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On Nonuniqueness of Quantum Channel for Fixed Input-Output States: Case of Decoherence Channel

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Abstract: For a fixed pair of input and output states in the space H_A of a system A, a quantum channel, i.e., a linear, completely positive and trace-preserving map, between them is not unique, in general. Here, this point is discussed specifically for a decoherence channel, which maps from a pure input state to a completely decoherent state like the thermal state. In particular, decoherence channels of two different types are analyzed: one is unital and the other is not, and both of them can be constructed through reduction of B in the total extended space $H_A \otimes H_B$, where H_B is the space of an ancillary system B that is a replica of A. The nonuniqueness is seen to have its origin in the unitary symmetry in the extended space. It is shown in an example of a two-qubit system how such symmetry is broken in the objective subspace H_A due to entanglement between A and B. A comment is made on possible relevance of the present work to nanothermodynamics in view of quantum Darwinism.

Keywords: channel nonuniqueness; decoherence channel; unitary symmetry; two-qubit

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1. Introduction

Decoherence is a central concept in the quantum theory of open systems and measurements (see [1–4] and the references therein). It plays key roles in diverse topics, including foundations of thermostatistics, quantum information processing and the emergence of the classical from the quantum. From the dynamical viewpoint, it is concerned with influences of the environmental system interacting with an open subsystem and arises when the environmental degrees of freedom are eliminated. This is actually a highly complex issue, having been a major subject of investigation ever since the concept was introduced about a half-century ago [5]. Among others, quantum Darwinism [6–8] should be noted, which assumes decoherence as a basic premise and aims to describe classical "objectivity", that is, different observers can simultaneously determine the same state of the system under consideration without disturbance.

A quantum channel is a map from a density matrix of an input quantum state to an output one in which dynamical details are contained. It is linear, completely positive and trace-preserving. Since it is not necessarily unitary, it can map from a pure state to a mixed state with loss of coherence. Today, the concept of quantum channels is regarded as fundamental as quantum states. This seems to be mathematically natural: to understand the property of a given set is to understand that of a set of maps defined on it. Accordingly, discussions have been developed about the entropies of quantum channels for quantification of channel complexity [9] and entanglement of channels [10]. As a higher category, the notion of quantum supermaps/superchannels between different quantum channels has also been introduced [11]. These may be of importance in designing quantum heat engines and quantum circuits, for example.

Now, for a given pair of input and output states, the quantum channel between them is not unique, in general. That is, for the same input, different channels may give the same output, offering flexibility to quantum engineering. This fact naturally gives rise to the following question: how such channels are related to each other. In the present paper, we discuss this issue for decoherence channels, which are specific channels mapping an input pure state to an output mixed state with perfect loss of coherence like the thermal state. In particular, we consider a couple of different decoherence channels: one is unital, and the other is not. For this purpose, we extend the state space by introducing an ancillary system that is a replica of the objective system. By performing nonlocal transformations of the total input pure state in the extended space and then reducing the ancillary degrees of freedom, we describe those two channels and explicitly show how the nonuniqueness is related to the unitary symmetry in the extended space. We explicitly analyze an example of a two-qubit system and see how such symmetry is broken in the space of the objective system due to entanglement and reduction.

This paper is organized as follows. In Section 2, some basic concepts are recapitulated, such as the Kraus representation of a completely positive trace-preserving map, the Schmidt decomposition and an ancillary system for extension of the state space and purification. Then, the decoherence channel and its nonuniqueness are discussed. In Section 3, a two-qubit (i.e., a bipartite two-level system) is analyzed as an explicit example. It is shown how different decoherence channels are related to each other. In Section 4, the results of the present work are briefly summarized, and a comment is made on nanothermodynamics in connection with quantum Darwinism.

2. Nonuniqueness of Decoherence Channel

Consider the objective system A in a quantum state described by a density matrix $\rho_{A,0}$ that should be positive semidefinite and satisfy the normalization condition $\mathrm{tr}_A \rho_{A,0} = 1$, where the symbol "$\mathrm{tr}_A$" stands for the trace in the state space H_A of A. A quantum channel Φ is a map, which is required to be linear, completely positive and trace-preserving, i.e., $\mathrm{tr}_A[\Phi(X_A)] = \mathrm{tr}_A X_A$ with X_A being an arbitrary trace-class operator/matrix. In the Kraus representation [12], it is written as follows:

$$\rho_A = \Phi(\rho_{A,0}) \equiv \sum_i V_i \rho_{A,0} V_i^\dagger. \tag{1}$$

Here, V_i's are certain operators on H_A satisfying

$$\sum_i V_i^\dagger V_i = I_A \tag{2}$$

with I_A being the identity operator on H_A. (For the notational convenience, the label "A" is abbreviated for V_i's.) Due to this partition-of-unity property, the set $\{V_i^\dagger V_i\}_i$ is said to form a positive operator-valued measure. $\rho_{A,0}$ and ρ_A in Equation (1) are referred to as input and output states, respectively. In quantum measurement theory, Equation (1) represents an operation of an unsharp (or fuzzy) measurement, which generalizes von Neumann's projective measurement. There, V_i's are the operators associated with an observable of interest. The fundamental nature of quantum mechanics is the unitarity of dynamics, whereas Φ in Equation (1) is not unitary. This point is made clear if the state space is extended from H_A to $H_A \otimes H_B$ by the introduction of an ancillary system B. The total input state is taken to be $\rho_{A,0} \otimes |\phi\rangle_{BB}\langle\phi|$; that is, B is in a pure state. (Here and hereafter, the symbol of tensor product is commonly used for state spaces, vectors and operators/matrices.) The total system is supposed to be isolated and to be governed by unitary dynamics now. Write a unitary transformation as follows: $\rho_{A,0} \otimes |\phi\rangle_{B\,B}\langle\phi| \to U_{AB}\, \rho_{A,0} \otimes |\phi\rangle_{B\,B}\langle\phi| U_{AB}^\dagger$, where the operator is assumed to be nonlocal (i.e., unfactorizable), i.e., $U_{AB} \neq U_A \otimes U_B$, introducing entanglement between A and B. Then, to look only at A in the output, B is eliminated through the partial trace of the total output density matrix over B: $\rho_A = \mathrm{tr}_B\big(U_{AB}\,\rho_{A,0} \otimes |\phi\rangle_{B\,B}\langle\phi|U^\dagger_{AB}\big)$. In terms of a certain complete orthonormal system $\{|v_i\rangle_B\}_i$ in H_B for the calculation of the partial trace, the expression

$$V_i = {}_B\langle v_i|U_{AB}|\phi\rangle_B \tag{3}$$

is obtained for the operators in Equation (1).

The decoherence channel is a special case of Φ in Equation (1) that maps an input pure state $\rho_{A,0} = |\psi\rangle_{A\,A}\langle\psi|$ to an output mixed state of perfect decoherence with no off-diagonal elements:

$$\rho_A = \Phi(|\psi\rangle_{A\,A}\langle\psi|) = \sum_i p_i|u_i\rangle_{A\,A}\langle u_i|, \tag{4}$$

provided that the rank of this matrix should be larger than unity. Here, $\{|u_i\rangle_A\}_i$ is a complete orthonormal system in H_A and is actually the set of normalized eigenstates of ρ_A itself with the eigenvalues p_i's. From the positive semidefiniteness of and normalization condition on the density matrix, it follows that $0 \leq p_i < 1$ and $\sum_i p_i = 1$. It is instructive to see this in an inverse way in connection with the concept of purification. Given a density matrix ρ_A of a completely decoherent state, it is possible to represent it as a pure state in an extended space:

$$|\Psi\rangle_{AB}= \sum_i \sqrt{p_i}|u_i\rangle_A\otimes|v_i\rangle_B, \tag{5}$$

where $\{|v_i\rangle_B\}_i$ is not necessarily the same as that used in Equation (3). From this state, ρ_A in Equation (4) is in fact recovered as follows:

$$\rho_A = \mathrm{tr}_B(|\Psi\rangle_{AB\,AB}\langle\Psi|). \tag{6}$$

Equation (5), called the Schmidt decomposition, manifests how A and B are entangled in order for perfect decoherence to be realized. This procedure indicates a possibility of employing a replica of A as B.

Let $|u_i\rangle_A$ ($|v_i\rangle_B$) be the i-th energy eigenstate with the eigenvalue E_i of the Hamiltonian of A (B), and set $p_i = [1/Z(\beta)]\exp(-\beta E_i)$, $Z(\beta) = \sum_i \exp(-\beta E_i)$, where $\beta = 1/(k_B T)$ with k_B and T being the Boltzmann constant and temperature, respectively. In this case, ρ_A becomes a canonical density matrix of the Gibbsian state that is a perfectly decoherent state with no off-diagonal elements in the energy eigenbasis (see References [13,14] and the works cited therein). In other words, B is regarded as the heat reservoir. Since B is a replica of A that may be a small system in a canonical ensemble, B can be far from the thermodynamic limit. Therefore, B is an "economical description" of the heat reservoir. It should, however, be noted that, in this case of small B, ensemble equivalence, i.e., equivalence between theories of microcanonical, canonical and grandcanonical ensembles, is not established because of the absence of the thermodynamic limit. Thus, in what follows, the ancillary system B is taken to be a replica of the objective subsystem A. It is also noted that, in the vanishing temperature limit, only the lowest-state ($i \equiv 0$) element survives in the thermal state. In conformity with this, the lowest state

$$\rho_{AB,0} = |\Omega\rangle_{AB\,AB}\langle\Omega| \tag{7}$$

with $|\Omega\rangle_{AB} \equiv |u_0\rangle_A \otimes |v_0\rangle_B$ is employed in the rest of this section as the total input pure state. A more general input state will be treated in the example analyzed in the next section.

After recapitulating these basic issues, let us start discussing the nonuniqueness of the decoherence channel in Equation (4) with $|\psi\rangle_A = |u_0\rangle_A$.

In general, i.e., not limited to the one in Equation (4), a quantum channel for a given pair of input and output states is not unique. This nonuniqueness comes from the reduction of the ancillary system, which leads to the loss of information. In other words, it may be related to symmetry in the extended state space that is broken in the objective subspace.

The following two different forms of the map in Equation (1) are considered:

$$V_i^{(\mathrm{I})} = (I_A - |u_0\rangle_{AA}\langle u_0|)\,\delta_{i0} + \sqrt{p_i}|u_i\rangle_{AA}\langle u_0|, \tag{8}$$

$$V_i^{(\mathrm{II})} = \sqrt{p_i}\,(I_A - |u_0\rangle_{AA}\langle u_0| - |u_i\rangle_{AA}\langle u_i| + |u_0\rangle_{AA}\langle u_i| + |u_i\rangle_{AA}\langle u_0|), \tag{9}$$

where p_i are the ones given below Equation (4). Both of these form positive operator-valued measures as in Equation (2). However, a salient feature of difference between them is that

$$\sum_i V_i^{(\mathrm{I})}\,V_i^{(\mathrm{I})\dagger} = I_A - |u_0\rangle_{AA}\langle u_0| + \sum_i p_i\,|u_i\rangle_{AA}\langle u_i|, \tag{10}$$

$$\sum_i V_i^{(\mathrm{II})}\,V_i^{(\mathrm{II})\dagger} = I_A. \tag{11}$$

That is, the channel $\Phi^{(\mathrm{II})}$ associated with $V_i^{(\mathrm{II})}$ is unital, i.e., $\Phi^{(\mathrm{II})}(I_A) = I_A$, whereas $\Phi^{(\mathrm{I})}$ associated with $V_i^{(\mathrm{I})}$ is not. In spite of this difference, both of them are seen to satisfy Equation (4) with $|\psi\rangle_A = |u_0\rangle_A$.

An important property of a unital channel is that the entropy (the von Neumann entropy or the Rényi entropy indexed by α with $0 < \alpha \leq 2$) always increases due to the channel for any input state. In other words, the entropy monotonically increases for repeated applications of a unital channel to any state, whereas such monotonicity does not hold if a channel is not unital [15] (see also Reference [16]).

Equation (8) is related to the theory of Takahashi and Umezawa, who have developed a real-time field theory at finite temperature termed thermo-field dynamics [17]. In that theory, the thermal vacuum of a quantum field is constructed from the zero-temperature vacuum in the extended space by the Bogoliubov transformation. Its generalization has been discussed in Reference [18]. The unitary operator on the extended space presented there is as follows:

$$U_{AB}^{(\mathrm{I})} = \exp(\theta\,G_{AB}), \tag{12}$$

where G_{AB} is the anti-Hermitian operator given by

$$G_{AB} = |J\rangle_{AB\,AB}\langle\Omega| - |\Omega\rangle_{AB\,AB}\langle J| \tag{13}$$

with $|\Omega\rangle_{AB}$ being given below Equation (7) and

$$|J\rangle_{AB} = \frac{1}{\sqrt{1-p_0}} \sum_{i\neq 0} \sqrt{p_i}\,|u_i\rangle_A \otimes |v_i\rangle_B \tag{14}$$

with $\theta = \cos^{-1}\sqrt{p_0}$. Equation (8) is in fact given by $V_i^{(\mathrm{I})} = {}_B\langle v_i|U_{AB}^{(\mathrm{I})}|v_0\rangle_B$.

On the other hand, regarding Equation (9), which has been presented in Reference [19], a general form is not known yet for the unitary operator associated with it, unfortunately. (In the next section, one such example will be presented for a two-qubit system.) Suppose at present that $U_{AB}^{(\mathrm{II})}$ is found in a specific case. Then, there always uniquely exists the unitary operator R_{AB} satisfying

$$U_{AB}^{(\mathrm{I})}\,R_{AB} = U_{AB}^{(\mathrm{II})}. \tag{15}$$

This makes manifest how the nonuniqueness of the decoherence channel is related to the nonlocal unitary symmetry between the transformations in the extended space. In the next section, we show in an explicit example how the nonuniqueness comes from the breakdown of such symmetry in H_A due to entanglement between A and B.

3. Two-Qubit

The nonuniqueness of the decoherence channel and related issues are best illustrated in a two-qubit system, which plays important roles in quantum processors using superconductors [20] and semiconductors [21], for example. In this case, the total space $H_A \otimes H_B$ is 4-dimensional and can be spanned by the basis

$$\begin{aligned} |\chi_0\rangle_{AB} &= |u_0\rangle_A \otimes |v_0\rangle_B, \\ |\chi_1\rangle_{AB} &= |u_0\rangle_A \otimes |v_1\rangle_B, \\ |\chi_2\rangle_{AB} &= |u_1\rangle_A \otimes |v_0\rangle_B, \\ |\chi_3\rangle_{AB} &= |u_1\rangle_A \otimes |v_1\rangle_B, \end{aligned} \tag{16}$$

satisfying ${}_{AB}\langle\chi_\alpha|\chi_\beta\rangle_{AB} = \delta_{\alpha\beta}$ and $\sum_{\alpha=0}^{3}|\chi_\alpha\rangle_{AB\,AB}\langle\chi_\alpha| = I_A \otimes I_B$.

The input of A considered here is a general qubit state $|\psi\rangle_{AA}\langle\psi|$ with the superposition of the lower and upper states

$$|\psi\rangle_A = a_0|u_0\rangle_A + a_1|u_1\rangle_A, \tag{17}$$

where the complex coefficients obey the normalization condition: $|a_0|^2 + |a_1|^2 = 1$. Two different sets of operators we consider here are as follows:

$$\begin{aligned} V_0^{(\mathrm{I})} &= a_0^*\sqrt{p_0}|u_0\rangle_{AA}\langle u_0| + a_1^*\sqrt{p_0}|u_0\rangle_{AA}\langle u_1| - a_1|u_1\rangle_{AA}\langle u_0| + a_0|u_1\rangle_{AA}\langle u_1|, \\ V_1^{(\mathrm{I})} &= a_0^*\sqrt{p_1}|u_1\rangle_{AA}\langle u_0| + a_1^*\sqrt{p_1}|u_1\rangle_{AA}\langle u_1|, \end{aligned} \tag{18}$$

and

$$\begin{aligned} V_0^{(\mathrm{II})} &= a_0^*\sqrt{p_0}|u_0\rangle_{AA}\langle u_0| + a_1^*\sqrt{p_0}|u_0\rangle_{AA}\langle u_1| - a_1\sqrt{p_0}|u_1\rangle_{AA}\langle u_0| \\ &\quad + a_0\sqrt{p_0}|u_1\rangle_{AA}\langle u_1|, \\ V_1^{(\mathrm{II})} &= -a_1\sqrt{p_1}|u_0\rangle_{AA}\langle u_0| + a_0\sqrt{p_1}|u_0\rangle_{AA}\langle u_1| + a_0^*\sqrt{p_1}|u_1\rangle_{AA}\langle u_0| \\ &\quad + a_1^*\sqrt{p_1}|u_1\rangle_{AA}\langle u_1|, \end{aligned} \tag{19}$$

where $0 < p_i < 1$ $(i = 0, 1)$ and $p_0 + p_1 = 1$. It is straightforward to ascertain that both of these yield

$$\begin{aligned} \Phi^{(Q)}(|\psi\rangle_{AA}\langle\psi|) &= \sum_{i=0,1} V_i^{(Q)}|\psi\rangle_{A\,A}\langle\psi|\, V_i^{(Q)\dagger} \\ &= \sum_{i=0,1} p_i|u_i\rangle_{AA}\langle u_i| \; (Q = \mathrm{I}, \mathrm{II}) \end{aligned} \tag{20}$$

for $|\psi\rangle_A$ in Equation (17) as well as the trace-preserving condition

$$\sum_{i=0,1} V_i^{(Q)\dagger} V_i^{(Q)} = I_A \; (Q = \mathrm{I}, \mathrm{II}). \tag{21}$$

Therefore, both Equations (18) and (19) define decoherence channels for the same input and output states. However, $\sum_{i=0,1} V_i^{(\mathrm{II})} V_i^{(\mathrm{II})\dagger} = I_A$ and therefore $\Phi^{(\mathrm{II})}$ is unital, whereas $\Phi^{(\mathrm{I})}$ is not, since $\sum_{i=0,1} V_i^{(\mathrm{I})} V_i^{(\mathrm{I})\dagger} = p_0|u_0\rangle_{AA}\langle u_0| + (1 + p_1)|u_1\rangle_{AA}\langle u_1|$.

To understand this nonuniqueness, let us consider the unitary operators corresponding to these two channels on the extended space $H_A \otimes H_B$ of the two-qubit system. As examples, here we present the following ones:

$$\begin{aligned} U_{AB}^{(\mathrm{I})} &= a_0^*\sqrt{p_0}|\chi_0\rangle_{AB\,AB}\langle\chi_0| + a_1\sqrt{p_1}|\chi_0\rangle_{AB\,AB}\langle\chi_1| \\ &\quad + a_1^*\sqrt{p_0}|\chi_0\rangle_{AB\,AB}\langle\chi_2| - a_0\sqrt{p_1}|\chi_0\rangle_{AB\,AB}\langle\chi_3| \\ &\quad + a_0^*|\chi_1\rangle_{AB\,AB}\langle\chi_1| + a_1^*|\chi_1\rangle_{AB\,AB}\langle\chi_3| - a_1|\chi_2\rangle_{AB\,AB}\langle\chi_0| + a_0|\chi_2\rangle_{AB\,AB}\langle\chi_2| \\ &\quad + a_0^*\sqrt{p_1}|\chi_3\rangle_{AB\,AB}\langle\chi_0| - a_1\sqrt{p_0}|\chi_3\rangle_{AB\,AB}\langle\chi_1| \\ &\quad + a_1^*\sqrt{p_1}|\chi_3\rangle_{AB\,AB}\langle\chi_2| + a_0\sqrt{p_0}|\chi_3\rangle_{ABA\,B}\langle\chi_3|, \end{aligned} \tag{22}$$

$$
\begin{aligned}
U_{AB}^{(\mathrm{II})} = {} & a_0^* \sqrt{p_0} |\chi_0\rangle_{AB\,AB}\langle\chi_0| + a_1 \sqrt{p_1} |\chi_0\rangle_{AB\,AB}\langle\chi_1| \\
& + a_1^* \sqrt{p_0} |\chi_0\rangle_{AB\,AB}\langle\chi_2| - a_0 \sqrt{p_1} |\chi_0\rangle_{AB\,AB}\langle\chi_3| \\
& - a_1 \sqrt{p_1} |\chi_1\rangle_{AB\,AB}\langle\chi_0| + a_0^* \sqrt{p_0} |\chi_1\rangle_{AB\,AB}\langle\chi_1| \\
& + a_0 \sqrt{p_1} |\chi_1\rangle_{AB\,AB}\langle\chi_2| + a_1^* \sqrt{p_0} |\chi_1\rangle_{AB\,AB}\langle\chi_3| \\
& - a_1 \sqrt{p_0} |\chi_2\rangle_{AB\,AB}\langle\chi_0| - a_0^* \sqrt{p_1} |\chi_2\rangle_{AB\,AB}\langle\chi_1| \\
& + a_0 \sqrt{p_0} |\chi_2\rangle_{AB\,AB}\langle\chi_2| - a_1^* \sqrt{p_1} |\chi_2\rangle_{AB\,AB}\langle\chi_3| \\
& + a_0^* \sqrt{p_1} |\chi_3\rangle_{AB\,AB}\langle\chi_0| - a_1 \sqrt{p_0} |\chi_3\rangle_{AB\,AB}\langle\chi_1| \\
& + a_1^* \sqrt{p_1} |\chi_3\rangle_{AB\,AB}\langle\chi_2| + a_0 \sqrt{p_0} |\chi_3\rangle_{AB\,AB}\langle\chi_3|.
\end{aligned}
\tag{23}
$$

The input state of the ancillary system is still taken to be its lower state in order to avoid unnecessary complications. Therefore, the total input pure state here is $\rho_{0,AB} = |\psi\rangle_{A\,A}\langle\psi| \otimes |v_0\rangle_{B\,B}\langle v_0|$. Then, it can be verified that the reduced operators $V_i^{(Q)} = {}_B\langle v_i| U_{AB}^{(Q)} |v_0\rangle_B$ ($Q = \mathrm{I}, \mathrm{II}$; $i = 0, 1$) in fact give rise to Equations (18) and (19).

With these explicit forms of the unitary operators on the extended space, now the unitary operator R_{AB} satisfying $U_{AB}^{(\mathrm{I})} R_{AB} = U_{AB}^{(\mathrm{II})}$ can immediately be calculated. Its explicit form is given by

$$
\begin{aligned}
R_{AB} = {} & \left(|a_0|^2 + |a_1|^2 \sqrt{p_0} \right) |\chi_0\rangle_{AB\,AB}\langle\chi_0| + a_0^* a_1^* \sqrt{p_1} |\chi_0\rangle_{AB\,AB}\langle\chi_1| \\
& + a_0 a_1^* (1 - \sqrt{p_0}) |\chi_0\rangle_{AB\,AB}\langle\chi_2| + a_1^{*2} \sqrt{p_1} |\chi_0\rangle_{AB\,AB}\langle\chi_3| \\
& - a_0 a_1 \sqrt{p_1} |\chi_1\rangle_{AB\,AB}\langle\chi_0| + \left(|a_0|^2 \sqrt{p_0} + |a_1|^2 \right) |\chi_1\rangle_{AB\,AB}\langle\chi_1| \\
& + a_0^2 \sqrt{p_1} |\chi_1\rangle_{AB\,AB}\langle\chi_2| - a_0\, a_1^* (1 - \sqrt{p_0}) |\chi_1\rangle_{AB\,AB}\langle\chi_3| \\
& + a_0^* a_1 (1 - \sqrt{p_0}) |\chi_2\rangle_{AB\,AB}\langle\chi_0| - a_0^{*2} \sqrt{p_1} |\chi_2\rangle_{AB\,AB}\langle\chi_1| \\
& + \left(|a_0|^2 \sqrt{p_0} + |a_1|^2 \right) |\chi_2\rangle_{AB\,AB}\langle\chi_2| - a_0^*\, a_1^* \sqrt{p_1} |\chi_2\rangle_{AB\,AB}\langle\chi_3| \\
& - a_1^2 \sqrt{p_1} |\chi_3\rangle_{AB\,AB}\langle\chi_0| - a_0^* a_1 (1 - \sqrt{p_0}) |\chi_3\rangle_{AB\,AB}\langle\chi_1| \\
& + a_0 a_1 \sqrt{p_1} |\chi_3\rangle_{AB\,AB}\langle\chi_2| + \left(|a_0|^2 + |a_1|^2 \sqrt{p_0} \right) |\chi_3\rangle_{AB\,AB}\langle\chi_3|.
\end{aligned}
\tag{24}
$$

This unitary operator is nonlocal, that is, $R_{AB} \neq R_A \otimes R_B$. To see it, let us apply R_{AB} on an unentangled state, say $|\chi_2\rangle_{AB}$ in Equation (16), to have

$$
\begin{aligned}
R_{AB} |\chi_2\rangle_{AB} = {} & a_0\, a_1^* \, (1 - \sqrt{p_0}) |\chi_0\rangle_{AB} + a_0^2 \sqrt{p_1} |\chi_1\rangle_{AB} \\
& + \left(|a_0|^2 \sqrt{p_0} + |a_1|^2 \right) |\chi_2\rangle_{AB} + a_0 a_1 \sqrt{p_1} |\chi_3\rangle_{AB},
\end{aligned}
\tag{25}
$$

and then evaluate the purity $\mathrm{tr}_A\, \sigma_A^2$, where σ_A is the reduced density matrix $\sigma_A \equiv \mathrm{tr}_B \left(R_{AB} |\chi_2\rangle_{AB\,AB}\langle\chi_2| R_{AB}^\dagger \right)$. The result turns out to be simplified in the special case $|a_0|^2 = |a_1|^2 = 1/2$: $\mathrm{tr}_A\, \sigma_A^2 = \left(3 + p_0^2\right)/4 < 1$, implying that A is in a mixed state and the state in Equation (25) is entangled. Therefore, R_{AB} is in fact nonlocal. Accordingly, a unitary relation like Equation (15) does not exist in H_A. It is, however, noted that since R_{AB} is input-state specific, the input state remains unchanged (and thus unentangled): $R_{AB} |\psi\rangle_A \otimes |v_0\rangle_B = |\psi\rangle_A \otimes |v_0\rangle_B$, as it should do.

4. Concluding Remarks

For a given pair of input and output states of a system, the quantum channel between them is not unique, in general. In the present work, we have discussed this point for a specific case of the decoherence channel that maps from a pure input state to a completely decoherent output state. We have considered two different decoherence channels: one is unital and the other is not. By introducing an ancillary system that is a replica of the objective subsystem of interest, the unitary symmetry between the two in the extended space is identified. The nonuniqueness is then seen to be due to the breakdown of the symmetry in the objective subspace.

Although only two specific decoherence channels are treated, here, the structure of unitary symmetry and its breaking in the subspace as the origin of channel nonuniqueness may not be limited to the decoherence channels.

A remaining challenge is to examine if smallness permitted for the size of the ancillary system as a replica of the objective system, which is an economical description of its thermal reservoir, has some relevance to the foundations of nanothermodynamics [22–24]. There, a central concept is subdivision of a macroscopic system into small systems. Each single piece is an objective system in contact with the neighbor(s)/fragment(s) regarded as the reservoir(s). Thus, the structured environment is essential. Each fragment of the environment can be thought of as a replica or a collection of some replicas discussed in the present work. Here seems to be a point of quantum Darwinism [6–8] analogous to nanothermodynamics characterized by the modified Gibbsian state. In quantum Darwinism, decoherence due to the heterogeneous environment and proliferation of (classical) information over it as the collection of all fragments may explain "objectivity" (see Section 1). Since the thermal states are generally considered to possess objectivity of the high level, it is natural to anticipate that quantum Darwinism plays a role as a possible foundation of nanothermodynamics. A main obstacle here is the fact that proliferation of (classical) information or energy transport over the whole environment as the total collection of the fragments is hard to be realized, in general. This point has recently been discussed for the Gibbsian state [25]. There, it is shown *not to be impossible* to simultaneously establish both thermality and objectivity. As a future subject, it may be important to clarify under what conditions replications of the system A can be related to the proliferation of classical information and quantum cloning of the states associated with it.

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