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Topological Quantization of Fractional Quantum Hall Conductivity

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Abstract: We derive a novel topological expression for the Hall conductivity. To that degree we consider the quantum Hall effect (QHE) in a system of interacting electrons. Our formalism is valid for systems in the presence of an external magnetic field, as well as for systems with a nontrivial band topology. That is, the expressions for the conductivity derived are valid for both the ordinary QHE and for the intrinsic anomalous QHE. The expression for the conductivity applies to external fields that may vary in an arbitrary way, and takes into account disorder. Properties related to symmetry and topology are revealed in the fractional quantization of the Hall conductivity. It is assumed that the ground state of the system is degenerate. We represent the QHE conductivity as $\frac{e^2}{h} \times \frac{\mathcal{N}}{K}$, where K is the degeneracy of the ground state, while $\mathcal{N}$ is the topological invariant composed of the Wigner-transformed multi-leg Green functions, which takes discrete values.

Keywords: fractional quantum hall effect; Wigner–Weyl calculus

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1. Introduction

In this we derive a novel expression for the Hall conductivity. The Hamiltonian of the system contains a potential term that describes an interaction between two particles. When this interaction potential is set to zero, the obtained expression for the conductivity reduces to a known expression valid in the absence of interaction. This stands as a validity check of our result.

The quantum Hall effect (QHE) is a phenomenon observed in electrons confined to a plane in a magnetic field. It is perhaps one of the most tangible observations of quantum theory in experiment. Originally the Hall conductivity was found experimentally to take integer values of the inverse of the *quantum of resistivity* or the *Klitzing constant*, equal to $2\pi\hbar/e^2$ [1]. Granted the quantization of physical quantities on the atomic scale should not be surprising, but the Hall conductivity is a macroscopic quantity in a system involving many particles. This observation of the Hall conductivity being quantized can be explained theoretically by the role of topology in quantum many-body systems.

Later it was discovered that the QHE has two starkly different types: the first is the *integer quantum Hall effect* (IQHE) discussed above. The second is the *fractional quantum Hall effect* (FQHE), a phenomenon where the Hall conductivity can take very specific fractional values of the conductivity quantum. The most prominent fractions found experimentally are $1/3$ and $2/5$, but many dozens of different fractions have been observed. Such fractional quantization of the conductivity can be accounted for by interactions between electrons.

To explain the QHE in theoretical terms Thouless, Kohomoto, Nightingale and den Nijs (TKNN) derived a formula called the *TKNN formula* for the quantized Hall conductivity in their seminal paper [2]. The TKNN formula contains an integer factor in front of the conductivity quantum, given by a sum of *Chern numbers* commonly referred to as the *TKNN invariant* . A pedagogical overview of the theory can be found in references [3–7].

The TKNN formula is the statement that the Hall conductivity is a topological invariant of the system [2], proposed for systems subject to a constant external magnetic field. In this case the invariant is the TKNN invariant, related to the Hall conductivity by the TKNN formula. In [2] the Hall conductivity for lattice models has been expressed as an integral of the Berry curvature over the magnetic Brillouin zone. The nontrivial topology makes only integer multiples of the Hall conductivity possible.

The TKNN invariant has two major drawbacks: (i) it is not defined for systems where interactions occur, and (ii) it can only be applied to systems subject to a constant magnetic field, or homogeneous Chern insulators. The first is overcome through an alternate form of the TKNN invariant applicable to Chern insulators, expressed in terms of the two point Green function. In this approach the topological invariant for systems with interactions is obtained. The simplest such topological invariant is composed of a two point Green function. The invariance property leads to the stability of the Fermi surface in $3+1D$ systems, and has been shown to be admissible for interacting systems. Nonetheless it is still not valid for non-homogeneous systems.

Progress has been made towards this goal. It has been shown in references [8–10] that in the absence of electron interactions the TKNN invariant for the intrinsic anomalous QHE (AQHE) is expressible in terms of the momentum space Green function, and importantly, this expression is unchanged when the given system is modified smoothly. While this representation was derived originally only for non-interacting systems, it has since been suggested [9,10] that it can be generalized to describe interactions simply by replacing the non-interacting two point Green function with the full two point Green function that includes corrections due to interactions.

This has been proven in the framework of $2+1D$ QED [11,12]. The influence of interactions on the Integer QHE has been investigated recently [13]. It was shown that the QHE conductivity is the topological invariant composed of the Wigner-transformed two-point Green's function of the interacting model. This result refers to the Hall conductivity in non-homogeneous systems, in particular for systems subject to a varying magnetic field. The basic expression for the QHE conductivity has been suggested [14] for the Hall conductivity, constituting a topological invariant containing the Wigner transformed two point Green functions. Applications to graphene in presence of elastic deformation have been considered in [15]. Even more, as it was mentioned above, in [13,16] it was proved that for the systems with interactions the QHE conductivity is expressed as in [14] but with the complete Green function, which includes loop corrections.

Similar methods can be used to describe the QHE in $3+1$ D systems, which opens the door to a number of research goals addressed in this article. The first is to apply these methods to the QHE in Weyl semi-metals. The machinery developed for the representation of the QHE current in terms of Green functions has also been extended to the chiral separation effect (CSE) [17]. However, the question about the role of interactions in the CSE still remains open. The family of non-dissipative transport effects contains more members, such as the chiral torsional effect, chiral magnetic effect, chiral vortical effect, Hall viscosity, and more. An additional research goal is to construct the topological representation for the conductivity for each of these effects in terms of the Green functions. A similar representation for the fractional Hall effect also awaits investigation. In the latter case it might be necessary to build more involved topological invariants, composed of multi-leg Green functions. Such complicated topological invariants may also be relevant for considering various other topological phenomena in QCD.

First, to summarize some background theory. In the presence of a magnetic field the Hall conductivity is given by [3]

$$\sigma_H = \frac{\mathcal{N}}{2\pi},$$

where $\mathcal{N}$ is related to the number of filled Landau states. (Here the conductivity is expressed in units of $e^2/\hbar$.) A similar expression for the intrinsic QHE conductivity in topological insulators is derived in [8,9,18] in terms of the two-point Green function $G(p)$ (in the absence of interactions):

$$\mathcal{N} = -\frac{\epsilon_{ijk}}{3!\,4\pi^2}\int d^3p\,\mathrm{Tr}\,G(p)\frac{\partial G^{-1}(p)}{\partial p_i}\frac{\partial G(p)}{\partial p_j}\frac{\partial G^{-1}(p)}{\partial p_k}\,. \quad (1)$$

In [19], the expression in (1) was generalized to include interactions in the case of a varying magnetic field. In that expression the non-homogeneous nature of the system is characterized by the full two-point Green function expressed in terms of the Weyl symbol $G_W(x,p)$. Its explicit form is

$$\mathcal{N} = -\frac{T\epsilon_{ijk}}{A\,3!\,4\pi^2}\int d^3x \int d^3p \,\mathrm{tr}\, G_W(x,p) \star \frac{\partial Q_W(x,p)}{\partial p^i} \star \frac{\partial G_W(x,p)}{\partial p^j} \star \frac{\partial Q_W(x,p)}{\partial p^k}\,,$$

where $T \to 0$ is temperature, A is the area of the system, $G_W(x,p)$ is the Wigner transformation of the two-point Green's function $\hat{G} = \hat{Q}^{-1}$, while Q_W is the Wigner transformation of $\hat{Q}$. The star product $\star$ entering the above expression is the Moyal product of the conventional Wigner–Weyl calculus.

In the presence of interactions the conductivity of integer QHE is given by the above expression, where the complete Green function is substituted. It makes heavy use of the version of the Wigner–Weyl calculus used in these notes, which is described fully in [20]. However, this treatment is not valid for the FQHE .

The absence of correction terms to the IQHE due to Coulomb interactions and impurities (in the presence of a constant magnetic field) has been widely discussed some time ago in refs. [21–24] (see also [25–29]). In particular, in [22] the systems with both inter-electron interactions and disorder were considered, and the corresponding topological expression for the Hall conductivity was derived. It may be applied both to the IQHE and to the FQHE. Although the expression given in [22] was not applied for a practical calculation of the Hall conductivity, its topological nature itself is proof that the FQHE in the presence of a constant magnetic field is robust with respect to smooth modifications of the system. This proof is important for a more practical consideration of materials with the FQHE. Still, a substantial gap remains between the relevant theoretical models and real experiments in which magnetic fields are never precisely homogeneous. Rather variations of the magnetic field are always present. For the latter case, a theoretical proof that the FQHE conductivity is robust with respect to smooth modifications of the system, has still not been given. In this article we fill this gap and present this very proof.

In our approach we use a specific version of the Wigner–Weyl (WW) calculus developed earlier for field theoretical models of solid state physics. Originally the WW formalism was formulated by Groenewold [30] and Moyal [31] as a way of expressing results of quantum mechanics in terms of classical functions in phase space instead of operators. A transformation from a given operator to a classical function exists in general called the Weyl transformation. Later the WW formalism was applied to quantum field theory (QFT) and condensed matter physics. This WW calculus allows us to express the FQHE conductivity through a certain topological invariant composed of multi-particle Green functions. A number of results from the WW formalism are assumed. For a full discussion and derivation of these results the reader is recommended to consult [20]. A summary of the background and essential results are given in Appendix B. For a detailed description of the fractional QHE we advise the reader to consult [32–47].

This paper is organized as follows. In Section 2 we state the main result of the paper, namely the derived expression for the Hall conductivity in the presence of interactions. We also summarize in this section some of the notation used throughout this paper. In Section 3 the Hall conductivity of a system in the presence of a varying magnetic field is derived, but for a system with a *fixed number of N different* particles, and it is shown that the obtained expression is a topological invariant of the system. Section 4 contains the full derivation of the main result of this paper: an expression for the Hall conductivity for a system with a varying number of particles, but a fixed chemical potential. We also show in this section that the expression reduces to the analogous conductivity in the absence of interactions given in Section 3, when the interaction piece of the Hamiltonian is neglected. In Section 5 we summarize our results and discuss the implications of our findings in the wider context of related results on the Hall conductivity, topological quantum field theory and condensed matter physics.

2. Statement of the Main Result

We consider a system that has a varying number of particles but fixed chemical potential. A number of identities that involve creation and annihilation operators are used in this section. Their derivations can be found in Appendix A.

The Hamiltonian operator for the whole interacting system is

$$\hat{H} = \int d^2x\, a^\dagger(x)\mathscr{H}_0\, a(x) + \int d^2x\, d^2y\, a^\dagger(x)a(x)\mathscr{V}(x-y)a^\dagger(y)a(y) + \Delta\,. \tag{2}$$

Here $\mathscr{H}_0$ is the one-particle Hamiltonian defined with respect to the Fermi level, i.e., it is equal to the true one particle Hamiltonian minus a chemical potential, μ. The term $\mathscr{V}(x-y)$ is a potential term representing an inter-particle interaction. If Δ is a constant, its presence in $\hat{H}$ does not affect observable quantities. With this freedom, Δ is chosen in a way that the ground state of the system has negative energy while all excited states carry positive energy values. It is easily verified that

$$\hat{H}a^\dagger(x_1)\ldots a^\dagger(x_N)|\varnothing\rangle = \left(\sum_{a=1}^{N}\mathscr{H}_0(x_a) + \sum_{a,b=1}^{N}\mathscr{V}(x_a-x_b) + \Delta\right)a^\dagger(x_1)\ldots a^\dagger(x_N)|\varnothing\rangle\,.$$

Note that the particle-number operator, $\hat{N}$ commutes with the Hamiltonian. Therefore, $\hat{H}$ and $\hat{N}$ share common eigenstates. As a result, the ground state in particular corresponds to a definite value for the number of particles in the state. The ground state may be degenerate. However, at least in non-marginal cases, a degenerate ground state does not correspond to different eigenvalues for $\hat{N}$.

The statement immediately below is the main result of this paper: For a system with a Hamiltonian of the form of Equation (2), the Hall conductivity in the units of $e^2/\hbar$ averaged over the system area A is

$$\sigma_{xy} = \frac{\mathcal{N}}{2\pi K}\,, \tag{3}$$

where K is the degeneracy of the ground state while $\mathcal{N}$ is a topologically invariant quantity given by

$$\begin{aligned}
\mathcal{N} &= -\frac{1}{2A}\sum_{N=0,\ldots}\frac{1}{(2\pi)^{2N}}\sum_{b,c=1}^{N}\int d\omega\left(\prod_{a=1}^{N}d^2p_a\,d^2x_a\right)\epsilon^{jk} \\
&\mathrm{tr}\left[G_W^{(N)}(\omega,\{p_a\},\{x_a\})\star\frac{\partial Q_W^{(N)}(\omega,\{p_a\},\{x_a\})}{\partial\omega}\star\frac{\partial G_W^{(N)}(\omega,\{p_a\},\{x_a\})}{\partial p_b^j}\star\frac{\partial Q_W^{(N)}(\omega,\{p_a\},\{x_a\})}{\partial p_c^k}\right] \\
&= -\frac{1}{2A}\frac{1}{(2\pi)^{2N_0}}\sum_{b,c=1}^{N_0}\int d\omega\left(\prod_{a=1}^{N_0}d^2p_a\,d^2x_a\right)\epsilon^{jk} \\
&\mathrm{tr}\left[G_W^{(N_0)}(\omega,\{p_a\},\{x_a\})\star\frac{\partial Q_W^{(N_0)}(\omega,\{p_a\},\{x_a\})}{\partial\omega}\star\frac{\partial G_W^{(N_0)}(\omega,\{p_a\},\{x_a\})}{\partial p_b^j}\star\frac{\partial Q_W^{(N_0)}(\omega,\{p_a\},\{x_a\})}{\partial p_c^k}\right].
\end{aligned} \tag{4}$$

Here, N_0 is the number of particles in the ground state of the system. The $\star$ operator is defined as

$$\begin{aligned}
&A_W(\{x_a\},\{p_a\})\star B_W(\{x_a\},\{p_a\}) \\
=&A_W(\{x_a\},\{p_a\})\exp\left[\frac{i}{2}\sum_{a=1}^{N}\sum_{i=1}^{2}\left(\frac{\overleftarrow{\partial}}{\partial x_a^i}\frac{\overrightarrow{\partial}}{\partial p_a^i}-\frac{\overleftarrow{\partial}}{\partial p_a^i}\frac{\overrightarrow{\partial}}{\partial x_a^i}\right)\right]B_W(\{x_a\},\{p_a\})\,.
\end{aligned}$$

The Weyl symbols $Q_W^{(N)}(\omega,\{p_a\},\{x_a\})$ and $G_W^{(N)}(\omega,\{p_a\},\{x_a\})$ that appear in (4) are functions of $2N+1$ variables $\omega, p_1, x_1, \ldots, p_N, x_N$. Specifically, $Q_W^{(N)}(\omega,\{p_a\},\{x_a\})$ is the Weyl symbol of the operator $\hat{Q}^{(N)}$ defined by

$$Q_W^{(N)}(\omega,\{p_a\},\{x_a\}) = \frac{1}{N!}\int\left(\prod_{a=1}^{N} dq_a e^{iq_a x_a}\right)\langle\{p_a+q_a/2\}|\hat{Q}^{(N)}|\{p_a-q_a/2\}\rangle$$

where $|\{p_a\}\rangle$ denotes the multi-particle state defined by

$$|\{p_a\}\rangle \equiv a_1^\dagger(p_1)\ldots a_N^\dagger(p_N)\,|\varnothing\rangle\ ,$$

and the operator $\hat{Q}^{(N)}$ is defined by

$$\hat{Q}^{(N)} = (i\omega - \hat{H})\hat{\Pi}_N\ , \tag{5}$$

with $\hat{H}$ given explicitly in (2) being the field-theoretical Hamiltonian. Its matrix elements $\langle\{p_a\}|\hat{H}|\{q_a\}\rangle$ are between states with N particles having momenta that belong to the sets $\{p_a\}$ and $\{q_a\}$. Here $\hat{\Pi}_N$ is the projection operator onto N particle states defined by

$$\hat{\Pi}_N = \frac{1}{N!}\int dp_1 ... dp_N\,|\{p_a\}\rangle\,\langle\{p_a\}|\ ,$$

or equivalently

$$\hat{\Pi}_N = \frac{1}{N!}\int dx_1\ldots dx_N\, a^\dagger(x_1)\ldots a^\dagger(x_N)\,|0\rangle\,\langle 0|\, a(x_N)\ldots a(x_1)\,.$$

$G_W^{(N)}(\omega,\{p_a\},\{x_a\})$ is the Weyl symbol of the operator $\hat{G}^{(N)}$ defined by

$$G_W^{(N)}(\omega,\{p_a\},\{x_a\}) = \frac{1}{N!}\int\left(\prod_{a=1}^{N} dq_a e^{iq_a x_a}\right)\langle\{p_a+q_a/2\}|\hat{G}^{(N)}|\{p_a-q_a/2\}\rangle\ ,$$

where

$$\hat{G}^{(N)} = \frac{1}{i\omega-\hat{H}}\hat{\Pi}_N\,.$$

3. Fixed Number of Different Particles

3.1. Derivation of the Expression for Hall Conductance

In this section, the Hall conductivity of a system in the presence of a varying magnetic field is discussed. We seek an expression for the Hall conductivity for a system of N *different* particles. By different it is meant that the particles themselves are different, and to that extent neither symmeterization or anti-symmeterization is applied to the state. To that degree the results obtained in this section are intermediate, however the techniques developed are crucial for obtaining the main result in the next section, where a system of identical fermions is considered. In all expressions from now on, $\hbar = c = 1$ is assumed unless stated explicitly otherwise.

Let the operator $\hat{Q}$ be defined as

$$\hat{Q} = i\omega - \hat{H}\ , \tag{6}$$

where $\hat{H}$ is the multi-particle Hamiltonian inclusive of interaction terms:

$$\hat{H} = \sum_{a=1}^{N}\hat{H}_0(x_a, -i\partial_{x_a}) + \frac{1}{2}\sum_{\substack{a,b=1\\ a\neq b}}^{N} V(x_a - x_b)\,, \tag{7}$$

where $\hat{H}_0$ is the free-particle Hamiltonian and indices $a, b = 1, \ldots, N$ label the particles themselves. We assume that the ground state (either degenerate or unique) corresponds to a negative value of energy, while all excited states have positive values of energy. This may always be achieved simply by adding a constant to the single particle Hamiltonian $\hat{H}_0$ that appears in Equation (7). The inverse operator of $\hat{Q}$ is

$$\hat{G} = \frac{1}{i\omega - \hat{H}}\,, \tag{8}$$

where the notation on the right of Equation (8) is intended to denote the inverse of the operator $i\omega - \hat{H}$.

The Wigner transformation of the operator $\hat{Q}$ is defined as a function of the $2N+1$ variables $\omega, \{p_a\}, \{x_a\}$ $(a = 1, ..., N)$ in terms of its matrix elements in momentum space as

$$Q_W(\omega, \{p_a\}, \{x_a\}) = \int \left(\prod_{a=1}^{N} dq_a\, e^{iq_a x_a} \right) \langle \{p_a + \tfrac{q_a}{2}\} | \hat{Q} | \{p_a - \tfrac{q_a}{2}\} \rangle\,. \tag{9}$$

Here, $|\{p_a - \frac{q_a}{2}\}\rangle$ refers to a state comprised of N different fermions, defined by

$$|\{p_a\}\rangle \equiv a_1^\dagger(p_1) \ldots a_N^\dagger(p_N)\, |0\rangle\ , \tag{10}$$

where the suffix $_{1,2,\ldots,N}$ labels the particle, following the convention in [48] (See [48] §59 p. 225. Here the discussion is about boson states but the notation, described in detail here, applies also to fermion states, as described later on in [48] §65 pp. 248–259. Dirac writes $|\alpha_1^a\, \alpha_2^b\ \ldots\ \alpha_{u'}^g\rangle$, each α corresponding to a particle, where the suffixes $1, 2, 3, \ldots, u'$ label the particles themselves, while $a, b, c \ldots, g$ denote the indices $^{(1)}, {}^{(2)}, {}^{(3)}, \ldots$ in the basic kets for one particle, or in equivalent terms $a, b, c \ldots, g$ label the actual states in which the particles lie). The operators themselves are creation and annihilation operators of a single fermion that satisfy the familiar anticommutation relations

$$\{a_r(p), a_s^\dagger(p')\} = \delta(p - p')\delta_{rs}\,, \quad \{a_r(p), a_s(p')\} = \{a_r^\dagger(p), a_s^\dagger(p')\} = 0\,. \tag{11}$$

In a precisely analogous way, the Wigner symbol of $\hat{G}$ is

$$G_W(\omega, \{p_a\}, \{x_a\}) = \int \left(\prod_{a=1}^{N} dq_a\, e^{iq_a x_a} \right) \langle \{p_a + \tfrac{q_a}{2}\} | \hat{G} | \{p_a - \tfrac{q_a}{2}\} \rangle\,. \tag{12}$$

The goal of this paper is two fold. Firstly, to show that the Hall conductivity averaged over the system area A is given by

$$\sigma_{xy} = \frac{\mathcal{N}}{2\pi K}, \tag{13}$$

where K is the degeneracy of the ground state, $\mathcal{N}$ is given by

$$\begin{aligned}\mathcal{N} = & -\frac{1}{2A} \frac{1}{(2\pi)^{2N}} \sum_{b,c=1}^{N} \int d\omega \left(\prod_{a=1}^{N} d^2 p_a\, d^2 x_a \right) \\ & \epsilon^{jk}\, \mathrm{tr}\Bigg[G_W(\omega, \{p_a\}, \{x_a\}) \star \frac{\partial Q_W(\omega, \{p_a\}, \{x_a\})}{\partial \omega} \star \frac{\partial G_W(\omega, \{p_a\}, \{x_a\})}{\partial p_b^j} \\ & \star \frac{\partial Q_W(\omega, \{p_a\}, \{x_a\})}{\partial p_c^k} \Bigg]\,, \end{aligned} \tag{14}$$

and

$$\star = \exp\left(\frac{i}{2} \sum_{a=1}^{N} \overleftrightarrow{\Delta}_a \right), \qquad \overleftrightarrow{\Delta}_a = \frac{\overleftarrow{\partial}}{\partial x_a^i} \frac{\overrightarrow{\partial}}{\partial p_{a,i}} - \frac{\overleftarrow{\partial}}{\partial p_a^i} \frac{\overrightarrow{\partial}}{\partial x_{a,i}}\,, \qquad (i = 1, 2)\,.$$

The second goal is to show that $\mathcal{N}$ is topologically invariant.

The identity $G_W \star Q_W = 1$ that shall be proven below, together with the product rule of differentiation and the commutative property of derivatives, allows ordinary derivatives and the $\star$ operator to be interchanged. In particular,

$$\frac{\partial}{\partial p_b} G_W \star Q_W + G_W \star \frac{\partial Q_W}{\partial p_b} = 0 \qquad (b = 1, \ldots, N)\,,$$

such that the following relation holds:

$$\frac{\partial G_W}{\partial p_b} = -G_W \star \frac{\partial Q_W}{\partial p_b} \star G_W, \qquad (b = 1, \ldots, N)\,. \tag{15}$$

By substituting (15) in (14) it is obtained that

$$\begin{aligned}\mathcal{N} =& \frac{1}{2A} \frac{1}{(2\pi)^{2N}} \sum_{b,c=1}^{N} \int d\omega \left(\prod_{a=1}^{N} d^2 p_a \, d^2 x_a \right) \\ & \epsilon^{jk} \operatorname{tr} \Bigg(G_W(\omega, \{p_a\}, \{x_a\}) \star \frac{\partial Q_W(\omega, \{p_a\}, \{x_a\})}{\partial \omega} \star G_W(\omega, \{p_a\}, \{x_a\}) \\ & \star \frac{\partial Q_W(\omega, \{p_a\}, \{x_a\})}{\partial p_b^j} \star G_W(\omega, \{p_a\}, \{x_a\}) \star \frac{\partial Q_W(\omega, \{p_a\}, \{x_a\})}{\partial p_c^k} \Bigg).\end{aligned} \tag{16}$$

By (6) and (9),

$$\frac{\partial Q_W(\omega, \{p_a\}, \{x_a\})}{\partial p_b^j} = \int \left(\prod_{a=1}^{N} dq_a \, e^{i q_a x_a} \right) \frac{\partial}{\partial p_b^j} \langle \{p_a + \tfrac{q_a}{2}\} | i\omega - \hat{H} | \{p_a - \tfrac{q_a}{2}\} \rangle\,. \tag{17}$$

The following identities from the standard bra-ket formalism may be invoked:

$$-\, i \frac{\partial}{\partial p_j^b} |p\rangle = \hat{x}_b^j |p\rangle, \qquad -i \frac{\partial}{\partial p_j^b} \langle p| = -\langle p| \hat{x}_b^j\,. \tag{18}$$

For an explanation of the origins the identities quoted in Equation (18), the reader may consult ref. [48] for example. Accordingly (17) becomes

$$\begin{aligned}\frac{\partial Q_W(\omega, \{p_a\}, \{x_a\})}{\partial p_b^j} &= \int \left(\prod_{a=1}^{N} dq_a \, e^{i q_a x_a} \right) i \langle \{p_a + \tfrac{q_a}{2}\} | [\hat{x}_b^j, \hat{H}] | \{p_a - \tfrac{q_a}{2}\} \rangle \\ &= - \int \left(\prod_{a=1}^{N} dq_a \, e^{i q_a x_a} \right) \langle \{p_a + \tfrac{q_a}{2}\} | \hat{J}_b^j | \{p_a - \tfrac{q_a}{2}\} \rangle\,.\end{aligned} \tag{19}$$

where in the last step the relation $[\hat{x}_b^j, \hat{H}] = i\hat{J}_b^j$ was substituted, where $\hat{J}_b^j$ is the operator associated with the j component of the electric current. *Note that in our calculations we define electric current in units of electric charge e, and we use natural units where $c = \hbar = 1$.*

Let (19) be the definition of the Weyl symbol $J_{b\,W}^j$, such that (16) can be cast in the form

$$\begin{aligned}\mathcal{N} =& \frac{1}{2A} \frac{1}{(2\pi)^{2N}} \sum_{b,c=1}^{N} \int d\omega \left(\prod_{a=1}^{N} d^2 p_a \, d^2 x_a \right) \\ & \epsilon^{jk} \operatorname{tr} \Bigg(G_W(\omega, \{p_a\}, \{x_a\}) \star \frac{\partial Q_W(\omega, \{p_a\}, \{x_a\})}{\partial \omega} \star G_W(\omega, \{p_a\}, \{x_a\}) \\ & \star J_{b\,W}^j \star G_W(\omega, \{p_a\}, \{x_a\}) \star J_{c\,W}^k \Bigg).\end{aligned}$$

After invoking the relation

$$A_W(x,p) \star B_W(x,p) := (AB)_W(x,p) = A_W(x,p) \exp\left(\frac{i}{2}\left(\overleftarrow{\partial_x}\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\overrightarrow{\partial_x}\right)\right) B_W(x,p)\,,$$

we obtain

$$\mathcal{N} = \frac{1}{2A(2\pi)^{2N}} \sum_{b,c=1}^{N} \int d\omega \left(\prod_{a=1}^{N} d^2p_a\, d^2x_a\right) \epsilon^{jk} \mathrm{tr}\left(\hat{G}\, \frac{\partial \hat{Q}}{\partial \omega}\, \hat{G}\, \hat{J}_b^j\, \hat{G}\, \hat{J}_c^k\right)_W .$$

Next, by substituting the formal definition of a Weyl symbol we find that

$$\begin{aligned}\mathcal{N} =& \frac{1}{2A(2\pi)^{2N}} \sum_{b,c=1}^{N} \int d\omega \left(\prod_{a=1}^{N} d^2p_a\, d^2q_a\, d^2x_a\, e^{iq_a x_a}\right) \\ & \epsilon^{jk} \mathrm{tr} \langle \{p_a + \frac{q_a}{2}\} |\, \hat{G}\, \frac{\partial \hat{Q}}{\partial \omega}\, \hat{G}\, \hat{J}_b^j\, \hat{G}\, \hat{J}_c^k\, | \{p_a - \frac{q_a}{2} \rangle\}\,,\end{aligned} \tag{20}$$

where $|\{p_a + \frac{q_a}{2}\}\rangle$ is the N fermion state defined in (10) but with $\{p_a\}$ replaced with $\{p_a + q_a/2\}$. Equation (20) can be expressed using a complete set of antisymmetric N fermion states, using the result proved in (A8), as

$$\begin{aligned}\mathcal{N} =& \frac{1}{2A(2\pi)^{2N}} \sum_{b,c=1}^{N} \int d\omega \left(\prod_{a=1}^{N} d^2p_a\, d^2q_a\, d^2x_a\, d^2p_{a,1}\, d^2p_{a,2}\, d^2p_{a,3}\, e^{iq_a x_a}\right) \\ & \epsilon^{jk}\, \mathrm{tr} \langle \{p_a + \tfrac{q_a}{2}\} |\, \hat{G} \frac{\partial \hat{Q}}{\partial \omega} \hat{G}\, | \{p_{a,1}\} \rangle \langle \{p_{a,1}\} |\, \hat{J}_b^j\, | \{p_{a,2}\} \rangle \langle \{p_{a,2}\} |\, \hat{G}\, | \{p_{a,3}\} \rangle \langle \{p_{a,3}\} |\, \hat{J}_c^k\, | \{p_a - \tfrac{q_a}{2}\} \rangle.\end{aligned} \tag{21}$$

The outcome from evaluating the x integrals is a product of δ functions, namely one factor of $(2\pi)\delta(q_a)$ corresponding to each integrand labeled by a. These δ functions make each q_a integral trivial. In all, after evaluating the x_a and subsequently the q_a integrals, (21) reduces to

$$\begin{aligned}\mathcal{N} =& \frac{1}{2A} \sum_{b,c=1}^{N} \int d\omega \left(\prod_{a=1}^{N} d^2p_a\, d^2p_{a,1}\, d^2p_{a,2}\, d^2p_{a,3}\right) \\ & \epsilon^{jk} \mathrm{tr} \langle \{p_a\} |\, \hat{G} \frac{\partial \hat{Q}}{\partial \omega} \hat{G}\, | \{p_{a,1}\} \rangle \langle \{p_{a,1}\} |\, \hat{J}_b^j\, | \{p_{a,2}\} \rangle \langle \{p_{a,2}\} |\, \hat{G}\, | \{p_{a,3}\} \rangle \langle \{p_{a,3}\} |\, \hat{J}_c^k\, | \{p_a\} \rangle\,.\end{aligned}$$

The next steps are first to plug in the explicit forms in (6) and (8), from which $\partial \hat{Q}/\partial \omega = i$. Subsequently complete sets of eigenstates of $\hat{H}$ are inserted into the expression, assuming that each set is discrete and belongs to discrete eigenvalues. This yields

$$\begin{aligned}\mathcal{N} =& \frac{i}{2A} \sum_{b,c=1}^{N} \int d\omega \left(\prod_{a=1}^{N} d^2p_a\, d^2p_{a,1}\, d^2p_{a,2}\, d^2p_{a,3}\right) \\ & \epsilon^{jk}\, \mathrm{tr} \sum_{E,E',E''} \langle \{p_a\} |\, \frac{1}{i\omega - \hat{H}}\, |E\rangle \langle E|\, \frac{1}{i\omega - \hat{H}}\, |E''\rangle \langle E'' | p_1 \rangle \langle \{p_{a,1}\} |\, \hat{J}_b^j\, | \{p_{a,2}\} \rangle \\ & \langle \{p_{a,2}\} |\, \frac{1}{i\omega - \hat{H}}\, |E'\rangle \langle E' | \{p_{a,3}\} \rangle \langle p_{a,3} |\, \hat{J}_c^k\, | \{p_a\} \rangle\,.\end{aligned} \tag{22}$$

By their very definition of being eigenstates of $\hat{H}$ it stands to reason that the inverse operator $(i\omega - \hat{H})^{-1}$, denoted as $1/(i\omega - \hat{H})$ in the expression above, has the eigenvalue equation $(i\omega - \hat{H})^{-1} |E\rangle = \frac{1}{i\omega - E} |E\rangle$. To that extent (22) becomes

$$\mathcal{N} = \frac{i}{2A} \sum_{b,c=1}^{N} \int d\omega \left(\prod_{a=1}^{N} d^2p_a \, d^2p_{a,1} \, d^2p_{a,2} \, d^2p_{a,3} \right) \sum_{E,E',E''} \frac{1}{i\omega - E} \frac{1}{i\omega - E'} \frac{1}{i\omega - E''}$$
$$\epsilon^{jk} \, \mathrm{tr} \, \langle \{p_a\}|E\rangle \langle E|E''\rangle \langle E''|\{p_{a,1}\}\rangle \langle \{p_{a,1}\}| \hat{J}^j_b |\{p_{a,2}\}\rangle$$
$$\langle \{p_{a,2}\}|E'\rangle \langle E'|\{p_{a,3}\}\rangle \langle \{p_{a,3}\}| \hat{J}^k_c |\{p_a\}\rangle \ ,$$

and since the trace operator allows the freedom to change the order of inner products cyclically, this can be written equally as

$$\mathcal{N} = \frac{i}{2A} \sum_{b,c=1}^{N} \int d\omega \left(\prod_{a=1}^{N} d^2p_a \, d^2p_{a,1} \, d^2p_{a,2} \, d^2p_{a,3} \right) \sum_{E,E',E''} \frac{1}{i\omega - E} \frac{1}{i\omega - E'} \frac{1}{i\omega - E''}$$
$$\epsilon^{jk} \, \mathrm{tr} \, \langle E|E''\rangle \langle E''|\{p_{a,1}\}\rangle \langle \{p_{a,1}\}| \hat{J}^j_b |\{p_{a,2}\}\rangle$$
$$\langle \{p_{a,2}\}|E'\rangle \langle E'|\{p_{a,3}\}\rangle \langle \{p_{a,3}\}| \hat{J}^k_c |\{p_a\}\rangle \langle \{p_a\}|E\rangle \ .$$

The integrals are simplified using the identity derived in (A8), which fixes each of the intermediate outer products to be the identity operator, namely

$$\int \left(\prod_{a=1}^{N} dp_a \right) |\{p_a\}\rangle \langle \{p_a\}| = 1 \, ,$$
$$\int \left(\prod_{a=1}^{N} dp_{a,1} \right) |\{p_{a,1}\}\rangle \langle \{p_{a,1}\}| = 1 \, ,$$
$$\int \left(\prod_{a=1}^{N} dp_{a,2} \right) |\{p_{a,2}\}\rangle \langle \{p_{a,2}\}| = 1 \, ,$$
$$\int \left(\prod_{a=1}^{N} dp_{a,3} \right) |\{p_{a,3}\}\rangle \langle \{p_{a,3}\}| = 1 \, .$$

As well the relation $\langle E|E''\rangle = \delta_{E,E''}$ eliminates the sum over E''. Putting everything together,

$$\mathcal{N} = \frac{i}{2A} \sum_{b,c=1}^{N} \sum_{E,E'} \int d\omega \frac{1}{(i\omega - E)^2 (i\omega - E')} \mathrm{tr} \, \epsilon^{jk} \langle E| \hat{J}^j_b |E'\rangle \langle E'| \hat{J}^k_c |E\rangle \ .$$

The ω integral, with the line of the integration range $[-\infty, \infty]$, is evaluated by extending l to be a closed semi-circle, C in the upper complex plane. Two possibilities arise: (i) C encloses both of the points iE and iE' on the imaginary axis if $E > 0$ and $E' > 0$, or (ii) C encloses only one of them, if say $E > 0$ $E' < 0$, or the converse. If $E < 0$ and $E' < 0$, integrating over the variable $-\omega$ (namely minus ω) instead, produces the same integral described in (i) with the same contour C. In both cases the integral can be evaluated using Cauchy's integral formula. For case (i) the integral vanishes, but for case (ii) there is a non-zero contribution. The result is

$$\mathcal{N} = -\frac{2\pi i}{A} \sum_{b,c=1}^{N} \sum_{E,E'} \frac{\theta(-E)\theta(E')}{(E - E')^2} \epsilon^{jk} \, \mathrm{tr} \, \langle E| \hat{J}^j_b |E'\rangle \langle E'| \hat{J}^k_c |E\rangle \ . \quad (23)$$

Just as the presence of the term $\theta(E)\theta(-E')$ indicates, the integral was solved assuming that $E > 0$, $E' < 0$. Had the converse been assumed, precisely the same formula in (23) would hold, seeing as interchanging E and E', given the trace operator in front, leaves the expression unaltered.

Appropriately the sum over the labels b and c gets absorbed by replacing $\sum_{b=1}^{N}\hat{J}_b^j = \hat{J}^j$ and $\sum_{c=1}^{N}\hat{J}_c^k = \hat{J}^k$, resulting in

$$\mathcal{N} = -\frac{2\pi i}{A}\sum_{E,E'}\frac{\theta(-E)\theta(E')}{(E-E')^2}\epsilon^{jk}\,\mathrm{tr}\,\langle E|\hat{J}^j|E'\rangle\langle E'|\hat{J}^k|E\rangle$$

$$= -\frac{2\pi i}{KA}\sum_{E,E'}\frac{\theta(-E)\theta(E')}{(E-E')^2}\,\mathrm{tr}\big(\langle E|\hat{J}^x|E'\rangle\langle E'|\hat{J}^y|E\rangle - \langle E|\hat{J}^y|E'\rangle\langle E'|\hat{J}^x|E\rangle\big)\,. \tag{24}$$

Here, the ket $|E\rangle$ with $E<0$ is an N fermion eigenstate of $\hat{H}$.

There are two separate cases to consider. The first applied when there is only one such state: the ground state of the system. In this case $K=1$. Let this state be denoted by $|0\rangle$ instead, and let the eigenvalue of $\hat{H}$ that it belongs to be denoted by E_0, such that $\hat{H}|0\rangle = E_0|0\rangle$. Correspondingly (24) reads

$$\mathcal{N} = -\frac{2\pi i}{A}\sum_{E}\frac{1}{(E-E_0)^2}\,\mathrm{tr}\big(\langle 0|\hat{J}^x|E\rangle\langle E|\hat{J}^y|0\rangle - \langle 0|\hat{J}^y|E\rangle\langle E|\hat{J}^x|0\rangle\big)\,.$$

However, as stated in (13), $\sigma_{xy} = \mathcal{N}/2\pi$ (in natural units), thus

$$\sigma_{xy} = \frac{i}{A}\sum_{E}\frac{1}{(E-E_0)^2}\,\mathrm{tr}\big(\langle 0|\hat{J}^y|E\rangle\langle E|\hat{J}^x|0\rangle - \langle 0|\hat{J}^x|E\rangle\langle E|\hat{J}^y|0\rangle\big)\,, \tag{25}$$

which is the conventional expression for Hall conductivity.

Now assume that there are K degenerate ground states. In this case the system at zero temperature does not remain in a pure quantum state. Instead, the true state is described by a density matrix. In such a state, with a diagonal density matrix (corresponding to the probabilities of all ground states being equal) the conventional expression for the Hall conductivity of Equation (25) has the modified form

$$\sigma_{xy} = \frac{i}{KA}\sum_{n}\sum_{E}\frac{1}{(E-E_0)^2}\,\mathrm{tr}\big(\langle n|\hat{J}^y|E\rangle\langle E|\hat{J}^x|n\rangle - \langle n|\hat{J}^x|E\rangle\langle E|\hat{J}^y|n\rangle\big)\,.$$

Here the sum is over the degenerate ground states $|n\rangle$. This expression can be rewritten as $\sigma_{xy} = \dfrac{\mathcal{N}}{2\pi K}$ with $\mathcal{N}$ given by Equation (24).

3.2. Proof of Topological Invariance

In this section, the fact that Equation (14) is a topological invariant is proved. Let it be written in compact form as

$$\begin{aligned}\mathcal{N} =& -\frac{1}{2A(2\pi)^{2N}}\int d\omega\int\left(\prod_{a=1}^{N}d^2p_a d^2x_a\right)\mathrm{tr}\epsilon^{3jk} \\ &\left[G_W(\omega,\{p_a\},\{x_a\})\star\frac{\partial Q_W(\omega,\{p_a\},\{x_a\})}{\partial\omega}\star\sum_a\frac{\partial G_W(\omega,\{p_a\},\{x_a\})}{\partial p_a^j}\right. \\ &\left.\star\sum_b\frac{\partial Q_W(\omega,\{p_a\},\{x_a\})}{\partial p_b^k}\right]\,.\end{aligned} \tag{26}$$

We introduce a convenient notation

$$D_3 = \frac{\partial}{\partial\omega}\,,\qquad D_i = \sum_a\frac{\partial}{\partial p_a^i},\quad (i=1,2),$$

and write Equation (26) as

$$\mathcal{N} = -\frac{1}{6A(2\pi)^{2N}} \int d\omega \int \left(\prod_{a=1}^{N} d^2 p_a d^2 x_a \right)$$
$$\operatorname{tr} \epsilon^{ijk} \Big[G_W(\omega, \{p_a\}, \{x_a\}) \star D_i Q_W(\omega, \{p_a\}, \{x_a\}) \star D_j G_W(\omega, \{p_a\}, \{x_a\})$$
$$\star D_k Q_W(\omega, \{p_a\}, \{x_a\}) \Big] . \tag{27}$$

It is instructive to write (27) using the identity $D_j G_W(\omega, \{p_a\}, \{x_a\}) = -G_W \star D_j Q_W \star G_W$, to obtain a more symmetric form for $\mathcal{N}$ as

$$\mathcal{N} = \frac{1}{6A(2\pi)^{2N}} \int d\omega \int \left(\prod_{a=1}^{N} d^2 p_a d^2 x_a \right)$$
$$\operatorname{tr} \epsilon^{ijk} G_W(\omega, \{p_a\}, \{x_a\}) \star D_i Q_W(\omega, \{p_a\}, \{x_a\}) \star G_W(\omega, \{p_a\}, \{x_a\}) \star D_j Q_W(\omega, \{p_a\}, \{x_a\})$$
$$\star G_W(\omega, \{p_a\}, \{x_a\}) \star D_k Q_W(\omega, \{p_a\}, \{x_a\})$$
$$= \frac{1}{6A(2\pi)^{2N}} \int d\omega \int \left(\prod_{a=1}^{N} d^2 p_a d^2 x_a \right) \operatorname{tr} \epsilon^{ijk} \Big[K_{i,W}(\omega, \{p_a\}, \{x_a\}) \star K_{j,W}(\omega, \{p_a\}, \{x_a\})$$
$$\star K_{k,W}(\omega, \{p_a\}, \{x_a\}) \Big] ,$$

where $K_{i,W} \equiv G_W(\omega, \{p_a\}, \{x_a\}) \star D_i Q_W(\omega, \{p_a\}, \{x_a\})$. Now it is straightforward to apply an arbitrary variation of the Green function $G \to G + \delta G$. The resulting variation of $\mathcal{N}$ is

$$\delta\mathcal{N} = \frac{3}{6A(2\pi)^{2N}} \int d\omega \int \left(\prod_{a=1}^{N} d^2 p_a d^2 x_a \right) \operatorname{tr} \epsilon^{ijk} \Big[K_{i,W}(\omega, \{p_a\}, \{x_a\}) \star K_{j,W}(\omega, \{p_a\}, \{x_a\})$$
$$\star K_{k,W}(\omega, \{p_a\}, \{x_a\}) \Big]$$
$$= \frac{3}{6A(2\pi)^{2N}} \int d\omega \int \left(\prod_{a=1}^{N} d^2 p_a d^2 x_a \right) \operatorname{tr} \epsilon^{ijk} \Big[\delta K_{i,W}(\omega, \{p_a\}, \{x_a\}) \star K_{j,W}(\omega, \{p_a\}, \{x_a\})$$
$$\star K_{k,W}(\omega, \{p_a\}, \{x_a\}) \Big] ,$$

where $\delta K_{i,W}(\omega, \{p_a\}, \{x_a\}) = \delta G_W \star D_i Q_W + G_W \star D_i \delta Q_W = -G_W \star \delta Q_W \star G_W \star D_i Q_W + G_W \star D_i \delta Q_W$. Putting everything together we find that

$$\delta\mathcal{N} = -\frac{3}{6A(2\pi)^{2N}} \int d\omega \int \left(\prod_{a=1}^{N} d^2 p_a d^2 x_a \right) \operatorname{tr} \epsilon^{ijk}$$
$$\Big[(-G_W \star \delta Q_W \star G_W \star D_i Q_W + G_W \star D_i \delta Q_W) \star G_W \star D_j Q_W \star G_W \star D_k Q_W \Big] . \tag{28}$$

The trace can be re-ordered as

$$\begin{aligned}
&\mathrm{tr}\,\epsilon^{ijk}\Big[(-G_W\star\delta Q_W\star G_W\star D_iQ_W+G_W\star D_i\delta Q_W)\star G_W\star D_jQ_W\star G_W\star D_kQ_W\Big]\\
=&\mathrm{tr}\,\epsilon^{ijk}\Big[(-\delta Q_W\star G_W\star D_iQ_W+D_i\delta Q_W)\star G_W\star D_jQ_W\star G_W\star D_kQ_W\star G_W\Big]\\
=&\mathrm{tr}\,\epsilon^{ijk}\Big[(-\delta Q_W\star G_W\star D_iQ_W\star G_W+D_i\delta Q_W\star G_W)\star D_jQ_W\star G_W\star D_kQ_W\star G_W\Big]\\
=&-\mathrm{tr}\,\epsilon^{ijk}\Big[(\delta Q_W\star D_iG_W+D_i\delta Q_W\star G_W)\star D_jQ_W\star D_kG_W\Big]\\
=&-\mathrm{tr}\,\epsilon^{ijk}\Big[D_i(\delta Q_W\star G_W)\star D_jQ_W\star D_kG_W\Big]\\
=&-\mathrm{tr}\,\epsilon^{ijk}D_i\Big[(\delta Q_W\star G_W)\star D_jQ_W\star D_kG_W\Big]\,.
\end{aligned}$$

Finally, by substituting this back into (28) we end up with

$$\delta\mathcal{N}=+\frac{3}{6A(2\pi)^{2N}}\int d\omega\int\left(\prod_{a=1}^{N}d^2p_ad^2x_a\right)\mathrm{tr}\,\epsilon^{ijk}D_i\Big[(\delta Q_W\star G_W)\star D_jQ_W\star D_kG_W\Big].$$

This is of course zero, since the integrand comprises a total derivative. In conclusion,

$$\delta\mathcal{N}=0\,.$$

The implication is that $\mathcal{N}$ is topologically invariant.

4. System with Varying Number of Identical Particles and Fixed Chemical Potential

4.1. Derivation of the Topological Expression

Suppose that the system has a varying number of particles but a fixed chemical potential. If the ground state of the system is non-degenerate, the Hall conductivity is still given by the familiar Kubo expression

$$\sigma_{12}=\frac{i}{A}\sum_{n\neq0}\frac{\langle0|\hat{J}_2|n\rangle\langle n|\hat{J}_1|0\rangle-\langle0|\hat{J}_1|n\rangle\langle n|\hat{J}_2|0\rangle}{(E_n-E_0)^2}\,. \tag{29}$$

If the ground state is degenerate, then the linear response of the system (remaining in thermal equilibrium at zero temperature) to an external electric field gives rise to the following expression for the Hall conductivity:

$$\sigma_{12}=\frac{i}{KA}\sum_{k=1}^{K}\sum_{n}\frac{\langle0_k|\hat{J}_2|n\rangle\langle n|\hat{J}_1|0_k\rangle-\langle0_k|\hat{J}_1|n\rangle\langle n|\hat{J}_2|0_k\rangle}{(E_n-E_0)^2}\,.$$

Here the sum $\sum_{k=1}^{K}$ is over the K degenerate ground states $|0_k\rangle$, while the sum $\sum_n$ is over excited states of the system.

For a sufficiently weak magnetic field,

$$\hat{J}_i=\frac{1}{i}[\hat{x}_i,\hat{H}]\,. \tag{30}$$

Here the Hamiltonian operator for the case of varying particle number is

$$\hat{H}=\int d^2x\,a^\dagger(x)(H_0-\mu)\,a(x)+\int d^2x\,d^2y\,a^\dagger(x)a(x)\mathscr{V}(x-y)a^\dagger(y)a(y)\;,+\Delta \tag{31}$$

with Δ a constant term chosen in such a way that the ground states (i.e., the states with minimal values of the total energy) have negative energy, while all excited states belong to energy eigenvalues that are positive. The position operator is

$$\hat{x}^i = \int d^2x\, a^\dagger(x) x^i a(x) \,. \tag{32}$$

The physical meaning of this operator is that it is a measure of spatial inhomogeneity. Namely, for a system in which particles are distributed homogeneously in space, its value is equal to zero. At the same time, the value of this operator is nonzero if the particles are distributed in a non-uniform way. In the marginal case, when N particles are placed around the coordinates of vector X^i, the corresponding eigenvalue of $\hat{x}^i$ is close to NX^i.

In the following we denote

$$\mathscr{H}_0 = H_0 - \mu \,,$$

where μ is the chemical potential. These forms for $\hat{H}$ and $\hat{x}$ in (31) and (32) are justified as follows. From the expression in (31), the operator $\hat{H}$ acts on states comprised of N quanta of energy as described by (33) below:

$$\begin{aligned}(\hat{H}-\Delta)a^\dagger(x_1)\dots a^\dagger(x_N)|\varnothing\rangle &= \int d^2x\, a^\dagger(x)\mathscr{H}_0(x)\, a(x) a^\dagger(x_1)\dots a^\dagger(x_N)|\varnothing\rangle \\ &+ \int d^2x\, d^2y\, a^\dagger(x)a(x)\mathscr{V}(x-y)a^\dagger(y)a(y)a^\dagger(x_1)\dots a^\dagger(x_N)|\varnothing\rangle \,.\end{aligned} \tag{33}$$

The right-hand side can be recast by re-arranging the order of creation and annihilation operators. Repeated use of the anticommutation relation in (11), namely $\{a(x), a^\dagger(x_1)\} = \delta(x-x_1)$, leads to

$$\begin{aligned}&a^\dagger(x)\, a(x)a^\dagger(x_1)\dots a^\dagger(x_N)\,|\varnothing\rangle \\ &=\delta(x-x_1)a^\dagger(x)a^\dagger(x_2)\dots a^\dagger(x_N)\,|\varnothing\rangle - \delta(x-x_2)a^\dagger(x)a^\dagger(x_1)a^\dagger(x_3)\dots a^\dagger(x_N)\,|\varnothing\rangle + \cdots \\ &\quad\cdots + (-1)^{N-1}\delta(x-x_N)a^\dagger(x)a^\dagger(x_1)a^\dagger(x_2)\dots a^\dagger(x_{N-1})\,|\varnothing\rangle \\ &= \left(\sum_{a=1}^{N}\delta(x-x_a)(-1)^{a-1}\right)a^\dagger(x_1)a^\dagger(x_2)\dots a^\dagger(x_N)\,|\varnothing\rangle \,.\end{aligned} \tag{34}$$

This in turn implies that

$$\begin{aligned}&a^\dagger(x)\, a(x)a^\dagger(y)\, a(y)a^\dagger(x_1)\dots a^\dagger(x_N)\,|\varnothing\rangle \\ &=\left(\sum_{a,b=1}^{N}\delta(x-x_a)\delta(y-x_b)(-1)^{a+b}\right)a^\dagger(x_1)a^\dagger(x_2)\dots a^\dagger(x_N)\,|\varnothing\rangle \,.\end{aligned} \tag{35}$$

Thus, by substituting (34) and (35) in (33) the outcome is the eigenvalue equation

$$\begin{aligned}&\hat{H}a^\dagger(x_1)\dots a^\dagger(x_N)|\varnothing\rangle = \\ &=\left(\sum_{a=1}^{N}\mathscr{H}_0(x_a) + \sum_{a,b=1}^{N}\mathscr{V}(x_a-x_b) + \Delta\right)a^\dagger(x_1)\dots a^\dagger(x_N)|\varnothing\rangle \,.\end{aligned} \tag{36}$$

Based on the expression in (32) the operator $\hat{x}$ acts on N particle states as

$$\begin{aligned}\hat{x}^i a^\dagger(x_1)\dots a^\dagger(x_N)|\emptyset\rangle &= \int dx\, x^i a^\dagger(x)a(x)a^\dagger(x_1)\dots a^\dagger(x_N)|\emptyset\rangle \\ &= \int dx\, x^i \sum_{a=1}^{N}(-1)^a\delta(x-x_a)a^\dagger(x_1)\dots a^\dagger(x_N)|\emptyset\rangle \\ &= \sum_{a=1}^{N} x_a^i a^\dagger(x_1)\dots a^\dagger(x_N)|\emptyset\rangle\,, \end{aligned} \tag{37}$$

where in the second step the identity in (34) was used. Equation (37) can be re-cast as

$$\hat{x}^i a^\dagger(x_1)\dots a^\dagger(x_N)|\emptyset\rangle = x^i a^\dagger(x_1)\dots a^\dagger(x_N)|\emptyset\rangle\,,$$

where $x^i \equiv \sum_{a=1}^N x_a^i$ is the i component of the vector sum of the position vectors of all of the N particles.

By substituting (30) in (29) we arrive at

$$\sigma_{12} = \frac{1}{iKA}\sum_{k=1}^{K}\sum_{n\neq 0_k}\frac{\langle 0_k|[\hat{x}_2,\hat{H}]|n\rangle\langle n|[\hat{x}_1,\hat{H}]|0_k\rangle - \langle 0_k|[\hat{x}_1,\hat{H}]|n\rangle\langle n|[\hat{x}_2,\hat{H}]|0_k\rangle}{(E_n-E_0)^2}\,. \tag{38}$$

Say that the ground states of the system, $|0_k\rangle$ is a sum over states containing $N = 0,1,2,\dots$ particles, with the form

$$|0_k\rangle \equiv \sum_N \frac{1}{\sqrt{N!}}\int d^2x_1\dots d^2x_N \psi_N^{(0_k)}(x_1,\dots,x_N)a^\dagger(x_1)\dots a^\dagger(x_N)|\emptyset\rangle\,, \tag{39}$$

where $|\emptyset\rangle$ denotes the vacuum state, in which there are no particles at all. Since the number operator commutes with the Hamiltonian, it is reasonable to suppose that $\psi_N^{(0_k)}(x_1,\dots,x_N)$ is non-zero only for the value of $N = N_{0_k}$. Moreover, N_{0_k} does not depend on k except when the marginal case is encountered, when a particle can be added to the system without changing the energy of the system. Excited states are decomposed in a similar fashion:

$$|n\rangle \equiv \sum_N \frac{1}{\sqrt{N!}}\int d^2x_1\dots d^2x_N \psi_N^{(n)}(x_1,\dots,x_N)a^\dagger(x_1)\dots a^\dagger(x_N)|\emptyset\rangle. \tag{40}$$

In the same way, only one value of N contributes to this sum. Nevertheless in the discussion to follow, sums over N are retained in the expressions in order to have expressions in a forms that are easily generalized to the cases where the Hamiltonian does not conserve particle number.

The normalization condition $\langle n|n\rangle = 1$ is assumed, implying that

$$\begin{aligned}&\sum_{N,N'}\frac{1}{\sqrt{N!}\sqrt{N'!}}\int d^2x_1\dots d^2x_N\, d^2x'_1\dots d^2x'_{N'}\psi_{N'}^{(n)\dagger}(x'_1,\dots,x'_{N'})\psi_N^{(n)}(x_1,\dots,x_N)\\ &\langle\emptyset|a(x'_{N'})\dots a(x'_1)a^\dagger(x_1)\dots a^\dagger(x_N)|\emptyset\rangle\\ =&1\,.\end{aligned}$$

We invoke (A5) in order to write this as

$$\begin{aligned}&\sum_N\frac{1}{N!}\int d^2x_1\dots d^2x_N\, d^2x'_1\dots d^2x'_N\psi_N^{(n)\dagger}(x'_1,\dots,x'_N)\psi_N^{(n)}(x_1,\dots,x_N)\\ &\sum_{i_1\dots i_N}\epsilon^{i_1\dots i_N}\delta(x_1-x'_{i_1})\dots\delta(x_N-x'_{i_N})\\ =&1\,,\end{aligned}$$

or equally

$$\sum_N \frac{1}{N!} \sum_{i_1 \ldots i_N} \int d^2x_1 \ldots d^2x_N \, \psi_N^{(n)\dagger}(x_{i_1}, \ldots, x_{i_N}) \psi_N^{(n)}(x_1, \ldots, x_N) \epsilon^{i_1 \ldots i_N} = 1 \, .$$

The factor $1/N!$ is canceled by the antisymmetric sum over $N!$ identical terms, through the contraction with the Levi-Civita symbol, to finally yield

$$\sum_N \int d^2x_1 \ldots d^2x_N \, \psi_N^{(n)\dagger}(x_1, \ldots, x_N) \psi_N^{(n)}(x_1, \ldots, x_N) = 1 \, .$$

The total Fock space, $\boldsymbol{H}$ of the system may be decomposed into a direct sum of sub-spaces $\boldsymbol{H}^{(N)}$, each containing a fixed number of particles as

$$\boldsymbol{H} = \boldsymbol{H}^{(0)} \cup \ldots \boldsymbol{H}^{(N)} \cup \ldots$$

The functions $\psi_N^{(n)}$ are defined on $\boldsymbol{H}^{(N)}$. In a case where only one value of N contributes to $|n\rangle$, the latter may be denoted as $|\psi_N^{(n)}\rangle$. In line with the convention of notation in standard quantum mechanics, the coordinate representation of the functions $\psi_N^{(n)}(x_1, \ldots, x_N)$ may be expressed as the inner product of $|\psi_N^{(n)}\rangle$ with a basis of coordinate eigenstates as

$$\psi_N^{(n)}(x_1, \ldots, x_N) = \frac{1}{\sqrt{N!}} \langle x_1, \ldots, x_N | \psi_N^{(n)} \rangle \, .$$

In the framework of this structure of the Fock space, the goal is to derive a new expression for the Hall conductivity, starting from (39), as a sum over terms where each term is the contribution coming from a state with N particles. For this purpose let the N -particle Hamiltonian be defined as

$$\mathscr{H}_N = \sum_a (H_0(x_a, -i\partial_{x_a}) - \mu) + \frac{1}{2} \sum_{a \neq b} \mathscr{V}(x_a - x_b) + \Delta ,$$

such that (36) reads

$$\hat{H} a^\dagger(x_1) \ldots a^\dagger(x_N) |\varnothing\rangle = \mathscr{H}_N a^\dagger(x_1) \ldots a^\dagger(x_N) |\varnothing\rangle \, .$$

By a similar set of steps as Section 3.1, it can be shown that $\sigma_{12} = \dfrac{\mathcal{N}}{2\pi K}$ where $\mathcal{N}$ is given by

$$\begin{aligned}
\mathcal{N} = & -\frac{1}{2A} \sum_{N=0,\ldots} \frac{1}{(2\pi)^{2N}} \sum_{b,c=1}^{N} \int d\omega \left(\prod_{a=1}^{N} d^2p_a \, d^2x_a \right) \epsilon^{jk} \\
& \mathrm{tr}\Big[G_W^{(N)}(\omega, \{p_a\}, \{x_a\}) \star \frac{\partial Q_W^{(N)}(\omega, \{p_a\}, \{x_a\})}{\partial \omega} \star \frac{\partial G_W^{(N)}(\omega, \{p_a\}, \{x_a\})}{\partial p_b^j} \star \frac{\partial Q_W^{(N)}(\omega, \{p_a\}, \{x_a\})}{\partial p_c^k} \Big] \\
= & -\frac{1}{2A} \frac{1}{(2\pi)^{2N_0}} \sum_{b,c=1}^{N_0} \int d\omega \left(\prod_{a=1}^{N_0} d^2p_a \, d^2x_a \right) \epsilon^{jk} \\
& \mathrm{tr}\Big[G_W^{(N_0)}(\omega, \{p_a\}, \{x_a\}) \star \frac{\partial Q_W^{(N_0)}(\omega, \{p_a\}, \{x_a\})}{\partial \omega} \star \frac{\partial G_W^{(N_0)}(\omega, \{p_a\}, \{x_a\})}{\partial p_b^j} \\
& \star \frac{\partial Q_W^{(N_0)}(\omega, \{p_a\}, \{x_a\})}{\partial p_c^k} \Big] ,
\end{aligned} \tag{41}$$

where $\star$ is given by (A22), while $Q_W^{(N)}(\omega, \{p_a\}, \{x_a\})$ and $G_W^{(N)}(\omega, \{p_a\}, \{x_a\})$ are functions of $2N + 1$ variables $\omega, p_1, x_1, \ldots, p_N, x_N$. These functions are the Weyl symbols of the corresponding operators:

$$Q_W^{(N)}(\omega, \{p_a\}, \{x_a\}) = \frac{1}{N!} \int \left(\prod_{a=1}^{N} dq_a e^{iq_a x_a} \right) \langle \{p_a + q_a/2\} | \hat{Q}^{(N)} | \{p_a - q_a/2\} \rangle \,, \tag{42}$$

where the multi-particle state $|\{p_a\}\rangle$ is defined above in (10), and similarly

$$G_W^{(N)}(\omega, \{p_a\}, \{x_a\}) = \frac{1}{N!} \int \left(\prod_{a=1}^{N} dq_a e^{iq_a x_a} \right) \langle \{p_a + q_a/2\} | \hat{G}^{(N)} | \{p_a - q_a/2\} \rangle \,,$$

where

$$\hat{Q}^{(N)} = (i\omega - \hat{H}) \hat{\Pi}_N, \qquad \hat{G}^{(N)} = \frac{1}{i\omega - \hat{H}} \hat{\Pi}_N,$$

and where

$$\hat{\Pi}_N = \frac{1}{N!} \int \left(\prod_{a=1}^{N} dp_a \right) |\{p_a\}\rangle \langle \{p_a\}| \tag{43}$$

is the projector onto N-particle states, with $\hat{H}$ given explicitly in (31), being the field-theoretical Hamiltonian. Its matrix elements $\langle \{p_a\} | \hat{H} | \{q_a\} \rangle$ are between states with N particles having momenta that belong to the sets $\{p_a\}$ and $\{q_a\}$. The proof that (38) is equivalent to (41) is the topic of Section 4.2. The proof that the given expression for $\mathcal{N}$ is a topological invariant closely follows the proof given in Section 3.2 for the case of different particles. The presence of identical particles results in extra factors $1/N!$ and an antisymmetric basis of states, but this does not affect the logic behind the derivation.

4.2. The Proof of the Statement That (38) Is Equivalent to (41)

The proof of this statement that Equation (38) is equivalent to Equation (41) proceeds along analogous lines to the argument in Section 3.1 as mentioned above.

From the definition of G_W given in (8) with (12), its derivative with respect to p_b^j is

$$\frac{\partial G_W(\omega, \{p_a\}, \{x_a\})}{\partial p_b^j} = \frac{1}{N!} \int \left(\prod_{a=1}^{N} dq_a \, e^{iq_a x_a} \right) \frac{\partial}{\partial p_b^j} \langle \{p_a + \tfrac{q_a}{2}\} | (i\omega - \hat{H})^{-1} | \{p_a - \tfrac{q_a}{2}\} \rangle \,. \tag{44}$$

For a one-particle state we have [48]

$$-i \frac{\partial}{\partial p_j} |p\rangle = \hat{x}^j |p\rangle, \qquad -i \frac{\partial}{\partial p_j} \langle p| = -\langle p| \hat{x}^j \,,$$

At the same time a multi-particle state has the form

$$|\{p\}\rangle = \frac{1}{\sqrt{N!}} \sum_{i_1 \ldots i_N} \epsilon^{i_1 \ldots i_N} |p_{i_1}\rangle \otimes \ldots \otimes |p_{i_N}\rangle = a_1^\dagger \ldots a_N^\dagger |\varnothing\rangle \,. \tag{45}$$

The action of an annihilation operator on a multi-particle state of the form (45) is

$$a_l |\{p\}\rangle = \frac{1}{\sqrt{N!}} \sum_{k=1 \ldots N} (-1)^{k+1} \sum_{i_1 \ldots i_N} \epsilon^{i_1 \ldots i_N} |p_{i_1}\rangle \otimes \ldots \otimes \langle p_l | p_{i_k} \rangle \otimes \ldots \otimes |p_{i_N}\rangle \,,$$

while a creation operator acts on (45) as

$$
\begin{aligned}
&a_l^\dagger|\{p\}\rangle = \frac{1}{\sqrt{(N+1)!}} \sum_{k=1\ldots N} (-1)^{k+1} \\
&\sum_{i_1\ldots i_N} \epsilon^{i_1\ldots i_N} |p_{i_1}\rangle \otimes \ldots \otimes |p_l\rangle \otimes |p_{i_k}\rangle \otimes \ldots \otimes |p_{i_N}\rangle \\
&= a_l^\dagger a_1^\dagger \ldots a_N^\dagger |\varnothing\rangle \,,
\end{aligned} \tag{46}
$$

(here $|p\rangle \otimes |\varnothing\rangle \equiv |p\rangle$). It follows that a derivative acts on (45) as

$$
\sum_{b=1\ldots N} \frac{\partial}{\partial p_b^j} |\{p\}\rangle = \frac{1}{\sqrt{N!}} \sum_{k=1\ldots N} \sum_{i_1\ldots i_N} \epsilon^{i_1\ldots i_N} \ldots \otimes \frac{\partial}{\partial p_{i_k}^j} |p_{i_k}\rangle \otimes \ldots \otimes |p_{i_N}\rangle = i\hat{x}^j |\{p\}\rangle \,.
$$

The last equality is established as follows:

$$
\begin{aligned}
i\hat{x}^j|\{p\}\rangle &= \frac{1}{\sqrt{N!}} \int dp a^\dagger(p) \left(-i\frac{\partial}{\partial p^j}\right) a(p) \sum_{i_1\ldots i_N} \epsilon^{i_1\ldots i_N} \ldots \otimes |p_{i_k}\rangle \otimes \ldots \otimes |p_{i_N}\rangle \\
= &\frac{1}{\sqrt{N!}} \int dp a^\dagger(p) \left(-i\frac{\partial}{\partial p^j}\right) \sum_{k=1\ldots N} (-1)^{k+1} \sum_{i_1\ldots i_N} \epsilon^{i_1\ldots i_N} \ldots \otimes \langle p|p_{i_k}\rangle \otimes \ldots \otimes |p_{i_N}\rangle \\
= &\frac{1}{\sqrt{N!}} \int dp a^\dagger(p) \sum_{k=1\ldots N} (-1)^{k+1} \sum_{i_1\ldots i_N} \epsilon^{i_1\ldots i_N} \ldots \otimes \left(i\frac{\partial}{\partial p_{i_k}^j}\right) \langle p|p_{i_k}\rangle \otimes \ldots \otimes |p_{i_N}\rangle \\
= &\frac{1}{\sqrt{N!}} \sum_{k=1\ldots N} (-1)^{k+1} \left(i\frac{\partial}{\partial p_{i_k}^j}\right) a^\dagger(p_{i_k}) \sum_{i_1\ldots i_N} \epsilon^{i_1\ldots i_N} \ldots \otimes |p_{i_{k-1}}\rangle \otimes |p_{i_{k+1}}\rangle \otimes \ldots \otimes |p_{i_N}\rangle \\
= &\frac{1}{\sqrt{N!}} \sum_{k=1\ldots N} \sum_{i_1\ldots i_N} \epsilon^{i_1\ldots i_N} \ldots \otimes \frac{\partial}{\partial p_{i_k}^j} |p_{i_k}\rangle \otimes \ldots \otimes |p_{i_N}\rangle \,.
\end{aligned} \tag{47}
$$

After summation over b Equation (44) becomes

$$
\begin{aligned}
\sum_{b=1\ldots N} \frac{\partial G_W(\omega,\{p_a\},\{x_a\})}{\partial p_b^j} = &\frac{1}{N!} \int \left(\prod_{a=1}^N dq_a\, e^{iq_a x_a}\right) \\
&i\langle\{p_a + \tfrac{q_a}{2}\}|\,[\,\hat{x}^j\,,\,(i\omega - \hat{H})^{-1}\,]\,|\{p_a - \tfrac{q_a}{2}\}\rangle \,.
\end{aligned} \tag{48}
$$

Based on the identity

$$
[\hat{B}, \hat{A}^{-1}]\hat{A} = -\hat{A}^{-1}[\hat{B}, \hat{A}] \qquad \Leftrightarrow \qquad [\hat{B}, \hat{A}^{-1}] = -\hat{A}^{-1}[\hat{B}, \hat{A}]\hat{A}^{-1} \,,
$$

then

$$
[\,\hat{x}^j\,,\,(i\omega - \hat{H})^{-1}\,] = -(i\omega - \hat{H})^{-1}[\,\hat{x}^j\,,\,(i\omega - \hat{H})\,]\,(i\omega - \hat{H})^{-1},
$$

or equivalently

$$
[\,\hat{x}^j\,,\,(i\omega - \hat{H})^{-1}\,] = -\hat{G}\,[\,\hat{x}^j\,,\,\hat{Q}\,]\,\hat{G} = \hat{G}\,[\,\hat{x}^j\,,\,\hat{H}\,]\,\hat{G} \,. \tag{49}
$$

By substituting (49) in (48) we obtain

$$
\begin{aligned}
\sum_{b=1\ldots N} \frac{\partial G_W(\omega,\{p_a\},\{x_a\})}{\partial p_b^j} = &\frac{1}{N!} \int \left(\prod_{a=1}^N dq_a\, e^{iq_a x_a}\right) i\langle\{p_a + \tfrac{q_a}{2}\}|\,\hat{G}\,[\,\hat{x}^j\,,\,\hat{H}\,]\,\hat{G}\,|\{p_a - \tfrac{q_a}{2}\}\rangle \\
= &i\Big(\hat{G}\,[\,\hat{x}^j\,,\,\hat{H}\,]\,\hat{G}\Big)_W \,,
\end{aligned} \tag{50}
$$

where the term $\big(\hat{G}\,[\,\hat{x}^j\,,\,\hat{H}\,]\,\hat{G}\big)_W$ in Equation (50) is defined to be the Weyl symbol of the operator $\hat{G}\,[\,\hat{x}^j\,,\,\hat{H}\,]\,\hat{G}$. Using a similar argument it may be derived that

$$\sum_{b=1...N} \frac{\partial Q_W(\omega,\{p_a\},\{x_a\})}{\partial p^j_b} = \frac{1}{N!}\int \left(\prod_{a=1}^N dq_a\, e^{iq_a x_a}\right) i\langle\{p_a+\tfrac{q_a}{2}\}|\,[\,\hat{x}^j\,,\,\hat{H}\,]\,|\{p_a-\tfrac{q_a}{2}\}\rangle$$
$$= i\Big([\,\hat{x}^j\,,\,\hat{H}\,]\Big)_W\,, \tag{51}$$

where the term $\big([\,\hat{x}^j\,,\,\hat{H}\,]\big)_W$ in Equation (51) is defined to be the Weyl symbol of the operator $[\,\hat{x}^j\,,\,\hat{H}\,]$.

We conclude that

$$\sum_{b=1...N} \frac{\partial G_W(\omega,\{p_a\},\{x_a\})}{\partial p^j_b} = -G_W(\omega,\{p_a\},\{x_a\})\star \sum_{b=1...N} \frac{\partial Q_W(\omega,\{p_a\},\{x_a\})}{\partial p^j_b}\star G_W(\omega,\{p_a\},\{x_a\})\,. \tag{52}$$

This means, in particular, that

$$\sum_{b=1...N} \frac{\partial 1_W(\omega,\{p_a\},\{x_a\})}{\partial p^j_b} = 0\,.$$

The last identity is verified by calculating the Weyl symbol of unity operator:

$$1_W(\omega,\{p_a\},\{x_a\}) = \frac{1}{N!}\int \left(\prod_{a=1}^N dq_a\, e^{iq_a x_a}\right)\langle\{p_a+\tfrac{q_a}{2}\}|\,\{p_a-\tfrac{q_a}{2}\}\rangle\,.$$

For its derivative we obtain

$$\sum_{b=1...N} \frac{\partial 1_W(\omega,\{p_a\},\{x_a\})}{\partial p^j_b} = \frac{1}{N!}\int \left(\prod_{a=1}^N dq_a\, e^{iq_a x_a}\right) i\langle\{p_a+\tfrac{q_a}{2}\}|\,[\,\hat{x}^j\,,\,\hat{1}\,]\,|\{p_a-\tfrac{q_a}{2}\}\rangle$$
$$= i\Big([\,\hat{x}^j\,,\,\hat{1}\,]\Big)_W = 0\,.$$

Now we substitute Equations (50) and (51) in (41) to obtain

$$\mathcal{N} = -\frac{1}{2A}\sum_{N=0,...}\frac{1}{(2\pi)^{2N}}\int d\omega \left(\prod_{a=1}^N d^2p_a\, d^2x_a\right)\epsilon^{jk}$$
$$\mathrm{tr}\Big[G^{(N)}_W(\omega,p_a,x_a)\star\frac{\partial Q^{(N)}_W(\omega,\{p_a\},\{x_a\})}{\partial\omega}\star\Big(\hat{G}\,[\,\hat{x}^j\,,\,\hat{H}\,]\,\hat{G}\Big)_W\star\Big([\,\hat{x}^k\,,\,\hat{H}\,]\Big)_W\Big]\,. \tag{53}$$

Next (A13) can be invoked to replace the star product of Weyl symbols with a single Weyl symbol corresponding to a product of operators as

$$\mathcal{N} = \frac{1}{2A}\sum_{N=0,...}\frac{1}{(2\pi)^{2N}}\int d\omega \left(\prod_{a=1}^N d^2p_a\, d^2x_a\right)\epsilon^{jk}$$
$$\left(\hat{G}^{(N)}\,\frac{\partial\hat{Q}^{(N)}}{\partial\omega}\,\hat{G}^{(N)}\,i[\hat{x}^j,\hat{H}]\,\hat{G}^{(N)}\,i[\hat{x}^k,\hat{H}]\right)_W\,.$$

Next we substitute the formal definition of a Weyl symbol given above in (9), (where in the case of varying particle number $|\{p_a\pm\frac{q_a}{2}\}\rangle = a^\dagger_{1\pm}\dots a^\dagger_{N\pm}\,|0\rangle$) to find

$$\mathcal{N} = \frac{1}{2A}\sum_{N=0,...}\frac{1}{(2\pi)^{2N}N!}\int d\omega \left(\prod_{a=1}^N d^2p_a\, d^2q_a\, d^2x_a\, e^{iq_a x_a}\right)$$
$$\epsilon^{jk}\,\langle\{p_a+\tfrac{q_a}{2}\}|\,\hat{G}^{(N)}\,\frac{\partial\hat{Q}^{(N)}}{\partial\omega}\,\hat{G}^{(N)}\,i[\hat{x}^j,\hat{H}]\,\hat{G}^{(N)}\,i[\hat{x}^k,\hat{H}]\,|\{p_a+\tfrac{q_a}{2}\}\rangle\,.$$

This can be expressed using a complete set of antisymmetric N fermion states, using the result proved in (A8). Even more, this particular result is valid for varying particle numbers, due to the fact that all states are deliberately expressed in terms of creation and annihilation operators. The implication, after invoking (A8), is that

$$\begin{aligned}\mathcal{N} =& \frac{1}{2A}\sum_{N=0,\ldots}\frac{1}{(2\pi)^{2N}}\frac{1}{N!^4}\int d\omega\left(\prod_{a=1}^{N} d^2p_a\, d^2q_a\, d^2x_a\, d^2p_{a,1}\, d^2p_{a,2}\, d^2p_{a,3}\, e^{iq_a x_a}\right)\\ &\epsilon^{jk}\, \langle\{p_a+\tfrac{q_a}{2}\}|\, \hat{G}^{(N)}\frac{\partial\hat{Q}^{(N)}}{\partial\omega}\hat{G}^{(N)}\,|\{p_{a,1}\}\rangle\,\langle\{p_{a,1}\}|\, i[\hat{x}^j,\hat{H}]\,|\{p_{a,2}\}\rangle\\ &\langle\{p_{a,2}\}|\,\hat{G}^{(N)}\,|\{p_{a,3}\}\rangle\,\langle\{p_{a,3}\}|\; i[\hat{x}^k,\hat{H}]\,|\{p_a-\tfrac{q_a}{2}\}\rangle\ .\end{aligned} \tag{54}$$

The x integrals yield a product of δ functions, namely one factor of $(2\pi)\delta(q_a)$ for each a, which render the q_a integrals trivial. To that extent (54) reduces to

$$\begin{aligned}\mathcal{N} =& \frac{1}{2A}\sum_{N=0,1,\ldots}\frac{1}{N!^4}\int d\omega\left(\prod_{a=1}^{N} d^2p_a\, d^2p_{a,1}\, d^2p_{a,2}\, d^2p_{a,3}\right)\epsilon^{jk}\,\mathrm{tr}\\ &\langle\{p_a\}|\,\hat{G}^{(N)}\frac{\partial\hat{Q}^{(N)}}{\partial\omega}\hat{G}^{(N)}\,|\{p_{a,1}\}\rangle\,\langle\{p_{a,1}\}|\, i[\hat{x}^j,\hat{H}]\,|\{p_{a,2}\}\rangle\\ &\langle\{p_{a,2}\}|\,\hat{G}^{(N)}\,|\{p_{a,3}\}\rangle\,\langle\{p_{a,3}\}|\; i[\hat{x}^k,\hat{H}]\,|\{p_a\}\rangle\ .\end{aligned}$$

By Substituting the explicit forms in (5), from which $\partial\hat{Q}^{(N)}/\partial\omega = i$, and inserting complete sets of eigenstates of $\hat{H}$ assuming that each set is discrete belonging to discrete eigenvalues, we obtain

$$\begin{aligned}&\mathcal{N} = -\frac{i}{2A}\sum_{N=0,1,\ldots}\frac{1}{N!^4}\int d\omega\left(\prod_{a=1}^{N} d^2p_a\, d^2p_{a,1}\, d^2p_{a,2}\, d^2p_{a,3}\right)\\ &\epsilon_{jk}\,\mathrm{tr}\sum_{E,E',E''}\langle\{p_a\}|\,\frac{1}{i\omega-\hat{H}}\hat{\Pi}_N\,|E\rangle\,\langle E|\,\frac{1}{i\omega-\hat{H}}\hat{\Pi}_N\,|E''\rangle\,\langle E''|\{p_{a,1}\}\rangle\,\langle\{p_{a,1}\}|\,[\hat{x}^j\ ,\ \hat{H}]\hat{\Pi}_N\,|\{p_{a,2}\}\rangle\\ &\langle\{p_{a,2}\}|\,\frac{1}{i\omega-\hat{H}}\hat{\Pi}_N\,|E'\rangle\,\langle E'|\{p_{a,3}\}\rangle\,\langle\{p_{a,3}\}|\,[\hat{x}^k\ ,\ \hat{H}]\hat{\Pi}_N\,|\{p_a\}\rangle\ .\end{aligned} \tag{55}$$

Here by $|E\rangle$ we denote the eigenstates of the Hamiltonian corresponding to the eigenvalue E. We assume here for simplicity that all eigenvalues are not degenerate. However, the extension to the case of degenerate eigenvalues is straightforward. For the next step of the argument it is necessary to show that the Hamiltonian $\hat{H}$ commutes with the projection operator onto N particle states, $\hat{\Pi}_N$. The form of the projection operator assumed is

$$\hat{\Pi}_N = \frac{1}{N!}\int dx_1\ldots dx_N\, a^\dagger(x_1)\ldots a^\dagger(x_N)\,|0\rangle\,\langle 0|\, a(x_N)\ldots a(x_1)\,. \tag{56}$$

This is consistent with the requirement that, given a state $|\psi\rangle = |x_1\ldots x_{N'}\rangle = a^\dagger(x_1)\ldots a^\dagger(x'_N)\,|0\rangle$, then $\hat{\Pi}_N\,|\psi\rangle = \delta_{NN'}\,|\psi\rangle$, which is easily shown to be true by invoking Theorem A1. Based on (56) and the form of the Hamiltonian in (31), then it can be shown that the two commute:

$$[\hat{H},\hat{\Pi}_N] = 0.$$

To show that they commute substitute their explicit forms:

$$
\begin{aligned}
[\hat{H},\hat{\Pi}_N] =& \int dX\,\mathscr{H}_0(X)\frac{1}{N!}\int dx_1\ldots dx_N\,[a^\dagger(X)a(X),a^\dagger(x_1)\ldots a^\dagger(x_N)\,|0\rangle\,\langle 0|\,a(x_N)\ldots a(x_1)] \\
&+\int dX\,dY\,\mathscr{V}(X-Y)\frac{1}{N!}\int dx_1\ldots dx_N \\
&[a^\dagger(X)a(X)a^\dagger(Y)a(Y),a^\dagger(x_1)\ldots a^\dagger(x_N)\,|0\rangle\,\langle 0|\,a(x_N)\ldots a(x_1)] \\
=& \int dX\,\mathscr{H}_0(X)\frac{1}{N!}\int dx_1\ldots dx_N\,\Big\{a^\dagger(X)a(X)a^\dagger(x_1)\ldots a^\dagger(x_N)\,|0\rangle\,\langle 0|\,a(x_N)\ldots a(x_1) \\
&-a^\dagger(x_1)\ldots a^\dagger(x_N)\,|0\rangle\,\langle 0|\,a(x_N)\ldots a(x_1)a^\dagger(X)a(X)\Big\} \\
&+\int dX\,dY\,\mathscr{V}(X-Y)\frac{1}{N!}\int dx_1\ldots dx_N \\
&\Big\{a^\dagger(X)a(X)a^\dagger(Y)a(Y)a^\dagger(x_1)\ldots a^\dagger(x_N)\,|0\rangle\,\langle 0|\,a(x_N)\ldots a(x_1) \\
&-a^\dagger(x_1)\ldots a^\dagger(x_N)\,|0\rangle\,\langle 0|\,a(x_N)\ldots a(x_1)a^\dagger(X)a(X)a^\dagger(Y)a(Y)\Big\}\,.
\end{aligned}
\tag{57}
$$

By substituting (A11)–(A12) into (57) and integrating over X and Y the result is

$$
\begin{aligned}
[\hat{H},\hat{\Pi}_N] =& \sum_{i=1}^{N}\mathscr{H}_0(x_i)\frac{1}{N!}\int dx_1\ldots dx_N \\
&\Big\{a^\dagger(x_1)\ldots a^\dagger(x_N)\,|0\rangle\,\langle 0|\,a(x_N)\ldots a(x_1)-a^\dagger(x_1)\ldots a^\dagger(x_N)\,|0\rangle\,\langle 0|\,a(x_N)\ldots a(x_1)\Big\} \\
&+\sum_{\substack{i,j=1\\ i\neq j}}^{N}\mathscr{V}(x_i-x_j)\frac{1}{N!}\int dx_1\ldots dx_N \\
&\Big\{a^\dagger(x_1)\ldots a^\dagger(x_N)\,|0\rangle\,\langle 0|\,a(x_N)\ldots a(x_1)-a^\dagger(x_1)\ldots a^\dagger(x_N)\,|0\rangle\,\langle 0|\,a(x_N)\ldots a(x_1)\Big\} \\
=&\,0\,.
\end{aligned}
$$

Even more, by their very definition of being eigenstates of $\hat{H}$, the inverse operator $(i\omega-\hat{H})^{-1}$ [denoted in (55) as $1/(i\omega-\hat{H})$] has the eigenvalue equation $(i\omega-\hat{H})^{-1}\,|E\rangle = \frac{1}{i\omega-E}\,|E\rangle$. Importantly, the energy eigenstates correspond to definite values of the particle number N. We denote this number by $N(E)$. Hence, this fact and the fact that $\hat{\Pi}_N$ and $\hat{H}$ commute, mean that (55) becomes

$$
\begin{aligned}
\mathcal{N} =& -\frac{i}{2A}\sum_{N=0,1,\ldots}\frac{1}{N!^4}\int d\omega\left(\prod_{a=1}^{N}d^2p_a\,d^2p_{a,1}\,d^2p_{a,2}\,d^2p_{a,3}\right)\sum_{E,E',E''}\frac{1}{i\omega-E}\,\frac{1}{i\omega-E'}\,\frac{1}{i\omega-E''} \\
&\epsilon_{jk}\,\mathrm{tr}\,\langle\{p_a\}|E\rangle\,\langle E|E''\rangle\,\langle E''|\{p_{a,1}\}\rangle\,\langle\{p_{a,1\}}|\,[\hat{x}^j\,,\,\hat{H}]\,|\{p_{a,2}\}\rangle \\
&\langle\{p_{a,2}\}|E'\rangle\,\langle E'|\{p_{a,3}\}\rangle\,\langle\{p_{a,3}\}|\,[\hat{x}^k\,,\,\hat{H}]\,|\{p_a\}\rangle\ ,
\end{aligned}
$$

and since the trace is unaffected by a change in order of inner products, this can equally be written as

$$
\begin{aligned}
\mathcal{N} =& -\frac{i}{2A}\sum_{N=0,1,\ldots}\frac{1}{N!^4}\int d\omega\left(\prod_{a=1}^{N}d^2p_a\,d^2p_{a,1}\,d^2p_{a,2}\,d^2p_{a,3}\right)\sum_{E,E',E''}\frac{1}{i\omega-E}\,\frac{1}{i\omega-E'}\,\frac{1}{i\omega-E''} \\
&\mathrm{tr}\,\epsilon_{jk}\,\langle E|E''\rangle\,\langle E''|\{p_{a,1\}}\rangle\,\langle\{p_{a,1}\}|\,[\hat{x}^j\,,\,\hat{H}]\,|\{p_{a,2}\}\rangle \\
&\langle\{p_{a,2}\}|E'\rangle\,\langle E'|\{p_{a,3}\}\rangle\,\langle\{p_{a,3}\}|\,[\hat{x}^k\,,\,\hat{H}]\,|\{p_a\}\rangle\,\langle p|E\rangle\ .
\end{aligned}
$$

The integrals simplify through the identities in (A8), and the relation $\langle E|E''\rangle = \delta_{E,E''}$ eliminates the sum over E''. Ergo

$$\mathcal{N} = -\frac{i}{2A}\sum_{N=0,1,\ldots}\sum_{E,E'}\int d\omega \frac{1}{(i\omega - E)^2(i\omega - E')}\operatorname{tr}\epsilon_{jk}\langle E|\,[\hat{x}^j\,,\,\hat{H}]\,|E'\rangle\,\langle E'|\,[\hat{x}^k\,,\,\hat{H}]\,|E\rangle\ .$$

Here, both states with energies E and E' correspond to the same value of particle number $N = N(E)$. The ω integral over the integration range $l = [-\infty,\infty]$, is evaluated by deforming l to the closed contour being a semi circle, C in the upper complex plane. Two possibilities arise: (i) C encloses both of the points iE and iE' on the imaginary axis if $E > 0$ and $E' > 0$, or (ii) C encloses only one of them, if say $E > 0$ $E' < 0$, or the converse. If $E < 0$ and $E' < 0$, integrating over the variable $-\omega$ (minus ω) instead, produces the same integral described in (i) with the same contour C. In both cases the integral can be evaluated using Cauchy's integral formula. For case (i) the integral vanishes, but for case (ii) the contribution is non-zero. The result is

$$\mathcal{N} = \frac{-i}{A}\sum_{N=0,1,\ldots} 2\pi\sum_{E,E'}\frac{\theta(E)\theta(-E')}{(E-E')^2}\epsilon^{jk}\operatorname{tr}\langle E|\,[\hat{x}^j\,,\,\hat{H}]\,|E'\rangle\,\langle E'|\,\hat{x}^k\,,\,\hat{H}]\,|E\rangle\ . \tag{58}$$

As the term $\theta(E)\theta(-E')$ itself indicates, the integral was solved assuming that $E > 0$, $E' < 0$. Had the converse been assumed, precisely the same formula in the form (58) would hold, since interchanging E and E' (noting the trace operator in front) leaves the expression unaltered.

According to our choice of value for the constant term Δ entering the field Hamiltonian of Equation (31), only the ground states have negative energy $E' < 0$. Let the ground states be denoted by $|0_k\rangle$, $k = 1,\ldots,K$ instead, and let the eigenvalue of $\hat{H}$ that they belong to be denoted by E_0, such that $\hat{H}\,|0_k\rangle = E_0\,|0_k\rangle$. Accordingly (58) reads

$$\mathcal{N} = \frac{-i}{A}\sum_{k=1}^{K} 2\pi\sum_{E\neq E_0}\frac{1}{(E-E_0)^2}\epsilon_{ij}\operatorname{tr}\langle 0_k|\,[\hat{x}^j\,,\,\hat{H}]\,|E\rangle\,\langle E|\,\hat{x}^i\,,\,\hat{H}]\,|0_k\rangle\ ,$$

where in the last step the two inner products inside the trace were swapped, since this is the order that the Hall conductivity is conventionally written. The eigenstates $|E\rangle$ and can be identified with $|n\rangle$ in the coordinate representation defined above in (39) and (40):

$$\mathcal{N} = -\frac{2\pi i}{A}\sum_{k=1}^{K}\sum_{n}\frac{1}{(E-E_0)^2}\epsilon_{ij}\operatorname{tr}\langle 0_k|\,[\hat{x}^j\,,\,\hat{H}]\,|n\rangle\,\langle n|\,\hat{x}^i\,,\,\hat{H}]\,|0_k\rangle\ . \tag{59}$$

Equation (59) is precisely analogous to the result in (38).

This completes the proof. It is clear in the sum over N in Equation (41) that only the term with $N = N_0$ remains. This is a direct consequence of our choice for the value of Δ, according to which only the ground state has negative energy.

4.3. The Case of a Non-Interacting System

The aim of this subsection is to show that the expression derived in Equation (14) is equivalent to an analogous formula but with two-point Green functions instead of N-point Green functions.

It was shown in Section 4.2 that the expression for the Hall conductivity (53), viz

$$
\begin{aligned}
\mathcal{N} =& \frac{1}{2A} \sum_{N=0,\ldots} \frac{1}{(2\pi)^{2N}} \sum_{b,c=1}^{N} \int d\omega \left(\prod_{a=1}^{N} d^2 p_a \, d^2 x_a \right) \\
& \epsilon^{jk} \operatorname{tr} \Bigg[G_W^{(N)}(\omega, \{p_a\}, \{x_a\}) \star \frac{\partial Q_W^{(N)}(\omega, \{p_a\}, \{x_a\})}{\partial \omega} \\
& \star G_W^{(N)}(\omega, \{p_a\}, \{x_a\}) \star \frac{\partial Q_W^{(N)}(\omega, \{p_a\}, \{x_a\})}{\partial p_b^j} \\
& \star G_W^{(N)}(\omega, \{p_a\}, \{x_a\}) \star \frac{\partial Q_W^{(N)}(\omega, \{p_a\}, \{x_a\})}{\partial p_c^k} \Bigg] ,
\end{aligned} \tag{60}
$$

with $G^{(N)}$ and $Q_W^{(n)}$ given by (42) and (43) and $\hat{H}$ given by (31) without the interaction term, is equivalent to (58). To emphasize that the Hamiltonian in (58) is the field theoretical Hamiltonian, let it be written in the notation

$$
\mathbb{H} = \sum_q a_q^\dagger a_q \mathcal{E}_q \,, \tag{61}
$$

and let the field theoretical position operator be denoted by

$$
\mathbb{X}_j = \sum_{k,n} a_k X_{kn} a^\dagger{}_n \,. \tag{62}
$$

The reader may consult [48] for an explanation of the origins of the right-hand side of Equation (62).

Here by a_q we denote the annihilation operator corresponding to the one particle state with energy $\mathcal{E}_q$. For convenience in this section we do not define the field theoretical Hamiltonian with a chemical potential subtracted from $\mathcal{E}_q$. This redefinition does not change the expressions given below. At any rate in the absence of interactions, we set $\Delta = 0$. The conductivity in (60) was shown in Section 4.2 to be equivalent to (58), which in the notation of (61) and (62) is given by

$$
\begin{aligned}
\mathcal{N} &= -\frac{2\pi i}{A} \sum_{n\neq 0} \frac{1}{(E - E_0)^2} \epsilon_{ij} \operatorname{tr} \langle\langle 0 | [\mathbb{X}^j \,, \, \mathbb{H}] | n \rangle\rangle \langle\langle n | \mathbb{X}^i \,, \, \mathbb{H}] | 0 \rangle\rangle \,. \\
&= -\frac{2\pi i}{A} \sum_{n\neq 0} \frac{\langle\langle 0 | [\mathbb{X}_2, \mathbb{H}] | n \rangle\rangle \langle\langle n | [\mathbb{X}_1, \mathbb{H}] | 0 \rangle\rangle - \langle\langle 0 | [\mathbb{X}_1, \mathbb{H}] | n \rangle\rangle \langle\langle n | [\mathbb{X}_2, \mathbb{H}] | 0 \rangle\rangle}{(E_n - E_0)^2} \,,
\end{aligned} \tag{63}
$$

where here, to distinguish from single particle bra and ket vectors, $|0\rangle\rangle$ denotes the non-degenerate multi-particle ground state

$$
|0\rangle\rangle = a_1^\dagger \ldots a_N^\dagger |\varnothing\rangle\rangle \,,
$$

where $|\varnothing\rangle\rangle$ denotes the true vacuum and

$$
a_k^\dagger = \int dx \; \psi_k(x) a^\dagger(x) \,.
$$

Here $\psi_k(x)$ is the wave function of the kth one-particle state. We enumerate one-particle states in such a way that $\mathcal{E}_1 \leq \mathcal{E}_2 \leq \ldots \leq \mathcal{E}_N < \mu$, where μ is the chemical potential. By $|n\rangle\rangle$ we denote the excited multi-particle states that have the same number of particles, but which have total energy larger than that of the ground state. Using the anticommutation relations in (11),

$$\left[\sum_{k,n} a_k^\dagger X_{kn} a_n, \sum_q a_q^\dagger \mathcal{E}_q a_q\right] = \sum_{k,n}\sum_q a_k^\dagger X_{kn} a_n a_q^\dagger \mathcal{E}_q a_q - a_q^\dagger \mathcal{E}_q a_q a_k^\dagger X_{kn} a_n$$
$$= \sum_{k,n}\sum_q X_{kn}\mathcal{E}_q \left(a_k^\dagger a_n a_q^\dagger a_q - a_q^\dagger a_q a_k^\dagger a_n\right)$$
$$= \sum_{k,n}\sum_q X_{kn}\mathcal{E}_q \left(a_k^\dagger \delta_{nq} a_q - a_k^\dagger a_q^\dagger a_n a_q - a_q^\dagger a_q a_k^\dagger a_n\right)$$
$$= \sum_{k,n}\sum_q X_{kn}\mathcal{E}_q \left(a_k^\dagger \delta_{nq} a_q + a_q^\dagger a_k^\dagger a_n a_q - a_q^\dagger a_q a_k^\dagger a_n\right)$$
$$= \sum_{k,n}\sum_q X_{kn}\mathcal{E}_q \left(a_k^\dagger \delta_{nq} a_q - a_q^\dagger a_k^\dagger a_q a_n - a_q^\dagger a_q a_k^\dagger a_n\right)$$
$$= \sum_{k,n}\sum_q X_{kn}\mathcal{E}_q \left(a_k^\dagger \delta_{nq} a_q - a_q^\dagger \delta_{kq} a_n + a_q^\dagger a_q a_k^\dagger a_n - a_q^\dagger a_q a_k^\dagger a_n\right)$$
$$= \sum_{k,n}\sum_q X_{kn}\mathcal{E}_q \left(a_k^\dagger \delta_{nq} a_q - a_q^\dagger \delta_{kq} a_n\right)$$
$$= \sum_{k,n}\left(X_{kn}\mathcal{E}_n a_k^\dagger a_n - X_{kn}\mathcal{E}_k a_k^\dagger a_n\right) .$$

The right-hand side is precisely the matrix elements of the commutator $[\hat{x}, \hat{H}]$, where $\hat{x}$ and $\hat{H}$ are the ordinary one-particle position and Hamiltonian operators. Hence

$$\left[\sum_{k,n} a_k^\dagger X_{kn} a_n, \sum_q a_q^\dagger \mathcal{E}_q a_q\right] = \sum_{k,n} a_k^\dagger [\hat{x}, \hat{H}]_{kn} a_n . \tag{64}$$

After combining Equations (61), (62) and (64) it follows that

$$[\mathbb{X}_i, \mathbb{H}] = \sum_{k,n} a_k^\dagger [\hat{x}_i, \hat{H}]_{kn} a_n . \tag{65}$$

With the result (65) an expression may be derived for $\langle\!\langle 0_k | [\mathbb{X}_i, \mathbb{H}] | n \rangle\!\rangle$. The ground state $|0_k\rangle\!\rangle$ consisting of N particles has the form

$$|0\rangle\!\rangle = a_1^\dagger \dots a_N^\dagger \, |\varnothing\rangle \ . \tag{66}$$

The state $|n\rangle\!\rangle$ is taken to be the state that differs from the ground state (66) by one out of the N particles, say particle j ($j = 1, \dots N$), which gets excited and jumps to a state of higher energy. Correspondingly the creation operator $a_j^\dagger$ is replaced with $a_l^\dagger$, $l = N+1, \dots, \infty$, such that $|n\rangle\!\rangle$ has the form

$$|n\rangle\!\rangle = a_1^\dagger \dots a_{j-1}^\dagger a_{j+1}^\dagger \dots a_N^\dagger a_l^\dagger \, |\varnothing\rangle =: |l, j\rangle\!\rangle \, , \qquad (j = 1, \dots, N \, , \quad l \geq N+1) \, . \tag{67}$$

Then, by (65), (66) and (67),

$$\langle\!\langle 0 | [\mathbb{X}_i, \mathbb{H}] | n \rangle\!\rangle = \langle\varnothing| \, a_N \dots a_1 \sum_{k,n} a_k^\dagger [\hat{x}_i, \hat{H}]_{kn} a_n a_1^\dagger \dots a_{j-1}^\dagger a_{j+1}^\dagger \dots a_N^\dagger a_l^\dagger \, |\varnothing\rangle$$
$$= \sum_{k,n} [\hat{x}_i, \hat{H}]_{kn} \, \langle\varnothing| \, a_N \dots a_1 a_k^\dagger a_n a_1^\dagger \dots a_{j-1}^\dagger a_{j+1}^\dagger \dots a_N^\dagger a_l^\dagger \, |\varnothing\rangle$$
$$= \sum_{k,n} \langle k | \, [\hat{x}_i, \hat{H}] \, | n \rangle \, \langle\varnothing| \, a_N \dots a_1 a_k^\dagger a_n a_1^\dagger \dots a_{j-1}^\dagger a_{j+1}^\dagger \dots a_N^\dagger a_l^\dagger \, |\varnothing\rangle \ .$$

The operator $a_l^\dagger$ may be anticommuted to the left past the $N-1$ operators standing in front of it to obtain

$$\langle\!\langle 0|[\mathbb{X}_i,\mathbb{H}]|n\rangle\!\rangle = (-1)^{N-1}\sum_{k,n}\langle k|\,[\hat{x}_i,\hat{H}]\,|n\rangle\,\langle\varnothing|\,a_N\dots a_1 a_k^\dagger a_n a_l^\dagger a_1^\dagger\dots a_{j-1}^\dagger a_{j+1}^\dagger\dots a_N^\dagger\,|\varnothing\rangle\ . \tag{68}$$

Note two important observations of the inner product in (68). First, there stands one annihilation operator a_j to the right of the bra of the vacuum state $\langle 0_k|$, so for it not to vanish there must be present one creation operator $a_j^\dagger$. Since $j = 1,\dots,N$ and $l \geq N+1$, then $a_l^\dagger$ is never equal to $a_j^\dagger$ but $a_k^\dagger$ is equal to $a_j^\dagger$ corresponding to the $k = j$ term in the sum. Secondly, the creation operator $a_l^\dagger$ will act on the vacuum to create one particle in the state l. Consequently the inner product will vanish without the presence of one annihilation operator a_l, which forces $a_n = a_l\delta_{nl}$. Putting all this together, the non-vanishing contribution to (68) is found to be

$$\begin{aligned}\langle\!\langle 0|[\mathbb{X}_i,\mathbb{H}]|j,l\rangle\!\rangle =& (-1)^{N-1}\sum_{k,n}\delta_{k,j}\delta_{n,l}\,\langle k|\,[\hat{x}_i,\hat{H}]\,|n\rangle \\ &\langle\varnothing|\,a_N\dots a_1 a_k^\dagger a_n a_l^\dagger a_1^\dagger\dots a_{j-1}^\dagger a_{j+1}^\dagger\dots a_N^\dagger\,|\varnothing\rangle \\ =& (-1)^{N-1}\,\langle j|\,[\hat{x}_i,\hat{H}]\,|l\rangle\,\langle\varnothing|\,a_N\dots a_1 a_j^\dagger a_l a_l^\dagger a_1^\dagger\dots a_{j-1}^\dagger a_{j+1}^\dagger\dots a_N^\dagger\,|\varnothing\rangle\ .\end{aligned}$$

Further, since $l \neq j$ ($l \geq N+1$ and $j = 1,\dots N$), $a_j^\dagger$ and a_l can be anticommuted past each other to yield

$$\langle\!\langle 0|[\mathbb{X}_i,\mathbb{H}]|j,l\rangle\!\rangle = (-1)^{N}\,\langle j|\,[\hat{x}_i,\hat{H}]\,|l\rangle\,\langle\varnothing|\,a_N\dots a_1 a_l a_j^\dagger a_l^\dagger a_1^\dagger\dots a_{j-1}^\dagger a_{j+1}^\dagger\dots a_N^\dagger\,|\varnothing\rangle\ ,$$

and subsequently $a_j^\dagger$ and $a_l^\dagger$ may be anticommuted past each other to bring it to the form

$$\begin{aligned}\langle\!\langle 0|[\mathbb{X}_i,\mathbb{H}]|j,l\rangle\!\rangle =& (-1)^{N+1}\,\langle j|\,[\hat{x}_i,\hat{H}]\,|l\rangle\,\langle\varnothing|\,a_N\dots a_1 a_l a_l^\dagger a_j^\dagger a_1^\dagger\dots a_{j-1}^\dagger a_{j+1}^\dagger\dots a_N^\dagger\,|\varnothing\rangle \\ =& (-1)^{N+j}\,\langle j|\,[\hat{x}_i,\hat{H}]\,|l\rangle\,\langle\varnothing|\,a_N\dots a_1 a_l a_l^\dagger a_1^\dagger\dots a_{j-1}^\dagger a_j^\dagger a_{j+1}^\dagger\dots a_N^\dagger\,|\varnothing\rangle\ ,\end{aligned} \tag{69}$$

where in the last step $a_j^\dagger$ was anticommuted past operators to appear between $a_{j-1}^\dagger$ and $a_{j+1}^\dagger$.

The inner product on the right-hand side of (69) comprises an inner product of $N+1$ different creation operators acting on the ket of the vacuum $|\varnothing\rangle$, with $N+1$ corresponding annihilation operators acting on the bra of the vacuum $\langle\varnothing|$. It can be shown by induction to equal unity, namely

$$\langle\varnothing|\,a_N\dots a_1 a_1^\dagger\dots a_N^\dagger\,|\varnothing\rangle = 1\ . \tag{70}$$

To prove (70) by induction, we start with the $n = 1$ case:

$$\langle\varnothing|\,a_1 a_1^\dagger\,|\varnothing\rangle = \langle\varnothing|\,\{a_1,a_1^\dagger\}\,|\varnothing\rangle - \langle\varnothing|\,a_1^\dagger a_1\,|\varnothing\rangle = \langle\varnothing|\,1\,|\varnothing\rangle - \langle\varnothing|\,0\,|\varnothing\rangle = \langle\varnothing|\varnothing\rangle$$

by (11). Hence our claim is true for the $n = 1$ case. For the inductive step, we assume that it is true for $n = N$, then we write down (70) for the $n = N+1$ case, viz

$$\langle\varnothing|\,a_{N+1}a_N\dots a_1 a_1^\dagger\dots a_N^\dagger a_{N+1}^\dagger\,|\varnothing\rangle\ .$$

By anticommuting a_{N+1} to the right N places and anticommuting $a_{N+1}^\dagger$ to the left N places, we bring it to the form

$$\begin{aligned}\langle\varnothing|\,a_N\dots a_1 a_{N+1}a_{N+1}^\dagger a_1^\dagger\dots a_N^\dagger\,|\varnothing\rangle =& \langle\varnothing|\,a_N\dots a_1\{a_{N+1},a_{N+1}^\dagger\}a_1^\dagger\dots a_N^\dagger\,|\varnothing\rangle \\ &- \langle\varnothing|\,a_N\dots a_1 a_{N+1}^\dagger a_{N+1} a_1^\dagger\dots a_N^\dagger\,|\varnothing\rangle\ ,\end{aligned}$$

then we invoke (11) on the 1st term, and in the second term we anticommute operators $a_{N+1}^\dagger$ and a_{N+1} to obtain

$$\langle\varnothing|\, a_N \dots a_1 a_{N+1} a^\dagger_{N+1} a^\dagger_1 \dots a^\dagger_N \,|\varnothing\rangle = \langle\varnothing|\, a_N \dots a_1 a^\dagger_1 \dots a^\dagger_N \,|\varnothing\rangle$$
$$- \langle\varnothing|\, a^\dagger_{N+1} a_N \dots a_1 a^\dagger_1 \dots a^\dagger_N a_{N+1} \,|\varnothing\rangle \,.$$

The second term vanishes and the 1st term is the $n = N$ case, which is unity by the inductive hypothesis. Therefore, $\langle\varnothing|\, a_N \dots a_1 a_{N+1} a^\dagger_{N+1} a^\dagger_1 \dots a^\dagger_N \,|\varnothing\rangle = 1$ and the claim is proven.

On the basis of (70) that was just proven, (69) simplifies to

$$\langle\!\langle 0|[\mathbb{X}_i, \mathbb{H}]|j,l\rangle\!\rangle = (-1)^{N+j} \langle j|\, [\hat{x}_i, \hat{H}] \,|l\rangle \;. \tag{71}$$

Equation (71) was established on the basis that the state $\langle\!\langle n|$ comprises a single particle that gets excited from the ground state to a higher state. It can be shown that the analogous state in which two or more particles are in excited states do not contribute. Consider the case where the state $|n\rangle\!\rangle$ is the state that differs from the ground state (66) by two particles one out of the N particles, say particles j_1 and j_2 ($j_1, j_2 = 1, \dots N$), which get excited and jump to a state of higher energy. Correspondingly the creation operators $a^\dagger_{j_1}, a^\dagger_{j_2}$ are replaced with $a^\dagger_{l_1}, a^\dagger_{l_2}$ respectively ($l_1, l_2 = N+1, \dots, \infty$), such that $|n\rangle\!\rangle$ has the form

$$|n\rangle\!\rangle = a^\dagger_1 \dots a^\dagger_{j_1-1} a^\dagger_{j_1+1} \dots a^\dagger_{j_2-1} a^\dagger_{j_2+1} \dots a^\dagger_N a^\dagger_{l_1} a^\dagger_{l_2} \,|\varnothing\rangle =: |l_1, l_2; j_1, j_2\rangle\!\rangle \,, \tag{72}$$

where in (72), $j_1, j_2 = 1, \dots, N$ and $l_1, l_2 \geq N+1$. Then by (65), (72) and (67),

$$\begin{aligned}
\langle\!\langle 0|[\mathbb{X}_i, \mathbb{H}]|n\rangle\!\rangle &= \langle\varnothing|\, a_N \dots a_1 \sum_{k,n} a^\dagger_k [\hat{x}_i, \hat{H}]_{kn} a_n a^\dagger_1 \dots a^\dagger_{j_1-1} a^\dagger_{j_1+1} \dots a^\dagger_{j_2-1} a^\dagger_{j_2+1} \dots a^\dagger_N a^\dagger_{l_1} a^\dagger_{l_2} \,|\varnothing\rangle \\
&= \sum_{k,n} [\hat{x}_i, \hat{H}]_{kn} \langle\varnothing|\, a_N \dots a_1 a^\dagger_k a_n a^\dagger_1 \dots a^\dagger_{j_1-1} a^\dagger_{j_1+1} \dots a^\dagger_{j_2-1} a^\dagger_{j_2+1} \dots a^\dagger_N a^\dagger_{l_1} a^\dagger_{l_2} \,|\varnothing\rangle \\
&= \sum_{k,n} \langle k|\, [\hat{x}_i, \hat{H}] \,|n\rangle \\
&\quad \langle\varnothing|\, a_N \dots a_1 a^\dagger_k a_n a^\dagger_1 \dots a^\dagger_{j_1-1} a^\dagger_{j_1+1} \dots a^\dagger_{j_2-1} a^\dagger_{j_2+1} \dots a^\dagger_N a^\dagger_{l_1} a^\dagger_{l_2} \,|\varnothing\rangle \;.
\end{aligned}$$

In this expression $a^\dagger_k a_n$ can be replaced with $\{a^\dagger_k, a_n\} - a_n a^\dagger_k = \delta_{nk} - a_n a^\dagger_k$ by (11) to obtain

$$\begin{aligned}
\langle\!\langle 0|[\mathbb{X}_i, \mathbb{H}]|n\rangle\!\rangle = &\sum_{k,n} \langle k|\, [\hat{x}_i, \hat{H}] \,|n\rangle\, \delta_{nk} \\
&\langle\varnothing|\, a_N \dots a_1 a^\dagger_1 \dots a^\dagger_{j_1-1} a^\dagger_{j_1+1} \dots a^\dagger_{j_2-1} a^\dagger_{j_2+1} \dots a^\dagger_N a^\dagger_{l_1} a^\dagger_{l_2} \,|\varnothing\rangle \\
&- \sum_{k,n} \langle k|\, [\hat{x}_i, \hat{H}] \,|n\rangle \\
&\langle\varnothing|\, a_N \dots a_1 a_n a^\dagger_k a^\dagger_1 \dots a^\dagger_{j_1-1} a^\dagger_{j_1+1} \dots a^\dagger_{j_2-1} a^\dagger_{j_2+1} \dots a^\dagger_N a^\dagger_{l_1} a^\dagger_{l_2} \,|\varnothing\rangle \;.
\end{aligned}$$

The first inner product vanishes. The ket on the right comprises two particles corresponding to the creation operators $a^\dagger_{l_1}, a^\dagger_{l_2}$ ($l_1, l_2 \geq N+1$), but there are no such particles in the bra on the right, as there are no corresponding annihilation operators a_{l_1}, a_{l_2}. The second inner product vanishes for similar reasons. Say that $a^\dagger_k$ is identified with $a^\dagger_{j_1}$ and a_n with a_{l_1}. There would still be a particle created by $a^\dagger_{l_2}$ in the left-hand ket with no corresponding particle in the bra on the right, and there would still be a particle annihilated by a_{j_2} in the right-hand bra with no corresponding particle created in the ket on the left. The argument holds even if the particles are permuted. In conclusion, the inner product between the ground state and states with two excited particles vanishes. Furthermore, an analogous argument shows that the inner product between the ground state and states with more than two excited particles vanishes.

Finally, (71) may be substituted in (63). The sum over states labeled by n becomes a sum over j, namely a sum over single particles that jump from the ground state to a higher excitation. The difference in energy between the state $|n\rangle$ and the ground state $|0_k\rangle$ (denoted by $E_n - E_0$ in the denominator), is equal to the difference in energy between particle j in an excited state and in the ground state, which shall be denoted $E_l - E_j$. The resulting expression for the topological invariant is

$$\mathcal{N} = -\frac{2\pi i}{A} \sum_{l=N+1}^{\infty} \sum_{j=1}^{N} \frac{\langle j| [\hat{x}_1, \hat{H}] |l\rangle \langle l| [\hat{x}_2, \hat{H}] |j\rangle - \langle j| [\hat{x}_2, \hat{H}] |l\rangle \langle l| [\hat{x}_1, \hat{H}] |j\rangle}{(E_l - E_j)^2},$$

and for the conductivity itself

$$\sigma = -\frac{i}{A} \sum_{l=N+1}^{\infty} \sum_{j=1}^{N} \frac{\langle j| [\hat{x}_1, \hat{H}] |l\rangle \langle l| [\hat{x}_2, \hat{H}] |j\rangle - \langle j| [\hat{x}_2, \hat{H}] |l\rangle \langle l| [\hat{x}_1, \hat{H}] |j\rangle}{(E_l - E_j)^2}. \tag{73}$$

In ref. [14] the expression in (73) has been shown to be equal to

$$\begin{aligned}\mathcal{N} =& \frac{1}{2A} \frac{1}{(2\pi)^2} \int d\omega d^2p\, d^2x \\ & \epsilon^{jk} \operatorname{tr}\left[G_W^{(1)}(\omega, p, x) \star \frac{\partial Q_W^{(1)}(\omega, p, x)}{\partial \omega} \star G_W^{(1)}(\omega, p, x) \star \frac{\partial Q_W^{(1)}(\omega, p, x)}{\partial p^j} \right. \\ & \left. \star\, G_W^{(1)}(\omega, p, x) \star \frac{\partial Q_W^{(1)}(\omega, p, x)}{\partial p^k} \right].\end{aligned}$$

5. Conclusions

To conclude, in this paper we propose a topological description of the fractional Hall effect. The corresponding conductivity (averaged over the system area) in units of $e^2/\hbar$ has the form

$$\sigma_H = \frac{1}{2\pi} \frac{\mathcal{N}}{K},$$

where K is the degeneracy of the ground state while $\mathcal{N}$ is the topological invariant composed of the multi-particle Green functions. While the original expression for $\mathcal{N}$ contains a summation over all possible numbers of fermionic legs, in actual fact only the term with $2N_0$ legs contributes, where N_0 is the number of electrons in the ground state of the system.

The expression for the topological invariant $\mathcal{N}$ contains a generalization of the Wigner transformation of multi-leg Green functions, and a generalization of Moyal product. The form of this expression resembles that of the integer Hall effect. The main difference between the two is that now $\mathcal{N}$ is expressed in terms of multi-particle Green functions instead of one-particle Green functions. We have demonstrated that in the absence of interactions, our expression for the conductivity reduces to that of the IQHE (in terms of one-particle Green functions). In the general case of an interacting system, the value of $\mathcal{N}$ is discrete and is likely also to be given by an integer. This result is an alternative proof of the topological nature of the FQHE. Moreover, unlike the topological expression in ref. [22], our result is valid for the case of varying external fields. As well as the expression in ref. [22], our expressions in Equations (3) and (4) are of no use for practical calculations of the FQHE conductivity. Nevertheless they stand as a rigorous proof of the robustness of $\mathcal{N}$ with respect to smooth modification of the system.

Topological invariance of the fractional QHE conductivity means that any smooth modification of the system cannot bring it to a state with a different value for the conductivity, unless a phase transition is encountered. In simpler terms the transition between different FQHE states cannot be a crossover. This is a direct consequence implied by our result.

It has been proven (see, for example, [16]) that perturbatively inter-electron interactions cannot drive transitions of the system from an IQHE to a FQHE state. This means that the FQHE is essentially a non-perturbative phenomenon. Non-perturbative contributions to the multi-leg Green functions are responsible for the fractional QHE conductivity. It would be insightful to formulate phenomenological models of interacting electronic systems that directly yield expressions for the corresponding multi-leg Green functions. Values of the QHE conductivity could then be calculated directly within these phenomenological models. How to relate such models to the popular concept of composite fermions [46,47] is worthy of investigation. This is beyond the scope of this paper.

As commented above, there are two factors entering the expression for QHE conductivity: the degeneracy of the ground state and a topological invariant composed of Green functions. The latter has a topological interpretation. An investigation into the relationship between our invariant to standard algebraic topology is an important task for the future. The degeneracy of the ground state, at first sight, has no direct geometrical or topological interpretation. However, this issue also deserves attention.

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Appendix A. Multi Particle States

Appendix A.1. Fermion Creation/Annihilation Operators and One Fermion States

A single-particle state $|p\rangle$ is a momentum eigenstate with momentum p, constructed by acting on $|0\rangle$ with a creation operator as

$$a^\dagger(p)\,|0\rangle = |p\rangle \ .$$

Consistent with this definition the operator $\int dp\,|p\rangle\,\langle p|$ behaves like an identity operator when it acts on one-particle states:

$$\begin{aligned}\left(\int dp\ \ |p\rangle\,\langle p|\right)|q\rangle =& \int dp\ a^\dagger(p)\,|0\rangle\,\langle 0|\,a(p)a^\dagger(q)\,|0\rangle \\ =& \int dp\ a^\dagger(p)\{a(p),a^\dagger(q)\}\,|0\rangle = \int dp\ a^\dagger(p)\ \delta(p-q)\,|0\rangle \\ =& a^\dagger(q)\,|0\rangle = |q\rangle \ .\end{aligned}$$

Appendix A.2. Two Fermion States

A similar reasoning can be used to find an analogous expression for the identity operator that acts on a *two-fermion state*, comprising two identical fermions in two distinguishable states. Re-arranging the fermions to be in different states introduces a minus sign if the permutation is odd. This is the very antisymmetric property that characterizes states of more than one fermions. More generally this property guarantees that the number of fermions in any given state is either zero and one. This phenomenon is the Pauli-exclusion principle. Accordingly expressions for two-fermion states must be anti-symmetric under odd-permutations of fermions between states. To that extent the order of terms within expressions must be preserved as they appear here in the discussion. Below in Appendix A.3 a more sophisticated notation fixes the ordering of fermions between states, such that attention to the order of terms when writing expressions is redundant. While this notation indeed is needed to assign fermions to states in the right order, the proper assignment can be done for two fermions by merely writing terms in the correct order, with the advantage that the antisymmetry is more obvious without extra cumbersome notation.

A two-fermion state in the momentum representation has the form

$$
\begin{aligned}
|p_1\ p_2\rangle =& a^\dagger(p_1)a^\dagger(p_2)\,|0\rangle = \frac{1}{2}(|p_1\rangle\,|p_2\rangle - |p_2\rangle\,|p_1\rangle)\,, \\
\langle p_1\ p_2| =& \langle 0|\,a(p_2)a(p_1)\,.
\end{aligned}
$$

The first term describes a fermion with momentum p_1 in the 1st state and a fermion with momentum p_2 in the 2nd state, while in the second term the two fermions have swapped states. Acting on $|q_1\ q_2\rangle$, a two-fermion state with momenta q_1, q_2, with the operator $\frac{1}{2}\int dp_1\,dp_2\ \ |p_1\ p_2\rangle\,\langle p_1\ p_2|$ produces

$$
\begin{aligned}
&\frac{1}{2}\int dp_1\,dp_2\ \ |p_1\ p_2\rangle\,\langle p_1\ p_2|q_1\ q_2\rangle \\
=&\frac{1}{2}\int dp_1\,dp_2\ \ |p_1\ p_2\rangle\,\langle 0|\,a(p_2)a(p_1)a^\dagger(q_1)a^\dagger(q_2)\,|0\rangle\ \ .
\end{aligned} \tag{A1}
$$

Generally speaking

$$
\begin{aligned}
&a(p_2)a(p_1)a^\dagger(q_1)a^\dagger(q_2)\,|0\rangle \\
=&a(p_2)\{a(p_1),a^\dagger(q_1)\}a^\dagger(q_2)\,|0\rangle - a(p_2)a^\dagger(q_1)a(p_1)a^\dagger(q_2)\,|0\rangle \\
=&\{a(p_2),a^\dagger(q_2)\}\delta(p_1-q_1)\,|0\rangle - \{a(p_2),a^\dagger(q_1)\}\{a(p_1),a^\dagger(q_2)\}\,|0\rangle' \\
=&\delta(p_1-q_1)\delta(p_2-q_2)\,|0\rangle - \delta(p_2-q_1)\delta(p_1-q_2)\,|0\rangle
\end{aligned}
$$

such that (A1) becomes

$$
\begin{aligned}
&\frac{1}{2}\int dp_1\,dp_2\ \ |p_1\ p_2\rangle\,\langle p_1\ p_2|q_1\ q_2\rangle \\
=&\frac{1}{2}\int dp_1\,dp_2\ \ |p_1\ p_2\rangle\,\langle 0|\,\Big(\delta(p_1-q_1)\delta(p_2-q_2) - \delta(p_2-q_1)\delta(p_1-q_2)\Big)\,|0\rangle \\
=&\frac{1}{2}(|q_1\ q_2\rangle - |q_2\ q_1\rangle)\,\langle 0|0\rangle \\
=&\,|q_1\ q_2\rangle\ \ ,
\end{aligned}
$$

where the final step follows from the very antisymmetric property of multi-fermion states, namely $|q_1\ q_2\rangle = -\,|q_2\ q_1\rangle$.

The upshot is that the identity operator for the group of two-fermion states is

$$
\mathbb{1} = \frac{1}{2}\int dp_1\,dp_2\ |p_1\ p_2\rangle\,\langle p_1\ p_2|\ \ .
$$

Appendix A.3. N Fermion States

Before moving on to states of more than two fermions, new notation is needed to represent the permutation of particles between different states, which closely follows the convention in [48]. A given state of N fermions is represented by the ket

$$
|\psi^{i_1}\ \psi^{i_2}\ldots\psi^{i_N}\rangle = \frac{1}{N!}\sum_{i_1\ldots i_N}\epsilon_{i_1\ldots i_N}\,|\psi^{i_1}\rangle\otimes|\psi^{i_2}\rangle\otimes\cdots\otimes|\psi^{i_N}\rangle = A\,|\psi^1\rangle\otimes|\psi^2\rangle\otimes\cdots\otimes|\psi^N\rangle\,.
$$

Indices ${}^{s_1},{}^{s_2},\cdots,{}^{s_N} = {}^{(1)},{}^{(2)},\cdots\ {}^{(N)}$ label one-particle base ket-vectors. In the broad scheme it is an abstract label not necessarily assigned to be any specific physical observable (e.g., an eigenvalue of momentum). The base ket-vector labels ${}^{s_1}, {}^{s_2}, \cdots, {}^{s_N}$ assign an order to the one-particle kets. This ordering does not refer to any physical configuration of the particles themselves, rather it enables an antisymmetric sum over configurations for fermions (and a symmetric sum for bosons) without the need to write the order of terms explicitly like in Appendix A.2 with just two fermions. On the right the operator A produces an antisymmetric sum over configurations of N different base ket-vectors.

Such an antisymmetric sum possesses the required properties of fermion states: states vanish if more than one identical fermion occupies a given state, and a single fermion state has odd-integer spin. The former is the Pauli exclusion principle and the latter is a result of the spin-statistics theorem. Conversely, mutli-boson states are represented by a symmetric sum that carries the required property that any number of identical bosons can co-exist in a given state, and by the spin-statistics theorem, bosons carry integer spin. Bosons and fermions do not literally assume a particular order in nature any more than they are confined to a particular location, like classical particles. The ordering described here is purely a mathematical ordering of terms to give multi-particle states the correct properties: Fermi–Dirac statistics for fermions and Bose–Einstein statistics for bosons.

To ensure that at most one fermion lies in any given state, the following anti-commutation relations are assumed:

$$\{a(p_1), a(p_2)\} = 0, \qquad \{a^\dagger(p_1), a^\dagger(p_2)\} = 0, \qquad \{a(p_1), a^\dagger(p_2)\} = \delta(p_1 - p_2)\,.$$

An N-fermion state can be expressed as an antisymmetric tensor product of single-particle momentum eigenstates as

$$|p_1 \ldots p_N\rangle = \frac{1}{N!}\sum_{i_1\cdots i_N} \epsilon^{i_1\cdots i_N} |p_{i_1}\rangle \otimes \cdots \otimes |p_{i_N}\rangle = a^\dagger(p_1)\cdots a^\dagger(p_N)\,|0\rangle\ . \tag{A2}$$

It is clear from (A2) that having a label for the particle itself and a separate label for the basis ket means that assigning fermions to basis kets in an antisymmetric way is effortless with no need to pay attention to the order of terms. The anti-symmetric tensor is contracted on indices of basis kets. The order of assignment of N particles into N different kets is different between different terms, where two terms differing by an odd-permutation have opposite signs.

An operator that behaves like an identity on N fermion states is achieved by the same approach as Appendix A.2. Suppose that the identity is

$$\frac{1}{N!}\int \prod_{n=1}^{N} dp_k\ |p_1 \ldots p_N\rangle \langle p_1 \ldots p_N|\ , \tag{A3}$$

with $|p_1 \ldots p_N\rangle$ given by (A2). Acting on the analogous state $|q_1 \ldots q_N\rangle$ produces

$$\begin{aligned}&\frac{1}{N!}\left(\int \prod_{k=1}^{N} dp_k\ |p_1 \ldots p_N\rangle \langle p_1 \ldots p_N|\right) |q_1 \ldots q_N\rangle \\ =&\frac{1}{N!}\int \prod_{k=1}^{N} dp_k\ |p_1 \ldots p_N\rangle \langle 0|\, a(p_N)\ldots a(p_1) a^\dagger(q_1)\cdots a^\dagger(q_N)\,|0\rangle\ .\end{aligned} \tag{A4}$$

A way to write $\langle 0|\, a(p_N)\ldots a(p_1) a^\dagger(q_1)\cdots a^\dagger(q_N)\,|0\rangle$ that makes the integrals easily solvable is needed. The following identity can be proved that serves this purpose.

Theorem A1. *Inner product of multi-fermion states I*

$$\langle 0|\, a(p_N)\ldots a(p_1) a^\dagger(q_1)\cdots a^\dagger(q_N)\,|0\rangle = \sum_{i_1\ldots i_N} \epsilon^{i_1\ldots i_N} \delta(p_1 - q_{i_1})\delta(p_2 - q_{i_2})\ldots\delta(p_N - q_{i_N})\,. \tag{A5}$$

Proof. The most straightforward way to prove this statement is by induction. We start by verifying it for $N = 2$. The left-hand side is

$$\begin{aligned}
&\langle 0|\, a(p_2)a(p_1)a^\dagger(q_1)a^\dagger(q_2)\,|0\rangle \\
&= \langle 0|\, a(p_2)\{a(p_1), q^\dagger(q_1)\}a^\dagger(q_2)\,|0\rangle - \langle 0|\, a(p_2)a^\dagger(q_1)a(p_1)a^\dagger(q_2)\,|0\rangle \\
&= \delta(p_1 - q_1)\,\langle 0|\,\{a(p_2), a^\dagger(q_2)\}\,|0\rangle - \langle 0|\,\{a(p_2), a^\dagger(q_1)\}\{a(p_1), a^\dagger(q_2)\}\,|0\rangle \\
&= \delta(p_1 - q_1)\delta(p_2 - q_2) - \delta(p_1 - q_2)\delta(p_2 - q_1) \\
&= \sum_{i_1 i_2} \epsilon^{i_1 i_2}\delta(p_1 - q_{i_1})\delta(p_2 - q_{i_2})\,.
\end{aligned}$$

However, this is precisely the right of (A5) for $N = 2$. The implication is that (A5) is true for $N = 2$. Now for the inductive step of the prove. Assume that it is true for $N - 1$. We anticommute $a(p_1)$ one place to the right to obtain

$$\begin{aligned}
\langle 0|\, a(p_N)\ldots a(p_1)a^\dagger(q_1)\cdots a^\dagger(q_N)\,|0\rangle = \;& \delta(p_1 - q_1)\,\langle 0|\, a(p_N)\ldots a(p_2)a^\dagger(q_2)\cdots a^\dagger(q_N)\,|0\rangle \\
& - \langle 0|\, a(p_N)\ldots a(p_2)a^\dagger(q_1)a(p_1)a^\dagger(q_2)\cdots a^\dagger(q_N)\,|0\rangle\;.
\end{aligned}$$

In the second line we anticommute $a(p_1)$ one place to the right again to obtain

$$\begin{aligned}
& \langle 0|\, a(p_N)\ldots a(p_1)a^\dagger(q_1)\cdots a^\dagger(q_N)\,|0\rangle \\
= \;& \delta(p_1 - q_1)\,\langle 0|\, a(p_N)\ldots a(p_2)a^\dagger(q_2)\cdots a^\dagger(q_N)\,|0\rangle \\
& - \delta(p_1 - q_2)\,\langle 0|\, a(p_N)\ldots a(p_2)a^\dagger(q_1)a^\dagger(q_3)\cdots a^\dagger(q_N)\,|0\rangle \\
& + \langle 0|\, a(p_N)\ldots a(p_2)a^\dagger(q_1)a^\dagger(q_2)a(p_1)a^\dagger(q_3)\cdots a^\dagger(q_N)\,|0\rangle\;,
\end{aligned}$$

and we continue this process to eventually end up with

$$\begin{aligned}
& \langle 0|\, a(p_N)\ldots a(p_1)a^\dagger(q_1)\cdots a^\dagger(q_N)\,|0\rangle \\
= \;& \delta(p_1 - q_1)\,\langle 0|\, a(p_N)\ldots a(p_2)a^\dagger(q_2)\cdots a^\dagger(q_N)\,|0\rangle \\
& - \delta(p_1 - q_2)\,\langle 0|\, a(p_N)\ldots a(p_2)a^\dagger(q_1)a^\dagger(q_3)\cdots a^\dagger(q_N)\,|0\rangle \\
& + \delta(p_1 - q_3)\,\langle 0|\, a(p_N)\ldots a(p_2)a^\dagger(q_1)a^\dagger(q_2)a^\dagger(q_4)\cdots a^\dagger(q_N)\,|0\rangle \\
& \vdots \\
& + (-1)^{N-1}\delta(p_1 - q_N)\,\langle 0|\, a(p_N)\ldots a(p_2)a^\dagger(q_1)a^\dagger(q_2)\cdots a^\dagger(q_{N-1})\,|0\rangle\;.
\end{aligned}$$

The term next to the delta functions in each line is precisely the left of (A5) for $N - 1$, and since (A5) is assumed true for $N - 1$, the right-hand side can be substituted to yield

$$
\begin{aligned}
&\langle 0|\, a(p_N)\dots a(p_1)a^\dagger(q_1)\cdots a^\dagger(q_N)\,|0\rangle \\
=\;& \sum_{i_2\dots i_N} \delta(p_1-q_1)\epsilon^{i_2\dots i_N}\delta(p_2-q_{i_2})\delta(p_3-q_{i_3})\dots\delta(p_N-q_{i_N})\Big|_{i_k\neq 1} \\
&- \sum_{i_2\dots i_N} \delta(p_1-q_2)\epsilon^{i_2\dots i_N}\delta(p_2-q_{i_2})\delta(p_3-q_{i_3})\dots\delta(p_N-q_{i_N})\Big|_{i_k\neq 2} \\
&+ \sum_{i_2\dots i_N} \delta(p_1-q_3)\epsilon^{i_2\dots i_N}\delta(p_2-q_{i_2})\delta(p_3-q_{i_3})\dots\delta(p_N-q_{i_N})\Big|_{i_k\neq 3} \\
&\vdots \\
&+(-1)^{N-1} \sum_{i_2\dots i_N} \delta(p_1-q_N)\epsilon^{i_2\dots i_N}\delta(p_2-q_{i_2})\delta(p_3-q_{i_3})\dots\delta(p_N-q_{i_N})\Big|_{i_k\neq N}\,.
\end{aligned}
$$

The exclusion of a specific i_k is excluded from the contraction of the Levi-Civita symbol can be expressed as, for example,

$$
\sum_{i_2\dots i_N} \epsilon^{i_2\dots i_N}\delta(p_2-q_{i_2})\dots\delta(p_N-q_{i_N})\big|_{i_k\neq 1} = \sum_{i_2\dots i_N} \epsilon^{1i_2\dots i_N}\delta(p_2-q_{i_2})\dots\delta(p_N-q_{i_N}),
$$

where the Levi-Civita symbol on the left has $N-1$ indices and that on the right has N indices. Hence (*) becomes

$$
\begin{aligned}
&\langle 0|\, a(p_N)\dots a(p_1)a^\dagger(q_1)\cdots a^\dagger(q_N)\,|0\rangle \\
=\;& \sum_{i_2\dots i_N} \delta(p_1-q_1)\epsilon^{1i_2\dots i_N}\delta(p_2-q_{i_2})\delta(p_3-q_{i_3})\dots\delta(p_N-q_{i_N}) \\
&- \sum_{i_2\dots i_N} \delta(p_1-q_2)\epsilon^{2i_2\dots i_N}\delta(p_2-q_{i_2})\delta(p_3-q_{i_3})\dots\delta(p_N-q_{i_N}) \\
&+ \sum_{i_2\dots i_N} \delta(p_1-q_3)\epsilon^{3i_2\dots i_N}\delta(p_2-q_{i_2})\delta(p_3-q_{i_3})\dots\delta(p_N-q_{i_N}) \\
&\vdots \\
&+(-1)^{N-1} \sum_{i_2\dots i_N} \delta(p_1-q_N)\epsilon^{Ni_2\dots i_N}\delta(p_2-q_{i_2})\delta(p_3-q_{i_3})\dots\delta(p_N-q_{i_N}) \\
=\;& \sum_{k=1}^{N}(-1)^{k-1} \sum_{i_2\dots i_N} \epsilon^{ki_2\dots i_N}\delta(p_1-q_k)\delta(p_2-q_{i_2})\delta(p_3-q_{i_3})\dots\delta(p_N-q_{i_N})\,.
\end{aligned}
$$

In each term in the sum, we move the index ^{k} in the Levi-Civita symbol $k-1$ places to the right and insert the accompanying factor of $(-1)^{k-1}$, we re-label dummy indices appropriately and we replace the sum over k with the Einstein summation convention to end up with

$$
\begin{aligned}
&\langle 0|\, a(p_N)\dots a(p_1)a^\dagger(q_1)\cdots a^\dagger(q_N)\,|0\rangle \\
&= \sum_{i_1\dots i_N} \epsilon^{i_1 i_2\dots i_N}\delta(p_1-q_{i_1})\delta(p_2-q_{i_2})\delta(p_3-q_{i_3})\dots\delta(p_N-q_{i_N})\,.
\end{aligned}
\tag{A6}
$$

This is precisely the right-hand side of (A5). This completes the proof of the assertion in Equation (A5) by induction. □

Theorem A2. *Inner product of multi-fermion states II*

$$\begin{aligned} &\langle 0|\, a(p_N)\dots a(p_1)a^\dagger(q_1)\cdots a^\dagger(q_{N'})\,|0\rangle \\ = &\textstyle\sum_{i_1\dots i_N} \epsilon^{i_1\dots i_N}\delta(p_1-q_{i_1})\delta(p_2-q_{i_2})\dots\delta(p_N-q_{i_N})\,\delta_{NN'}. \end{aligned} \tag{A7}$$

Proof. There are three possibilities:
(i) $N' = N$
(ii) $N' > N$
(iii) $N' < N$.
Case (i): If $N = N'$ the theorem is proved by theorem A1.

Case (ii): $N' > N$. The proof of this statement is by induction and follows similar lines as the proof of theorem A1. We start by proving it for $N = 2$:

$$\begin{aligned} &\langle 0|\, a(p_2)a(p_1)a^\dagger(q_1)a^\dagger(q_2)\dots a^\dagger_{N'}\,|0\rangle \\ =&\langle 0|\, a(p_2)\{a(p_1),q^\dagger(q_1)\}a^\dagger(q_2)\dots a^\dagger_{N'}\,|0\rangle - \langle 0|\, a(p_2)a^\dagger(q_1)a(p_1)a^\dagger(q_2)\dots a^\dagger_{N'}\,|0\rangle \end{aligned}$$

$$\begin{aligned} =&\delta(p_1-q_1)\,\langle 0|\,\{a(p_2),a^\dagger(q_2)\}a^\dagger(q_3)\dots a^\dagger_{N'}\,|0\rangle \\ &-\langle 0|\,\{a(p_2),a^\dagger(q_1)\}\{a(p_1),a^\dagger(q_2)\}a^\dagger(q_3)\dots a^\dagger_{N'}\,|0\rangle \\ &+\langle 0|\,\{a(p_2),a^\dagger(q_1)\}a^\dagger(q_2)a(p_1)a^\dagger(q_3)\dots a^\dagger_{N'}\,|0\rangle \end{aligned}$$

$$\begin{aligned} =&\delta(p_1-q_1)\delta(p_2-q_2)\,\langle 0|\, a^\dagger(q_3)\dots a^\dagger_{N'}\,|0\rangle - \delta(p_2-q_1)\delta(p_1-q_2)\,\langle 0|\, a^\dagger(q_3)\dots a^\dagger_{N'}\,|0\rangle \\ &+\delta(p_2-q_1)\,\langle 0|\, a^\dagger(q_2)a(p_1)a^\dagger(q_3)\dots a^\dagger_{N'}\,|0\rangle \end{aligned}$$

$$\begin{aligned} =&\begin{cases} \delta(p_1-q_1)\delta(p_2-q_2)-\delta(p_2-q_1)\delta(p_1-q_2) & N'=2 \\ 0 & N'>2 \end{cases} \\ =&\begin{cases} \epsilon^{i_1 i_2}\delta(p_1-q_{i_1})\delta(p_2-q_{i_2}) & N'=2 \\ 0 & N'>2 \end{cases}. \end{aligned}$$

However, this is precisely the right of (A7) for $N = 2$. The implication is that (A7) is true for $N = 2$. □

Now for the inductive step of the prove. We assume that it is true for $N - 1$. We anticommute $a(p_1)$ one place to the right to obtain

$$\begin{aligned} &\langle 0|\, a(p_N)\dots a(p_1)a^\dagger(q_1)\cdots a^\dagger(q_N)a^\dagger(q_{N+1})\dots a^\dagger(q_{N'})\,|0\rangle \\ =\;\; &\delta(p_1-q_1)\,\langle 0|\, a(p_N)\dots a(p_2)a^\dagger(q_2)\cdots a^\dagger(q_N)a^\dagger(q_{N+1})\dots a^\dagger(q_{N'})\,|0\rangle \\ &-\langle 0|\, a(p_N)\dots a(p_2)a^\dagger(q_1)a(p_1)a^\dagger(q_2)\cdots a^\dagger(q_N)a^\dagger(q_{N+1})\dots a^\dagger(q_{N'})\,|0\rangle\;\; . \end{aligned}$$

In the second line we anticommute $a(p_1)$ one place to the right again to obtain

$$\begin{aligned} &\langle 0|\, a(p_N)\dots a(p_1)a^\dagger(q_1)\cdots a^\dagger(q_N)\,|0\rangle \\ =\;\; &\delta(p_1-q_1)\,\langle 0|\, a(p_N)\dots a(p_2)a^\dagger(q_2)\cdots a^\dagger(q_N)a^\dagger(q_{N+1})\dots a^\dagger(q_{N'})\,|0\rangle \\ &-\delta(p_1-q_2)\,\langle 0|\, a(p_N)\dots a(p_2)a^\dagger(q_1)a^\dagger(q_3)\cdots a^\dagger(q_N)a^\dagger(q_{N+1})\dots a^\dagger(q_{N'})\,|0\rangle \\ &+\langle 0|\, a(p_N)\dots a(p_2)a^\dagger(q_1)a^\dagger(q_2)a(p_1)a^\dagger(q_3)\cdots a^\dagger(q_N)a^\dagger(q_{N+1})\dots a^\dagger(q_{N'})\,|0\rangle\;\; , \end{aligned}$$

and we continue this process to eventually end up with

$$
\begin{aligned}
& \langle 0|\, a(p_N) \ldots a(p_1) a^\dagger(q_1) \cdots a^\dagger(q_N) a^\dagger(q_{N+1}) \ldots a^\dagger(q_{N'})\, |0\rangle \\
= \;& \delta(p_1 - q_1)\, \langle 0|\, a(p_N) \ldots a(p_2) a^\dagger(q_2) \cdots a^\dagger(q_N) a^\dagger(q_{N+1}) \ldots a^\dagger(q_{N'})\, |0\rangle \\
& - \delta(p_1 - q_2)\, \langle 0|\, a(p_N) \ldots a(p_2) a^\dagger(q_1) a^\dagger(q_3) \cdots a^\dagger(q_N) a^\dagger(q_{N+1}) \ldots a^\dagger(q_{N'})\, |0\rangle \\
& + \delta(p_1 - q_3)\, \langle 0|\, a(p_N) \ldots a(p_2) a^\dagger(q_1) a^\dagger(q_2) a^\dagger(q_4) \cdots a^\dagger(q_N) a^\dagger(q_{N+1}) \ldots a^\dagger(q_{N'})\, |0\rangle \\
& \vdots \\
& + (-1)^{N-1} \delta(p_1 - q_N)\, \langle 0|\, a(p_N) \ldots a(p_2) a^\dagger(q_1) a^\dagger(q_2) \cdots a^\dagger(q_{N-1})\, a^\dagger(q_{N+1}) \ldots a^\dagger(q_{N'})\, |0\rangle \ ,
\end{aligned}
$$

The term next to the delta functions in each line is precisely the left of (A5) for $N-1$, $N'-1$ and since (A5) is assumed true for $N-1, N'-1$ the right-hand side can be substituted to yield

$$
\begin{aligned}
& \langle 0|\, a(p_N) \ldots a(p_1) a^\dagger(q_1) \cdots a^\dagger(q_N) a^\dagger(q_{N+1}) \ldots a^\dagger(q_{N'})\, |0\rangle \\
= \;& \sum_{i_2 \ldots i_N} \delta(p_1 - q_1) \epsilon^{i_2 \ldots i_N} \delta(p_2 - q_{i_2}) \delta(p_3 - q_{i_3}) \ldots \delta(p_N - q_{i_N}) \Big|_{i_k \neq 1} \delta_{N-1, N'-1} \\
& - \sum_{i_2 \ldots i_N} \delta(p_1 - q_2) \epsilon^{i_2 \ldots i_N} \delta(p_2 - q_{i_2}) \delta(p_3 - q_{i_3}) \ldots \delta(p_N - q_{i_N}) \Big|_{i_k \neq 2} \delta_{N-1, N'-1} \\
& + \sum_{i_2 \ldots i_N} \delta(p_1 - q_3) \epsilon^{i_2 \ldots i_N} \delta(p_2 - q_{i_2}) \delta(p_3 - q_{i_3}) \ldots \delta(p_N - q_{i_N}) \Big|_{i_k \neq 3} \delta_{N-1, N'-1} \\
& \vdots \\
& + (-1)^{N-1} \sum_{i_2 \ldots i_N} \delta(p_1 - q_N) \epsilon^{i_2 \ldots i_N} \delta(p_2 - q_{i_2}) \delta(p_3 - q_{i_3}) \ldots \delta(p_N - q_{i_N}) \Big|_{i_k \neq N} \delta_{N-1, N'-1} \, .
\end{aligned} \tag{*}
$$

The exclusion of a specific i_k from the contraction of the Levi-Civita symbol can be expressed as, for example, $\sum_{i_2 \ldots i_N} \epsilon^{i_2 \ldots i_N} \delta(p_2 - q_{i_2}) \ldots \delta(p_N - q_{i_N}) \big|_{i_k \neq 1} = \sum_{i_2 \ldots i_N} \epsilon^{1 i_2 \ldots i_N} \delta(p_2 - q_{i_2}) \ldots \delta(p_N - q_{i_N})$, where the Levi-Civita symbol on the left has $N-1$ indices and that on the right has N indices. Hence (*) becomes

$$
\begin{aligned}
& \langle 0|\, a(p_N) \ldots a(p_1) a^\dagger(q_1) \cdots a^\dagger(q_N)\, |0\rangle \\
= \;& \sum_{i_2 \ldots i_N} \delta(p_1 - q_1) \epsilon^{1 i_2 \ldots i_N} \delta(p_2 - q_{i_2}) \delta(p_3 - q_{i_3}) \ldots \delta(p_N - q_{i_N}) \delta_{N, N'} \\
& - \sum_{i_2 \ldots i_N} \delta(p_1 - q_2) \epsilon^{2 i_2 \ldots i_N} \delta(p_2 - q_{i_2}) \delta(p_3 - q_{i_3}) \ldots \delta(p_N - q_{i_N}) \delta_{N, N'} \\
& + \sum_{i_2 \ldots i_N} \delta(p_1 - q_3) \epsilon^{3 i_2 \ldots i_N} \delta(p_2 - q_{i_2}) \delta(p_3 - q_{i_3}) \ldots \delta(p_N - q_{i_N}) \delta_{N, N'} \\
& \vdots \\
& + (-1)^{N-1} \sum_{i_2 \ldots i_N} \delta(p_1 - q_N) \epsilon^{N i_2 \ldots i_N} \delta(p_2 - q_{i_2}) \delta(p_3 - q_{i_3}) \ldots \delta(p_N - q_{i_N}) \delta_{N, N'} \\
= & \sum_{k=1}^{N} (-1)^{k-1} \sum_{i_2 \ldots i_N} \epsilon^{k i_2 \ldots i_N} \delta(p_1 - q_k) \delta(p_2 - q_{i_2}) \delta(p_3 - q_{i_3}) \ldots \delta(p_N - q_{i_N}) \delta_{N, N'} \, .
\end{aligned}
$$

In each term in the sum, we move the index ^{k} in the Levi-Civita symbol $k-1$ places to the right, we insert the accompanying factor of $(-1)^{k-1}$, we re-label dummy indices appropriately and we replace the sum over k with the Einstein summation convention to end up with

$$\begin{aligned}&\langle 0|\, a(p_N)\dots a(p_1)a^\dagger(q_1)\cdots a^\dagger(q_N)\,|0\rangle \\ &= \sum_{i_1\dots i_N} \epsilon^{i_1 i_2\dots i_N}\delta(p_1-q_{i_1})\delta(p_2-q_{i_2})\delta(p_3-q_{i_3})\dots\delta(p_N-q_{i_N})\delta_{N,N'}\,.\end{aligned}$$

This is precisely the right-hand side of (A7). This completes the proof of the assertion in (A7) by induction.

Having verified (A5), we substitute it in (A4) to obtain

$$\begin{aligned}&\frac{1}{N!}\left(\int\prod_{k=1}^{N} dp_k\,\, |p_1\dots p_N\rangle\,\langle p_1\dots p_N|\right)|q_1\dots q_N\rangle \\ =&\frac{1}{N!}\int\prod_{k=1}^{N} dp_k\,\, |p_1\dots p_N\rangle \sum_{i_1\dots i_N}\epsilon^{i_1 i_2\dots i_N}\delta(p_1-q_{i_1})\delta(p_2-q_{i_2})\delta(p_3-q_{i_3})\dots\delta(p_N-q_{i_N})\,.\end{aligned}$$

As promised, in this form it is now a straightforward matter to evaluate the p_k integrals. The result is

$$\frac{1}{N!}\left(\int\prod_{k=1}^{N} dp_k\,\, |p_1\dots p_N\rangle\,\langle p_1\dots p_N|\right)|q_1\dots q_N\rangle = \frac{1}{N!}\sum_{i_1\dots i_N}\epsilon^{i_1 i_2\dots i_N}\,|q_{i_1}\dots q_{i_N}\rangle\ .$$

From the definition in (A2), namely $|q_1\dots q_N\rangle = a^\dagger(q_1)\dots a^\dagger(q_N)\,|0\rangle$, it is obvious, that

$$\sum_{i_1\dots i_N}\epsilon^{i_1 i_2\dots i_N}\,|q_{i_1}\dots q_{i_N}\rangle = \sum_{i_1\dots i_N}\epsilon^{i_1 i_2\dots i_N}a^\dagger(q_{i_1})\dots a^\dagger(q_{i_N})\,|0\rangle = N!\,|q_1\dots q_N\rangle\,,$$

i.e., the contraction of the Levi-Civita on $|q_{i_1}\dots q_{i_N}\rangle$, which is already completely antisymmetric by definition, simply results in a sum over $N!$ permutations. The conclusion is that

$$\frac{1}{N!}\left(\int\prod_{k=1}^{N} dp_k\,\, |p_1\dots p_N\rangle\,\langle p_1\dots p_N|\right)|q_1\dots q_N\rangle = |q_1\dots q_N\rangle\ , \tag{A8}$$

i.e., the operator $\frac{1}{N!}\int\prod_{k=1}^{N} dp_k\,\, |p_1\dots p_N\rangle\,\langle p_1\dots p_N|$ acts as an identity on N fold multi-fermion states defined in (A2).

A corollary of (A5) is that

$$\begin{aligned}\langle q_1\dots q_N|p_1\dots p_N\rangle &= \langle 0|\, a(q_N)\dots a(q_1)a^\dagger(p_1)\dots a^\dagger(p_N)\,|0\rangle \\ &= \sum_{i_1\dots i_N}\epsilon^{i_1\dots i_N}\delta(p_1-q_{i_1})\dots\delta(p_N-q_{i_N})\,.\end{aligned}$$

Theorem A3. *Anticommutator of an arbitrary number of fermion operators*

$$\{b, a_1\dots a_N\} = \sum_{k=1}^{N}(-1)^{k-1}a_1\dots a_{k-1}\{b,a_k\}a_{k+1}\dots a_N + (1+(-1)^N)a_1\dots a_N b\,. \tag{A9}$$

Proof. The most straightforward way to prove this statement is by induction. We start by verifying it for $N=2$:

$$\begin{aligned}\{b, a_1 a_2\} =& b a_1 a_2 + a_1 a_2 b \\ =&\{b,a_1\}a_2 - a_1 b a_2 + a_1 a_2 b \\ =&\{b,a_1\}a_2 - a_1\{b,a_2\} + 2a_1 a_2 b\,.\end{aligned}$$

However, this is precisely the right of (A9) for $N = 2$. The implication is that (A9) is true for $N = 2$. Now for the inductive step of the prove. We assume that it is true for $N = n - 1$.

$$\{b, a_1 \dots a_n\} = ba_1 \dots a_n + a_1 \dots a_n b = \{b, a_1 \dots a_{n-1}\}a_n - a_1 \dots a_{n-1} b a_n + a_1 \dots a_n b\,.$$

However, if (A9) is true for $N = n - 1$, then

$$\begin{aligned}
&\{b, a_1 \dots a_n\} \\
&= \sum_{k=1}^{n-1} (-1)^{k-1} a_1 \dots a_{k-1} \{b, a_k\} a_{k+1} \dots a_{n-1} a_n \\
&\quad + (1 + (-1)^{n-1}) a_1 \dots a_{n-1} b a_n - a_1 \dots a_{n-1} b a_n + a_1 \dots a_n b \\
&= \sum_{k=1}^{n-1} (-1)^{k-1} a_1 \dots a_{k-1} \{b, a_k\} a_{k+1} \dots a_{n-1} a_n + (-1)^{n-1} a_1 \dots a_{n-1} b a_n + a_1 \dots a_{n-1} a_n b \\
&= \sum_{k=1}^{n-1} (-1)^{k-1} a_1 \dots a_{k-1} \{b, a_k\} a_{k+1} \dots a_{n-1} a_n \\
&\quad + (-1)^{n-1} a_1 \dots a_{n-1} \{b, a_n\} - (-1)^{n-1} a_1 \dots a_{n-1} a_n b + a_1 \dots a_n b \\
&= \sum_{k=1}^{n} (-1)^{k-1} a_1 \dots a_{k-1} \{b, a_k\} a_{k+1} \dots a_{n-1} a_n + (-1)^{n} a_1 \dots a_{n-1} a_n b + a_1 \dots a_n b \\
&= \sum_{k=1}^{n} (-1)^{k-1} a_1 \dots a_{k-1} \{b, a_k\} a_{k+1} \dots a_{n-1} a_n + (1 + (-1)^{n}) a_1 \dots a_n b\,.
\end{aligned}$$

However, this is none other than (A9) for $N = n$. Hence, if (A9) holds for $N = n - 1$ it must also be true for $N = n$. This proves (A9), by induction. □

Appendix A.4. Derived Identities Involving the Projection Operator onto $N > 1$ Particle States

The projection operator onto N particle states is defined as

$$\hat{\Pi}_N = \frac{1}{N!} \int dp_1 \dots dp_N \, |p_1 \dots p_N\rangle \langle p_1 \dots p_N| \;.$$

The form of the projection operator that shall be assumed is

$$\hat{\Pi}_N = \frac{1}{N!} \int dx_1 \dots dx_N \, a^\dagger(x_1) \dots a^\dagger(x_N) \, |0\rangle \langle 0| \, a(x_N) \dots a(x_1)\,.$$

This is consistent with the requirement that, given a state $|\psi\rangle = |x_1 \dots x_{N'}\rangle = a^\dagger(x_1) \dots a^\dagger(x'_N) \, |0\rangle$, then $\hat{\Pi}_N \, |\psi\rangle = \delta_{NN'} \, |\psi\rangle$, which is easily shown to be true by invoking theorem A1. Based on (56) and the form of the Hamiltonian in (31), then it can be shown that the two commute:

$$[\hat{H}, \hat{\Pi}_N] = 0.$$

To show that they commute we substitute their explicit forms:

$$[\hat{H}, \hat{\Pi}_N] = \int dX\, \mathscr{H}_0(X) \frac{1}{N!} \int dx_1 \dots dx_N \, [a^\dagger(X)a(X), a^\dagger(x_1)\dots a^\dagger(x_N) |0\rangle \langle 0| a(x_N)\dots a(x_1)]$$
$$+ \int dX\, dY\, \mathscr{V}(X-Y) \frac{1}{N!} \int dx_1 \dots dx_N$$
$$[a^\dagger(X)a(X)a^\dagger(Y)a(Y), a^\dagger(x_1)\dots a^\dagger(x_N) |0\rangle \langle 0| a(x_N)\dots a(x_1)]$$
$$= \int dX\, \mathscr{H}_0(X) \frac{1}{N!} \int dx_1 \dots dx_N \Big\{ a^\dagger(X)a(X)a^\dagger(x_1)\dots a^\dagger(x_N) |0\rangle \langle 0| a(x_N)\dots a(x_1)$$
$$- a^\dagger(x_1)\dots a^\dagger(x_N) |0\rangle \langle 0| a(x_N)\dots a(x_1) a^\dagger(X)a(X) \Big\}$$
$$+ \int dX\, dY\, \mathscr{V}(X-Y) \frac{1}{N!} \int dx_1 \dots dx_N$$
$$\Big\{ a^\dagger(X)a(X)a^\dagger(Y)a(Y)a^\dagger(x_1)\dots a^\dagger(x_N) |0\rangle \langle 0| a(x_N)\dots a(x_1)$$
$$- a^\dagger(x_1)\dots a^\dagger(x_N) |0\rangle \langle 0| a(x_N)\dots a(x_1) a^\dagger(X)a(X)a^\dagger(Y)a(Y) \Big\} . \quad \text{(A10)}$$

Note that

$$a^\dagger(X)a(X)a^\dagger(x_1)\dots a^\dagger(x_N) |0\rangle$$
$$= \textstyle\sum_{i=1}^{N} \delta(X - x_i) a^\dagger(x_1)\dots a^\dagger(x_{i-1}) a^\dagger(X) a^\dagger(x_{i+1})\dots a^\dagger(x_N) |0\rangle \quad \text{(A11)}$$

$$\langle 0| a(x_N)\dots a(x_1) a^\dagger(X)a(X) = \sum_{i=1}^{N} \delta(X - x_i) \langle 0| a(x_N)\dots a(x_{i+1}) a(X) a(x_{i-1})\dots a(x_1)$$

$$a^\dagger(Y)a(Y)a^\dagger(x_1)\dots a^\dagger(x_N) |0\rangle$$
$$= \sum_{\substack{i,j=1 \\ i\neq j}}^{N} \delta(X - x_i)\delta(Y - x_j) a^\dagger(x_1)\dots a^\dagger(x_{i-1}) a^\dagger(X) a^\dagger(x_{i+1})\dots a^\dagger(x_{j-1}) a^\dagger(Y) a^\dagger(x_{j+1})\dots a^\dagger(x_N) |0\rangle$$

and

$$\langle 0| a(x_N)\dots a(x_1) a^\dagger(X)a(X)a^\dagger(Y)a(Y)$$
$$= \sum_{\substack{i,j=1 \\ i\neq j}}^{N} \delta(X - x_i)\delta(Y - x_j) \langle 0| a(x_N)\dots a(x_{j-1}) a(Y) a(x_{j+1})\dots a(x_{i+1}) a(X) a(x_{i-1})\dots a(x_1)\, , \quad \text{(A12)}$$

such that by substituting (A11)–(A12) into (A10) and integrating over X and Y the result is

$$
\begin{aligned}
[\hat{H}, \hat{\Pi}_N] = &\sum_{i=1}^{N} \mathscr{H}_0(x_i) \frac{1}{N!} \int dx_1 \dots dx_N \\
&\Big\{ a^\dagger(x_1) \dots a^\dagger(x_N) \left|0\right\rangle \left\langle 0\right| a(x_N) \dots a(x_1) - a^\dagger(x_1) \dots a^\dagger(x_N) \left|0\right\rangle \left\langle 0\right| a(x_N) \dots a(x_1) \Big\} \\
&+ \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \mathscr{V}(x_i - x_j) \frac{1}{N!} \int dx_1 \dots dx_N \\
&\Big\{ a^\dagger(x_1) \dots a^\dagger(x_N) \left|0\right\rangle \left\langle 0\right| a(x_N) \dots a(x_1) - a^\dagger(x_1) \dots a^\dagger(x_N) \left|0\right\rangle \left\langle 0\right| a(x_N) \dots a(x_1) \Big\} \\
= 0 \, .
\end{aligned}
$$

Appendix B. Miscellaneous Identities for Weyl Symbols with $N > 1$

The *Moyal product* of the Weyl symbols of two operators $\hat{A}$ and $\hat{B}$ is defined as

$$
\begin{aligned}
&A_W(\{x_a\}, \{p_a\}) \star B_W(\{x_a\}, \{p_a\}) \\
=&A_W(\{x_a\}, \{p_a\}) \exp\left[\frac{i}{2} \sum_{a=1}^{N} \sum_{i=1}^{2} \left(\overleftarrow{\frac{\partial}{\partial x_a^i}} \overrightarrow{\frac{\partial}{\partial p_a^i}} - \overleftarrow{\frac{\partial}{\partial p_a^i}} \overrightarrow{\frac{\partial}{\partial x_a^i}} \right) \right] B_W(\{x_a\}, \{p_a\}) \, .
\end{aligned} \tag{A13}
$$

The functional trace of a Weyl symbol of an operator on single-particle states is defined as [19] (see Equation (10)).

$$
\mathrm{Tr} A_W(x, p) \equiv \frac{1}{(2\pi)^D} \int dx \, dp \, \mathrm{tr} A_W(x, p) \, .
$$

The analogous expression for N identical particles is

$$
\mathrm{Tr} A_W(x_1, \dots x_N, p_1, \dots, p_N) \equiv \frac{1}{(2\pi)^{ND}} \int dx_1 \dots dx_N \, dp_1 \dots dp_N \, \mathrm{tr} A_W(x_1, \dots x_N, p_1, \dots, p_N) \, .
$$

The Wigner transformation of the operator $\hat{A}$ that acts on one-particle states is

$$
A_W(p, x) = \int dq \, e^{iqx} \langle p + \tfrac{q}{2} | \hat{A} | p - \tfrac{q}{2} \rangle \, , \tag{A14}
$$

The Wigner transformation of the product of two operators, by analogy with (A14), is

$$
(AB)_W(p, x) = \int dq \, e^{iqx} \langle p + \tfrac{q}{2} | \hat{A}\hat{B} | p - \tfrac{q}{2} \rangle = \int dq \, dQ \, e^{iqx} \langle p + \tfrac{q}{2} | \hat{A} \left|Q\right\rangle \left\langle Q\right| \hat{B} | p - \tfrac{q}{2} \rangle \, , \tag{A15}
$$

where the analogue of (A3) for $N = 1$ was substituted ($\int dQ \left|Q\right\rangle \left\langle Q\right| = 1$ when acting on one-particle states). Through the change of variables $q = u + v$, $Q = p - \frac{u}{2} + \frac{v}{2}$ with the associated Jacobian $\partial(q, Q)/\partial(u, v) = 1$, (A15) takes the form

$$
\begin{aligned}
(AB)_W(p, x) &= \int du \, dv \, e^{i(u+v)x} \langle p + \tfrac{u}{2} + \tfrac{v}{2} | \hat{A} \, | p - \tfrac{u}{2} + \tfrac{v}{2} \rangle \, \langle p - \tfrac{u}{2} + \tfrac{v}{2} | \, \hat{B} | p - \tfrac{u}{2} - \tfrac{v}{2} \rangle \\
&= \int du \, e^{iux} \langle p + \tfrac{u}{2} | \hat{A} \, | p - \tfrac{u}{2} \rangle \exp\left(\frac{v}{2} \overleftarrow{\partial}_p - \frac{u}{2} \overrightarrow{\partial}_p \right) \int dv \, e^{ivx} \, \langle p + \tfrac{v}{2} | \, \hat{B} | p - \tfrac{v}{2} \rangle \\
&= \int du \, e^{iux} \langle p + \tfrac{u}{2} | \hat{A} \, | p - \tfrac{u}{2} \rangle \exp\left(\frac{i}{2} \overleftarrow{\partial}_p \overrightarrow{\partial}_x - \frac{i}{2} \overleftarrow{\partial}_x \overrightarrow{\partial}_p \right) \int dv \, e^{ivx} \, \langle p + \tfrac{v}{2} | \, \hat{B} | p - \tfrac{v}{2} \rangle \\
&= A_W(x, p) \star B_W(x, p) \, .
\end{aligned} \tag{A16}
$$

Now to generalize this result to operators on $N > 1$ particle states, the Wigner transformation of the operator $\hat{A}$ is defined as a function of $2N+1$ variables ω, p_a, x_a ($a = 1, ..., N$), in terms of its matrix elements in momentum space:

$$\int dp_1 \dots dp_N \frac{1}{(2\pi)^{ND}} \int dx_1 \dots dx_N A_W(\{x_a\}, \{p_a\}) = \frac{1}{N!} \int dp_1 \dots dp_N \langle \{p_a\}|\hat{A}|\{p_a\}\rangle \; .$$

and

$$A_W(\omega, \{p_a\}, \{x_a\}) = \frac{1}{N!} \int \left(\prod_{a=1}^{N} dq_a \, e^{iq_a x_a} \right) \langle \{p_a + \tfrac{q_a}{2}\}|\hat{A}|\{p_a - \tfrac{q_a}{2}\}\rangle \; . \tag{A17}$$

The extra factor of $1/N!$ ensures that the N-particle extension of the result in (A16) is the same. That is, given two operators $\hat{A}$ and $\hat{B}$ that act on N-particle states, the Wigner transformation of the product of the two operators is the N-particle generalization of (A15):

$$\begin{aligned}(AB)_W(\{p_a\}, \{x_a\}) =& \frac{1}{N!} \int \left(\prod_{a=1}^{N} dq_a \, e^{iq_a x_a} \right) \langle \{p_a + \tfrac{q_a}{2}\}|\hat{A}\hat{B}|\{p_a - \tfrac{q_a}{2}\}\rangle \\ =& \frac{1}{N!^2} \int \left(\prod_{a=1}^{N} dq_a \, dQ_a \, e^{iq_a x_a} \right) \langle \{p_a + \tfrac{q_a}{2}\}|\hat{A}\,|\{Q_a\}\rangle \, \langle \{Q_a\}|\,\hat{B}|\{p_a - \tfrac{q_a}{2}\}\rangle \; .\end{aligned} \tag{A18}$$

Note the presence of the extra factor of $1/N!$ in the second equality, coming from (A3). Following the same steps as in (A16), Equation A18 yields

$$\begin{aligned}(AB)_W(\{p_a\}, \{x_a\}) =& \frac{1}{N!} \int \left(\prod_{a=1}^{N} du_a \, e^{iu_a x_a} \right) \langle \{p_a + \tfrac{u_a}{2}\}|\hat{A}\,|\{p_a - \tfrac{u_a}{2}\}\rangle \\ & \exp\left[\frac{i}{2} \sum_{a=1}^{N} \sum_{i=1}^{2} \left(\frac{\overleftarrow{\partial}}{\partial x_a^i} \frac{\overrightarrow{\partial}}{\partial p_a^i} - \frac{\overleftarrow{\partial}}{\partial p_a^i} \frac{\overrightarrow{\partial}}{\partial x_a^i} \right) \right] \\ & \frac{1}{N!} \int \left(\prod_{a=1}^{N} dv_a \, e^{iv_a x_a} \right) \langle \{p_a + \tfrac{v_a}{2}\}|\,\hat{B}|\{p_a - \tfrac{v_a}{2}\}\rangle \\ =& A_W(\{p_a\}, \{x_a\}) \star B_W(\{p_a\}, \{x_a\}) \; ,\end{aligned}$$

as required.

From the definition of (A17) an important property of $A_W(\omega, \{p_a\}, \{x_a\})$ emerges. Take for example, the appropriate expression for $N = 2$, viz

$$A_W(\omega, p_1, p_2, x_1, x_2) = \frac{1}{2} \int dq_1 \, dq_2 \, e^{iq_1 x_1 + iq_2 x_2} \, \langle p_1 + \tfrac{q_1}{2}, p_2 + \tfrac{q_2}{2}|\hat{A}|p_1 - \tfrac{q_1}{2}, p_2 - \tfrac{q_2}{2}\rangle \; , \tag{A19}$$

paying attention to the order of variables in the ket $|p_1 - \frac{q_1}{2}, p_2 - \frac{q_2}{2}\rangle$ and similarly for the bra vector $\langle p_1 + \frac{q_1}{2}, p_2 + \frac{q_2}{2}|$. It then follows from the definition in (A2) that exchanging the order of variables results in a change in sign:

$$|p_2 - \tfrac{q_2}{2}, p_1 - \tfrac{q_1}{2}\rangle = -|p_1 - \tfrac{q_1}{2}, p_2 - \tfrac{q_2}{2}\rangle \; , \qquad \langle p_2 + \tfrac{q_2}{2}, p_1 + \tfrac{q_1}{2}| = -\langle p_1 + \tfrac{q_1}{2}, p_2 + \tfrac{q_2}{2}| \; .$$

Under such a change of order of variables in the two-state vectors the expression in (A19) becomes

$$A_W(\omega, p_1, p_2, x_1, x_2) = \frac{(-1)^2}{2} \int dq_1 \, dq_2 \, e^{iq_1 x_1 + iq_2 x_2} \, \langle p_2 + \tfrac{q_2}{2}, p_1 + \tfrac{q_1}{2}|\hat{A}|p_2 - \tfrac{q_2}{2}, p_1 - \tfrac{q_1}{2}\rangle \; ,$$

with two minus signs that enter in the form on the right that cancel. Now by relabeling the dummy integration variables we obtain

$$A_W(\omega, p_1, p_2, x_1, x_2) = \int dq_2\, dq_1\, e^{iq_2 x_1 + iq_1 x_2} \langle p_2 + \tfrac{q_1}{2}, p_1 + \tfrac{q_2}{2} | \hat{A} | p_2 - \tfrac{q_1}{2}, p_1 - \tfrac{q_2}{2} \rangle$$
$$= \int dq_1\, dq_2\, e^{iq_1 x_2 + iq_2 x_1} \langle p_2 + \tfrac{q_1}{2}, p_1 + \tfrac{q_2}{2} | \hat{A} | p_2 - \tfrac{q_1}{2}, p_1 - \tfrac{q_2}{2} \rangle ,$$

which, upon comparison with (A19), is found to be precisely the same expression but with the variables p_1, p_2 interchanged and the variables x_1, x_2 interchanged. The upshot is that

$$A_W(\omega, p_1, p_2, x_1, x_2) = A_W(\omega, p_2, p_1, x_2, x_1) . \tag{A20}$$

It can be shown using analogous examples generalized to the case of states greater than two particles that

$$A_W(\omega, p_1, p_2, \ldots, p_N, x_1, x_2, \ldots x_N) = A_W\Big(\omega, p_{\sigma(1)}, p_{\sigma(2)}, \ldots, p_{\sigma(N)}, x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(N)}\Big) . \tag{A21}$$

where $\sigma(i)$ is the number in position i under a permutation σ.

The Weyl symbol of a product of operators, $(AB)_W(x, p)$ is defined as [19] (See Equation (5)).

$$(AB)_W(\{x_a\}, \{p_a\}) := A_W(\{x_a\}, \{p_a\}) \star B_W(\{x_a\}, \{p_a\}) ,$$

where the Moyal product or $\star$ product of the Weyl symbols of two operators $\hat{A}$ and $\hat{B}$ is defined as [19] (see Equation (9))

$$\begin{aligned} & A_W(\{x_a\}, \{p_a\}) \star B_W(\{x_a\}, \{p_a\}) \\ =& A_W(\{x_a\}, \{p_a\}) \exp\left[\frac{i}{2} \sum_{a=1}^{N} \sum_{i=1}^{2} \left(\frac{\overleftarrow{\partial}}{\partial x_a^i} \frac{\overrightarrow{\partial}}{\partial p_a^i} - \frac{\overleftarrow{\partial}}{\partial p_a^i} \frac{\overrightarrow{\partial}}{\partial x_a^i} \right)\right] B_W(\{x_a\}, \{p_a\}) . \end{aligned} \tag{A22}$$

where the subscript $a = 1, \ldots, N$ distinguishes between variables that belong to the N different particles and the label $i = 1, 2 = x, y$ refers to the component each variable in the lattice plane. To clarify the expression in (A22) take the case of $N = 2$:

$$\begin{aligned} & (AB)_W(\{x_a\}, \{p_a\}) \\ =& (AB)_W(x_1, x_2, p_1, p_2) \\ =& A_W(x_1, x_2, p_1, p_2) \star B_W(x_1, x_2, p_1, p_2) \\ =& A_W(x_1, x_2, p_1, p_2) \exp\left[\frac{i}{2} \sum_{i=1}^{2} \left(\frac{\overleftarrow{\partial}}{\partial x_1^i} \frac{\overrightarrow{\partial}}{\partial p_1^i} + \frac{\overleftarrow{\partial}}{\partial x_2^i} \frac{\overrightarrow{\partial}}{\partial p_2^i} - \frac{\overleftarrow{\partial}}{\partial p_1^i} \frac{\overrightarrow{\partial}}{\partial x_1^i} - \frac{\overleftarrow{\partial}}{\partial p_2^i} \frac{\overrightarrow{\partial}}{\partial x_2^i} \right)\right] B_W(x_1, x_2, p_1, p_2) . \end{aligned}$$

However, thanks to the property (A20), $A_W(x_1, x_2, p_1, p_2) = A_W(x_2, x_1, p_2, p_1)$ and similarly $B_W(x_1, x_2, p_1, p_2) = B_W(x_2, x_1, p_2, p_1)$, hence, since the interchange of the variables p_1, p_2 and interchange of the variables x_1, x_2 inside the $\star$ operator does not change the operator, such that

$$(AB)_W(x_1, x_2, p_1, p_2) = (AB)_W(x_2, x_1, p_2, p_1) . \tag{A23}$$

(A23) is the invariance property analogous to the one found in (A20) for just one Weyl symbol. Similar arguments using (A21) leads to a generalization of (A23) for $N > 2$, namely

$$(AB)_W(p_1, p_2, \ldots, p_N, x_1, x_2, \ldots, x_N) = (AB)_W\Big(p_{\sigma(1)}, p_{\sigma(2)}, \ldots, p_{\sigma(N)}, x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(N)}\Big) .$$

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