symmetry

Article

Refined Hermite–Hadamard Inequalities and Some Norm Inequalities

Kenjiro Yanagi

Department of Mathematics, Josai University, 1-1, Keyakidai, Sakado 350-0295, Japan; yanagi@yamaguchi-u.ac.jp

Abstract: It is well known that the Hermite–Hadamard inequality (called the HH inequality) refines the definition of convexity of function $f(x)$ defined on $[a,b]$ by using the integral of $f(x)$ from a to b. There are many generalizations or refinements of HH inequality. Furthermore HH inequality has many applications to several fields of mathematics, including numerical analysis, functional analysis, and operator inequality. Recently, we gave several types of refined HH inequalities and obtained inequalities which were satisfied by weighted logarithmic means. In this article, we give an N-variable Hermite–Hadamard inequality and apply to some norm inequalities under certain conditions. As applications, we obtain several inequalities which are satisfied by means defined by symmetry. Finally, we obtain detailed integral values.

Keywords: Hermite–Hadamard inequality; norm inequality

MSC: Primary 26D15; secondary 26B25

check for updates

Citation: Yanagi, K. Refined Hermite–Hadamard Inequalities and Some Norm Inequalities. *Symmetry* **2022**, *14*, 2522. https://doi.org/10.3390/sym14122522

Academic Editors: Nicusor Minculete and Shigeru Furuichi

Received: 15 October 2022
Accepted: 21 November 2022
Published: 29 November 2022

1. Introduction

A function, $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$, is said to be convex on $[a,b]$ if the inequality

$$f(\frac{x+y}{2}) \leq \frac{f(x)+f(y)}{2} \tag{1}$$

holds for all $x,y \in [a,b]$. If the inequality (1) reverses, then f is said to be concave on $[a,b]$. Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function on an interval $[a.b]$. Then,

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \tag{2}$$

This double inequality is known in the literature as the Hermite–Hadamard integral inequality for convex functions. It has many applications in different areas of pure and applied mathematics. For some references about this latter point, we can consult [1–10]. Recently, we obtained the following two refined Hermite–Hadamard inequalities in order to obtain inequalities stronger than (2).

Theorem 1 ([11])**.** *Let $f(x)$ be a convex function on $[a,b]$. Then, for any $m,n \in \mathbb{N}\cup\{0\}$*

$$\begin{aligned} & f(\frac{a+b}{2}) \leq L_{f,n}^{(1)}(a,b) \\ \leq\ & \frac{1}{b-a}\int_a^b f(t)dt = \int_0^1 f((1-t)a+tb)dt \\ \leq\ & L_{f,m}^{(2)}(a,b) \leq \frac{f(a)+f(b)}{2}, \end{aligned} \tag{3}$$

where

$$L_{f,n}^{(1)}(a,b) = \frac{1}{2^n}\sum_{k=1}^{2^n} f((1-\frac{2k-1}{2^{n+1}})a + \frac{2k-1}{2^{n+1}}b)$$

and

$$L^{(2)}_{f,m}(a,b) = \frac{1}{2^{m+1}}\{f(a)+f(b)+2\sum_{k=1}^{2^m-1} f((1-\frac{k}{2^m})a+\frac{k}{2^m}b)\}.$$

Theorem 2 ([11])**.** *Let $f(x)$ be a convex function on $[a,b]$. Then, for any $v \in [0,1]$ and $m,n \in \mathbb{N}\cup\{0\}$,*

$$\begin{aligned} & f(\frac{a+b}{2}) \le r^{(1)}_{f,v,n}(a,b) \\ \le\ & \frac{1}{b-a}\int_a^b f(t)dt = \int_0^1 f((1-t)a+tb)dt \\ \le\ & r^{(2)}_{f,v,m}(a,b) \le \frac{f(a)+f(b)}{2}, \end{aligned} \tag{4}$$

where

$$\begin{aligned} & r^{(1)}_{f,v,n}(a,b) \\ =\ & \frac{1}{2^n}\sum_{k=1}^{2^n}\{vf((1-\frac{(2k-1)v}{2^{n+1}})a+\frac{(2k-1)v}{2^{n+1}}b)\} \\ & +\frac{1}{2^n}\sum_{k=1}^{2^n}\{(1-v)f((1-v-\frac{(2k-1)(1-v)}{2^{n+1}})a+(v+\frac{(2k-1)(1-v)}{2^{n+1}}b)\} \end{aligned}$$

and

$$\begin{aligned} & r^{(2)}_{f,v,m}(a,b) \\ =\ & \frac{1}{2^{m+1}}\{vf(a)+(1-v)f(b)+f((1-v)a+vb)\} \\ & +\frac{1}{2^m}\sum_{k=1}^{2^m-1}\{vf((1-\frac{kv}{2^m})a+\frac{kv}{2^m}b) \\ & +(1-v)f((1-v-\frac{k(1-v)}{2^m})a+(v+\frac{k(1-v)}{2^m})b)\}. \end{aligned}$$

In Section 2, we try to obtain an N-variable Hermite–Hadamard inequality. As applications we obtain several inequalities satisfied by arithmetic mean, geometric mean, logarithmic mean, harmonic mean, and so on. These means have the properties of symmetry. In Section 3, we obtain some norm inequalities. In Section 4, we obtain integral values of the Hermite–Hadamard inequality under some norm conditions.

2. *N*-Variable Hermite–Hadamard Inequality

We need the following result.

Lemma 1. *Let $x_1, x_2, \ldots, x_N \in \mathbb{R}$ or $x_1, x_2, \ldots, x_N \in X$, where X is a linear space. Then,*

$$\sum_{i=1}^{N} x_i = \frac{1}{N-1}\sum_{i<j}(x_i+x_j).$$

Proof.

$$\begin{aligned}
\sum_{i=1}^{N} x_i &= \frac{1}{2}\left\{\sum_{i=1}^{N} x_i + \sum_{j=1}^{N} x_j\right\} = \frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}(x_i + x_j) \\
&= \frac{1}{2N}\left\{2\sum_{i=1}^{N} x_i + \sum_{i\neq j}(x_i + x_j)\right\} \\
&= \frac{1}{N}\sum_{i=1}^{N} x_i + \frac{1}{2N}\left\{\sum_{i<j}(x_i + x_j) + \sum_{i>j}(x_i + x_j)\right\} \\
&= \frac{1}{N}\sum_{i=1}^{N} x_i + \frac{1}{N}\sum_{i<j}(x_i + x_j).
\end{aligned}$$

Then,

$$(1 - \frac{1}{N})\sum_{i=1}^{N} x_i = \frac{1}{N}\sum_{i<j}(x_i + x_j).$$

That is

$$\sum_{i=1}^{N} x_i = \frac{1}{N-1}\sum_{i<j}(x_i + x_j).$$

□

We have the following N-variable Hermite–Hadamard inequality.

Theorem 3. *Let $f(x)$ be a convex function on $\mathbb{R}$ and let $x_1, x_2, \ldots, x_N \in \mathbb{R}$. Then, for any $m, n \in \mathbb{R} \cup \{0\}$,*

$$\begin{aligned}
f(\frac{1}{N}\sum_{i=1}^{N} x_i) &\leq \frac{2}{N(N-1)}\sum_{i<j} L_{f,n}^{(1)}(x_i, x_j) \\
&\leq \frac{2}{N(N-1)}\sum_{i<j}\int_0^1 f((1-t)x_i + t x_j)dt \\
&\leq \frac{2}{N(N-1)}\sum_{i<j} L_{f,m}^{(2)}(x_i, x_j) \\
&\leq \frac{1}{N}\sum_{i=1}^{N} f(x_i).
\end{aligned}$$

Proof. By Lemma 1 and the convexity of $f(x)$,

$$\begin{aligned}
f(\frac{1}{N}\sum_{i=1}^{N} x_i) &= f(\frac{1}{N(N-1)}\sum_{i<j}(x_i + x_j)) = f(\frac{2}{N(N-1)}\sum_{i<j}\frac{x_i + x_j}{2}) \\
&\leq \frac{2}{N(N-1)}\sum_{i<j} f(\frac{x_i + x_j}{2}).
\end{aligned}$$

By (3),

$$
\begin{aligned}
& \frac{2}{N(N-1)} \sum_{i<j} f(\frac{x_i + x_j}{2}) \\
\leq \; & \frac{2}{N(N-1)} \sum_{i<j} L_{f,n}^{(1)}(x_i, x_j) \\
\leq \; & \frac{2}{N(N-1)} \sum_{i<j} \int_0^1 f((1-t)x_i + tx_j)dt \\
\leq \; & \frac{2}{N(N-1)} \sum_{i<j} L_{f,m}^{(2)}(x_i, x_j) \leq \frac{2}{N(N-1)} \sum_{i<j} \frac{f(x_i) + f(x_j)}{2}
\end{aligned}
$$

By using Lemma 1 again, we have the last inequality. □

When $f(x) = -\log x$, we have the following corollary.

Corollary 1. *Let $f(x) = -\log x$ and let $x_i > 0$ $(1 \leq i \leq N)$. We suppose that $x_i \neq x_j$ for $i \neq j$. Then,*

$$-\log \frac{1}{N} \sum_{i=1}^{N} x_i \leq \frac{2}{N(N-1)} \sum_{i<j} \left\{ \frac{x_i \log x_i}{x_j - x_i} - \frac{x_j \log x_j}{x_j - x_i} + 1 \right\} \leq -\frac{1}{N} \sum_{i=1}^{N} \log x_i.$$

That is

$$\frac{1}{N} \sum_{i=1}^{N} x_i \geq \exp\left\{ \frac{2}{N(N-1)} \sum_{i<j} \{ \frac{x_i \log x_i}{x_i - x_j} + \frac{x_j \log x_j}{x_j - x_i} - 1 \} \right\} \geq \left(\prod_{i=1}^{N} x_i \right)^{1/N}.$$

When $f(x) = e^x$, we have the following corollary.

Corollary 2. *Let $f(x) = e^x$. We suppose that $x_i \neq x_j$ for $i \neq j$. Then,*

$$exp\{ \frac{1}{N} \sum_{i=1}^{N} x_i \} \leq \frac{2}{N(N-1)} \sum_{i<j} \frac{e^{x_j} - e^{x_i}}{x_j - x_i} \leq \frac{1}{N} \sum_{i=1}^{N} e^{x_i}.$$

When $f(x) = x^{-1}$, we have the following corollary.

Corollary 3. *Let $f(x) = x^{-1}$ and let $x_i > 0$ $(1 \leq i \leq N)$. We suppose that $x_i \neq x_j$ for $i \neq j$. Then,*

$$\left(\frac{1}{N} \sum_{i=1}^{N} x_i \right)^{-1} \leq \frac{2}{N(N-1)} \sum_{i<j} \frac{\log x_j - \log x_i}{x_j - x_i} \leq \frac{1}{N} \sum_{i=1}^{N} x_i^{-1}.$$

That is

$$\frac{1}{N} \sum_{i=1}^{N} x_i \geq \left(\frac{2}{N(N-1)} \sum_{i<j} \left(\frac{x_j - x_i}{\log x_j - \log x_i} \right)^{-1} \right)^{-1} \geq \left(\frac{1}{N} \sum_{i=1}^{N} x_i^{-1} \right)^{-1}.$$

When $f(x) = x^2$, we have the following corollary.

Corollary 4. *Let $f(x) = x^2$. Then,*

$$\left(\frac{1}{N}\sum_{i=1}^{N} x_i\right)^2 \leq \frac{2}{3N(N-1)}\sum_{i<j}(x_j^2 + x_j x_i + x_i^2) \leq \frac{1}{N}\sum_{i=1}^{N} x_i^2.$$

3. Some Norm Inequalities

We put $a = 0$ and $b = 1$ in (2). Then, we have

$$f(\frac{1}{2}) \leq \int_0^1 f(t)dt \leq \frac{f(0)+f(1)}{2}.$$

Furthermore by (3), we have

$$\begin{aligned} & f(\frac{1}{2}) \leq \frac{1}{2^n}\sum_{k=1}^{2^n} f(\frac{2k-1}{2^{n+1}}) \leq \int_0^1 f(t)dt \\ \leq & \frac{1}{2^{m+1}}\{f(0)+f(1)+2\sum_{k=1}^{2^m-1} f(\frac{k}{2^m})\} \leq \frac{f(0)+f(1)}{2}. \end{aligned}$$

Now, we suppose that $F(x)$ is a convex and monotone increasing function on $[0,\infty)$. We put $f(t) = F(\|(1-t)x + ty\|)$, where $x, y \in X$ and X is a Banach space with norm $\|\cdot\|$. Then, $f(t)$ is convex on $[0,1]$. Because for any $t, s \in [0,1]$ and for any $\alpha, \beta \geq 0$ satisfying $\alpha + \beta = 1$,

$$\begin{aligned} f(\alpha t + \beta s) &= F(\|x + (\alpha t + \beta s)(y - x)\|) \\ &= F(\|\alpha(x + t(y-x)) + \beta(x + s(y-x))\|) \\ &\leq F(\alpha\|x + t(y-x)\| + b\|x + s(y-x)\|) \\ &\leq \alpha F(\|x + t(y-x)\|) + \beta F(\|x + s(y-x)\|) \\ &= \alpha f(t) + \beta f(s). \end{aligned}$$

Then, we have

Theorem 4. *Let $F(x)$ is a convex and monotone increasing function on $[0,\infty)$. Let X be a Banach space. We put $f(t) = F(\|(1-t)x + ty\|)$, where $x, y \in X$. Then, for any $x_1, x_2, \ldots, x_N \in X$ and for any $m, n \in \mathbb{N} \cup \{0\}$, we have*

$$\begin{aligned} & F(\|\frac{1}{N}\sum_{i=1}^{N} x_i\|) \\ \leq & \frac{2}{N(N-1)}\sum_{i<j}\frac{1}{2^n}\sum_{k=1}^{2^n} F(\|(1-\frac{2k-1}{2^{n+1}})x_i + \frac{2k-1}{2^{n+1}}x_j\|) \\ \leq & \frac{2}{N(N-1)}\sum_{i<j}\int_0^1 F(\|(1-t)x_i + tx_j\|)dt \\ \leq & \frac{2}{N(N-1)}\sum_{i<j}\frac{1}{2^{m+1}}\{F(\|x_i\|) + F(\|x_j\|) \\ & +2\sum_{k=1}^{2^m-1} F(\|(1-\frac{k}{2^m})x_i + \frac{k}{2^m}x_j\|)\} \\ \leq & \frac{1}{N}\sum_{i=1}^{N} F(\|x_i\|). \end{aligned}$$

Proof. By Lemma 1 and the convexity and monotonicity of $F(x)$,

$$\begin{aligned} & F(\|\frac{1}{N}\sum_{i=1}^{N} x_i\|) = F(\|\frac{1}{N(N-1)}\sum_{i<j}(x_i+x_j)\|) \\ = \;& F(\|\frac{2}{N(N-1)}\sum_{i<j}\frac{x_i+x_j}{2}\|) \leq F(\frac{2}{N(N-1)}\sum_{i<j}\|\frac{x_i+x_j}{2}\|) \\ \leq \;& \frac{2}{N(N-1)}\sum_{i<j}F(\|\frac{x_i+x_j}{2}\|). \end{aligned}$$

The inequalities, from the first to the third, are given by (3). Furthermore, the last inequality is given by Lemma 1. □

We take examples of $F(x)$.

Example 1. (1) $F(x) = x^p$, *where* $p \geq 1$.
(2) $F(x) = e^x$.
(3) $F(x) = \cosh(x) = \frac{e^x+e^{-x}}{2}$.
(4) $F(x) = (x+1)\log(x+1)$.

4. Calculations of the Detailed Integral Values

We need the following two lemmas in order to prove some theorems.

Lemma 2. *Let* $\|\cdot\|$ *be the Hilbert norm on a Hilbert space* H. *Then, for any* $x, y \in H$ *we have*

$$\int_0^1 \|(1-t)x+ty\|^2 dt = \frac{1}{6}\{\|x\|^2+\|y\|^2+\|x+y\|^2\}$$

Proof.

$$\begin{aligned} & \int_0^1 \|(1-t)x+ty\|^2 dt = \int_0^1 \|x+t(y-x)\|^2 dt \\ = \;& \|x\|^2 + \frac{1}{2}\langle x, y-x\rangle + \frac{1}{2}\langle y-x, x\rangle + \frac{1}{3}\|y-x\|^2 \\ = \;& \|x\|^2 + \frac{1}{2}\langle x, y\rangle - \frac{1}{2}\|x\|^2 + \frac{1}{2}\langle y, x\rangle - \frac{1}{2}\|x\|^2 + \frac{1}{3}\|y-x\|^2 \\ = \;& \frac{1}{2}\langle x, y\rangle + \frac{1}{2}\langle y, x\rangle + \frac{1}{3}\langle y-x, y-x\rangle \\ = \;& \frac{1}{2}\langle x, y\rangle + \frac{1}{2}\langle y, x\rangle + \frac{1}{3}(\|y\|^2 - \langle y, x\rangle - \langle x, y\rangle + \|x\|^2) \\ = \;& \frac{1}{3}\|x\|^2 + \frac{1}{3}\|y\|^2 + \frac{1}{6}\langle x, y\rangle + \frac{1}{6}\langle y, x\rangle \\ = \;& \frac{1}{6}\|x\|^2 + \frac{1}{6}\|y\|^2 + \frac{1}{6}\|x+y\|^2. \end{aligned}$$

□

Lemma 3. *Let $\|\cdot\|$ be the Hilbert norm on a Hilbert space H. Then, for any $x, y \in H$ we have*

$$
\begin{aligned}
& \int_0^1 \|(1-t)x+ty\|dt = \int_0^1 \sqrt{\|x+t(y-x)\|^2}dt \\
= \ & \int_0^1 \sqrt{\delta_{yx}^2 t^2 + 2\nu_{yx}t + \|x\|^2}dt \\
= \ & \frac{1}{2}\left\{ \frac{\nu_{yx}(\|y\|-\|x\|)+\delta_{yx}^2\|y\|}{\delta_{yx}^2} \right\} \\
& + \frac{1}{2}\left\{ \left(\frac{\|x\|^2}{\delta_{yx}} - \frac{\nu_{yx}^2}{\delta_{yx}^3} \right) \log \frac{\nu_{yx}+\delta_{yx}^2+\|y\|\delta_{yx}}{\nu_{yx}+\|x\|\delta_{yx}} \right\} \\
= \ & \frac{1}{2}\left\{ \frac{(\mathrm{Re}\langle x,y\rangle - \|x\|^2)(\|y\|-\|x\|)+\delta_{yx}^2\|y\|}{\delta_{yx}^2} \right\} \\
& + \frac{1}{2}\left\{ \left(\frac{\|x\|^2}{\delta_{yx}} - \frac{(\mathrm{Re}\langle x,y\rangle - \|x\|^2)^2}{\delta_{yx}^3} \right) \log \frac{\|y\|^2 - \mathrm{R}\langle x,y\rangle + \|y\|\delta_{yx}}{\mathrm{Re}\langle x,y\rangle - \|x\|^2 + \|x\|\delta_{yx}} \right\},
\end{aligned}
$$

where $\delta_{yx} = \|y-x\|$ and $\nu_{yx} = \mathrm{Re}\langle x, y-x\rangle$.

Proof. Since

$$
\begin{aligned}
& \int_0^1 \sqrt{\|y-x\|^2 t^2 + 2\mathrm{Re}\langle x,y-x\rangle t + \|x\|^2}dt \\
= \ & \|y-x\| \int_0^1 \sqrt{t^2 + \frac{2\mathrm{Re}\langle x,y-x\rangle}{\|y-x\|^2}t + \frac{\|x\|^2}{\|y-x\|^2}}dt \\
= \ & \|y-x\| \int_0^1 \sqrt{(t + \frac{\mathrm{Re}\langle x,y-x\rangle}{\|y-x\|^2})^2 - \frac{(\mathrm{Re}\langle x,y-x\rangle)^2}{\|y-x\|^4} + \frac{\|x\|^2}{\|y-x\|^2}}dt,
\end{aligned}
$$

we may obtain the integral value of $\int_0^1 \sqrt{(t+a)^2+b^2}dt$, where

$$a = \frac{\mathrm{Re}\langle x,y-x\rangle}{\|y-x\|^2}$$

and

$$b^2 = -\frac{(\mathrm{Re}\langle x,y-x\rangle)^2}{\|y-x\|^4} + \frac{\|x\|^2}{\|y-x\|^2}.$$

Then,

$$
\begin{aligned}
& \int_0^1 \sqrt{(t+a)^2+b^2}dt \\
= \ & \int_a^{a+1} \sqrt{s^2+b^2}ds \\
= \ & \left[\frac{1}{2}(s\sqrt{s^2+b^2} + b^2 \log|s+\sqrt{s^2+b^2}|) \right]_a^{a+1} \\
= \ & \frac{1}{2}\left\{ (a+1)\sqrt{(a+1)^2+b^2} + b^2 \log|a+1+\sqrt{(a+1)^2+b^2}| \right\} \\
& - \frac{1}{2}\left\{ a\sqrt{a^2+b^2} + b^2 \log|a+\sqrt{a^2+b^2}| \right\}.
\end{aligned}
$$

Since

$$\sqrt{(a+1)^2+b^2} = \frac{\|y\|}{\|y-x\|}, \qquad \sqrt{a^2+b^2} = \frac{\|x\|}{\|y-x\|},$$

we obtain the result. □

Corollary 5. *Let $\|\cdot\|$ be the Hilbert norm on a Hilbert space H and let $F(x) = x^2$. Then, for any $x_1, x_2, \ldots, x_N \in H$ we have*

$$\|\frac{1}{N}\sum_{i=1}^{N} x_i\|^2 \le \frac{1}{3N}\left\{\sum_{i=1}^{N}\|x_i\|^2 + \frac{1}{N-1}\sum_{i<j}\|x_i + x_j\|^2\right\} \le \frac{1}{N}\sum_{i=1}^{N}\|x_i\|^2.$$

Proof. It is clear from Lemma 2. □

Corollary 6. *Let $\|\cdot\|$ be the Hilbert norm on a Hilbert space H and let $F(x) = x$. Then, for any $x_1, x_2, \ldots, x_N \in H$ we have*

$$\begin{aligned} & \|\sum_{i=1}^{N} x_i\| \\ \le \quad & \frac{1}{N-1}\sum_{i<j}\left\{\frac{(\mu_{ij} - \|x_i\|^2)(\|x_j\| - \|x_i\|) + \delta_{ji}^2\|x_j\|}{\delta_{ji}^2}\right\} \\ + \quad & \frac{1}{N-1}\sum_{i<j}\left\{\left(\frac{\|x_i\|^2}{\delta_{ji}} - \frac{(\mu_{ij} - \|x_i\|^2)^2}{\delta_{ji}^3}\right)\log\frac{\|x_j\|^2 - \mu_{ij} + \|x_j\|\delta_{ji}}{\mu_{ij} - \|x_i\|^2 + \|x_i\|\delta_{ji}}\right\} \\ \le \quad & \sum_{i=1}^{N}\|x_i\|, \end{aligned}$$

where $\delta_{ji} = \|x_j - x_i\|$ and $\mu_{ij} = \mathrm{Re}\langle x_i, x_j\rangle$.

Proof. It is clear from Lemma 3. □

Corollary 7. *Let $\|\cdot\|$ be the Hilbert–Schmidt norm on all of the Hilbert–Schmidt class operators and let $F(x) = x^2$. Then for any positive Hilbert–Schmidt operators $A_1, A_2, \ldots, A_N$ we have*

$$\|\frac{1}{N}\sum_{i=1}^{N} A_i\|^2 \le \frac{1}{3N}\left\{\sum_{i=1}^{N}\|A_i\|^2 + \frac{1}{N-1}\sum_{i<j}\|A_i - A_j\|^2\right\} \le \frac{1}{N}\sum_{i=1}^{N}\|A_i\|^2.$$

Proof. It is clear from Lemma 2. □

Corollary 8. *Let $\|\cdot\|$ be the Hilbert–Schmidt norm on all of the Hilbert–Schmidt class operators and let $F(x) = x$. Then for any positive Hilbert–Schmidt operators $A_1, A_2, \ldots, A_N$ we have*

$$\begin{aligned} & \|\sum_{i=1}^{N} A_i\| \\ \le \quad & \frac{1}{N-1}\sum_{i<j}\left\{\frac{(t_{ij} - \|A_i\|^2)(\|A_j\| - \|A_i\|) + \delta_{ji}^2\|A_j\|}{\delta_{ji}^2}\right\} \\ + \quad & \frac{1}{N-1}\sum_{i<j}\left\{\left(\frac{\|A_i\|^2}{\delta_{ji}} - \frac{(t_{ij} - \|A_i\|^2)^2}{\delta_{ji}^3}\right)\log\frac{\|A_j\|^2 - t_{ij} + \|A_j\|\delta_{ji}}{t_{ij} - \|A_i\|^2 + \|A_i\|\delta_{ji}}\right\} \\ \le \quad & \sum_{i=1}^{N}\|A_i\|, \end{aligned}$$

where $\delta_{ji} = \|A_j - A_i\|$ and $t_{ij} = \mathrm{Tr}[A_iA_j]$.

Proof. It is clear from Lemma 3. □

5. Conclusions

Though the Hermite–Hadamard inequality had been given in 2-variable inequality for convex function, we obtained N-variable Hermite–Hadamard inequality in Theorem 3. Furthermore, we obtained one of norm inequalities as applications of Theorem 4 represented by an N-variable Hermite–Hadamard inequality. Lastly, we calculated several detailed integral values of norm inequalities.

Funding: The author is partially supported by JSPS KAKENHI 19K03525.

Data Availability Statement: Not applicable.

Acknowledgments: The author would like to thank the reviewers for their important suggestions and careful reading of the manuscript.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Cerone, P.; Dragomir, S.S. Ostrowski type inequalities for functions whose derivatives satisfying certain convexity assumptions. *Demonstr. Math.* **2004**, *37*, 299–308. [CrossRef]
2. Dragomir, S.S.; Agarwal, R.P. Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. *Appl. Math. Lett.* **1998**, *11*, 91–99 . [CrossRef]
3. Dragomir, S.S.; Cerone, P.; Sofo, A. Some remarks on the midpoint rule in numerical integration. *Studia Univ. Babes-Bolyai, Math.* **2000**, *45*, 63–74.
4. Dragomir, S.S.; Cerone, P.; Sofo, A. Some remarks on the trapezid rule in numerical integration. *Indian J. Pure Appl. Math.* **2000**, *31*, 475–494.
5. Furuichi, S.; Moradi, H.R. *Advances in Mathematical Inequalities*; De Gruyter: Berlin, Germany, 2020.
6. Mitroi-Symeonidis, F.C. About the precision in Jensen-Steffensen inequality. *An. Univ. Craiova Ser. Mat. Inform.* **2010**, *37*, 73–84.
7. Moslehian, M.S. Matrix Hermite-Hadamard type inequalities. *Houston J. Math.* **2013**, *39*, 177–189.
8. Pal, R.; Singh, M.; Moslehian, M.S.; Aujla, J.S. A new class of operator monotone functions via operator means. *Linear Multilinear Algebra* **2016**, *64*, 2463–2473. [CrossRef]
9. Simic, S.; Bin-Mohsin, B. Some generalizations of the Hermite-Hadamard integral inequality. *J. Inequal. Appl.* **2021**, *72*, 1–7.
10. Yanagi, K. Refined Hermite-Hadamard inequality and its application. *Linear Nonlinear Anal.* **2021**, *7*, 173–184.
11. Yanagi, K. Refined Hermite-Hadamard inequality and weighted logarithmic mean. *Linear Nonlinear Anal.* **2020**, *6*, 167–177.