-covering with $x\mathbb{P}_1$-coloring of a graph $\mathbb{C}_n$ with t_n belonging to $\mathbb{P}_k$. Since we have k such paths, hence $\sigma_k(\mathbb{C}_n, k) = k\sigma(\mathbb{P}_{n-k}, k)$.

Taking into account both cases, Theorem 1 and induction hypothesis we obtain

$$\sigma(\mathbb{C}_n, k) = \sigma_1(\mathbb{C}_n, k) + \sigma_k(\mathbb{C}_n, k) = x\sigma(\mathbb{P}_{n-1}, k) + k\sigma(\mathbb{P}_{n-k}, k) = x f_{n-1}(k,x) + k f_{n-k}(k,x).$$

Thus, the theorem is proved. □

As a consequence of Theorem 2 we obtain a new graph interpretation of the classical Lucas polynomials $l_n(x)$.

Corollary 1. *Let $n \geq 3, x \geq 1$ be integers. Then $\sigma(\mathbb{C}_n, 2) = l_n(x)$.*

Proof. Let us consider$(\mathbb{P}_2, \mathbb{P}_1)$-covering with $x\mathbb{P}_1$-coloring of an n-vertex cycle C_n. By Theorem 2 we have $\sigma(\mathbb{C}_n, 2) = xf_{n-1}(2,x) + 2f_{n-2}(2,x)$. Since $f_n(2,x) = f_{n+1}(x)$ then we have the equality $\sigma(\mathbb{C}_n, 2) = xf_n(x) + 2f_{n-1}(x)$ and by the well-known identity $xf_n(x) + 2f_{n-1}(x) = l_n(x)$ we obtain $\sigma(\mathbb{C}_n, 2) = l_n(x)$. □

Let us denote $\sigma(\mathbb{C}_n, k) = l_n(k,x)$ and call this parameter as the distance Lucas polynomial.

Theorem 3. *Let $k \geq 2, n \geq 3, x \geq 1$ be integers. Then $l_n(k,x) = xl_{n-1}(k,x) + l_{n-k}(k,x)$.*

Proof. We prove this theorem by induction on n. Let $k \geq 2, n \geq 3, x \geq 1$ be integers.

For $n = 3$ and $k = 2$, we can easily check that $l_3(2,x) = xl_2(2,x) + l_1(2,x)$. Namely, using Theorem 2 and Tables 1 and 2, we get that $l_3(2,x) = xf_2(2,x) + 2f_1(2,x) = x(x^2 + 1) + 2x = x^3 + 3x$. In turn, on the right side we get $xl_2(2,x) + l_1(2,x) = x[xf_1(2,x) + 2f_0(2,x)] + xf_0(2,x) + 2f_{-1}(2,x) = x^3 + 3x$.

If $n = 3$ and $k = 3$, then we obtain $l_3(3,x) = xf_2(3,x) + 3f_0(3,x) = x^3 + 3$ and on the other hand $xl_2(3,x) + l_0(3,x) = x[xf_1(3,x) + 3f_{-1}(3,x)] + xf_{-1}(3,x) + 3f_{-3}(3,x) = x^3 + 3$. Thus, for $n = 3$ the theorem is true.

Assume that the theorem is valid for all integers n. We will prove that it is also true for $n+1$, i.e., $l_{n+1}(k,x) = xl_n(k,x) + l_{n-k+1}(k,x)$ is satisfied. Using Theorem 2, recurrence (1) and induction hypothesis we obtain

$l_{n+1}(k,x) = xf_n(k,x) + kf_{n-k+1}(k,x) = x[xf_{n-1}(k,x) + f_{n-k}(k,x)] + k[xf_{n-k}(k,x) + f_{n-2k+1}(k,x)] = x[xf_{n-1}(k,x) + kf_{n-k}(k,x)] + xf_{n-k}(k,x) + kf_{n-2k+1}(k,x) = xl_n(k,x) + l_{n-k+1}(k,x)$.

Thus, the theorem is proved. □

Based on Theorem 3 we can define the distance Lucas polynomials $l_n(k,x)$ as follows.

Let $k \geq 2$, $n \geq 0$ be integers. The distance Lucas polynomials $l_n(k,x)$ are given by the recurrence relation

$$l_n(k,x) = xl_{n-1}(k,x) + l_{n-k}(k,x) \quad \text{for} \quad n \geq k \tag{3}$$

with initial conditions $l_0(k,x) = k,\ l_n(k,x) = x^n$ for $n = 1, 2, \ldots, k-1$.

Table 3 presents some distance Lucas polynomials $l_n(k,x)$ for special values of k and n.

Table 3. Distance Lucas polynomials $l_n(k,x)$.

k \ n	0	1	2	3	4	5	6
$l_n(2,x)$	2	x	x^2+2	x^3+3x	x^4+4x^2+2	x^5+5x^3+5x	$x^6+6x^4+9x^2+2$
$l_n(3,x)$	3	x	x^2	x^3+3	x^4+4x	x^5+5x^2	x^6+6x^3+3
$l_n(4,x)$	4	x	x^2	x^3	x^4+4	x^5+5x	x^6+6x^2
$l_n(5,x)$	5	x	x^2	x^3	x^4	x^5+5	x^6+6x

It is obvious that for $k = 2$ we have $l_n(2,x) = l_n(x)$, and therefore $l_n(2,1) = l_n$. For $n \geq k$ and $x = 1$ we get $l_n(k,1) = L(k,n)$, where $L(k,n)$ is the nth generalized Lucas number defined in [2]. Moreover, by recursion (3), one can easy check that

$$l_{k+m}(k,x) = x^{k+m} + (k+m)x^m \quad \text{for} \quad m = 0, 1, \ldots, k-1 \tag{4}$$

By graph interpretation of the distance Lucas polynomials almost immediately follows a direct formula for these polynomials.

Theorem 4. *Let $k \geq 2, n \geq 1$ be integers. Then*

$$l_n(k,x) = \sum_{j=0}^{\lfloor \frac{n}{k} \rfloor} \frac{n}{n-(k-1)j} \binom{n-(k-1)j}{j} x^{n-kj}. \tag{5}$$

For $k = 2$ by the formula (5) and relation $l_n(2, x) = l_n$ we get

$$l_n(x) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-j} \binom{n-j}{j} x^{n-2j},$$

which is the well-known direct formula for Lucas polynomials.

Using steps method described in [22] we can observe some Pascal-like relations between coefficients of the distance Lucas polynomials. For a fixed $k \geq 2$ and each $n \geq 0$ let us arrange coefficients of the distance Lucas polynomials $l_n(k, x)$ in ascending order and form a left-shifted array of these coefficients. Building steps of height $k - 1$ we can see that the sum of elements on steps beginning in a row corresponding to $n = ki$, where $i = 1, 2, \ldots$, is equal to $(k+1)2^{i-1}$. Moreover, adding two consecutive elements on steps i.e., an element in the nth row and the jth column and an element in the $(n - k + 1)$st row and the $(j + 1)$st column we obtain an element in the $(n + 1)$st row and the $(j + 1)$st column. In Tables 4 and 5 we present cases $k = 3$ and $k = 4$, respectively.

In Table 4 we have marked the steps of height 2 starting in rows 3, 6, 9 (rows are counted from $n = 0$), adding the elements (red numbers) on the steps we obtain sums $4 \times 2^0, 4 \times 2^1, 4 \times 2^2$ in turn. Analogously, in Table 5 we have marked the steps of height 3 starting in rows 4 and 8, appropriate sums on the steps are 5×2^0 and 5×2^1, respectively. We have marked in blue the rule of generating elements in Tables 4 and 5.

Table 4. Coefficients of $l_n(3, x)$ in ascending order.

n	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
0	3									
1	0	1								
2	0	0	1							
3	3	0	0	1						
4	0	4	0	0	1					
5	0	0	5	0	0	1				
6	3	0	0	6	0	0	1			
7	0	7	0	0	7	0	0	1		
8	0	0	12	0	0	8	0	0	1	
9	3	0	0	18	0	0	9	0	0	1

Table 5. Coefficients of $l_n(4, x)$ in ascending order.

n	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
0	4									
1	0	1								
2	0	0	1							
3	0	0	0	1						
4	4	0	0	0	1					
5	0	5	0	0	0	1				
6	0	0	6	0	0	0	1			
7	0	0	0	7	0	0	0	1		
8	4	0	0	0	8	0	0	0	1	
9	0	9	0	0	0	9	0	0	0	1

3. Generating Function, Extension for Negative Integers and Some Identities

Using the standard method we can derive a generating function for the distance Lucas polynomials $l_n(k, n)$.

Theorem 5. *Let $n \geq 0, k \geq 2$ be integers. The generating function of the distance Lucas polynomials sequence $\{l_n(k,x)\}$ is given by the formula $h(t) = \frac{k+(1-k)xt}{1-xt-t^k}$.*

Proof. Let $h(t) = \sum_{n=0}^{\infty} l_n(k,x)t^n$. By recurrence relation (3) we get

$$h(t) = l_0(k,x) + l_1(k,x)t + \ldots + l_{k-1}(k,x)t^{k-1} + \sum_{n=k}^{\infty} l_n(k,x)t^n$$

$$= k + xt + \ldots + x^{k-1}t^{k-1} + \sum_{n=k}^{\infty} [\, x l_{n-1}(k,x) + l_{n-k}(k,x)\,] t^n$$

$$= k + xt + \ldots + x^{k-1}t^{k-1} + xt \sum_{n=k}^{\infty} l_{n-1}(k,x)t^{n-1} + t^k \sum_{n=k}^{\infty} l_{n-k}(k,x)t^{n-k}$$

$$= k + xt + \ldots + x^{k-1}t^{k-1} + xt \sum_{n=k-1}^{\infty} l_n(k,x)t^n + t^k \sum_{n=0}^{\infty} l_n(k,x)t^n$$

$$= k + xt + \ldots + x^{k-1}t^{k-1} + xt(\sum_{n=0}^{\infty} l_n(k,x)t^n - k - xt - \ldots - x^{k-2}t^{k-2}) + t^k \sum_{n=0}^{\infty} l_n(k,x)t^n$$

$$= k + (1-k)xt + xt \sum_{n=0}^{\infty} f_n(k,x)t^n + t^k \sum_{n=0}^{\infty} f_n(k,x)t^n = k + (1-k)xt + xth(t) + t^k h(t).$$

Thus,

$$h(t) = \frac{k+(1-k)xt}{1-xt-t^k},$$

which ends the proof. □

Note that for $k = 2$ by Theorem 5 and the fact that $l_n(2,x) = l_n(x)$ we obtain a function $h(t) = \frac{2-xt}{1-xt-t^2}$ being a generating function for the classical Lucas polynomials $l_n(x)$.

The distance Lucas polynomials $l_n(k,x)$ can be extended to negative integers n. Let $k \geq 2$, $n \geq 0$ be integers. Then

$$l_{-n}(k,x) = l_{-n+k}(k,x) - x l_{-n+k-1}(k,x) \text{ for } n \geq 0 \tag{6}$$

with initial conditions $l_0(k,x) = k,\ l_n(k,x) = x^n$, for $n = 1, 2\ldots, k-1$.

Table 6 includes the first few elements of $l_{-n}(k,x)$ polynomials for special k and negative n.

Table 6. Distance Lucas polynomials $l_{-n}(k,x)$.

k \ n	-6	-5	-4	-3	-2	-1	0
$l_n(2,x)$	$x^6+6x^4+9x^2+2$	$-x^5-5x^3-5x$	x^4+4x+2	$-x^3-3x$	x^2+2	$-x$	2
$l_n(3,x)$	$-2x^3+3$	$-5x$	$2x^2$	3	$-2x$	0	3
$l_n(4,x)$	$3x^2$	0	4	$-3x$	0	0	4
$l_n(5,x)$	0	5	$-4x$	0	0	0	5

Notice that setting $k = 2$ in (6), we get the well-known extension of Lucas polynomials for negative numbers

$$l_{-n}(2,x) = (-1)^n l_n(2,x).$$

For $k = 2$ and $x = 1$ we obtain the extension of classical Lucas numbers for negative numbers.

Theorem 6. *Let $k \geq 2, n \geq 0$ be integers. Then*
(i) $x\sum_{i=1}^{n} l_{ik-1}(k,x) = l_{nk}(k,x) - l_0(k,x)$,
(ii) $x\sum_{i=0}^{n} l_{ik}(k,x) = l_{nk+1}(k,x) + (k-1)x$,
(iii) $x\sum_{i=0}^{n} l_i(k,x) = \sum_{i=n+2-k}^{n+1} l_i(k,x) + (k-1)x - k$ *for* $n \geq k-2$,
(iv) $l_n(k,x) = \sum_{i=0}^{k-1} x^i l_{n-k-i}(k,x) + x^k l_{n-k}(k,x)$,
(v) $xl_n(k,x) = x^2 l_{n-1}(k,x) + l_{n-k+1}(k,x) - l_{n-2k+1}(k,x)$ *for* $n \geq 2k-1$,
(vi) $l_n(k,x) = f_n(k,x) + (k-1)f_{n-k}(k,x)$.

Proof. At the beginning we prove the identity (i). Using the recurrence relation (3) we obtain

$$xl_{n-1}(k,x) = l_n(k,x) - l_{n-k}(k,x), n \geq k.$$

Hence, for integers $k-1, 2k-1, \ldots, nk-1$, we get

$$\begin{gathered}
xl_{k-1}(k,x) = l_k(k,x) - l_0(k,x) \\
xl_{2k-1}(k,x) = l_{2k}(k,x) - l_k(k,x) \\
xl_{3k-1}(k,x) = l_{3k}(k,x) - l_{2k}(k,x) \\
\vdots \\
xl_{nk-1}(k,x) = l_{nk}(k,x) - l_{(n-1)k}(k,x)
\end{gathered}$$

Adding these equalities we have

$$x\sum_{i=1}^{n} l_{ik-1}(k,x) = l_{nk}(k,x) - l_0(k,x).$$

Thus the identity (i) is proved.

Now we prove the identity (ii) by induction on n. If $n = 0$, then we have $xl_0(k,x) = xk = x + xk - x = l_1(k,x) + (k-1)$. Hence, the identity is true for $n = 0$. Assume that $n \geq 1$ and the equality (ii) is true for an arbitrary n. We will prove that it holds for $n+1$.

By induction hypothesis and the recurrence relation (3) we have

$$\begin{aligned}
x\sum_{i=0}^{n+1} l_{ik}(k,x) &= x\sum_{i=0}^{n} l_{ik}(k,x) + xl_{(n+1)k}(k,x) = l_{nk+1}(k,x) + (k-1)x + xl_{(n+1)k}(k,x) \\
&= l_{(n+1)k+1}(k,x) + (k-1)x.
\end{aligned}$$

Thus the identity (ii) is proved.

Analogously we can prove the identity (iii).

To prove the identity (iv) we use the definition of distance Lucas polynomials (3) by $k-1$ times. Then we obtain

$$\begin{aligned}
l_n(k,x) &= xl_{n-1}(k,x) + l_{n-k}(k,x) = x^2 l_{n-2}(k,x) + xl_{n-k-1}(k,x) + l_{n-k}(k,x) \\
&= x^2 l_{n-2}(k,x) + xl_{n-k-1}(k,x) + l_{n-k}(k,x) \\
&= x^3 l_{n-3}(k,x) + x^2 l_{n-k-2}(k,x) + xl_{n-k-1}(k,x) + l_{n-k}(k,x) = \vdots \\
&= \sum_{i=0}^{k-1} x^i l_{n-k-i}(k,x) + x^k l_{n-k}(k,x).
\end{aligned}$$

Hence the identity (iv) holds.

Using the recurrence relation (3) once again we can prove the identity (v). Let $n \geq 2k-1, k \geq 2$ be integers. Then

$$\begin{aligned} & x^2 l_{n-1}(k,x) + l_{n-k+1}(k,x) - l_{n-2k+1}(k,x) \\ &= x^2 l_{n-1}(k,x) + x l_{n-k}(k,x) + l_{n-2k+1}(k,x) - l_{n-2k+1}(k,x) \\ &= x^2 l_{n-1}(k,x) + x l_{n-k}(k,x) = x l_n(k,x). \end{aligned}$$

Thus the identity (v) is proved.

The last identity (vi) follows from the recursion of the Theorem 2 and the definition of the Fibonacci polynomials.

Thus the theorem is proved. $\square$

Note that for $k = 2$ we obtain the identities for Lucas polynomials and for $x = 1$, we obtain well-known identities for Lucas numbers. Moreover, for $x = 1$ and $n \geq k$, we obtain some new identities for generalized Lucas numbers $L(k,n)$.

Proving analogously as Theorem 6, we get the following identities for polynomials $l_{-n}(k,x)$ for negative integers.

Theorem 7. *Let $n \geq 1, k \geq 2$ be integers. Then*

(vii) $x \sum_{i=1}^{n} l_{-ik}(k,x) = -l_{-nk-k+1}(k,x) - (k-1)x,$

$(viii)$ $x \sum_{i=1}^{n} l_{-i}(k,x) = -\sum_{i=-n-k+1}^{-n} l_i(k,x) - (k-1)x + k,$

(ix) $x \sum_{i=1}^{n} l_{-ik+1}(k,x) = -l_{-nk-k+2}(k,x)$ *for* $k \geq 3$.

4. Matrix Generators

In this section we show how to get the distance Lucas polynomials by a matrix method. We use a notion of general Q- matrix introduced by J. Ivie [23]. The technique described in [23] originally refered to generalized Fibonacci numbers but it was extended to the polynomial case by many authors (see for example [7,10,24]).

A general Q-matrix associated with recurrence relation (3) is a square $k \times k$ matrix of the form

$$Q_k(x) = \begin{bmatrix} x & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Now for a fixed integer $k \geq 2$ let us define a matrix $B_k(x)$ of order k being the matrix of initial conditions

$$B_k(x) = \begin{bmatrix} l_{k-1}(k,x) & l_{k-2}(k,x) & \cdots & l_1(k,x) & l_0(k,x) \\ l_{k-2}(k,x) & l_{k-3}(k,x) & \cdots & l_0(k,x) & l_{-1}(k,x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ l_1(k,x) & l_0(k,x) & \cdots & l_{-k+3}(k,x) & l_{-k+2}(k,x) \\ l_0(k,x) & l_{-1}(k,x) & \cdots & l_{-k+2}(k,x) & l_{-k+1}(k,x) \end{bmatrix}.$$

Theorem 8. *Let $k \geq 2$, $n \geq 1$ be integers. Then*

$$B_k(x)Q_k^n(x) = \begin{bmatrix} l_{n+k-1}(k,x) & l_{n+k-2}(k,x) & \cdots & l_{n+1}(k,x) & l_n(k,x) \\ l_{n+k-2}(k,x) & l_{n+k-3}(k,x) & \cdots & l_n(k,x) & l_{n-1}(k,x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ l_{n+1}(k,x) & l_n(k,x) & \cdots & l_{n-k+3}(k,x) & l_{n-k+2}(k,x) \\ l_n(k,x) & l_{n-1}(k,x) & \cdots & l_{n-k+2}(k,x) & l_{n-k+1}(k,x) \end{bmatrix}. \tag{7}$$

Proof (by induction on n). Let $k \geq 2$ be an integer. If $n = 1$, then by simple calculations and recursion (3) we get (7). Assume now that the statement is true for all integers $1, \ldots, n$. We will show that it is also true for an integer $n + 1$.

Since $B_k(x)Q_k^{n+1}(x) = B_k(x)Q_k^n(x)Q_k(x)$, thus by our assumption and the recurrence relation (3) we obtain

$$B_k(x)Q_k^{n+1}(x) = \begin{bmatrix} l_{n+k-1}(k,x) & l_{n+k-2}(k,x) & \cdots & l_{n+1}(k,x) & l_n(k,x) \\ l_{n+k-2}(k,x) & l_{n+k-3}(k,x) & \cdots & l_n(k,x) & l_{n-1}(k,x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ l_{n+1}(k,x) & l_n(k,x) & \cdots & l_{n-k+3}(k,x) & l_{n-k+2}(k,x) \\ l_n(k,x) & l_{n-1}(k,x) & \cdots & l_{n-k+2}(k,x) & l_{n-k+1}(k,x) \end{bmatrix} \begin{bmatrix} x & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} xl_{n+k-1}(k,x)+l_n(k,x) & l_{n+k-1}(k,x) & \cdots & l_{n+1}(k,x) \\ xl_{n+k-2}(k,x)+l_{n-1}(k,x) & l_{n+k-2}(k,x) & \cdots & l_n(k,x) \\ \vdots & \vdots & \ddots & \vdots \\ xl_{n+1}(k,x)+l_{n-k+2}(k,x) & l_{n+1}(k,x) & \cdots & l_{n-k+3}(k,x) \\ xl_n(k,x)+l_{n-k+1}(k,x) & l_n(k,x) & \cdots & l_{n-k+2}(k,x) \end{bmatrix}$$

$$= \begin{bmatrix} l_{n+k}(k,x) & l_{n+k-1}(k,x) & \cdots & l_{n+2}(k,x) & l_{n+1}(k,x) \\ l_{n+k-1}(k,x) & l_{n+k-2}(k,x) & \cdots & l_{n+1}(k,x) & l_n(k,x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ l_{n+2}(k,x) & l_{n+1}(k,x) & \cdots & l_{n-k+4}(k,x) & l_{n-k+3}(k,x) \\ l_{n+1}(k,x) & l_n(k,x) & \cdots & l_{n-k+3}(k,x) & l_{n-k+2}(k,x) \end{bmatrix}. \quad \square$$

Theorem 9. *Let $k \geq 2$, $n \geq 1$ be integers. Then*

$$det\,(\,B_k(x)) = (-1)^{\lfloor \frac{k}{2} \rfloor}[(k-1)^{k-1}x^k + k^k]. \tag{8}$$

Proof. By definition of the matrix $B_k(x)$, initial conditions for the distance Lucas polynomials, the recursion (6) and observation (4) follows

$$B_k(x) = \begin{bmatrix} x^{k-1} & x^{k-2} & \cdots & x & k \\ x^{k-2} & x^{k-3} & \cdots & k & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x & k & \cdots & 0 & 0 \\ k & 0 & \cdots & 0 & (1-k)x \end{bmatrix}.$$

To calculate determinant of the matrix $B_k(x)$ we initially write it in the form

$$det(B_k(x)) = (-1)^{\lfloor \frac{k}{2} \rfloor} \begin{vmatrix} k & x & \cdots & x^{k-2} & x^{k-1} \\ 0 & k & \cdots & x^{k-3} & x^{k-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & k & x \\ (1-k)x & 0 & \cdots & 0 & k \end{vmatrix}.$$

Continuing calculations, by properties of determinants, we finally get the equality 8 □

Corollary 2. *Let $k \geq 2$, $n \geq 1$ be integers. Then*

$$det\left(B_k(x)\, Q_k^n(x)\right) = (-1)^{n(k+1)+\lfloor \frac{k}{2} \rfloor}[(k-1)^{k-1}x^k + k^k]. \tag{9}$$

Proof. It follows immediatelly by Theorem 9, Cauchy's theorem for determinants and the fact that $\det Q_k^n(x) = (-1)^{n(k+1)}$. □

Note that for $k = 2$ by Theorem 8, Corollary 2 and the equallity $l_n(2,x) = l_n(x)$ we obtain

$$det\begin{bmatrix} l_{n+1}(x) & l_n(x) \\ l_n(x) & l_{n-1}(x) \end{bmatrix} = (-1)^{n+1}(x^2+4),$$

that gives the well-known Cassini's identity for the classical Lucas polynomials

$$l_{n+1}(x)l_{n-1}(x) - l_n^2(x) = (-1)^{n+1}(x^2+4).$$

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