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Quasi-Hadamard Product and Partial Sums for Sakaguchi-Type Function Classes Involving q-Difference Operator

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Abstract: We create two Sakaguchi-type function classes that are starlike and convex with respect to their symmetric points, including a q-difference operator, which may have symmetric or assymetric properties, in the open unit disc. We first obtain sufficient coefficient bounds for these functions. In view of these bounds, we obtain quasi-Hadamard products and several partial sums for these function classes. Moreover, the special values of the parameters provided the corresponding consequences of the partial sums.

Keywords: Sakaguchi-type functions; q-difference operator; quasi-Hadamard product; partial sums

MSC: 30C45; 30C50

check for updates

Citation: Çetinkaya, A.; Cotîrlă, L.-I. Quasi-Hadamard Product and Partial Sums for Sakaguchi-Type Function Classes Involving q-Difference Operator. *Symmetry* **2022**, *14*, 709. https://doi.org/10.3390/sym14040709

Academic Editors: Şahsene Altınkaya and Sergei D. Odintsov

Received: 15 March 2022
Accepted: 25 March 2022
Published: 31 March 2022

1. Introduction

$\mathcal{A}$ can be used to denote the family of holomorphic (analytic) functions with the expansion

$$f(\epsilon) = \epsilon + \sum_{\ell=2}^{\infty} a_\ell \epsilon^\ell \tag{1}$$

in the open-unit disc $\mathbb{D} := \{\epsilon : |\epsilon| < 1\}$. If a function f is one-to-one in $\mathbb{D}$, then it is called univalent in $\mathbb{D}$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ comprising all univalent functions in $\mathcal{A}$. Comprehensive details on univalent functions can be found in [1].

Quantum calculus is an approach to examining the calculus without using the limits. The most important step in q-calculus was discoverd by Jackson, who defined the useful formulas of q-integral and q-derivative operators (see [2–4]). Later, q-calculus has attracted the attention of researchers due to its applications in several areas of mathematics, such as combinatorics, ordinary fractional calculus, basic hypergeometric functions, orthogonal polynomials, and, more recently, in geometric function theory.

In 1909, Jackson [2] introduced the operator

$$(D_q f)(\epsilon) = \frac{f(\epsilon) - f(q\epsilon)}{(1-q)\epsilon}, \ (\epsilon \neq 0), \quad (D_q f)(0) = f'(0)$$

which is said to be q-derivative (or q-difference) operator of a function f. By taking q-derivative of the function f in the form (1), we can see that

$$(D_q f)(\epsilon) = 1 + \sum_{\ell=2}^{\infty} [\ell]_q a_\ell \epsilon^{\ell-1},$$

where

$$[\ell]_q = \frac{1-q^\ell}{1-q}$$

is called q-number of ℓ. The parameter q is assumed to be within the range $(0,1)$. Clearly, $D_q \to \frac{d}{d\epsilon}$ as $q \to 1^-$. For more details, one can see the books and papers on q-derivative [5–22] and references therein.

Sakaguchi [23] defined function $f \in \mathcal{A}$ as starlike with respect to its symmetric points if, for each r less than and sufficiently close to one, and each ξ on the circle $|\epsilon| = r$, the angular velocity of $f(\epsilon)$ at about the point $f(-\xi)$ is positive at $\epsilon = -\xi$ as ϵ traverses the circle $|\epsilon| = r$ in the positive direction, i.e.,

$$\mathrm{Re}\left(\frac{\epsilon f'(\epsilon)}{f(\epsilon) - f(-\xi)}\right) > 0 \ \text{ for } \ \epsilon = \xi, |\xi| = r.$$

Denote by $\mathcal{S}_s^*$ the class of starlike functions with respect to symmetric points is given by

$$\mathrm{Re}\left(\frac{\epsilon f'(\epsilon)}{f(\epsilon) - f(-\epsilon)}\right) > 0, \ \ (\epsilon \in \mathbb{D}).$$

The above function is univalent in $\mathbb{D}$ because $(f(\epsilon) - f(-\epsilon))/2$ is a starlike function in $\mathbb{D}$.

Denote by $\mathcal{C}_s$ the class of convex functions with respect to symmetric points, characterized by (Das and Singh [24])

$$\mathrm{Re}\left(\frac{(\epsilon f'(\epsilon))'}{(f(\epsilon) - f(-\epsilon))'}\right) > 0, \ \ (\epsilon \in \mathbb{D}).$$

In [25], Owa et al. generalized the aforementioned classes and defined class $\mathcal{S}_s^*(\sigma, t)$ by

$$\mathrm{Re}\left(\frac{(1-t)\epsilon f'(\epsilon)}{f(\epsilon) - f(t\epsilon)}\right) > \sigma, \ \ |t| \leq 1, t \neq 1$$

for some σ $(0 \leq \sigma < 1)$ and for every $\epsilon \in \mathbb{D}$. They also defined class $\mathcal{C}_s(\sigma, t)$, where $f \in \mathcal{C}_s(\sigma, t)$ if and only if $\epsilon f' \mathcal{S}_s^*(\sigma, t)$.

Motivated by q-difference operator, we define two new Sakaguchi-type function classes as follows.

Definition 1. *Let $q \in (0,1)$, $0 \leq \sigma < 1$, $|t| \leq 1, t \neq 1$ and $\epsilon \in \mathbb{D}$, a function $f \in \mathcal{A}$ is a member of the class $\mathcal{S}_s^q(\sigma, t)$ if and only if*

$$\mathrm{Re}\left(\frac{(1-t)\epsilon D_q(f(\epsilon))}{f(\epsilon) - f(t\epsilon)}\right) > \sigma. \tag{2}$$

We call $\mathcal{S}_s^q(\sigma, t)$ the class of q-starlike functions with respect to symmetric points of order σ.

Definition 2. *Let $q \in (0,1)$, $0 \leq \sigma < 1$, $|t| \leq 1, t \neq 1$ and $\epsilon \in \mathbb{D}$, a function $f \in \mathcal{A}$ is a member of the class $\mathcal{C}_s^q(\sigma, t)$ if and only if*

$$\mathrm{Re}\left(\frac{(1-t)D_q(\epsilon D_q f(\epsilon))}{D_q(f(\epsilon) - f(t\epsilon))}\right) > \sigma. \tag{3}$$

We call $\mathcal{C}_s^q(\sigma, t)$ the class of q-convex functions with respect to symmetric points of order σ.

For special parameter values, these classes reduce to the following known classes:

(1) Letting $q \to 1^-$ in Definitions 1 and 2, we obtain the classes $\mathcal{S}_s^*(\sigma, t)$ and $\mathcal{C}_s(\sigma, t)$ defined by Owa et al. [25].

(2) Letting $q \to 1^-$, $\sigma = 0$ and $t = -1$ in Definitions 1 and 2, we obtain the classes $\mathcal{S}_s^*$ (Sakaguchi [23]) and $\mathcal{C}_s$ (Das and Singh [24]).

(3) Letting $q \to 1^-$ and $t = 0$ in Definitions 1 and 2, we obtain the classes $\mathcal{S}^*(\sigma)$ of starlike functions of order σ and $\mathcal{C}(\sigma)$ of convex functions of order σ.

$\mathcal{T}$ can be used to indicate the class of analytic functions with negative coefficients in the form

$$f(\epsilon) = a_1\epsilon - \sum_{\ell=2}^{\infty} a_\ell \epsilon^\ell, \ \ (a_1 > 0, a_\ell \geq 0). \tag{4}$$

We also define the classes

$$\mathcal{TS}_s^q(\sigma,t) := \mathcal{T} \cap \mathcal{S}_s^q(\sigma,t),$$

$$\mathcal{TC}_s^q(\sigma,t) := \mathcal{T} \cap \mathcal{C}_s^q(\sigma,t).$$

For special parameter values, these classes reduce to the following classes with negative coefficients:

(1) Letting $q \to 1^-$, we obtain the classes $\mathcal{TS}_s^q(\sigma,t) =: \mathcal{TS}_s^*(\sigma,t)$ and $\mathcal{TC}_s^q(\sigma,t) =: \mathcal{TC}_s(\sigma,t)$.

(2) Letting $q \to 1^-$ and $t = 0$, we obtain the classes $\mathcal{TS}_s^q(\sigma,t) =: \mathcal{TS}^*(\sigma)$ of starlike functions of order σ and $\mathcal{TC}_s^q(\sigma,t) =: \mathcal{TC}(\sigma)$ of convex functions of order σ defined by Silverman [26].

For the functions f given by (4) and $g(\epsilon) = b_1\epsilon - \sum_{\ell=1}^{\infty} b_\ell \epsilon^\ell$ $(b_1 > 0, b_\ell \geq 0)$, the quasi-Hadamard product is defined by

$$f(\epsilon) * g(\epsilon) = a_1 b_1 \epsilon - \sum_{\ell=2}^{\infty} a_\ell b_\ell \epsilon^\ell.$$

Owa [27] defined the quasi-Hadamard product of two or more functions, and later Kumar [28] studied quasi-Hadamard products of certain function classes. Let the functions f_i $(i = 1, \ldots, m)$ and g_j $(j = 1, \ldots, k)$ with the series expansions

$$f_i(\epsilon) = a_{1,i}\epsilon - \sum_{\ell=2}^{\infty} a_{\ell,i}\epsilon^\ell, \ \ (a_{1,i} > 0; a_{\ell,i} \geq 0) \tag{5}$$

$$g_j(\epsilon) = b_{1,j}\epsilon - \sum_{\ell=2}^{\infty} b_{\ell,j}\epsilon^\ell, \ \ (b_{1,j} > 0; b_{\ell,j} \geq 0) \tag{6}$$

be analytic in $\mathbb{D}$. Using h, denote the product $f_1 * f_2 * \ldots * f_m * g_1 * g_2 * \ldots * g_k$, which is defined by

$$h(\epsilon) = \left\{ \prod_{i=1}^{m} a_{1,i} \prod_{j=1}^{k} b_{1,j} \right\} \epsilon - \sum_{\ell=2}^{\infty} \left\{ \prod_{i=1}^{m} a_{\ell,i} \prod_{j=1}^{k} b_{\ell,j} \right\} \epsilon^\ell. \tag{7}$$

The studies of partial sums was first initiated by Sheil-Small [29] in 1970. He proved that $\inf \operatorname{Re}\{f(\epsilon)/f_k(\epsilon)\}$ for $f \in \mathcal{C}(0)$ occurs when $k = 1$. In [30], Silvia studied the sharp lower bounds on $\operatorname{Re}\{f(\epsilon)/f_k(\epsilon)\}$ of the starlike, and convex functions of order σ. Furthermore, Silverman [31] introduced several type-partial sums for starlike, and convex functions. In view of these previous works, we seek the ratios of a function in the form (4) to its sequence of partial sums $f_k(\epsilon) = a_1\epsilon - \sum_{\ell=2}^{k} a_\ell \epsilon^\ell$ when the coefficients of the function f are adequately small.

To do this, we first introduce sufficient coefficient estimates for the function classes $\mathcal{TS}_s^q(\sigma,t)$ and $\mathcal{TC}_s^q(\sigma,t)$. In Section 3, we introduce a quasi-Hadamard product of for these function classes using their coefficient estimates. In Section 4, we obtain the ratios of the function in the form (4) to its sequence of partial sums f_k when the coefficients of the function f in the classes $\mathcal{TS}_s^q(\sigma,t)$ and $\mathcal{TC}_s^q(\sigma,t)$ are sufficiently small, and obtain lower bounds for the ratios of $\operatorname{Re}\{f(\epsilon)/f_k(\epsilon)\}$.

2. Coefficient Bounds

We first provide sufficient coefficient estimates for the classes $\mathcal{TS}_s^q(\sigma,t)$ and $\mathcal{TC}_s^q(\sigma,t)$.

Lemma 1. *If a function f given by (4) holds*

$$\sum_{\ell=2}^{\infty}\big(|[\ell]_q - v_\ell| + (1-\sigma)|v_\ell|\big)|a_\ell| \le (1-\sigma)a_1, \tag{8}$$

where $v_\ell = 1+t+t^2+\ldots+t^{\ell-1}$, then f is a member of the class $\mathcal{S}_s^q(\sigma,t)$.

Proof. Assume that (8) holds; then, we need to prove that

$$\left|\frac{(1-t)\epsilon D_q(f(\epsilon))}{f(\epsilon)-f(t\epsilon)} - 1\right| < 1-\sigma.$$

Thus, we observe

$$\begin{aligned}\frac{(1-t)\epsilon D_q(f(\epsilon))}{f(\epsilon)-f(t\epsilon)} - 1 &= \frac{-\sum_{\ell=2}^{\infty}([\ell]_q - v_\ell)a_\ell\epsilon^\ell}{a_1\epsilon - \sum_{\ell=2}^{\infty}a_\ell v_\ell \epsilon^\ell}\\ &= \frac{-\sum_{\ell=2}^{\infty}([\ell]_q - v_\ell)a_\ell\epsilon^{\ell-1}}{a_1 - \sum_{\ell=2}^{\infty}a_\ell v_\ell \epsilon^{\ell-1}},\end{aligned}$$

which provides

$$\left|\frac{(1-t)\epsilon D_q(f(\epsilon))}{f(\epsilon)-f(t\epsilon)} - 1\right| \le \frac{\sum_{\ell=2}^{\infty}|[\ell]_q - v_\ell||a_\ell|}{a_1 - \sum_{\ell=2}^{\infty}|a_\ell||v_\ell|}.$$

Therefore, if (8) holds, then we have

$$\left|\frac{(1-t)\epsilon D_q(f(\epsilon))}{f(\epsilon)-f(t\epsilon)} - 1\right| < 1-\sigma.$$

Hence, the proof is completed. □

Lemma 2. *If a function f given by (4) holds*

$$\sum_{\ell=2}^{\infty}[\ell]_q\big(|[\ell]_q - v_\ell| + (1-\sigma)|v_\ell|\big)|a_\ell| \le (1-\sigma)a_1, \tag{9}$$

where $v_\ell = 1+t+t^2+\ldots+t^{\ell-1}$, then f is a member of the class $\mathcal{TC}_s^q(\sigma,t)$.

Proof. In view of the Alexander [32] relation $f \in \mathcal{TC}_s^q(\sigma,t)$ if and only if $\epsilon f' \in \mathcal{TS}_s^q(\sigma,t)$. Thus, by using (3) and (4), we obtain the result. □

To further prove these results, we need to define a class $\mathcal{TS}_{c,s}^q(\sigma,t)$ as follows:

Lemma 3. *A function f in the form (4) is a member of the class $\mathcal{TS}_{c,s}^q(\sigma,t)$ if*

$$\sum_{\ell=2}^{\infty}[\ell]_q^c\big(|[\ell]_q - v_\ell| + (1-\sigma)|v_\ell|\big)|a_\ell| \le (1-\sigma)a_1, \tag{10}$$

where $v_\ell = 1+t+t^2+\ldots+t^{\ell-1}$, satisfies for all fixed non-negative real numbers c.

We observe that for all real number c, the class $\mathcal{TS}_{c,s}^q(\sigma,t)$ consists of the functions in the form

$$f(\epsilon) = a_1\epsilon - \sum_{\ell=2}^{\infty}\frac{(1-\sigma)a_1}{[\ell]_q^c\big(|[\ell]_q - v_\ell| + (1-\sigma)|v_\ell|\big)}\varphi_\ell\epsilon^\ell,$$

where $a_1 > 0$, $\varphi_\ell \geq 0$, $\sum_{\ell=2}^{\infty} \varphi_\ell \leq 1$. For such functions, the following inclusion relation holds:

(i) For $c = 1$, $\mathcal{TS}_{1,s}^{q}(\sigma, t) \equiv \mathcal{TC}_{s}^{q}(\sigma, t)$.
(ii) For $c = 0$, $\mathcal{TS}_{0,s}^{q}(\sigma, t) \equiv \mathcal{TS}_{s}^{q}(\sigma, t)$.
(iii) $\mathcal{TS}_{c_1,s}^{q}(\sigma, t) \subset \mathcal{TS}_{c_2,s}^{q}(\sigma, t), \quad (c_1 > c_2 \geq 0)$.
(iv) $\mathcal{TS}_{c,s}^{q}(\sigma, t) \subset \ldots \subset \mathcal{TS}_{2,s}^{q}(\sigma, t) \subset \mathcal{TS}_{1,s}^{q}(\sigma, t) \subset \mathcal{TS}_{0,s}^{q}(\sigma, t)$.

3. Quasi-Hadamard Products

Here, we present three theorems related to the quasi-Hadamard product for functions in the classes $\mathcal{TS}_{s}^{q}(\sigma, t)$ and $\mathcal{TC}_{s}^{q}(\sigma, t)$.

Theorem 1. *Let the functions f_i $(i = 1, 2, \ldots, m)$, given by (5), be a member of the class $\mathcal{TS}_{s}^{q}(\sigma, t)$. Then, the product $f_1 * f_2 * \ldots * f_m$ belongs to the class $\mathcal{TS}_{m-1,s}^{q}(\sigma, t)$.*

Proof. To prove the theorem, we need to show that

$$\sum_{\ell=2}^{\infty} \left[[\ell]_q^{m-1} \big(|[\ell]_q - v_\ell| + (1-\sigma)|v_\ell| \big) \prod_{i=1}^{m} a_{\ell,i} \right] \leq (1-\sigma) \prod_{i=1}^{m} a_{1,i}.$$

Since $f_i \in \mathcal{TS}_{s}^{q}(\sigma, t)$, we have

$$\sum_{\ell=2}^{\infty} \big(|[\ell]_q - v_\ell| + (1-\sigma)|v_\ell| \big) a_{\ell,i} \leq (1-\sigma) a_{1,i} \tag{11}$$

for all $i = 1, 2, \ldots, m$; thus,

$$\big(|[\ell]_q - v_\ell| + (1-\sigma)|v_\ell| \big) a_{\ell,i} \leq (1-\sigma) a_{1,i}$$

or

$$a_{\ell,i} \leq \frac{1-\sigma}{\big(|[\ell]_q - v_\ell| + (1-\sigma)|v_\ell| \big)} a_{1,i}.$$

The right hand side of the last inequality is no bigger than $[\ell]_q^{-1} a_{1,i}$ and we obtain

$$a_{\ell,i} \leq [\ell]_q^{-1} a_{1,i} \tag{12}$$

for every $i = 1, 2, \ldots, m$.

By making use of the inequality (12) for $i = 1, 2, \ldots, m-1$ and the inequality (11) for $i = m$, we obtain

$$\begin{aligned}
&\sum_{\ell=2}^{\infty} \left[[\ell]_q^{m-1} \big(|[\ell]_q - v_\ell| + (1-\sigma)|v_\ell| \big) \prod_{i=1}^{m} a_{\ell,i} \right] \\
&\leq \sum_{\ell=2}^{\infty} \left[[\ell]_q^{m-1} \big(|[\ell]_q - v_\ell| + (1-\sigma)|v_\ell| \big) a_{\ell,m} \left\{ [\ell]_q^{-(m-1)} \prod_{i=1}^{m-1} a_{1,i} \right\} \right] \\
&= \sum_{\ell=2}^{\infty} \big(|[\ell]_q - v_\ell| + (1-\sigma)|v_\ell| \big) a_{\ell,m} \left\{ \prod_{i=1}^{m-1} a_{1,i} \right\} \\
&\leq (1-\sigma) \prod_{i=1}^{m} a_{1,i}.
\end{aligned}$$

Since $\mathcal{TS}_{m-1,s}^{q}(\sigma, t) \subset \mathcal{TS}_{m-2,s}^{q}(\sigma, t) \subset \ldots \subset \mathcal{TS}_{0,s}^{q}(\sigma, t) \equiv \mathcal{TS}_{s}^{q}(\sigma, t)$; therefore,

$$f_1 * f_2 * \ldots * f_m \in \mathcal{TS}_{m-1,s}^{q}(\sigma, t).$$

Thus, the proof is completed. □

With $q \to 1^-$, Theorem 1 leads to the next result.

Corollary 1. *Let the functions f_i $(i = 1, 2, \ldots, m)$, given by (5), be a member of the class $\mathcal{TS}_s^*(\sigma, t)$. Then, the product $f_1 * f_2 * \ldots * f_m$ belongs to the class $\mathcal{TS}_{m-1,s}(\sigma, t)$.*

Theorem 2. *Let the functions f_i $(i = 1, 2, \ldots, m)$, given by (5), be a member of the class $\mathcal{TC}_s^q(\sigma, t)$. Then, the product $f_1 * f_2 * \ldots * f_m$ belongs to the class $\mathcal{TS}^q_{2m-1,s}(\sigma, t)$.*

Proof. We need to prove that

$$\sum_{\ell=2}^{\infty}\left[[\ell]_q^{2m-1}\big(|[\ell]_q - v_\ell| + (1-\sigma)|v_\ell|\big)\prod_{i=1}^{m} a_{\ell,i}\right] \le (1-\sigma)\prod_{i=1}^{m} a_{1,i}.$$

Since $f_i \in \mathcal{TC}_s^q(\sigma, t)$, we have

$$\sum_{\ell=2}^{\infty}[\ell]_q\big(|[\ell]_q - v_\ell| + (1-\sigma)|v_\ell|\big)a_{\ell,i} \le (1-\sigma)a_{1,i} \tag{13}$$

for each $i = 1, 2, \ldots, m$; thus,

$$[\ell]_q\big(|[\ell]_q - v_\ell| + (1-\sigma)|v_\ell|\big)a_{\ell,i} \le (1-\sigma)a_{1,i}$$

or

$$a_{\ell,i} \le \frac{1-\sigma}{[\ell]_q\big(|[\ell]_q - v_\ell| + (1-\sigma)|v_\ell|\big)}a_{1,i}.$$

The right side is no bigger than $[\ell]_q^{-2}a_{1,i}$. Thus,

$$a_{\ell,i} \le [\ell]_q^{-2}a_{1,i} \tag{14}$$

for every $i = 1, 2, \ldots, m$.

By making use of the inequality (14) for $i = 1, 2, \ldots, m-1$ and the inequality (13) for $i = m$, we obtain

$$\begin{aligned}
&\sum_{\ell=2}^{\infty}\left[[\ell]_q^{2m-1}\big(|[\ell]_q - v_\ell| + (1-\sigma)|v_\ell|\big)\prod_{i=1}^{m} a_{\ell,i}\right] \\
&\le \sum_{\ell=2}^{\infty}\left[[\ell]_q^{2m-1}\big(|[\ell]_q - v_\ell| + (1-\sigma)|v_\ell|\big)a_{\ell,m}\left\{[\ell]_q^{-2(m-1)}\prod_{i=1}^{m-1} a_{1,i}\right\}\right] \\
&= \sum_{\ell=2}^{\infty}[\ell]_q\big(|[\ell]_q - v_\ell| + (1-\sigma)|v_\ell|\big)a_{\ell,m}\left\{\prod_{i=1}^{m-1} a_{1,i}\right\} \\
&\le (1-\sigma)\prod_{i=1}^{m} a_{1,i}.
\end{aligned}$$

Since $\mathcal{TS}^q_{2m-1,s}(\sigma, t) \subset \mathcal{TS}^q_{2m-2,s}(\sigma, t) \subset \ldots \subset \mathcal{TS}^q_{1,s}(\sigma, t) \equiv \mathcal{TC}^q_s(\sigma, t)$; hence,

$$f_1 * f_2 * \ldots * f_m \in \mathcal{TS}^q_{2m-1,s}(\sigma, t).$$

This is the desired result. □

Using $q \to 1^-$, Theorem 2 gives the following result.

Corollary 2. *Let the functions f_i $(i = 1, 2, \ldots, m)$, given by (5), be a member of the class $\mathcal{TC}_s(\sigma, t)$. Then, the product $f_1 * f_2 * \ldots * f_m$ belongs to the class $\mathcal{TS}_{2m-1,s}(\sigma, t)$.*

Theorem 3. *Let the functions f_i $(i = 1, 2, \ldots, m)$, given by (5), be in the class $\in \mathcal{TC}_s^q(\sigma, t)$, and let the functions g_j $(j = 1, 2, \ldots, k)$ given by (6) be in the class $\in \mathcal{TS}_s^q(\sigma, t)$. Then, the product $f_1 * f_2 * \ldots * f_m * g_1 * g_2 * \ldots * g_k$ is in the class $\mathcal{TS}^q_{2m+k-1,s}(\sigma, t)$.*

Proof. We need to prove that

$$\sum_{\ell=2}^{\infty}\left[[\ell]_q^{2m+k-1}\left(|[\ell]_q - v_\ell| + (1-\sigma)|v_\ell|\right)\left\{\prod_{i=1}^{m} a_{\ell,i}\prod_{j=1}^{k} b_{\ell,j}\right\}\right] \le (1-\sigma)\left\{\prod_{i=1}^{m} a_{1,i}\prod_{j=1}^{k} b_{1,j}\right\}.$$

Since $f_i \in \mathcal{TC}_s^q(\sigma, t)$, we have

$$\sum_{\ell=2}^{\infty}[\ell]_q\left(|[\ell]_q - v_\ell| + (1-\sigma)|v_\ell|\right)a_{\ell,i} \le (1-\sigma)a_{1,i}$$

for each $i = 1, 2, \ldots, m$; thus,

$$a_{\ell,i} \le \frac{1-\sigma}{[\ell]_q\left(|[\ell]_q - v_\ell| + (1-\sigma)|v_\ell|\right)}a_{1,i}$$

and the right-hand side of the last inequality is no bigger than $[\ell]_q^{-2}a_{1,i}$. Thus,

$$a_{\ell,i} \le [\ell]_q^{-2}a_{1,i} \tag{15}$$

for each $i = 1, 2, \ldots, m$. Similarly, since $g_j \in \mathcal{TS}_s^q(\sigma, t)$, we have

$$\sum_{\ell=2}^{\infty}\left(|[\ell]_q - v_\ell| + (1-\sigma)|v_\ell|\right)b_{\ell,j} \le (1-\sigma)b_{1,j}. \tag{16}$$

Hence, we can observe

$$b_{\ell,j} \le [\ell]_q^{-1}b_{1,j} \tag{17}$$

for every $j = 1, 2, \ldots, k$.

By using inequality (15) for $i = 1, 2, \ldots, m$, the inequality (17) for $j = 1, 2, \ldots, k-1$ and the inequality (16) for $j = k$, we obtain

$$\begin{aligned}
&\sum_{\ell=2}^{\infty}\left[[\ell]_q^{2m+k-1}\left(|[\ell]_q - v_\ell| + (1-\sigma)|v_\ell|\right)\left\{\prod_{i=1}^{m} a_{\ell,i}\prod_{j=1}^{k} b_{\ell,j}\right\}\right] \\
&\le \sum_{\ell=2}^{\infty}\left[[\ell]_q^{2m+k-1}\left(|[\ell]_q - v_\ell| + (1-\sigma)|v_\ell|\right)b_{\ell,k}\left\{[\ell]_q^{-2m}[\ell]_q^{-(k-1)}\prod_{i=1}^{m} a_{1,i}\prod_{j=1}^{k-1} b_{1,j}\right\}\right] \\
&= \sum_{\ell=2}^{\infty}\left(|[\ell]_q - v_\ell| + (1-\sigma)|v_\ell|\right)b_{\ell,k}\left\{\prod_{i=1}^{m} a_{1,i}\prod_{j=1}^{k-1} b_{1,j}\right\} \\
&\le (1-\sigma)\left\{\prod_{i=1}^{m} a_{1,i}\prod_{j=1}^{k} b_{1,j}\right\}.
\end{aligned}$$

Since $\mathcal{TS}^q_{2m+k-1,s}(\sigma, t) \subset \mathcal{TS}^q_{2m+k-2,s}(\sigma, t) \subset \ldots \subset \mathcal{TC}_s^q(\sigma, t) \subset \mathcal{TS}_s^q(\sigma, t)$; thus,

$$f_1 * f_2 * \ldots * f_m * g_1 * g_2 * \ldots * g_k \in \mathcal{TS}^q_{2m+k-1,s}(\sigma, t),$$

we can achieve the result. □

Using $q \to 1^-$, the Theorem 3 gives the following result.

Corollary 3. *Let the functions f_i $(i = 1,2,\ldots,m)$, given by (5), be in the class $\in \mathcal{TC}_s(\sigma,t)$, and let the functions g_j $(j = 1,2,\ldots,k)$ given by (6) be in the class $\in \mathcal{TS}^*_s(\sigma,t)$. Then, the product $f_1 * f_2 * \ldots * f_m * g_1 * g_2 * \ldots * g_k$ belongs to the class $\mathcal{TS}_{2m+k-1,s}(\sigma,t)$.*

4. Partial Sums

Here, we determine sharp lower bounds for the ratios of $\mathrm{Re}\{f(\epsilon)/f_k(\epsilon)\}$ belonging to the classes $\mathcal{TS}^q_s(\sigma,t)$ and $\mathcal{TC}^q_s(\sigma,t)$.

Theorem 4. *If $f \in \mathcal{TS}^q_s(\sigma,t)$ in the form (4) holds (8), then*

$$\mathrm{Re}\left(\frac{f(\epsilon)}{f_k(\epsilon)}\right) \geq 1 - \frac{1-\sigma}{|[k+1]_q - v_{k+1}| + (1-\sigma)|v_{k+1}|} \tag{18}$$

with $v_k = 1 + t + t^2 + \ldots + t^{k-1}$. This result is sharp.

Proof. Let

$$\lambda_\ell = \frac{|[\ell]_q - v_\ell| + (1-\sigma)|v_\ell|}{(1-\sigma)a_1}, \quad (\ell \geq 2)$$

then $\lambda_{\ell+1} > \lambda_\ell > 1$ $(\ell \geq 2)$, it follows from (8) that

$$\sum_{\ell=2}^{k} |a_\ell| + \lambda_{k+1} \sum_{\ell=k+1}^{\infty} |a_\ell| \leq \sum_{\ell=2}^{\infty} \lambda_\ell |a_\ell| \leq 1.$$

Thus we write

$$\begin{aligned} \psi_1(\epsilon) &= 1 + \lambda_{k+1}\left(\frac{f(\epsilon)}{f_k(\epsilon)} - 1\right) \\ &= 1 - \frac{\lambda_{k+1}\sum_{\ell=k+1}^{\infty} a_\ell \epsilon^\ell}{a_1\epsilon - \sum_{\ell=2}^{k} a_\ell \epsilon^\ell}. \end{aligned}$$

which is analytic in $\mathbb{D}$ with $\psi_1(0) = 1$. It suffices to show that $Re\psi_1(\epsilon) > 0$, or

$$\left|\frac{\psi_1(\epsilon) - 1}{\psi_1(\epsilon) + 1}\right| \leq 1;$$

then, we obtain

$$\left|\frac{\psi_1(\epsilon) - 1}{\psi_1(\epsilon) + 1}\right| \leq \frac{\lambda_{k+1} \sum\limits_{\ell=k+1}^{\infty} |a_\ell|}{2a_1 - 2\sum\limits_{\ell=2}^{k} |a_\ell| - \lambda_{k+1} \sum\limits_{\ell=k+1}^{\infty} |a_\ell|} \leq 1$$

which implies that

$$\sum_{\ell=2}^{k} |a_\ell| + \lambda_{k+1} \sum_{\ell=k+1}^{\infty} |a_\ell| \leq a_1. \tag{19}$$

To prove the inequality (18), it is sufficent to show that LHS of (19) is bounded above by $\sum\limits_{\ell=2}^{\infty} \lambda_\ell |a_\ell|$, that is,

$$\sum_{\ell=2}^{k} (\lambda_\ell - 1)|a_\ell| + \sum_{\ell=k+1}^{\infty} (\lambda_\ell - \lambda_{k+1})|a_\ell| \geq 0.$$

If we take the sharp function

$$f(\epsilon) = a_1\epsilon - \frac{(1-\sigma)a_1}{|[k+1]_q - v_{k+1}| + (1-\sigma)|v_{k+1}|}\epsilon^{k+1};$$

then, $f_k(\epsilon) = a_1\epsilon$ and

$$\frac{f(\epsilon)}{f_k(\epsilon)} \to 1 - \frac{1-\sigma}{|[k+1]_q - v_{k+1}| + (1-\sigma)|v_{k+1}|}$$

as $\epsilon \to 1^-$. Thus, the proof is completed. □

Setting $q \to 1^-$ and $t = 0$, we obtain the partial sums for the class $\mathcal{TS}^*(\sigma)$.

Corollary 4. *If $f \in \mathcal{TS}^*(\sigma)$ of the form (4); then,*

$$\operatorname{Re}\left(\frac{f(\epsilon)}{f_k(\epsilon)}\right) \geq 1 - \frac{1-\sigma}{k+1-\sigma}.$$

For $k = 1$, Corollary 4 reduces to the result obtained by Silvia.

Remark 1 ([30]). *If $f(\epsilon) = \epsilon - \sum_{\ell=2}^{\infty} a_\ell \epsilon^\ell \in \mathcal{TS}^*(\sigma)$; then,*

$$\operatorname{Re}\left(\frac{f(\epsilon)}{f_k(\epsilon)}\right) \geq \frac{1}{2-\sigma},\ k = 1, 2, \ldots .$$

Theorem 5. *If $f \in \mathcal{TC}_s^q(\sigma, t)$ in the form (4) holds (9), then*

$$\operatorname{Re}\left(\frac{f(\epsilon)}{f_k(\epsilon)}\right) \geq 1 - \frac{1-\sigma}{[k+1]_q(|[k+1]_q - v_{k+1}| + (1-\sigma)|v_{k+1}|)} \tag{20}$$

with $v_k = 1 + t + t^2 + \ldots + t^{k-1}$. This result is sharp.

Proof. Let

$$\eta_\ell = \frac{[\ell]_q(|[\ell]_q - v_\ell| + (1-\sigma)|v_\ell|)}{(1-\sigma)a_1}, \quad (\ell \geq 2)$$

then $\eta_{\ell+1} > \eta_\ell > 1$ $(\ell \geq 2)$, it follows from (9) that

$$\sum_{\ell=2}^{k} |a_\ell| + \eta_{k+1} \sum_{\ell=k+1}^{\infty} |a_\ell| \leq \sum_{\ell=2}^{\infty} \eta_\ell |a_\ell| \leq 1,$$

thus, we write the analytic function

$$\begin{aligned}\psi_2(\epsilon) &= 1 + \eta_{k+1}\left(\frac{f(\epsilon)}{f_k(\epsilon)} - 1\right) \\ &= 1 - \frac{\eta_{k+1}\sum_{\ell=k+1}^{\infty} a_\ell \epsilon^\ell}{a_1\epsilon - \sum_{\ell=2}^{k} a_\ell \epsilon^\ell}.\end{aligned}$$

in $\mathbb{D}$ with $\psi_2(0) = 1$. Therefore, we need to show that $Re\psi_2(\epsilon) > 0$, or, equivalently, we obtain

$$\left|\frac{\psi_1(\epsilon) - 1}{\psi_1(\epsilon) + 1}\right| \leq \frac{\eta_{k+1} \sum\limits_{\ell=k+1}^{\infty} |a_\ell|}{2a_1 - 2\sum\limits_{\ell=2}^{k} |a_\ell| - \eta_{k+1} \sum\limits_{\ell=k+1}^{\infty} |a_\ell|} \leq 1$$

which indicates that

$$\sum_{\ell=2}^{k} |a_\ell| + \eta_{k+1} \sum_{\ell=k+1}^{\infty} |a_\ell| \leq a_1. \tag{21}$$

Since the LHS of (21) is bounded above by $\sum\limits_{\ell=2}^{\infty} \eta_\ell |a_\ell|$, we can arrive at

$$\sum_{\ell=2}^{k} (\eta_\ell - 1)|a_\ell| + \sum_{\ell=k+1}^{\infty} (\eta_\ell - \eta_{k+1})|a_\ell| \geq 0;$$

hence, the proof is completed.

Consider the sharp function

$$f(\epsilon) = a_1\epsilon - \frac{(1-\sigma)a_1}{[k+1]_q(|[k+1]_q - v_{k+1}| + (1-\sigma)|v_{k+1}|)}\epsilon^{k+1};$$

then, $f_k(\epsilon) = a_1\epsilon$ and

$$\frac{f(\epsilon)}{f_k(\epsilon)} \to 1 - \frac{1-\sigma}{[k+1]_q(|[k+1]_q - v_{k+1}| + (1-\sigma)|v_{k+1}|)}$$

as $\epsilon \to 1^-$. Thus, the proof is completed. □

Setting $q \to 1^-$ and $t = 0$, we obtain the partial sums for the class $\mathcal{TC}(\sigma)$.

Corollary 5. *If $f \in \mathcal{TC}(\sigma)$ in the form (4), then*

$$\text{Re}\left(\frac{f(\epsilon)}{f_k(\epsilon)}\right) \geq 1 - \frac{1-\sigma}{(k+1)(k+1-\sigma)}.$$

For $k = 1$, Corollary 5 reduces to the result obtained by Silvia.

Remark 2 ([30]). *If $f(\epsilon) = \epsilon - \sum_{\ell=2}^{\infty} a_\ell \epsilon^\ell \in \mathcal{TC}(\sigma)$, then*

$$\text{Re}\left(\frac{f(\epsilon)}{f_k(\epsilon)}\right) \geq \frac{3-\sigma}{4-2\sigma},\ k = 1, 2, \ldots .$$

5. Concluding Remarks

Motivated by the recent applications of a q-difference operator in geometric function theory, we defined two new subclasses of Sakaguchi-type function classes, which are starlike and convex with respect to their symmetric points. We introduced sufficient coefficient bounds for these function classes. By using these bounds, we obtained quasi-Hadamard products and several partial sums for these function classes. We note that our results naturally include several results that are known for those classes, which are listed after Definitions 1 and 2.

Author Contributions: Both authors contributed equally. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare that they have no competing interest.

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