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Subordination Results for the Second-Order Differential Polynomials of Meromorphic Functions

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Abstract: The outcome of the research presented in this paper is the definition and investigation of two new subclasses of meromorphic functions. The new subclasses are introduced using a differential operator defined considering second-order differential polynomials of meromorphic functions in $U\backslash\{0\} = \{z \in \mathbb{C} : 0 < |z| < 1\}$. The investigation of the two new subclasses leads to establishing inclusion relations and the proof of convexity and convolution properties regarding each of the two subclasses. Further, involving the concept of subordination, the Fekete–Szegö problem is also discussed for the aforementioned subclasses. Symmetry properties derive from the use of the convolution and from the convexity proved for the new subclasses of functions.

Keywords: analytic function; meromorphic function; differential operator; differential polynomials; subordination; differential operator; convolution; Fekete–Szegö problem

MSC: 30C45

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1. Introduction

The results presented in this paper are obtained considering the general context of geometric function theory and involve the class of meromorphic functions, a differential operator and certain well-known and intensely used tools for investigation, namely the concepts of subordination and convolution.

The basic classes involved in this study are introduced in the unit disc of the complex plane $U = \{z \in \mathbb{C} : |z| < 1\}$.

Let A denote the class of analytic functions in U of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which satisfies the relations $f(0) = 0$, $f'(0) = 1$. The subclass of A containing the univalent functions in U is denoted by S. The subclasses of A referred to as the class of starlike functions and the class of convex functions are denoted by S^* and K, respectively, and are defined as $S^* = \left\{f \in A : \mathrm{Re}\frac{zf'(z)}{f(z)} > 0\right\}$ and $K = \left\{f \in A : \mathrm{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > 0\right\}$.

Another special class of analytic and univalent functions is the class of meromorphic functions denoted by Σ and containing functions of the form:

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n = \frac{1}{z} + a_1 z + a_2 z^2 + a_3 z^3 + \dots, \tag{2}$$

analytic and univalent in $U^* = U\backslash\{0\} = \{z \in \mathbb{C} : 0 < |z| < 1\}$.

A function $f \in \Sigma$ is said to be in the class S^* of meromorphic starlike functions in U^* if and only if

$$-R\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad (z \in U^*). \tag{3}$$

A function $f \in \Sigma$ is said to be in the class C of meromorphic convex functions in U^* if and only if

$$-R\left\{1+\frac{zf''(z)}{f'(z)}\right\} > 0, \quad (z \in U^*). \tag{4}$$

We note that $f \in C \Leftrightarrow -zf' \in S^*$.

Considering two meromorphic functions having the form presented in (2) written as

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \quad g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n,$$

$f * g$ denotes the Hadamard product (or convolution) of f and g defined as [1]:

$$(f * g)(z) := \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n, \quad z \in U^*.$$

The class of meromorphic functions generates many interesting results when considered in studies regarding geometric function theory. Subclasses of meromorphic functions with positive coefficients were introduced and studied [2–5]. Certain differential operators were involved in the studies related to meromorphic functions [6–10]; integral operators were also added to research [11–13], and linear operators are also present in investigations [14–16]. The results presented in this paper concern a differential operator, which is introduced here by using differential polynomials of meromorphic functions.

2. Preliminaries

Definition 1. *For $n \geq 1$, let f be a meromorphic function, $a \in \mathbb{C}$ and $a \neq 0, \infty$, the differential polynomials $f^k, k = 1, 2, 3, \ldots$. Then, the differential polynomials of meromorphic functions can be written as follows:*

$$f(z) = \frac{1}{z} + a_1 z + a_2 z^2 + a_3 z^3 + \ldots,$$

$$\begin{aligned} f'(z) &= \frac{-1}{z^2} + a_1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \ldots \\ &= \frac{-1}{z^2} + \sum_{n=1}^{\infty} n a_n z^{n-1}, \end{aligned} \tag{5}$$

$$\begin{aligned} f''(z) &= \frac{2}{z^3} + 2a_2 + 6a_3 z + 12a_4 z^2 + 20a_5 z^3 + \ldots \\ &= \frac{2}{z^3} + \sum_{n=1}^{\infty} n(n+1) a_{n+1} z^{n-1}, \end{aligned} \tag{6}$$

$$\begin{aligned} f'''(z) &= \frac{-6}{z^4} + 6a_3 + 24a_4 z + 60a_5 z^2 + 120a_6 z^3 + \ldots \\ &= \frac{-6}{z^4} + \sum_{n=1}^{\infty} n(n+1)(n+2) a_{n+2} z^{n-1} \end{aligned} \tag{7}$$

$$\vdots$$

$$f^k(z) = \frac{(-1)^k k!}{z^{k+1}} + \sum_{n=1}^{\infty} \underbrace{n(n+1)(n+2)\dots(n+k-1)}_{k} a_{n+k-1} z^{n-1}.$$

Following the same procedure as seen in [17] and used by many other authors, the following operator is defined:

Definition 2. *Considering $f \in \Sigma$ and $f''(z)$ given by (6), a new differential operator is defined as:*

$$D^0 z^2 f''(z) = z^2 f''(z)$$

$$D^1 z^2 f''(z) = (1-\lambda) z^2 f''(z) + \lambda \frac{\left(z^2 \left(z^2 f''(z)\right)\right)'}{z} = (1+\lambda) z^2 f''(z) + \lambda \left(z^2 f''(z)\right)'$$

$$= D_\lambda z^2 f''(z), \quad \lambda \geq 0 \tag{8}$$

$$D^k z^2 f''(z) = D_\lambda \left(D^{k-1} z^2 f''(z)\right). \tag{9}$$

Then, from (8) and (9), we see that

$$D^k z^2 f''(z) = \frac{2}{z} + \sum_{n=1}^{\infty} n(n+1)[1+(n+1)\lambda]^k a_{n+1} z^{n+1}, \quad z \in U^*. \tag{10}$$

For the investigation presented in this paper, the following two subclasses of meromorphic functions are introduced using the operator given by (10):

Definition 3. *In conjunction with (3) and (10),*

$$S^{*,k}(\lambda) = \{f : f \in \Sigma \ \ and \ \ D^k z^2 f''(z) \in S^*\}. \tag{11}$$

Definition 4. *In conjunction with (4) and (10),*

$$C^k(\lambda) = \{f : f \in \Sigma \ \ and \ \ z(D^k z^2 f''(z))' \in C\}. \tag{12}$$

In order to obtain the new results contained in Section 3, the following lemmas are necessary:

Lemma 1 ([18])**.** *Let p be analytic in U with $p(0) = 1$, and suppose that*

$$R\left\{p(z) - \frac{zp'(z)}{p(z)}\right\} > 0, \ \ z \in U$$

Then we have

$$R(p(z)) > 0 \ \ in \ \ U.$$

Lemma 2 ([1])**.** *If $g \in \mathcal{S}^*$ and $f \in \mathcal{C}$, then $f * g \in \mathcal{S}^*$.*

Having in mind the very recent papers where the well-known Fekete–Szegö inequalities are evaluated for different subclasses of meromorphic functions [19–21], in Section 4 of the paper, the Fekete–Szegö problem is investigated concerning the classes defined by (11) and (12). In order to conduct the study, the notion of subordination is applied, and the classes are redefined in terms of subordination.

Let f and g be two analytic functions in U. We say that f is subordinate to g, written $f(z) \prec g(z)$, if there exists a Schwarz function w, analytic in U with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$; see [22]. If g is univalent in U, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

Definition 5. *Consider $\Phi(z)$ an analytic function with $Re\Phi(z) > 0, z \in U$, satisfying $\Phi(0) = 1$, $\Phi'(0) > 1$. Such a function maps U onto a region that is symmetric with respect to the real axis and is starlike with respect to 1. Denote by $S^{*,k}(\Phi, \lambda)$ the class of functions $f \in S$ that satisfy the subordination given by:*

$$-\frac{z(D^k z^2 f''(z))'}{D^k z^2 f''(z)} \prec \Phi(z), \qquad (\lambda \in \mathbb{C} \setminus \{0\}). \tag{13}$$

Definition 6. *Consider $\Phi(z)$ an analytic function with $Re\Phi(z) > 0, z \in U$, satisfying $\Phi(0) = 1$, $\Phi'(0) > 1$. Such a function maps U onto a region which is symmetric with respect to the real axis and is starlike with respect to 1. Denote by $C^k(\Phi, \lambda)$ the class of functions $f \in S$ that satisfy the subordination given by:*

$$-\left\{1 + \frac{z(D^k z^2 f''(z))''}{(D^k z^2 f''(z))'}\right\} \prec \Phi(z), \qquad (\lambda \in \mathbb{C} \setminus \{0\}). \tag{14}$$

We also note that

$$D^k z^2 f''(z) \in C^k(\lambda) \Leftrightarrow z(D^k z^2 f''(z))' \in S^{*,k}(\lambda). \tag{15}$$

For obtaining the Fekete–Szegö inequalities related to the classes given in Definitions 5 and 6, the following lemmas are applied:

Lemma 3 ([23])**.** *If $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots$ is a function with a positive real part in U and $\sigma \in \mathbb{R}$, then*

$$\left|c_2 - \sigma c_1^2\right| \le 2\max\{1; |2\sigma - 1|\}.$$

Lemma 4 ([23])**.** *If $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots$ is a function with a positive real part in U and $\sigma \in \mathbb{R}$, then*

$$\left|c_2 - \sigma c_1^2\right| \le \begin{cases} -4\sigma + 2, & \sigma \le 0 \\ 2 \quad , & 0 \le \sigma \le 1 \\ 4\sigma - 2, & \sigma \ge 1 \end{cases}.$$

Now, after all the preliminary notations, definitions and lemmas are listed, the next two sections contain the new results, which the authors want to bring to researchers' attention. In Section 3, containment relations are established for the classes $S^{*,k}(\lambda)$ and $C^k(\lambda)$ introduced in Definitions 3 and 4. It is proved that the functions belonging to those classes are convex, and also, convolution properties are obtained using the functions $f, g \in S^{*,k}(\lambda)$ and $f, g \in C^k(\lambda)$, respectively. In Section 4, the Fekete–Szegö problem is considered for the classes $S^{*,k}(\Phi)$ and $C^k(\Phi)$ seen in Definitions 5 and 6.

3. Inclusion and Convolution Theorems

The results presented in this section refer to inclusion relations established for the classes $S^{*,k}(\lambda)$ and $C^k(\lambda)$ introduced in Definitions 3 and 4, respectively. Convexity properties are stated for classes $S^{*,k}(\lambda)$ and $C^k(\lambda)$, and convolution properties involving functions from the two classes are also proved easily by using the iterative-type operator seen in Definition 2.

The first two theorems from this section include containment relations obtained for classes $S^{*,k}(\lambda)$ and $C^k(\lambda)$ given by (12) and (13).

Theorem 1.

$$S^{*,k+1}(\lambda) \subset S^{*,k}(\lambda). \tag{16}$$

Proof. Let $z^2 f''(z) \in S^{*,k+1}(\lambda)$ and suppose that

$$R\left\{\frac{z(D^{k+2}z^2 f''(z))}{D^{k+1}z^2 f''(z)}\right\} > 0, \quad z \in U. \tag{17}$$

Set

$$p(z) = \left(\frac{z(D^{k+1}z^2 f''(z))}{D^k z^2 f''(z)}\right) > 0, \quad z \in U. \tag{18}$$

The analytic function $p(z)$ satisfies conditions $p(0) = 1$ and $p(z) \neq 0$ for all $z \in U$. Differentiating logarithmically (18), and after manipulations, we obtain

$$\frac{zp'(z)}{p(z)} = \frac{z(D^{k+1}z^2 f''(z))'}{D^k z^2 f''(z)} - \frac{z(D^{k+1}z^2 f''(z))'}{D^k z^2 f''(z)}.$$

Since $z(D^k z^2 f''(z))' = -z(D^{k+1}z^2 f''(z))$ coupled with (18) yields

$$p(z) - \frac{zp'(z)}{p(z)} = \frac{z(D^{k+2}z^2 f''(z))}{D^{k+1}z^2 f''(z)}$$

that is

$$R\left(p(z) - \frac{zp'(z)}{p(z)}\right) > 0, \quad z \in U.$$

Now, by applying Lemma 1, we obtain that $z^2 f''(z) \in S^{*,k}(\lambda)$, $z \in U^*$. □

Theorem 2. *Let $\lambda \in \mathbb{C} \setminus \{0\}$. Then*

$$C^{k+1}(\lambda) \subset C^k(\lambda). \tag{19}$$

Proof. Applying (15) and Theorem 1, the following can be written:

$$\begin{aligned} z^2 f''(z) \in C^{k+1}(\lambda) &\Leftrightarrow D^k z^2 f''(z) \in C(\lambda) \\ &\Leftrightarrow z(D^k z^2 f''(z))' \in S^*(\lambda) \\ &\Leftrightarrow D^k\left(z(z^2 f''(z))'\right) \in S^*(\lambda) \\ &\Leftrightarrow z(z^2 f''(z))' \in S^{*,k+1}(\lambda) \\ &\Rightarrow z(z^2 f''(z))' \in S^{*,k}(\lambda) \\ &\Leftrightarrow D^k\left(z(z^2 f''(z))'\right) \in S^*(\lambda) \\ &\Leftrightarrow z(D^k)' \in S^*(\lambda) \\ &\Leftrightarrow D^k \in C^k(\lambda) \\ &\Leftrightarrow z^2 f''(z) \in C^k(\lambda), \end{aligned}$$

which evidently proves Theorem 2. □

The next two theorems prove the property of the sets of functions $S^{*,k}(\lambda)$ and $C^k(\lambda)$ to be convex.

Theorem 3. *The set $S^{*,k}(\lambda)$ is convex.*

Proof. Consider the function $z^2 f''(z) \in S^{*,k+1}(\lambda)$ and suppose that

$$z^2 f_i''(z) = \frac{2}{z} + \sum_{n=1}^{\infty} n(n+1) a_{(n+1)i} z^{n+1}, \quad (i = 1, 2), \tag{20}$$

belongs to the class $S^{*,k}(\lambda)$. The proof requires to show that the function $h(z) = u_1 f_1(z) + u_2 f_2(z)$, with u_1 and u_2 nonnegative and $u_1 + u_2 = 1$, belongs to the class $S^{*,k}(\lambda)$.

$$h(z) = \frac{2}{z^3} + \sum_{n=1}^{\infty} n(n+1)(u_1 a_{(n+1)1} + u_2 a_{(n+1)2} z^{n+1}, \quad (i = 1, 2), \tag{21}$$

Then, from (11), we have

$$z(D^k z^2 h(z))' = \frac{-6}{z} + \sum_{n=1}^{\infty} n(n+1)(n+2)[1 + (n+1)\lambda]^k (u_1 a_{(n+1)1} + u_2 a_{(n+1)2}) z^{n+1}. \tag{22}$$

Hence

$$-R\Big(z(D^k z^2 h(z))'\Big) = -R\left(\frac{-6}{z} + u_1 \sum_{n=1}^{\infty} n(n+1)(n+2)[1 + (n+1)\lambda]^k a_{(n+1)1} z^{n+1}\right) \tag{23}$$
$$-R\left(\frac{-6}{z^3} + u_2 \sum_{n=1}^{\infty} n(n+1)(n+2)[1 + (n+1)\lambda]^k a_{(n+1)2} z^{n}\right).$$

Since $f_1, f_2 \in S^{*,k}(\lambda)$, this implies that

$$-R\left(\frac{-6}{z} + u_i \sum_{n=1}^{\infty} n(n+1)(n+2)[1 + (n+1)\lambda]^k a_{(n+1)1} z^{n+1}\right) > 1 - u_i. \tag{24}$$

Using (24) in (23), we obtain

$$-R(z(D^k z^2 h(z))') > 1 - (u_1 + u_2) \tag{25}$$

and since $u_1 + u_2 = 1$, the theorem is proved. □

Theorem 4. *The set $C^k(\lambda)$ is convex.*

Proof. From Theorem 3 and (15), it follows easily that Theorem 4 is true. □

In the next four theorems, results related to convolution properties for the classes $S^{*,k}(\lambda)$ and $C^k(\lambda)$ are derived.

Theorem 5. *Consider the function $z^2 f''(z) \in S^{*,k}(\lambda)$. The following equality holds:*

$$D^k z^2 f''(z) = z^{-1} \cdot \exp \int_0^z \left(\frac{2w(t)}{t(w(t) - 1)} dt. \right), \tag{26}$$

where w is analytic in U with $|w(z)| < 1$ and $w(z) = 0$.

Proof. For $z^2 f''(z) \in S^{*,k}(\lambda)$, we write the following:

$$\frac{-z(D^k z^2 f''(z))'}{D^k z^2 f''(z)} = \frac{1 + w(z)}{1 - w(z)},$$

where w is analytic in U with $|w(z)| < 1$ and $w(z) = 0$. From this, we obtain

$$\frac{z(D^k z^2 f''(z))'}{D^k z^2 f''(z)} + \frac{1}{z} = \frac{2w(z)}{z(w(z)-1)},$$

which upon integration yields

$$In(D^k z^2 f''(z)) = \exp \int_0^z \left(\frac{2w(t)}{t(w(t)-1)} dt \right). \tag{27}$$

Assertion (27) can easily be obtained from (26). □

Theorem 6. *If $z^2 f''(z) \in C^k(\lambda)$, then*

$$z(D^k z^2 f''(z))' = \xi', \tag{28}$$

and

$$\xi = z^{-1} \exp \int_0^z \left(\frac{2w(t)}{t(w(t)-1)} dt. \right),$$

where w is analytic in U with $|w(z)| < 1$ and $w(z) = 0$.

Proof. Suppose that $z^2 f''(z) \in C^k(\lambda)$; then (12) can be written as follows:

$$1 + \frac{-z(D^k z^2 f''(z))''}{(D^k z^2 f''(z))'} = \frac{1+w(z)}{1-w(z)}$$

where w is analytic in U with $|w(z)| < 1$ and $w(z) = 0$. From this, we obtain

$$1 + \frac{-z(D^k z^2 f''(z))''}{(D^k z^2 f''(z))'} + \frac{1}{z} = \frac{2w(z)}{z(w(z)-1)}.$$

By integrating the above relation, we obtain:

$$In(z(D^k z^2 f''(z))') = \exp \int_0^z \left(\frac{2w(t)}{t(w(t)-1)} dt \right). \tag{29}$$

It follows that

$$z(D^k z^2 f''(z))' = \int_0^z \xi', \tag{30}$$

$$z(D^k z^2 f''(z))' = \int_0^z \left(\exp \int_0^z \left(\frac{2w(t)}{t(w(t)-1)} dt \right) \right) dt. \tag{31}$$

The equality given by relation (28) of Theorem 6 is easily obtained from (18) and (31). □

Theorem 7. *Consider functions $z^2 f''(z)$ and $z^2 g''(z)$ from the class $S^{*,k}(\lambda)$. Then, $z^2 f''(z) * z^2 g''(z) \in S^{*,k}(\lambda)$.*

Proof. Knowing that $g''(z)$ is convex univalent in U, from (11), we have

$$-R\left\{ \frac{z(D^k z^2 g''(z))'}{D^k z^2 g''(z)} \right\} > 0. \tag{32}$$

By applying convolution properties, we deduce:

$$-R\left\{\frac{z(D^k z^2 f''(z))'}{D^k z^2 f''(z)}\right\} * -R\left\{\frac{z(D^k z^2 g''(z))'}{D^k z^2 g''(z)}\right\}$$
$$= R\left((\frac{z(D^k z^2 f''(z))'}{D^k z^2 f''(z)}) * (\frac{z(D^k z^2 f g''(z))'}{D^k z^2 g''(z)}) \right). \quad (33)$$

The proof is concluded by applying Lemma 2. □

Theorem 8. *Consider the functions $z^2 f''(z)$ and $z^2 g''(z)$ from the class $C^k(\lambda)$. Then, $z^2 f''(z) * z^2 g''(z) \in C^k(\lambda)$.*

Proof. Applying (15), Lemma 2 and Theorem 7, we have

$$\begin{aligned}
z^2 f''(z), z^2 g''(z) \in C^k(\lambda) &\Leftrightarrow D^k z^2 f''(z) \in C(\lambda), and\ \ D^k z^2 g''(z) \in C(\lambda) \\
&\Leftrightarrow z(D^k z^2 f''(z))' \in S^*(\lambda), and\ \ z(D^k z^2 g''(z))' \in S^*(\lambda) \\
&\Leftrightarrow D^k\big(z(z^2 f''(z))'\big) \in S^*(\lambda), and\ \ D^k\big(z(z^2 g''(z))'\big) \in S^*(\lambda) \\
&\Leftrightarrow z(z^2 f''(z))' \in S^{*,k}(\lambda), and\ \ z(z^2 g''(z))' \in S^{*,k}(\lambda) \\
&\Rightarrow \big(z(z^2 f''(z))' * z(z^2 g''(z))'\big) \in S^{*,k}(\lambda) \\
&\Leftrightarrow \big(D^k\big(z(z^2 f''(z))'\big) * D^k\big(z(z^2 g''(z))'\big) \in S^*(\lambda) \\
&\Leftrightarrow \big(D^k\big(z(z^2 f''(z))'\big) * D^k\big(z(z^2 g''(z))'\big) \in C^k(\lambda) \\
&\Leftrightarrow (z^2 f''(z) * z^2 g''(z)) \in C^k(\lambda),
\end{aligned}$$

which evidently proves Theorem 8. □

4. Fekete–Szegö Problem

Fekete–Szegö inequalities are obtained in this section considering functions from classes $S^{*,k}(\Phi)$ and $C^k(\Phi)$ given by Definitions 5 and 6, respectively. In order to obtain those results, similar methods to those seen in [24] are implemented, and the proof of the results is facilitated by the use of the iterative-type operator given by Definition 2.

Theorem 9. *Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots$, with $(B_1 \neq 0)$ and f'' given in (6) belonging to $S^{*,k}(\Phi, \lambda)$; then, for $\sigma \in \mathbb{C}$,*

$$|a_3 - \sigma a_2^2| \quad \leq \frac{|B_1|}{6[1+3\lambda]^k} \max\left\{1; \left|\frac{B_2}{B_1} + \sigma \frac{[1+3\lambda]^k B_1}{(1+\frac{1}{3})[1+2\lambda]^{2k}}\right|\right\}.$$

The result is sharp.

Proof. Consider the function $f''(z) \in S^{*,k}(\Phi, \lambda)$. In this case, an analytic Schwarz function w exists with $w(0) = 0$ and $|w(z)| < 1$ in U such that

$$-\frac{z(D^k z^2 f''(z))'}{D^k z^2 f''(z)} = \Phi(w(z)). \quad (34)$$

We define the function $p(z)$ by

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \ldots, \quad (35)$$

since $w(z)$ is a Schwartz function. Therefore,

$$\begin{aligned}\Phi(w(z)) &= \Phi\left(\frac{p(z)-1}{p(z)+1}\right) \\ &= \Phi\left(\frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \left(c_3 - d_1 c_2 + \frac{c_1^3}{4}\right)z^3 + \ldots\right]\right) \\ &= 1 + \frac{1}{2}B_1 c_1 z + \frac{1}{2}\left(B_1\left[c_2 - \frac{c_1^2}{2}\right] + \frac{1}{2}B_2 c_1^2\right)z^2 + \ldots .\end{aligned} \tag{36}$$

By substituting (34) in (36), we obtain

$$-\frac{z(D^k z^2 f''(z))'}{D^k z^2 f''(z)} = 1 + \frac{B_1 c_1}{2} z + \left(\frac{B_1}{2}\left[c_2 - \frac{c_1^2}{2}\right] + \frac{B_2 c_1^2}{4}\right)z^2 + \ldots . \tag{37}$$

From Equation (37), we obtain

$$\begin{aligned} a_2 &= -\frac{B_1 c_1}{4[1+2\lambda]^k}, \\ a_3 &= -\frac{1}{3[1+3\lambda]^k}\left(\frac{B_1}{2}\left[c_2 - \frac{c_1^2}{2}\right] + \frac{B_2 c_1^2}{4}\right). \end{aligned}$$

Therefore,

$$a_3 - \sigma a_2^2 = -\frac{B_1}{6[1+3\lambda]^k}\left\{c_2 - \nu c_1^2\right\}, \tag{38}$$

where

$$\nu = \frac{1}{2}\left[1 + \frac{B_2}{B_1} - \sigma\frac{[1+3\lambda]^k B_1}{(1+\frac{1}{3})[1+2\lambda]^{2k}}\right]. \tag{39}$$

The proof is concluded by applying Lemma 3. The sharpness of the results is obtained for the functions

$$-\frac{z(D^k z^2 f''(z))'}{D^k z^2 f''(z)} = \Phi(z^2)$$

and

$$-\frac{z(D^k z^2 f''(z))'}{D^k z^2 f''(z)} = \Phi(z).$$

All the assertions of Theorem 9 are now proved. □

Theorem 10. *Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots$ with $(B_1 \neq 0)$ and let f'' given in (6) belong to $C^k(\Phi, \lambda)$. Then, for $\sigma \in \mathbb{C}$,*

$$\left|a_3 - \sigma a_2^2\right| \le \tfrac{|B_1|}{8[1+3\lambda]^k}\max\left\{1; \left|\tfrac{B_2}{B_1} + \sigma\tfrac{8[1+3\lambda]^k B_1}{[1+2\lambda]^{2k}}\right|\right\}.$$

The result is sharp.

Proof. Consider the function $f''(z) \in C^k(\Phi, \lambda)$. In this case, an analytic Schwarz function w exists with $w(0) = 0$ and $|w(z)| < 1$ in U such that

$$-\left\{1 + \frac{z(D^k z^2 f''(z))''}{(D^k z^2 f''(z))'}\right\} = \Phi(w(z)). \tag{40}$$

Along Equations (35), (36) and (40), we have

$$-\left\{1+\frac{z(D^kz^2f''(z))''}{(D^kz^2f''(z))'}\right\}=1+\frac{B_1c_1}{2}z+\left(\frac{B_1}{2}\left[c_2-\frac{c_1^2}{2}\right]+\frac{B_2c_1^2}{4}\right)z^2+\ldots. \tag{41}$$

From Equation (41), we obtain

$$a_2=-\frac{B_1c_1}{[1+2\lambda]^k},$$
$$a_3=-\frac{1}{2[1+3\lambda]^k}\left\{\frac{1}{2}(\frac{B_1}{2}\left[c_2-\frac{c_1^2}{2}\right]+\frac{B_2c_1^2}{4})\right\}.$$

Therefore,

$$a_3-\sigma a_2^2=-\frac{B_1}{8[1+3\lambda]^k}\left\{c_2-\nu_1c_1^2\right\}, \tag{42}$$

where

$$\nu_1=\frac{1}{2}\left[1+\frac{B_2}{B_1}-\sigma\frac{8[1+3\lambda]^kB_1}{[1+2\lambda]^{2k}}\right]. \tag{43}$$

The proof is concluded by applying Lemma 3. Sharpness of the results is obtained for the functions

$$-\left\{1+\frac{z(D^kz^2f''(z))''}{(D^kz^2f''(z))'}\right\}=\Phi(z^2)$$

and

$$-\left\{1+\frac{z(D^kz^2f''(z))''}{(D^kz^2f''(z))'}\right\}=\Phi(z).$$

This completes the proof of Theorem 10. □

Taking $\lambda=0$ in Theorems 9 and 10, we obtain the following results for functions belonging to the classes $S^{*,k}(\Phi)$ and $C^k(\Phi)$.

Corollary 1. *Let $\Phi(z)=1+B_1z+B_2z^2+B_3z^3+\ldots$ with $(B_1\neq 0)$. If f'' given in (6) belongs to the class $S^{*,k}(\Phi)$, then, for σ, a real number,*

$$\left|a_3-\sigma a_2^2\right|\leq\frac{|B_1|}{6}\max\left\{1;\left|\frac{B_2}{B_1}+\sigma\frac{B_1}{(1+\frac{1}{3})}\right|\right\}.$$

Sharpness of the results is obtained for the functions

$$-\frac{z(D^kz^2f''(z))'}{D^kz^2f''(z)}=\Phi(z^2)\quad\textit{and}\quad-\frac{z(D^kz^2f''(z))'}{D^kz^2f''(z)}=\Phi(z).$$

Corollary 2. *Let $\Phi(z)=1+B_1z+B_2z^2+B_3z^3+\ldots$ with $(B_1\neq 0)$. If f'' given in (6) belongs to the class $C^k(\Phi)$, then, for σ, a real number,*

$$\left|a_3-\sigma a_2^2\right|\leq\frac{|B_1|}{8}\max\left\{1;\left|\frac{B_2}{B_1}+\sigma 8B_1\right|\right\}.$$

Sharpness of the results is obtained for the functions

$$-\left\{1+\frac{z(D^kz^2f''(z))''}{(D^kz^2f''(z))'}\right\}=\Phi(z^2)\quad\textit{and}\quad-\left\{1+\frac{z(D^kz^2f''(z))''}{(D^kz^2f''(z))'}\right\}=\Phi(z).$$

Theorem 11. *Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots$ with $B_1 > 0$, $B_2 \geq 0$ and f'' be given by (6), which belongs to $S^{*,k}(\Phi, \lambda)$. Then, for $\sigma \in \mathbb{R}$,*

$$\left|a_3 - \sigma a_2^2\right| \leq \begin{cases} \frac{B_2}{6[1+3\lambda]^k} + \frac{\sigma B_1^2}{6(1+\frac{1}{3})[1+2\lambda]^{2k}} & \text{if } \sigma \leq \chi_1, \\ \frac{B_1}{6[1+3\lambda]^k} & \text{if } \chi_1 \leq \sigma \leq \chi_2, \\ \frac{B_2}{6[1+3\lambda]^k} - \frac{\sigma B_1^2}{6(1+\frac{1}{3})[1+2\lambda]^{2k}} & \text{if } \sigma \geq \chi_2, \end{cases} \tag{44}$$

where

$$\chi_1 = -\frac{[1+2\lambda]^{2k}\{(B_1 + B_2) + B_1^2\}}{3[1+3\lambda]^k B_1^2},$$
$$\chi_2 = -\frac{[1+2\lambda]^{2k}\{(B_1 - B_2) + B_1^2\}}{3[1+3\lambda]^k B_1^2}.$$

Proof. Applying Lemma 4 to Equations (38) and (39), we obtain three cases:

Case (1): If $\sigma \leq \chi_1$, then

$$\begin{aligned} \left|a_3 - \sigma a_2^2\right| &\leq \frac{B_1}{6[1+3\lambda]^k}\{2 - 4\nu\} \\ &\leq \frac{B_1}{6[1+3\lambda]^k}\left\{\frac{B_2}{B_1} + \sigma\frac{[1+3\lambda]^k B_1}{(1+\frac{1}{3})[1+2\lambda]^{2k}}\right\} \\ &\leq \frac{B_2}{6[1+3\lambda]^k} + \frac{\sigma B_1^2}{6(1+\frac{1}{3})[1+2\lambda]^{2k}}. \end{aligned}$$

Case (2): If $\chi_1 \leq \sigma \leq \chi_2$, then

$$\left|a_3 - \sigma a_2^2\right| \leq \frac{B_1}{6[1+3\lambda]^k}.$$

Case (3): If $\sigma \geq \chi_2$, then

$$\begin{aligned} \left|a_3 - \sigma a_2^2\right| &\leq \frac{B_1}{6[1+3\lambda]^k}\{2 - 4\nu\} \\ &\leq \frac{B_1}{6[1+3\lambda]^k}\left\{\frac{B_2}{B_1} - \sigma\frac{[1+3\lambda]^k B_1}{(1+\frac{1}{3})[1+2\lambda]^{2k}}\right\} \\ &\leq \frac{B_2}{6[1+3\lambda]^k} - \frac{\sigma B_1^2}{6(1+\frac{1}{3})[1+2\lambda]^{2k}}. \end{aligned}$$

□

Theorem 12. *Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots$ with $B_1 > 0$, $B_2 \geq 0$ and f'' be given by (6), which belongs to $C^k(\Phi, \lambda)$. Then, for $\sigma \in \mathbb{R}$,*

$$\left|a_3 - \sigma a_2^2\right| \leq \begin{cases} \frac{B_2}{8[1+3\lambda]^k} + \frac{\sigma B_1^2}{[1+2\lambda]^{2k}}, & \sigma \leq \gamma_1, \\ \frac{B_1}{8[1+3\lambda]^k}, & \gamma_1 \leq \sigma \leq \gamma_2, \\ \frac{B_2}{8[1+3\lambda]^k} - \frac{\sigma B_1^2}{[1+2\lambda]^{2k}}, & \sigma \geq \gamma_2, \end{cases}$$

where

$$\gamma_1 = -\frac{[1+2\lambda]^{2k}\{(B_1 + B_2) + B_1^2\}}{2[1+3\lambda]^k B_1^2},$$
$$\gamma_2 = -\frac{[1+2\lambda]^{2k}\{(B_1 - B_2) + B_1^2\}}{2[1+3\lambda]^k B_1^2}.$$

Proof. Applying Lemma 4 to Equations (42) and (43), we have the following three cases:

Case (1): If $\sigma \le \gamma_1$, then

$$\begin{aligned}\left|a_3-\sigma a_2^2\right| &\le \tfrac{B_1}{8[1+3\lambda]^k}\{2-4\nu\}\\ &\le \tfrac{B_1}{8[1+3\lambda]^k}\left\{\tfrac{B_2}{B_1}+\sigma\tfrac{8[1+3\lambda]^k B_1}{[1+2\lambda]^{2k}}\right\}\\ &\le \tfrac{B_2}{8[1+3\lambda]^k}+\tfrac{\sigma B_1^2}{[1+2\lambda]^{2k}}.\end{aligned}$$

Case (2): If $\gamma_1 \le \sigma \le \gamma_2$, then

$$\left|a_3-\sigma a_2^2\right| \le \frac{B_1}{8[1+3\lambda]^k}.$$

Case (3): If $\sigma \ge \gamma_2$, then

$$\begin{aligned}\left|a_3-\sigma a_2^2\right| &\le \tfrac{B_1}{8[1+3\lambda]^k}\{4\nu-2\}\\ &\le \tfrac{B_1}{8[1+3\lambda]^k}\left\{\tfrac{B_2}{B_1}-\sigma\tfrac{8[1+3\lambda]^k B_1}{[1+2\lambda]^{2k}}\right\}\\ &\le \tfrac{B_2}{8[1+3\lambda]^k}-\tfrac{\sigma B_1^2}{[1+2\lambda]^{2k}}.\end{aligned}$$

□

Taking $\lambda = 0$ in Theorems 11 and 12, we obtain the following results for functions belonging to the classes $S^{*,k}(\Phi)$ and $C^k(\Phi)$, respectively.

Corollary 3. *Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots$ with $B_1 > 0$ and $B_2 \ge 0$. If f'' given by (6) belonging to $S^{*,k}(\Phi)$, then, for σ, a real number,*

$$\left|a_3-\sigma a_2^2\right| \le \begin{cases} \frac{B_2}{6}+\frac{\sigma B_1^2}{6(1+\frac{1}{3})} & \text{if } \sigma \le \chi_1, \\ \frac{B_1}{6} & \text{if } \chi_1 \le \sigma \le \chi_2, \\ \frac{B_2}{6}-\frac{\sigma B_1^2}{6(1+\frac{1}{3})} & \text{if } \sigma \ge \chi_2, \end{cases} \tag{45}$$

where

$$\chi_1 = -\frac{(B_1+B_2)+B_1^2}{3B_1^2},$$
$$\chi_2 = -\frac{(B_1-B_2)+B_1^2}{3B_1^2}.$$

Corollary 4. *Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots$ with $B_1 > 0$ and $B_2 \ge 0$. If f'' given by (6) belongs to $C^k(\Phi)$, then, for σ, a real number,*

$$\left|a_3-\sigma a_2^2\right| \le \begin{cases} \frac{B_2}{8}+\sigma B_1^2, & \sigma \le \gamma_1, \\ \frac{B_1}{8}, & \gamma_1 \le \sigma \le \gamma_2, \\ \frac{B_2}{8}-\sigma B_1^2, & \sigma \ge \gamma_2, \end{cases}$$

where

$$\gamma_1 = -\frac{(B_1+B_2)+B_1^2}{2B_1^2},$$

$$\gamma_2 = -\frac{(B_1-B_2)+B_1^2}{2B_1^2}.$$

5. Conclusions

After a few aspects regarding the lines of research involving meromorphic functions are highlighted, the investigation presented in this paper starts with the introduction of a new differential operator given in Definition 2. This operator is obtained using the second-order differential polynomials of meromorphic functions seen in Definition 1. A new subclass of meromorphic starlike functions $S^{*,k}(\lambda)$ and a new subclass of meromorphic convex functions $C^k(\lambda)$ are introduced in Definitions 3 and 4, respectively. The new subclasses are investigated in Section 3 for inclusion relations, convexity and convolution properties. Using the concept of subordination, classes $S^{*,k}(\lambda)$ and $C^k(\lambda)$ are redefined in Definitions 5 and 6, and the notations used for them become $S^{*,k}(\Phi)$ and $C^k(\Phi)$ with reference to the subordinating function Φ instead of the parameter λ. The study is completed by establishing in Section 4 Fekete–Szegö inequalities regarding the coefficients of the functions from classes $S^{*,k}(\Phi)$ and $C^k(\Phi)$.

As future directions of research where the results presented in this paper could be used, the connection between convexity properties and symmetry could be further explored. The convolution properties proved here suggest future studies where functions from the new subclasses $S^{*,k}(\lambda)$ and $C^k(\lambda)$ could be combined with other functions with remarkable geometric and symmetry properties. The means of the theory of differential subordination could be used for further investigations on the two subclasses $S^{*,k}(\Phi)$ and $C^k(\Phi)$, which could provide interesting subordination results with a nice geometric interpretation. The results obtained for multivalent meromorphic functions connected with the Liu–Srivastava operator using the theory of strong differential subordination [25] suggest that this theory could also be applied for the functions of classes $S^{*,k}(\lambda)$ and $C^k(\lambda)$ involving the new operator defined here. Additionally, considering the strong differential results obtained for Sălăgean and Ruscheweyh differential operators [26], the idea of applying the means of strong differential subordination to the differential operator given in Definition 2 seems interesting.

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References

1. Ruscheweyh, S. *Convolutions in Geometric Function Theory; Sé minaire de Mathématiques Supérieures*; Presses de l'Université de Montréal: Montreal, QC, Canada, 1982; Volume 83.
2. Cho, N.E.; Lee, S.H.; Owa, S. A class of meromorphic univalent functions with positive coefficients. *Kobe J. Math.* **1987**, *4*, 43–50.
3. Joshi, S.B.; Kulkarni, S.R.; Srivastava, H.M. Certain classes of meromorphic functions with positive and missing coefficients. *J. Math. Anal. Appl.* **1995**, *193*, 1–14. [CrossRef]
4. Dziok, J.; Murugusundaramoorthy, G.; Sokoł, J. On certain class of meromorphic functions with positive coefficients. *Acta Math. Sci.* **2012**, *32*, 1–16. [CrossRef]
5. Shinde, A.S.; Ingle, R.N.; Reddy, P.Y. On Certain Subclass of Meromorphic Functions with Positive Coefficients. *Palest. J. Math.* **2021**, *10*, 685–693.
6. Uralegaddi, B.A.; Somanatha, C. Certain differential operators for meromorphic functions. *Houst. J. Math.* **1991**, *17*, 279–284.
7. Oros, G.; Oros, G.I. Applications of Sălăgean Differential Operator at the Class of Meromorphic Functions. *Lib. Math.* **2006**, *XXVI*, 61–67.
8. Oros, G.I. Differential subordinations defined by using Săl ăgean differential operator at the class of meromorphic functions. *Acta Univ. Apulensis* **2006**, *11*, 219-224.

9. Elrifai, E.A.; Darwish, H.E.; Ahmed, A.R. On certain subclasses of meromorphic functions associated with certain differential operators. *Appl. Math. Lett.* **2012**, *25*, 952–958. [CrossRef]
10. Al-Khafaji, A.K. On Subclass of Meromorphic Analytic Functions Defined by a Differential Operator. *J. Phys. Conf. Ser.* **2021**, *1818*, 012198. [CrossRef]
11. Mohammed, A.; Darus, M. On New p-Valent Meromorphic Function Involving Certain Differential and Integral Operators. *Abstr. Appl. Anal.* **2014**, *2014*, 208530. [CrossRef]
12. Shah, S.G.A.; Noor, S.; Darus, M.; Ul Haq, W.; Hussain, S. On Meromorphic Functions Defined by a New Class of Liu-Srivastava Integral Operator. *Int. J. Anal. Appl.* **2020**, *18*, 1056–1065.
13. Totoi, E.-A.; Cotîrlă, L.-I. Preserving Classes of Meromorphic Functions through Integral Operators. *Symmetry* **2022**, *14*, 1545. [CrossRef]
14. Challab, K.; Darus, M.; Ghanim, F. On a certain subclass of meromorphic functions defined by a new linear differential operator. *J. Math. Fund. Sci.* **2017**, *49*, 269–282. [CrossRef]
15. Elhaddad, S.; Darus, M. On Meromorphic Functions Defined by a New Operator Containing the Mittag–Leffler Function. *Symmetry* **2019**, *11*, 210. [CrossRef]
16. Ghanim, F.; Al-Janaby, H.F.; Bazighifan, O. Geometric properties of the meromorphic functions class through special functions associated with a linear operator. *Adv. Contin. Discret. Models* **2022**, *2022*, 17. [CrossRef]
17. Al-Oboudi, F.M.; Al-Zkeri, H.A. On Some Classes of Meromorphic Starlike Functions Defined by a Differential Operator. *Glob. J. Pure Appl. Math.* **2007**, *3*, 1–11.
18. Nunokawa, M. On meromorphically convex and starlike functions. *RIMS Kokyuroku* **2000**, *1164*, 57–62.
19. Ahmad, B.; Khan, M.G.; Frasin, B.A.; Aouf, M.K.; Abdeljawad, T.; Mashwani, W.K.; Arif, M. On q-analogue of meromorphic multivalent functions in lemniscate of Bernoulli domain. *AIMS Math.* **2021**, *6*, 3037–3052. [CrossRef]
20. Breaz, D.; Cotîrlă, L.-I.; Umadevi, E.; Karthikeyan, K.R. Properties of Meromorphic Spiral-Like Functions Associated with Symmetric Functions. *J. Funct. Spaces* **2022**, *2022*, 3444854. [CrossRef]
21. Aouf, M.K.; El-Emam, F.Z. Fekete–Szegö Problems for Certain Classes of Meromorphic Functions Involving Al-Oboudi Differential Operator. *J. Math.* **2022**, *2022*, 4731417. [CrossRef]
22. Miller, S.S.; Mocanu, P.T. Differential Subordinations. In *Theory and Applications*; Marcel Dekker, Inc.: New York, NY, USA; Basel, Switzerland, 2000.
23. Ma, W.; Minda, D. A unified treatment of some special classes of univalent functions. In *Lecture Notes Analysis, I, Proceedings of the Conference on Complex Analysis, Tianjin, China, 19–23 June 1992*; International Press Inc.: Cambridge, MA, USA, 1994; pp. 157–169.
24. Alsoboh, A.; Darus, M. On Fekete-Szego problem associated with q-derivative operator. *J. Phys. Conf. Ser.* **2019**, *1212*, 012003. [CrossRef]
25. Cho, N.E.; Kwon, O.S.; Srivastava, H.M. Strong differential subordination and superordination for multivalently meromorphic functions involving the Liu–Srivastava operator. *Integral Transform. Spec. Funct.* **2010**, *21*, 589–601. [CrossRef]
26. Alb Lupaş, A.; Oros, G.I.; Oros, G. On special strong differential subordinations using Sălăgean and Ruscheweyh operators. *J. Comput. Anal. Appl.* **2012**, *14*, 266–270.