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Certain Generalizations of Quadratic Transformations of Hypergeometric and Generalized Hypergeometric Functions

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Abstract: There have been numerous investigations on the hypergeometric series ${}_2F_1$ and the generalized hypergeometric series ${}_pF_q$ such as differential equations, integral representations, analytic continuations, asymptotic expansions, reduction cases, extensions of one and several variables, continued fractions, Riemann's equation, group of the hypergeometric equation, summation, and transformation formulae. Among the various approaches to these functions, the transformation formulae for the hypergeometric series ${}_2F_1$ and the generalized hypergeometric series ${}_pF_q$ are significant, both in terms of applications and theory. The purpose of this paper is to establish a number of transformation formulae for ${}_pF_q$, whose particular cases would include Gauss's and Kummer's quadratic transformation formulae for ${}_2F_1$, as well as their two extensions for ${}_3F_2$, by making advantageous use of a recently introduced sequence and some techniques commonly used in dealing with ${}_pF_q$ theory. The ${}_pF_q$ function, which is the most significant function investigated in this study, exhibits natural symmetry.

Keywords: gamma function; Psi function; generalized hypergeometric function ${}_pF_q$; Gauss's summation theorem for ${}_2F_1$; summation theorems for ${}_pF_q$; transformation formulas for ${}_pF_q$; series rearrangement techniques

MSC: 33B15; 33C05; 33C20; 34A25

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1. Introduction and Preliminaries

The ${}_pF_q$ $(p,\, q \in \mathbb{Z}_{\geq 0})$ is the generalized hypergeometric series defined by (see, e.g., [1], Section 1.5):

$$\begin{aligned} {}_pF_q\left[\begin{matrix}\alpha_1,\,\ldots,\,\alpha_p\,; \\ \beta_1,\,\ldots,\,\beta_q\,;\end{matrix}\; z\right] &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \\ &= {}_pF_q(\alpha_1,\,\ldots,\,\alpha_p;\,\beta_1,\,\ldots,\,\beta_q;\,z), \end{aligned} \tag{1}$$

being a natural generalization of the Gaussian hypergeometric series ${}_2F_1$, where $(\lambda)_\nu$ denotes the Pochhammer symbol (for $\lambda,\, \nu \in \mathbb{C}$) defined by:

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C} \setminus \{0\}), \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{Z}_{\geq 1};\ \lambda \in \mathbb{C}). \end{cases} \tag{2}$$

where Γ is the familiar Gamma function (see, e.g., [1], Section 1.1) and it is assumed that $(0)_0 := 1$, an empty product as 1, and that the variable z, the numerator parameters $\alpha_1, \ldots,$ α_p, and the denominator parameters $\beta_1, \ldots, \beta_q$ take on complex values, provided that no zeros appear in the denominator of (1), that is, that:

$$(\beta_j \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0};\ j = 1, \ldots, q).$$

Here and elsewhere, let $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ be, respectively, the sets of integers, real numbers, and complex numbers. Further,

$$\mathbb{E}_{\leq \nu}, \quad \mathbb{E}_{<\nu}, \quad \mathbb{E}_{\geq \nu}, \quad \text{and} \quad \mathbb{E}_{>\nu}$$

be the sets of numbers in $\mathbb{E}$ less than or equal to ν, less than ν, greater than or equal to ν, and greater than ν, respectively, for some $\nu \in \mathbb{E}$, where $\mathbb{E}$ is either $\mathbb{Z}$ or $\mathbb{R}$.

Furthermore, in the following, an empty sum and an empty product are assumed to be, respectively, 0 and 1.

We recall certain identities and theorems:

The generalized binomial theorem (see, e.g., [2], p. 44, Equation (8)) is given as:

$$(1-z)^{-\lambda} = \sum_{n=0}^{\infty} (\lambda)_n \frac{z^n}{n!} = {}_1F_0(\lambda\,;\,—\,;\,z) \tag{3}$$

$$(|\arg(1-z)| < \pi, \ |z| < 1; \ \lambda \in \mathbb{C}).$$

The classical Gauss's summation theorem is recalled (see [3]; see, e.g., [2], p. 30, Equation (7)):

$${}_2F_1(\alpha,\,\beta\,;\,\gamma\,;\,1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \tag{4}$$

$$(\gamma \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}, \ \Re(\gamma-\alpha-\beta) > 0).$$

Setting $\alpha = -n$ $(n \in \mathbb{Z}_{\geq 0})$ in (4) provides the Chu–Vandermonde summation theorem (see, e.g., [4], p. 69):

$${}_2F_1(-n,\,\beta\,;\,\gamma\,;\,1) = \frac{(\gamma-\beta)_n}{(\gamma)_n} \quad (\gamma \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}, \ n \in \mathbb{Z}_{\geq 0}). \tag{5}$$

An extension of Gauss's summation Theorem (4) is recalled (see, e.g., [5], p. 534, Entry 7.4.4–10; see also [6], Equation (8)):

$${}_3F_2\left[\begin{matrix} a,\, b,\, d+1\,; \\ c,\, d\,; \end{matrix}\; 1\right] = \frac{\Gamma(c)\,\Gamma(c-a-b-1)}{\Gamma(c-a)\,\Gamma(c-b)}\left(c-a-b-1+\frac{ab}{d}\right) \tag{6}$$

$$(c,\, d \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}; \ \Re(c-a-b) > 1).$$

Setting either $a = d$ or $b = d$ in (6) is found to be equivalent to (4).

The following identities are derivable from (2) (see, e.g., [1], p. 5):

$$(\alpha)_{-n} := \frac{\Gamma(\alpha-n)}{\Gamma(\alpha)} = \frac{(-1)^n}{(1-\alpha)_n} \quad (\alpha \in \mathbb{C} \setminus \mathbb{Z}; \ n \in \mathbb{Z}_{\geq 0}); \tag{7}$$

$$(-s)_j = \begin{cases} 0 & (j > s), \\ \frac{(-1)^j\, s!}{(s-j)!} & (0 \leq j \leq s); \end{cases} \tag{8}$$

$$\frac{(b+\ell)_k}{(b)_k} = \frac{(b+k)_\ell}{(b)_\ell} \quad (\ell,\, k \in \mathbb{Z}_{\geq 0}; \ b \in \mathbb{C}). \tag{9}$$

By mainly reducing suitable parameters involving ${}_pF_q$ to construct certain summation formulas for ${}_pF_q$, Choi et al. [7] introduced the following sequence $\{A_j(\alpha,\ell)\}_{j=0}^{\ell}$ (for details, see [7], Equations (28) and (33)):

$$\sum_{j=0}^{\ell} A_j(\alpha,\ell) k(k-1)\cdots(k-j+1) =: (\alpha+k)_\ell = \frac{(\alpha)_\ell(\alpha+\ell)_k}{(\alpha)_k} \tag{10}$$

$$(k \in \mathbb{Z}_{\geq 0}, \ \ell \in \mathbb{Z}_{\geq 1}, \ \alpha \in \mathbb{C}),$$

and,

$$A_j(\alpha,\ell)=\binom{\ell}{j}\frac{(\alpha)_\ell}{(\alpha)_j}=\binom{\ell}{j}(\alpha+j)_{\ell-j} \tag{11}$$

$$(\ell\in\mathbb{Z}_{\geq 0},\ j=0,1,\dots,\ell;\ \alpha\in\mathbb{C}).$$

Using (8) and (11), we may obtain:

$$\sum_{j=0}^{\ell}A_j(\alpha,\ell)k(k-1)\cdots(k-j+1)=(\alpha)_\ell\,{}_2F_1(-\ell,\,-k\,;\,\alpha\,;\,1) \tag{12}$$

$$(\ell,\,k\in\mathbb{Z}_{\geq 0},\ \alpha\in\mathbb{C}).$$

One defines the generalized harmonic numbers $H_n(z)$ by:

$$H_n(z):=\sum_{j=1}^{n}\frac{1}{z+j}\quad(n\in\mathbb{Z}_{\geq 1},\,z\in\mathbb{C}\setminus\mathbb{Z}_{\leq -1}), \tag{13}$$

where $H_n:=H_n(0)$ are the familiar harmonic numbers.

The Psi (or digamma) function $\psi(z)$ is defined by (see, e.g., [1], Section 1.3):

$$\psi(z):=\frac{d}{dz}\log\Gamma(z)=\frac{\Gamma'(z)}{\Gamma(z)}\quad(z\in\mathbb{C}\setminus\mathbb{Z}_{\leq 0}), \tag{14}$$

where log is assumed to be taken as the principal branch. This Psi function has a number of useful identities, for example,

$$\psi(z+n)-\psi(z)=\sum_{j=1}^{n}\frac{1}{z+j-1}=H_n(z-1)\quad(n\in\mathbb{Z}_{\geq 1}). \tag{15}$$

We also have:

$$\frac{d}{dz}(z)_n=(z)_n\big[\psi(z+n)-\psi(z)\big]=(z)_n\,H_n(z-1)\quad(n\in\mathbb{Z}_{\geq 1}). \tag{16}$$

Among a number of transformation formulas for ${}_2F_1$ and ${}_pF_q$ (see, e.g., [5,8]), for our purpose, we begin by recalling Gauss's quadratic transformation formula for ${}_2F_1$ (see [3], p. 225, Equation (100); see also [9], p. 92, Equation (1); [10], p. 50):

$$(1-z)^{-2a}\,{}_2F_1\left(a,\,a+\frac{1}{2}\,;\,c\,;\,-\frac{4z}{(1-z)^2}\right)={}_2F_1(2a,\,2a-c+1\,;\,c\,;\,-z) \tag{17}$$

$$\Big(c\in\mathbb{C}\setminus\mathbb{Z}_{\leq 0};\ |z|<1,\ 4|z|<|1-z|^2,\ \ |\arg(1-z)|<\pi\Big).$$

By making a main use of (6), Rakha et al. [6] (p. 173, Equation (9)) established the following quadratic transformation formula between ${}_3F_2$ and ${}_4F_3$:

$$\begin{aligned}(1-z)^{-2a}\,{}_3F_2&\left[\begin{matrix}a,\,a+\frac{1}{2},\,d+1\,;\\ c+1,\,d\,;\end{matrix}\ -\frac{4z}{(1-z)^2}\right]\\ &={}_4F_3\left[\begin{matrix}2a,\,2a-c,\,a-A+1,\,a+A+1\,;\\ c+1,\,a-A,\,a+A\,;\end{matrix}\ -z\right]\end{aligned} \tag{18}$$

$$\Big(c\in\mathbb{C}\setminus\mathbb{Z}_{\leq -1};\ a\pm A,\,d\in\mathbb{C}\setminus\mathbb{Z}_{\leq 0};\ |z|<1,\ 4|z|<|1-z|^2,\ \ |\arg(1-z)|<\pi\Big),$$

where:

$$A:=\Big(a^2-2ad+cd\Big)^{\frac{1}{2}} \tag{19}$$

is assumed to be taken one of its two values.

Remark 1. In [6], the A and restrictions are not specified.

Kummer [11] (p. 78, Equation (52)) presented the following quadratic transformation formula (see also [4], p. 65, Theorem 24): Let $c+\frac{3}{2},\, 2c+2 \in \mathbb{C}\setminus\mathbb{Z}_{\le 0}$; $|z|<\frac{1}{2}$, $|z|<|1-z|$, $\;|\arg(1-z)|<\pi$. Then:

$$(1-z)^{-2a}\,{}_2F_1\left[\begin{matrix} a,\, a+\frac{1}{2}\,; \\ c+\frac{3}{2}\,; \end{matrix}\ \frac{z^2}{(1-z)^2}\right] = {}_2F_1\left[\begin{matrix} 2a,\, c+1\,; \\ 2c+2\,; \end{matrix}\ 2z\right]. \tag{20}$$

By primarily using (6), Rakha et al. [12] (p. 208, Equation (3)) extended (20) in the following quadratic transformation formulas between ${}_3F_2$ and ${}_4F_3$:

$$\begin{aligned}(1-z)^{-2a}\,{}_3F_2&\left[\begin{matrix} a,\, a+\frac{1}{2},\, d+1\,; \\ c+\frac{3}{2},\, d\,; \end{matrix}\ \frac{z^2}{(1-z)^2}\right]\\ &= {}_4F_3\left[\begin{matrix} 2a,\, c,\, 2d+\frac{B}{2}+\frac{1}{2},\, 2d-\frac{B}{2}+\frac{1}{2}\,; \\ 2c+2,\, 2d+\frac{B}{2}-\frac{1}{2},\, 2d-\frac{B}{2}-\frac{1}{2}\,; \end{matrix}\ 2z\right]\end{aligned} \tag{21}$$

$$\Big(c+\tfrac{3}{2},\, 2c+2,\, 2d-\tfrac{1}{2}\pm\tfrac{B}{2},\, d\in\mathbb{C}\setminus\mathbb{Z}_{\le 0};\\ |z|<\tfrac{1}{2},\, |z|<|1-z|,\ \ |\arg(1-z)|<\pi\Big),$$

where:

$$B := \left(16d^2-16cd-8d+1\right)^{\frac{1}{2}} \tag{22}$$

is assumed to be taken one of its two values.

As stated in the abstract, the transformation formulas for the generalized hypergeometric series ${}_pF_q$ have theoretical and practical significance. The primary goal of this article is to develop a number of transformation formulae for ${}_pF_q$, with special emphasis on (17), (18), (20), and (21), by making beneficial use of the sequence in (10) and other techniques widely utilized in dealing with ${}_pF_q$ theory.

2. Extensions of the Quadratic Transformation Formulas

This section provides several generalizations of the quadratic transformation formulas (18) as well as (17).

Theorem 1. *Let $\ell\in\mathbb{Z}_{\ge 1}$; $b,\, d\in\mathbb{C}\setminus\mathbb{Z}_{\le 0}$; $b-2a\in\mathbb{C}\setminus\mathbb{Z}_{\ge 1}$, $\Re(b-2a)>\ell$; $|z|<1$, $4|z|<|1-z|^2$, $|\arg(1-z)|<\pi$. Then:*

$$\begin{aligned}(1-z)^{-2a}\,{}_3F_2&\left[\begin{matrix} a,\, a+\frac{1}{2},\, d+\ell\,; \\ b,\, d\,; \end{matrix}\ -\frac{4z}{(1-z)^2}\right]\\ &= \sum_{s=0}^{\infty}\frac{(2a)_s(1-b+2a)_s(-z)^s}{(b)_s\, s!}\,{}_3F_2\left[\begin{matrix} -\ell,\, -s,\, 2a+s\,; \\ d,\, 1-b+2a\,; \end{matrix}\ 1\right].\end{aligned} \tag{23}$$

Proof. Let $\mathcal{L}_1$ be the left member of (23). Using (1), we have:

$$\mathcal{L}_1 = \sum_{r=0}^{\infty}\frac{(-1)^r\, 2^{2r}\,(a)_r(a+\frac{1}{2})_r(d+\ell)_r\, z^r}{(d)_r(b)_r\, r!}(1-z)^{-2r-2a},$$

which, upon using the following duplication formula:

$$(\lambda)_{2n} = 2^{2n} \left(\frac{\lambda}{2}\right)_n \left(\frac{\lambda}{2} + \frac{1}{2}\right)_n \quad (n \in \mathbb{Z}_{\geq 0}), \tag{24}$$

$$\mathcal{L}_1 = \sum_{r=0}^{\infty} \frac{(-1)^r \, (2a)_{2r} \, (d+\ell)_r \, z^r}{(d)_r (b)_r \, r!} (1-z)^{-2r-2a}. \tag{25}$$

Employing (3) in (25) gives:

$$\mathcal{L}_1 = \sum_{s=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-1)^r \, (2a)_{2r} \, (d+\ell)_r \, z^{r+s}}{(d)_r (b)_r \, r!} \frac{(2a+2r)_s}{s!},$$

which, upon using the following identity:

$$(\lambda)_{m+n} = (\lambda)_m \, (\lambda+m)_n \quad (m,\, n \in \mathbb{Z}_{\geq 0}), \tag{26}$$

yields:

$$\mathcal{L}_1 = \sum_{s=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-1)^r \, (2a)_{s+2r} \, (d+\ell)_r \, z^{r+s}}{(d)_r (b)_r \, r! \, s!}. \tag{27}$$

Recall the following double series manipulation:

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f(m,n) = \sum_{m=0}^{\infty} \sum_{n=0}^{m} f(m-n,n), \tag{28}$$

$f : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{C}$ being a function, provided that the involved double series is assumed to be absolutely convergent.

Using (28) in (27) provides:

$$\mathcal{L}_1 = \sum_{s=0}^{\infty} \sum_{r=0}^{s} \frac{(-1)^r \, (2a)_{s+r} \, (d+\ell)_r \, z^s}{(d)_r (b)_r \, r! \, (s-r)!},$$

which, upon using (8) and (26), offers:

$$\mathcal{L}_1 = \sum_{s=0}^{\infty} \frac{(2a)_s \, z^s}{s!} \sum_{r=0}^{s} \frac{(-s)_r \, (2a+s)_r}{(b)_r \, r!} \frac{(d+\ell)_r}{(d)_r}. \tag{29}$$

Using (10) in (29), we obtain:

$$\mathcal{L}_1 = \frac{1}{(d)_\ell} \sum_{s=0}^{\infty} \frac{(2a)_s z^s}{s!} \sum_{j=0}^{\ell} A_j(d,\ell) \sum_{r=j}^{s} \frac{(-s)_r (2a+s)_r}{(b)_r \, (r-j)!},$$

which, upon setting $r - j = r'$ in the third summation and dropping the prime on r, with the aid of (26), leads to:

$$\mathcal{L}_1 = \frac{1}{(d)_\ell} \sum_{s=0}^{\infty} \frac{(2a)_s z^s}{s!} \sum_{j=0}^{\ell} \frac{A_j(d,\ell)(-s)_j (2a+s)_j}{(b)_j} \sum_{r=0}^{s-j} \frac{(-s+j)_r (2a+s+j)_r}{(b+j)_r \, r!},$$

or, equivalently,

$$\mathcal{L}_1 = \frac{1}{(d)_\ell} \sum_{s=0}^{\infty} \frac{(2a)_s z^s}{s!} \sum_{j=0}^{\ell} \frac{A_j(d,\ell)(-s)_j (2a+s)_j}{(b)_j} \times {}_2F_1\left[\begin{matrix} -s+j,\, 2a+s+j\,; \\ b+j\,; \end{matrix} 1\right]. \tag{30}$$

Employing (4) or (5) in ${}_2F_1$ in (30) provides:

$$\mathcal{L}_1 = \frac{1}{(d)_\ell} \sum_{s=0}^{\infty} \frac{(2a)_s(1-b+2a)_s(-z)^s}{(b)_s s!} \sum_{j=0}^{\ell} \frac{A_j(d,\ell)(-s)_j(2a+s)_j(-1)^j}{(1-b+2a)_j}. \tag{31}$$

Using (11) in the inner summation in (31), with the aid of (8), we may get:

$$\mathcal{L}_1 = \sum_{s=0}^{\infty} \frac{(2a)_s(1-b+2a)_s(-z)^s}{(b)_s s!} \sum_{j=0}^{\ell} \frac{(-\ell)_j(-s)_j(2a+s)_j}{(d)_j(1-b+2a)_j j!},$$

which, in virtue of (1), leads to the right member of (23). □

Remark 2. The right member of (23) may be expressed in terms of the double hypergeometric function of the Srivastava–Daoust (see, e.g., [13]; [14], p. 454, Equation (4.1); [15], pp. 199–200, Equation (2.1)).

Theorem 2. *Let* $\ell_1, \ell_2 \in \mathbb{Z}_{\geq 1}$; $b, d_1, d_2 \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$; $b-2a \in \mathbb{C} \setminus \mathbb{Z}_{\geq 1}$, $\Re(b-2a) > \ell_1+\ell_2$; $|z|<1$, $4|z| < |1-z|^2$, $|\arg(1-z)| < \pi$. *Then,*

$$\begin{aligned}
&(1-z)^{-2a}\, {}_4F_3\left[\begin{matrix} a,\, a+\frac{1}{2},\, d_1+\ell_1,\, d_2+\ell_2\,; \\ b,\, d_1,\, d_2\,; \end{matrix} -\frac{4z}{(1-z)^2}\right] \\
&= \sum_{s=0}^{\infty} \frac{(2a)_s(1+2a-b)_s(-z)^s}{(b)_s\, s!} \\
&\quad \times \sum_{j=0}^{\ell_1} \frac{(-\ell_1)_j(-s)_j(d_2+\ell_2)_j(2a+s)_j}{(d_1)_j(d_2)_j(1+2a-b)_j\, j!}\, {}_3F_2\left[\begin{matrix} -\ell_2,\, -s+j,\, 2a+s+j\,; \\ d_2+j,\, 1+2a-b+j\,; \end{matrix} 1\right].
\end{aligned} \tag{32}$$

Proof. The proof would run in parallel with that of Theorem 1. The details are omitted. □

The following theorem provides a general quadratic transformation formula for a ${}_pF_q$, which includes (23) and (32) as particular cases.

Theorem 3. *Let* $r, \ell_j \in \mathbb{Z}_{\geq 1}$ $(j=1,\ldots,r)$; $b, d_j \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ $(j=1,\ldots,r)$; $b-2a \in \mathbb{C} \setminus \mathbb{Z}_{\geq 1}$, $\Re(b-2a) > \ell_1+\cdots+\ell_r$; $|z|<1$, $4|z| < |1-z|^2$, $|\arg(1-z)| < \pi$. *Then,*

$$\begin{aligned}
&(1-z)^{-2a}\, {}_{r+2}F_{r+1}\left[\begin{matrix} a,\, a+\dfrac{1}{2},\, d_1+\ell_1,\, \ldots,\, d_r+\ell_r\,; \\ b,\, d_1,\, \ldots,\, d_r\,; \end{matrix} -\frac{4z}{(1-z)^2}\right] \\
&= \sum_{s=0}^{\infty} \frac{(2a)_s(1+2a-b)_s(-z)^s}{(b)_s\, s!} \left[\prod_{\mu=1}^{r-1} \mathcal{S}(\ell_\mu, \mathbf{J}_{\mu-1}; a, b, d_1, \ldots, d_r; s)\right] \\
&\quad \times {}_3F_2\left[\begin{matrix} -\ell_r,\, -s+\mathbf{J}_{r-1}, 2a+s+\mathbf{J}_{r-1}\,; \\ d_r+\mathbf{J}_{r-1},\, 1+2a-b+\mathbf{J}_{r-1}\,; \end{matrix} 1\right].
\end{aligned} \tag{33}$$

Here,

$$\mathbf{J}_p := \sum_{\nu=1}^{p} j_\nu \quad (p \in \mathbb{Z}_{\geq 1}), \tag{34}$$

and,

$$\begin{aligned}&\mathcal{S}\left(\ell_\mu,\mathbf{J}_{\mu-1};a,b,d_1,\ldots,d_r;s\right)\\&:=\sum_{j_\mu=0}^{\ell_\mu}(-1)^{j_\mu}\binom{\ell_\mu}{j_\mu}\frac{\left(-s+\mathbf{J}_{\mu-1}\right)_{j_\mu}\left(2a+s+\mathbf{J}_{\mu-1}\right)_{j_\mu}}{\left(1+2a-b+\mathbf{J}_{\mu-1}\right)_{j_\mu}\left(d_\mu+\mathbf{J}_{\mu-1}\right)_{j_\mu}}\\&\quad\times\prod_{\nu=\mu+1}^{r}\frac{\left(d_\nu+\ell_\nu+\mathbf{J}_{\mu-1}\right)_{j_\mu}}{\left(d_\nu+\mathbf{J}_{\mu-1}\right)_{j_\mu}}\quad(\mu=1,\ldots,r-1).\end{aligned}$$

Proof. As in the proof of Theorems 1 and 2, by induction on r, we may justify (33). Thus, the involved specifics are omitted. □

Theorems 1 and 2 can be rewritten, respectively, as in Theorems 4 and 5.

Theorem 4. *Let $\ell\in\mathbb{Z}_{\geq1}$; $b,\,d\in\mathbb{C}\setminus\mathbb{Z}_{\leq0}$; $b-2a\in\mathbb{C}\setminus\mathbb{Z}_{\geq1}$, $\Re(b-2a)>\ell$; $|z|<1$, $4|z|<|1-z|^2$, $|\arg(1-z)|<\pi$. Then,*

$$\begin{aligned}&(1-z)^{-2a}\,{}_3F_2\left[\begin{matrix}a,\,a+\frac{1}{2},\,d+\ell\,;\\ b,\,d\,;\end{matrix}\;-\frac{4z}{(1-z)^2}\right]\\&=\sum_{k=0}^{\ell}\binom{\ell}{k}\frac{(2a)_{2k}\,(-z)^k}{(b)_k\,(d)_k}\,{}_2F_1\left[\begin{matrix}2a+2k,\,1-b+2a+k\,;\\ b+k\,;\end{matrix}\;-z\right].\end{aligned}\tag{35}$$

Proof. Let $\mathcal{R}_1$ be the right member of (23). We have:

$$\mathcal{R}_1=\sum_{k=0}^{\ell}\frac{(-\ell)_k}{(d)_k\,(1-b+2a)_k\,k!}\sum_{s=0}^{\infty}\frac{(-s)_k\,(2a)_{s+k}\,(1-b+2a)_s(-z)^s}{(b)_s\,s!}.\tag{36}$$

Using (8) in (36), we obtain:

$$\begin{aligned}\mathcal{R}_1&=\sum_{k=0}^{\ell}\binom{\ell}{k}\frac{1}{(d)_k\,(1-b+2a)_k}\\&\quad\times\sum_{s=0}^{\infty}\frac{s(s-1)\cdots(s-k+1)\,(2a)_{s+k}\,(1-b+2a)_s(-z)^s}{(b)_s\,s!},\end{aligned}$$

which gives:

$$\mathcal{R}_1=\sum_{k=0}^{\ell}\binom{\ell}{k}\frac{1}{(d)_k\,(1-b+2a)_k}\sum_{s=k}^{\infty}\frac{(2a)_{s+k}\,(1-b+2a)_s(-z)^s}{(b)_s\,(s-k)!}.\tag{37}$$

Setting $s-k=s'$ in the inner sum in (37) and dropping the prime on s, we find:

$$\mathcal{R}_1=\sum_{k=0}^{\ell}\frac{\binom{\ell}{k}}{(d)_k\,(1-b+2a)_k}\sum_{s=0}^{\infty}\frac{(2a)_{s+2k}\,(1-b+2a)_{s+k}\,(-z)^{s+k}}{(b)_{s+k}\,s!}.\tag{38}$$

Employing (26) in (38), we get:

$$\mathcal{R}_1=\sum_{k=0}^{\ell}\binom{\ell}{k}\frac{(2a)_{2k}\,(-z)^k}{(b)_k\,(d)_k}\sum_{s=0}^{\infty}\frac{(2a+2k)_s\,(1-b+2a+k)_s\,(-z)^s}{(b+k)_s\,s!},$$

which, in light of (1), leads to the right member of (35). □

Remark 3. The formulas in Theorems 1 and 4 are found to hold true for $\ell=0$. The particular case of (35) when $\ell=0$ reduces to yield Gauss's quadratic transformation formula for ${}_2F_1$ (see (17)).

Theorem 5. *Let $\ell_1, \ell_2 \in \mathbb{Z}_{\geq 1}$; $b, d_1, d_2 \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$; $b-2a \in \mathbb{C} \setminus \mathbb{Z}_{\geq 1}$, $\Re(b-2a) > \ell_1 + \ell_2$; $|z| < 1$, $4|z| < |1-z|^2$, $|\arg(1-z)| < \pi$. Then,*

$$\begin{aligned}(1-z)^{-2a}\,{}_4F_3&\left[\begin{matrix}a,\, a+\frac{1}{2},\, d_1+\ell_1,\, d_2+\ell_2\,;\\ b,\, d_1,\, d_2\,;\end{matrix}\; -\frac{4z}{(1-z)^2}\right]\\ &= \sum_{j=0}^{\ell_1}\sum_{k=0}^{\ell_2}\binom{\ell_1}{j}\binom{\ell_2}{k}\frac{(2a)_{2(j+k)}\,(d_2+\ell_2)_j\,(-z)^{j+k}}{(b)_{j+k}\,(d_1)_j\,(d_2)_{j+k}}\\ &\quad\times {}_2F_1\left[\begin{matrix}2a+2j+2k,\, 1+2a-b+j+k\,;\\ b+j+k\,;\end{matrix}\; -z\right].\end{aligned} \tag{39}$$

Proof. The proof would proceed in the same manner as Theorem 4. The specifics have been avoided. □

By comparing (18) and the resultant identity, which may be derived from setting $\ell = 1$, $b = c+1$ in (35), we obtain a transformation formula asserted in the following theorem.

Theorem 6. *Let $c \in \mathbb{C} \setminus \mathbb{Z}_{\leq -1}$, $a \pm A \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, $d \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$; $|z| < 1$; $c - 2a \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$, $\Re(c-2a) > 0$. Then,*

$$\begin{aligned}&{}_4F_3\left[\begin{matrix}2a,\, 2a-c,\, a-A+1,\, a+A+1\,;\\ c+1,\, a-A,\, a+A\,;\end{matrix}\; -z\right]\\ &= {}_2F_1\left[\begin{matrix}2a,\, 2a-c\,;\\ c+1\,;\end{matrix}\; -z\right] - \frac{2a(2a+1)z}{(c+1)d}\,{}_2F_1\left[\begin{matrix}2a+2,\, 2a-c+1\,;\\ c+2\,;\end{matrix}\; -z\right],\end{aligned} \tag{40}$$

where A with its assumption is the same as in (19).

3. Extensions of the Quadratic Transformation Formulas

This section establishes several generalizations the quadratic transformation formulas (20) and (21).

Theorem 7. *Let $\ell \in \mathbb{Z}_{\geq 1}$; $c+\frac{3}{2}, 2c+2, d \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, $c \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$, $\Re(c) > \ell - 1$; $|z| < \frac{1}{2}$, $|z| < |1-z|$, $|\arg(1-z)| < \pi$. Then,*

$$\begin{aligned}(1-z)^{-2a}\,{}_3F_2&\left[\begin{matrix}a,\, a+\frac{1}{2},\, d+\ell\,;\\ c+\frac{3}{2},\, d\,;\end{matrix}\; \frac{z^2}{(1-z)^2}\right]\\ &= \sum_{s=0}^{\infty}\frac{(2a)_s(1+c)_s(2z)^s}{(2c+2)_s\, s!}\,{}_3F_2\left[\begin{matrix}-\ell,\, -\frac{s}{2},\, -\frac{s-1}{2}\,;\\ d,\, -c-s\,;\end{matrix}\; 1\right].\end{aligned} \tag{41}$$

Proof. Let $\mathcal{L}_2$ be the left member of (41). We have:

$$\mathcal{L}_2 = \sum_{r=0}^{\infty}\frac{2^{2r}\,(a)_r(a+\frac{1}{2})_r(d+\ell)_r\,z^{2r}}{2^{2r}\,(c+\frac{3}{2})_r(d)_r\,r!}(1-z)^{-(2r+2a)},$$

which, upon using (3) and (24), gives:

$$\mathcal{L}_2 = \sum_{s=0}^{\infty}\sum_{r=0}^{\infty}\frac{(2a)_{2r+s}\,(d+\ell)_r\,z^{2r+s}}{2^{2r}\,(c+\frac{3}{2})_r(d)_r\,r!\,s!}, \tag{42}$$

Recall the following double series manipulation:

$$\sum_{s=0}^{\infty}\sum_{r=0}^{\infty} f(s,r) = \sum_{s=0}^{\infty}\sum_{r=0}^{[\frac{s}{2}]} f(s-2r,r), \tag{43}$$

$f : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{C}$ being a function, provided that the involved double series is assumed to be absolutely convergent.

Employing (43) in (42), with the aid of (8), we obtain:

$$\mathcal{L}_2 = \sum_{s=0}^{\infty} \sum_{r=0}^{[\frac{s}{2}]} \frac{(-s)_{2r}(2a)_s(d+\ell)_r\, z^s}{2^{2r}\, \left(c+\frac{3}{2}\right)_r (d)_r\, r!\, s!},$$

which, upon using (24) and (9), yields:

$$\mathcal{L}_2 = \sum_{s=0}^{\infty} \frac{(2a)_s\, z^s}{s!} \sum_{r=0}^{[\frac{s}{2}]} \frac{\left(-\frac{s}{2}\right)_r \left(\frac{-s+1}{2}\right)_r}{\left(c+\frac{3}{2}\right)_r\, r!} \frac{(d+r)_\ell}{(d)_\ell}. \tag{44}$$

Using (10) in (44), we find:

$$\mathcal{L}_2 = \frac{1}{(d)_\ell} \sum_{s=0}^{\infty} \frac{(2a)_s z^s}{s!} \sum_{j=0}^{\ell} A_j(d,\ell) \sum_{r=j}^{[\frac{s}{2}]} \frac{\left(-\frac{s}{2}\right)_r \left(\frac{-s+1}{2}\right)_r}{\left(c+\frac{3}{2}\right)_r (r-j)!},$$

which, upon setting $r - j = r'$ and dropping the prime on r, yields:

$$\mathcal{L}_2 = \frac{1}{(d)_\ell} \sum_{s=0}^{\infty} \frac{(2a)_s\, z^s}{s!} \sum_{j=0}^{\ell} \frac{A_j(d,\ell)\left(-\frac{s}{2}\right)_j \left(\frac{-s+1}{2}\right)_j}{\left(c+\frac{3}{2}\right)_j} {}_2F_1\left[\begin{matrix} -\frac{s}{2}+j,\ -\frac{s}{2}+\frac{1}{2}+j\,; \\ c+\frac{3}{2}+j\,; \end{matrix}\, 1\right]. \tag{45}$$

Now, proceeding the similar manner as in the proof of Theorem 1, we may get the identity (41). The remaining specifics are omitted. □

Theorem 8. *Let* $\ell_1,\ \ell_2 \in \mathbb{Z}_{\geq 1}$*;* $c+\frac{3}{2},\ 2c+2,\ d_1,\ d_2 \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0}$*,* $c \in \mathbb{C}\setminus\mathbb{Z}_{\geq 0}$*,* $\Re(c) > \ell_1+\ell_2-1$*;* $|z| < \frac{1}{2}$*,* $|z| < |1-z|$*,* $|\arg(1-z)| < \pi$*. Then,*

$$\begin{aligned} &(1-z)^{-2a}\, {}_4F_3\left[\begin{matrix} a,\ a+\frac{1}{2},\ d_1+\ell_1,\ d_2+\ell_2\,; \\ c+\frac{3}{2},\ d_1,\ d_2\,; \end{matrix}\, \frac{z^2}{(1-z)^2}\right] = \sum_{s=0}^{\infty} \frac{(2a)_s(1+c)_s(2z)^s}{(2c+2)_s\, s!} \\ &\times \sum_{j=0}^{\ell_1} \frac{(-\ell_1)_j\left(-\frac{s}{2}\right)_j \left(\frac{-s+1}{2}\right)_j (d_2+\ell_2)_j}{(d_1)_j (d_2)_j (-c-s)_j\, j!} {}_3F_2\left[\begin{matrix} -\ell_2,\ -\frac{s}{2}+j,\ \frac{-s+1}{2}+j\,; \\ d_2+j,\ -c-s+j\,; \end{matrix}\, 1\right]. \end{aligned} \tag{46}$$

Proof. The proof would continue in the same fashion as that of Theorem 7, but without the details. □

Theorem 9. *Let* $\ell_j \in \mathbb{Z}_{\geq 1}$ $(j = 1,\ldots,r)$*;* $c+\frac{3}{2},\ 2c+2,\ d_j \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0}$ $(j = 1,\ldots,r)$*,* $c \in \mathbb{C}\setminus\mathbb{Z}_{\geq 0}$*,* $\Re(c) > \ell_1+\cdots+\ell_r-1$*;* $|z| < \frac{1}{2}$*,* $|z| < |1-z|$*,* $|\arg(1-z)| < \pi$*. Then,*

$$\begin{aligned} &(1-z)^{-2a}\, {}_{r+2}F_{r+1}\left[\begin{matrix} a,\ a+\dfrac{1}{2},\ d_1+\ell_1,\ \ldots,\ d_r+\ell_r\,; \\ c+\dfrac{3}{2},\ d_1,\ \ldots,\ d_r\,; \end{matrix}\, \frac{z^2}{(1-z)^2}\right] \\ &= \sum_{s=0}^{\infty} \frac{(2a)_s(1+c)_s(2z)^s}{(2+2c)_s\, s!} \left[\prod_{\mu=1}^{r-1} \mathcal{T}(\ell_\mu, \mathbf{J}_{\mu-1}; c, d_1,\ldots,d_r; s)\right] \\ &\times {}_3F_2\left[\begin{matrix} -\ell_r,\ -\dfrac{s}{2}+\mathbf{J}_{r-1},\ \dfrac{-s+1}{2}+\mathbf{J}_{r-1}\,; \\ d_r+\mathbf{J}_{r-1},\ -c-s+\mathbf{J}_{r-1}\,; \end{matrix}\, 1\right]. \end{aligned} \tag{47}$$

Here, $\mathbf{J}_p$ *is the same as in* (34)*, and:*

$$\begin{aligned}&\mathcal{T}\left(\ell_\mu,\mathbf{J}_{\mu-1};c,d_1,\ldots,d_r;s\right)\\&:=\sum_{j_\mu=0}^{\ell_\mu}(-1)^{j_\mu}\binom{\ell_\mu}{j_\mu}\frac{\left(-\frac{s}{2}+\mathbf{J}_{\mu-1}\right)_{j_\mu}\left(\frac{-s+1}{2}+\mathbf{J}_{\mu-1}\right)_{j_\mu}}{\left(-c-s+\mathbf{J}_{\mu-1}\right)_{j_\mu}\left(d_\mu+\mathbf{J}_{\mu-1}\right)_{j_\mu}}\\&\times\prod_{\nu=\mu+1}^{r}\frac{\left(d_\nu+\ell_\nu+\mathbf{J}_{\mu-1}\right)_{j_\mu}}{\left(d_\nu+\mathbf{J}_{\mu-1}\right)_{j_\mu}}\quad(\mu=1,\ldots,r-1).\end{aligned}$$

Proof. As with the proofs of Theorems 7 and 8, we may justify, by induction on r, (47). As a result, the details are eliminated. □

As in Theorem 4, Theorem 7 can be rewritten in the following theorem.

Theorem 10. *Let* $\ell\in\mathbb{Z}_{\geq 0}$*;* $c+\frac{3}{2}$, $2c+2$, $d\in\mathbb{C}\setminus\mathbb{Z}_{\leq 0}$, $c\in\mathbb{C}\setminus\mathbb{Z}_{\geq 0}$, $\Re(c)>\ell-1$*;* $|z|<\frac{1}{2}$, $|z|<|1-z|$, $|\arg(1-z)|<\pi$*. Then,*

$$\begin{aligned}&(1-z)^{-2a}\,{}_3F_2\left[\begin{matrix}a,\,a+\frac{1}{2},\,d+\ell\,;\\ c+\frac{3}{2},\,d\,;\end{matrix}\,\frac{z^2}{(1-z)^2}\right]\\&=\sum_{k=0}^{\ell}\binom{\ell}{k}\frac{(a)_k\,(a+\frac{1}{2})_k\,z^{2k}}{(c+\frac{3}{2})_k\,(d)_k}\,{}_2F_1\left[\begin{matrix}2a+2k,\,c+1+k\,;\\ 2c+2+2k\,;\end{matrix}\,2z\right].\end{aligned}\tag{48}$$

Remark 4. The case $\ell=0$ of (48) is found to yield Kummer's quadratic transformation formula (20). □

As in Theorem 5, Theorem 8 can be rewritten in the following theorem.

Theorem 11. *Let* $\ell_1,\,\ell_2\in\mathbb{Z}_{\geq 1}$*;* $c+\frac{3}{2}$, $2c+2$, $d_1,\,d_2\in\mathbb{C}\setminus\mathbb{Z}_{\leq 0}$, $c\in\mathbb{C}\setminus\mathbb{Z}_{\geq 0}$, $\Re(c)>\ell_1+\ell_2-1$*;* $|z|<\frac{1}{2}$, $|z|<|1-z|$, $|\arg(1-z)|<\pi$*. Then,*

$$\begin{aligned}&(1-z)^{-2a}\,{}_4F_3\left[\begin{matrix}a,\,a+\frac{1}{2},\,d_1+\ell_1,\,d_2+\ell_2\,;\\ c+\frac{3}{2},\,d_1,\,d_2\,;\end{matrix}\,\frac{z^2}{(1-z)^2}\right]\\&=\sum_{j=0}^{\ell_1}\sum_{k=0}^{\ell_2}\binom{\ell_1}{j}\binom{\ell_2}{k}\frac{(d_2+\ell_2)_j}{(d_1)_j\,(d_2)_{j+k}}\\&\times\frac{(a)_{j+k}\left(a+\frac{1}{2}\right)_{j+k}z^{2(j+k)}}{\left(c+\frac{3}{2}\right)_{j+k}}\,{}_2F_1\left[\begin{matrix}2a+2j+2k,\,c+1+j+k\,;\\ 2c+2+2j+2k\,;\end{matrix}\,2z\right].\end{aligned}\tag{49}$$

By matching the right members of (21) and the case $\ell=1$ of (48), we may obtain a transformation formula between ${}_4F_3$ and ${}_2F_1$ asserted in the following theorem.

Theorem 12. *Let* $c+\frac{3}{2}$, $2c+2$, $2d-\frac{1}{2}\pm\frac{B}{2}$, $d\in\mathbb{C}\setminus\mathbb{Z}_{\leq 0}$, $\Re(c)>0$ *and* $|z|<\frac{1}{2}$*. Then,*

$$\begin{aligned}&{}_4F_3\left[\begin{matrix}2a,\,c,\,2d+\frac{B}{2}+\frac{1}{2},\,2d-\frac{B}{2}+\frac{1}{2}\,;\\ 2c+2,\,2d+\frac{B}{2}-\frac{1}{2},\,2d-\frac{B}{2}-\frac{1}{2}\,;\end{matrix}\,2z\right]\\&={}_2F_1\left[\begin{matrix}2a,\,c+1\,;\\ 2c+2\,;\end{matrix}\,2z\right]+\frac{a\,(a+\frac{1}{2})\,z^2}{d\,(c+\frac{3}{2})}\,{}_2F_1\left[\begin{matrix}2a+2,\,c+2\,;\\ 2c+4\,;\end{matrix}\,2z\right],\end{aligned}\tag{50}$$

where B with its assumption is the same as in (22)*.*

4. Remarks, Further Formulas, and Posing Problems

In this article, by making a convenient use of the $\{A_j(\alpha,\ell)\}_{j=0}^{\ell}$ in (10), we provided a number of transformation formulas among ${}_pF_q$, which include some known formulae as particular cases.

For the terminating Clausen hypergeometric series ${}_3F_2(1)$ in Theorems 1–3, and 7–9, the summation theorems of Dixon, Saalschütz, Watson, Whipple, and other summation theorems for ${}_3F_2(1)$ (see, e.g., [5]) cannot be applied.

We may also establish a number of formulas for ${}_pF_q$ by applying calculus to those identities in the previous sections. For example, differentiating both sides of (23) with respect to d, and using (16), we may obtain an identity in Theorem 13.

Theorem 13. *Let $\ell \in \mathbb{Z}_{\geq 1}$; $b, d \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0}$; $b-2a \in \mathbb{C}\setminus\mathbb{Z}_{\geq 1}$, $\Re(b-2a) > \ell$; $|z|<1$, $4|z| < |1-z|^2$, $|\arg(1-z)| < \pi$. Then,*

$$\begin{aligned}(1-z)^{-2a}\sum_{k=1}^{\infty}\frac{(a)_k\left(a+\frac{1}{2}\right)_k(d+\ell)_k}{k!\,(b)_k\,(d)_k}\left[H_k(d+\ell-1)-H_k(d-1)\right]\frac{(-1)^k\,2^{2k}\,z^k}{(1-z)^{2k}}\\ = -\sum_{j=1}^{\ell}\frac{(2a)_{2j}\,(-z)^j}{(\ell-j)!\,(b)_j\,(d)_j}\,H_j(d-1)\,{}_2F_1\left[\begin{matrix}2a+2j,\,1-b+2a+j\,;\\ b+j\,;\end{matrix}\;-z\right].\end{aligned} \tag{51}$$

Setting $\ell=1$ in Theorem 13 may provide a transformation formula in the following corollary.

Corollary 1. *Let $b, d \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0}$; $b-2a \in \mathbb{C}\setminus\mathbb{Z}_{\geq 1}$, $\Re(b-2a) > 1$; $|z|<1$, $4|z| < |1-z|^2$, $|\arg(1-z)| < \pi$. Then,*

$$\begin{aligned}{}_2F_1\left[\begin{matrix}a,\,a+\frac{1}{2}\,;\\ b\,;\end{matrix}\;-\frac{4z}{(1-z)^2}\right]-{}_3F_2\left[\begin{matrix}a,\,a+\frac{1}{2},\,d+1\,;\\ b,\,d\,;\end{matrix}\;-\frac{4z}{(1-z)^2}\right]\\ =\frac{2a\,(2a+1)\,z\,(1-z)^{2a}}{bd}\,{}_2F_1\left[\begin{matrix}2a+2,\,2-b+2a\,;\\ b+1\,;\end{matrix}\;-z\right].\end{aligned} \tag{52}$$

The following problems are posed:

- Rewrite the results in Theorems 3 and 9 in the same manner as those in Theorems 4, 5, 10, and 11.
- Using the identities in the previous sections, establish formulae as those in (51) and (52).
- As noted in Remark 2, express the right members of (23) and (41), respectively, in terms of the double hypergeometric function of the Srivastava–Daoust (see, e.g., [13]; [14], p. 454, Equation (4.1); [15], pp. 199–200, Equation (2.1)).

In this study, only equalities associated with the hypergeometric function and generalized hypergeometric functions were explored. In fact, inequalities involving hypergeometric and related functions have also been investigated and appeared in the literature. For example, in [16], an intriguing inequality for the hypergeometric function, which is related to cost-effective numerical density estimation of the hyper-gamma probability distribution was shown (see also the references cited therein). Further it is intriguing to introduce that, in [17], using the features of superquadratic functions, various interesting improvements and popularizations on time scales of the Hardy-type inequalities and their converses were presented.

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