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Entropy and Semi-Entropies of Regular Symmetrical Triangular Interval Type-2 Fuzzy Variables

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Abstract: Fuzzy entropy has a wide range of applications in uncertainty problems. Due to the dual-complexity of its characteristics and calculation, the study on type-2 fuzzy entropy is rare, let alone the semi-ones. Given this, the paper takes the lead in proposing the credibility-based type-2 entropy and semi-entropies delivered around a specific symmetric type-2 fuzzy variable. After presenting the relevant theorems and definitions, we give the corresponding examples of linear entropy and semi-entropies to illustrate and verify the favorable property of our study. This series of formulas on type-2 entropy proposed has a strong advantage in reducing computational complexity. It can be commonly applied to measure fuzziness and solve return-oriented and cost-oriented problems in the business field. A sequence of measures on type-2 fuzzy entropy developed in this paper delivers fresh insights into this field. It also provides a new reasonable tool for the decision-making on cost and investment control in companies.

Keywords: interval type-2 fuzzy variable; entropy; semi-entropy

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1. Introduction

In 1975, Zadeh [1] extended the fuzzy set theory to the type-2 fuzzy set theory, which means the problem appeared that the membership function of classical fuzzy sets can be fuzzified. Mendel et al. [2–5] broadened the theory of type-2 fuzzy sets, establishing a type-2 fuzzy logic system and the related uncertainty measures. Mendel [6] discovered that type-2 fuzzy set theory can be useful when such systems are used in situations where multi-uncertainties exist. Since type-2 fuzzy sets can predict uncertain information more accurately [7,8], they have been successfully applied in many fields, such as machine learning [9], image recognition [10] and network security [11].

Numerous statistical measures are utilized to characterize the relationships between fuzzy variables (e.g., expected value, variance) [12,13]. For example, Nieminen [14] gave a detailed description of the geometric structure of type-2 fuzzy sets. Wu [15] proposed various uncertainty measures for the interval type-2 fuzzy sets, including center of mass, cardinality, fuzziness, variance and skewness. Considering that information entropy can be used to measure the uncertainty degree of the system [16], Bolturk [17] proposed an interval-valued neutral hierarchical analysis based on the cosine similarity measures in a type-2 fuzzy setting. Moreover, Roy and Bhaumik [18] developed a triangular type-2 intuitionistic fuzzy matrix games approach used into water management. Fuzzy entropy can well weigh information and denote the uncertainty [19]. Apart from that, it can also be applied into the field of operation management, including the decisions about portfolio [20] and cost controlling [21]. As a result, among these fuzzy measures, fuzzy entropy has received attention and has been widely accepte [22].

This paper summarizes the related studies about type-2 fuzzy entropy and organized the main contents of Table 1, including the type of fuzzy variables and the formulas for calculating them. Obviously, from this table, we can see that all of the calculation formulas about type-2 fuzzy entropy in Table 1 involve complex calculation processes, implying that they might bring high calculation costs to obtain the final results [23]. Additionally, we can see that previous studies mostly focused on discrete fuzzy numbers, yet there are rarely studies about the continuous ones. While continuous fuzzy variables are more common, those such as triangles, trapezoids, and normal fuzzy sets all belong to continuous type [24]. Therefore, there is a necessity to study the continuous fuzzy entropy.

Table 1. The fuzzy-entropy measures for interval type-2 fuzzy variables (IT2-FVs).

Literature	Type of Fuzzy Variable	The Formula for Calculating Entropy
Burillo & Bustince [25]	D.	$\mathrm{H}(\widetilde{\mathrm{A}}) = \sum_{i=1}^{\mathrm{N}}\left[\bar{\mu}_{\widetilde{\mathrm{A}}}(\mathrm{x_i}) - \underline{\mu}_{\widetilde{\mathrm{A}}}(\mathrm{x_i})\right]$
Szmidt & Kacprzyk [26]	D.	$H(\widetilde{A}) = \frac{1}{N}\sum_{i=1}^{N}\frac{1-\max[1-\bar{\mu}_{\widetilde{A}}(x_i),\underline{\mu}_{\widetilde{A}}(x_i)]}{1-\min\left[1-\bar{\mu}_{\widetilde{A}}(x_i),\underline{\mu}_{\widetilde{A}}(x_i)\right]}$
Zeng & Li [27]	D.	$H(\widetilde{A}) = 1-\frac{1}{N}\sum_{i=1}^{N}\left\lvert\bar{\mu}_{\widetilde{A}}(x_i)+\underline{\mu}_{\widetilde{A}}(x_i)-1\right\rvert$
Vlachos & Sergiadis [28]	D.	$H(\widetilde{A}) = \frac{p\left(\widetilde{A}\cap\widetilde{A}^c\right)}{p\left(\widetilde{A}\cup\widetilde{A}^c\right)}$
Cornelis & Kerre [29]	D.	$H(\widetilde{A}) = \left[\frac{2}{N}\sum_{i=1}^{N}\min\left(\underline{\mu}_{\widetilde{A}}(x_i), 1-\bar{\mu}_{\widetilde{A}}(x)\right), \frac{2}{N}\sum_{i=1}^{N}\min\left(0.5, 1-\underline{\mu}_{\widetilde{A}}(x_i), \bar{\mu}_{\widetilde{A}}(x)\right)\right]$
Hwang & Miin [30]	D.	$H(\widetilde{A}) = \frac{1}{n}min\left\{\sum_{x\in X}\frac{\sum_{u\in J_x} min\{f_x(u),1-f_x(u)\}}{\sum_{u\in J_x} f_x(u)}, \sum_{x\in X}\frac{\sum_{u\in J_x} min\{1-f_x(u),f_x(u)\}}{\sum_{u\in J_x} 1-f_x(u)}\right\}$
Ozkan & Turksen [31]	D.	$H(x+\Delta x) = \sum_{k=1}^{nd}\left[\sum_{i=1}^{nc}-\mu_{i,k}(x+\Delta x)\ln\left(\mu_{i,k}(x+\Delta x)\right)\right]$
Zhang & Zheng [32]	C.	$H(\widetilde{A}) = \frac{1}{b-a}\int_a^b \frac{\min\left\{\bar{\mu}_{\widetilde{A}}(x),1-\mu_{\widetilde{A}}(x)\right\}}{\max\left\{\bar{\mu}_{\widetilde{A}}(x),1-\mu_{\widetilde{A}}(x)\right\}}dx$

D. and C. are the abbreviations of discrete and continuous, respectively.

In this regard, we found that Zhou et al. [33] put forward a series of concepts about type-1 fuzzy entropy based on the credibility measurement. Because of the nature of credibility, this kind of calculation formula can make fuzzy entropy meet self-duality, and the results of type-1 fuzzy entropy can be acquired by the inverse credibility distribution (ICD) of fuzzy sets, which can reduce the calculation difficulty. Compared with other formulas, the entropy based on the credibility measurement has irreplaceable advantages in the generality of application.

Inspired by the type-1 entropy defined by Zhou et al. [33], this paper introduces the credibility-based fuzzy entropy and semi-entropy for the type-2 fuzzy sets for reducing the complexity of calculations. Due to their inherent complexity, this paper takes the study of Li and Cai [34] as an example to study a special type-2 fuzzy sets (RSTIT2-FV) and give a series of methods to calculate their fuzzy entropies. This paper further fills the current lack of the study on continuous type-2 fuzzy entropy and semi-entropies, facilitating the popularization of type-2 fuzzy entropy at the application level in the future.

The rest of this paper is as follows. Section 2 reviews several concepts related to the RTIT2-FVs. Subsequently, Section 3 introduces the definition of the credibility-based type-2 fuzzy entropy, and some related theorems are proved and followed by some examples. Moreover, the definitions of the lower and upper semi-entropies with some examples are presented to quantify the uncertainty on the only one side. Section 4 proposes the formula of calculating the entropy and semi-entropies of a linear function constructed by RSTIT2-FVs. Finally, the conclusions are drawn in Section 5.

2. Preliminaries

In the following, some necessary concepts and operational laws of fuzzy set theory that lay the foundation for the following sections are reviewed successively, and some concepts and theorems of type-2 fuzzy set, i.e., regular symmetric triangular interval type-2 fuzzy variable (RSTIT2-FV), are briefly introduced.

2.1. Type-1 Fuzzy Variable

Definition 1. *(Zadeh [35]) A type-1 fuzzy set (T1-FS) B can be defined as*

$$B = \int_{x \in X} \frac{\mu_B(x)}{x},$$

where $\mu_B(x)$ is the real-valued membership function (MF), and X is the universe of x.

Definition 2. *(Liu [36]) Given that Θ is a nonempty set, $\Gamma(\Theta)$ is the power set of Θ, Pos is a possibility measure, and $\mathbb{R}$ is a real number set. Let the triad $(\Theta, \Gamma(\Theta), \mathrm{Pos})$ be a possibility space, then the map η: $(\Theta, \Gamma(\Theta), \mathrm{Pos}) \to \mathbb{R}$ is called a type-1 fuzzy variable (T1-FV).*

Definition 3. *(Liu [36]) The credibility distribution (CD) Φ and credibility function (CF) v for a T1-FV η can be calculated by*

$$\Phi_\eta(x) = \mathrm{Cr}\{\eta \le x\} = \frac{1}{2}\Big(\mathrm{sup}_{\eta \le x}\mu_\eta(x) + 1 - \mathrm{sup}_{\eta > x}\mu_\eta(x)\Big), v(x) = \mathrm{Cr}\{\delta = x\}$$

where $\mu_\eta(x)$ is the MF of η.

Definition 4. *(Dubois and Prade [37]) A T1-FV η is called an LR-type FV if its shape function and scalers $\alpha > 0, c > 0$ satisfy*

$$\mu_\eta(x) = \begin{cases} L\left(\dfrac{c-x}{\alpha}\right), & x \in (-\infty, c] \\ \\ R\left(\dfrac{x-c}{\alpha}\right), & x \in [c, +\infty), \end{cases} \tag{1}$$

where the shape function L (for left) and R (for right) are decreasing functions from $\mathbb{R}^+ \to [0,1]$ and satisfy L(0) = R(0) = 1 and L(1) = R(1) = 0.

Definition 5. *(Zhou et al. [33]) A LR-type FV η, which has a continuous and strictly increasing credibility distribution Φ_η, is called a regular LR-FV.*

Definition 6. *(Zhou et al. [33]) For a LR fuzzy number $\delta \sim (\sigma, \alpha, \beta)_{LR}$ with the MF μ in Equation (1), its credibility distribution (CD) can be worked out in view of Equation (3) as*

$$\Phi(x) = \begin{cases} \dfrac{1}{2}L\left(\dfrac{\sigma-x}{\alpha}\right), & \text{if } x \le \sigma \\ \\ 1 - \dfrac{1}{2}R\left(\dfrac{x-\sigma}{\beta}\right), & \text{if } x > \sigma. \end{cases} \tag{2}$$

Theorem 1. *(Zhou et al. [38]) Let δ_1, δ_2, $\cdots$, δ_n be independent regular LR fuzzy numbers, their CDs are, respectively, Φ_1, Φ_2, $\cdots$, Φ_n. If $f(x_1, x_2, \cdots, x_n)$ is a strictly increasing function with regard to x_1, x_2, $\cdots$, x_m and a strictly decreasing function with regard to x_{m+1}, x_{m+2},*

$\cdots, x_n$, then $\delta = f(\delta_1, \delta_2, \cdots, \delta_n)$ is a regular LR fuzzy number and has the inverse credibility distribution (ICD).

$$\Phi^{-1}(\gamma) = f(\Phi_1^{-1}(\gamma), \cdots, \Phi_m^{-1}(\gamma), \Phi_{m+1}^{-1}(1-\gamma), \cdots, \Phi_n^{-1}(1-\gamma)). \tag{3}$$

2.2. Type-2 Fuzzy Variable

Definition 7. *(Zadeh [35], De and Termini [39]) A type-2 fuzzy set (T2-FS), denoted as S and characterized by a type MF $\mu_S(x,u)$, can be expressed as*

$$S = \int_{x \in X} \int_{u \in J_x} \frac{\mu_S(x,u)}{(x,u)},$$

where $u \in J_x \subseteq [0,1], x \in X$ is called the primary MF of x, $\mu_S(x,u)$ is called the secondary MF of u.

Definition 8. *(Mendel and John [3]) For any $x \in X$, let the primary membership function of a T2-FV, S, be $J_x \subseteq [0,1]$, the FOU of S is the union of all the primary membership functions and, thus, can be expressed as*

$$FOU(S) = \bigcup_{x \in X} J_x.$$

The upper and lower bounds of the FOU are called the upper membership function (UMF) and lower membership function (LMF) of S, respectively.

Definition 9. *(Li and Cai [34]) An IT2-FV S is called a regular symmetric triangular IT2-FV (RSTIT2-FV) if its UMF and LMF in the following forms,*

$$\text{UMF} = \begin{cases} \dfrac{1}{l_U}x - \dfrac{c-l_U}{l_U}, & x \in [c-l_U, c) \\ -\dfrac{1}{l_U}x + \dfrac{c+l_U}{l_U}, & x \in [c, c+l_U] \\ 0, & otherwise, \end{cases} \tag{4}$$

and

$$\text{LMF} = \begin{cases} \dfrac{1}{l_L}x - \dfrac{c-l_L}{l_L}, & x \in [c-l_L, c) \\ -\dfrac{1}{l_L}x + \dfrac{c+l_L}{l_L}, & x \in [c, c+l_L] \\ 0, & otherwise, \end{cases} \tag{5}$$

and can be denoted as $\begin{pmatrix} c-l_U & c & c+l_U \\ c-l_L & c & c+l_L \end{pmatrix}$, where the spreads of the UMF and LMF, l_U and l_L, satisfy $l_U > l_L$, and the peak of them, 1, are reached when x is equal to c.

Definition 10. *(Li and Cai [34]) Let S be an RSTIT2-FV, η be a T1-FV, and their MFs satisfy*

$$\mu_\eta(x) = \frac{1}{2}UMF + \frac{1}{2}LMF.$$

Then, η is called the medium of S, and the analytical expression of $\mu_\eta(x)$ can be calculated as

$$\mu_\eta(x) = \begin{cases} \frac{1}{2}\left(\frac{1}{l_U}x - \frac{c-l_U}{l_U}\right), & x \in [c-l_U, c-l_L) \\ \frac{1}{2}\left(\frac{1}{l_U}x - \frac{c-l_U}{l_U}\right) + \frac{1}{2}\left(\frac{1}{l_L}x - \frac{c-l_L}{l_L}\right), & x \in [c-l_L, c) \\ \frac{1}{2}\left(-\frac{1}{l_U}x + \frac{c+l_U}{l_U}\right) + \frac{1}{2}\left(-\frac{1}{l_L}x + \frac{c+l_L}{l_L}\right), & x \in [c, c+l_L) \\ \frac{1}{2}\left(-\frac{1}{l_U}x + \frac{c+l_U}{l_U}\right), & x \in [c+l_L, c+l_U] \\ 0, & \textit{otherwise}. \end{cases} \tag{6}$$

Definition 11. *(Li and Cai [34]) By means of the medium η, the CD and ICD of an RSTIT2-FV can be defined as*

$$\Phi_S(x) = \mathrm{Cr}\{S \le x\}$$

$$= \begin{cases} 0, & x \in (-\infty, c-l_U] \\ \frac{1}{4}\left(\frac{1}{l_U}x - \frac{c-l_U}{l_U}\right), & x \in (c-l_U, c-l_L] \\ \frac{1}{4}\left(\frac{1}{l_U}x - \frac{c-l_U}{l_U}\right) + \frac{1}{4}\left(\frac{1}{l_L}x - \frac{c-l_L}{l_L}\right), & x \in (c-l_L, c+l_L] \\ 1 + \frac{1}{4}\left(\frac{1}{l_U}x - \frac{c+l_U}{l_U}\right), & x \in (c+l_L, c+l_U] \\ 1, & x \in (c+l_U, +\infty), \end{cases} \tag{7}$$

and

$$\Phi_S^{-1}(\alpha) = \begin{cases} 4l_U\alpha + c - l_U, & \alpha \in \left[0, \frac{l_U - l_L}{4l_U}\right) \\ \frac{4l_Ul_L\alpha - 2l_Ul_L}{l_U + l_L} + c, & \alpha \in \left[\frac{l_U - l_L}{4l_U}, 1 - \frac{l_U - l_L}{4l_U}\right) \\ 4l_U\alpha + c - 3l_U, & \alpha \in \left[1 - \frac{l_U - l_L}{4l_U}, 1\right]. \end{cases} \tag{8}$$

Remark 1. *Figure 1 gives the visualization of Φ_S^{-1}. Obviously, Φ_S^{-1} is a continuous and strictly increasing function.*

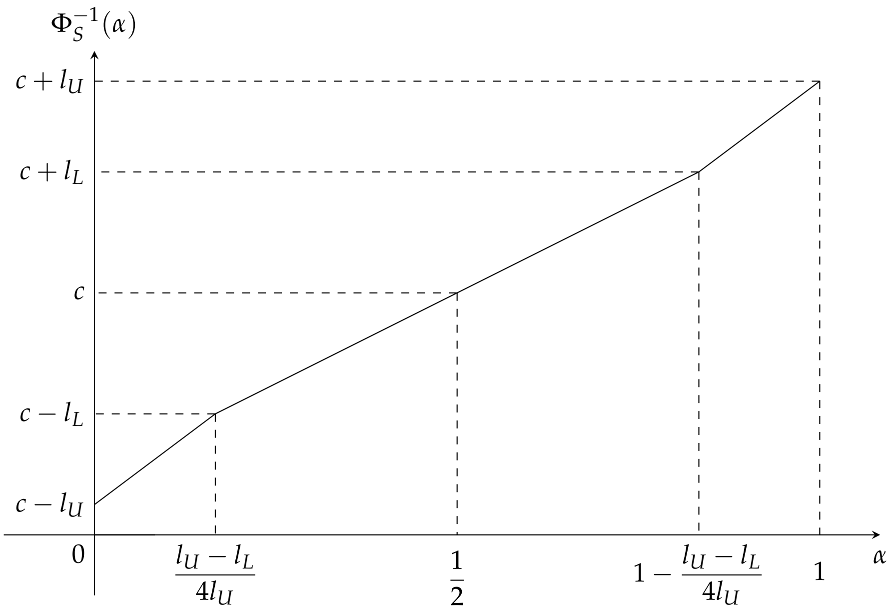

Figure 1. The inverse credibility distribution of S, Φ_S^{-1}.

Definition 12. *(Li and Cai [34]) Assume that $S_i, i = 1,2,\cdots,n$, are RSTIT2-FVs with the mediums of η_i, and f is a function from $\mathbb{R}^n$ to $\mathbb{R}$, then the credibility distribution of $S = f(S_1, S_2, \cdots, S_n)$, $\Phi_S(x) = \mathrm{Cr}\{S \le x\}$, is defined as*

$$\Phi_S(x) = \Phi_\eta(x),$$

where $\eta = f(\eta_1, \eta_2, \cdots, \eta_n)$ is the medium of S, and the inverse credibility distribution of S is the inverse function of $\Phi_S(\alpha)$, i.e., $\Phi_S^{-1}(\alpha)$.

Remark 2. *According to Definition 12, it can be easily deduced that*

$$\Phi_S^{-1}(\alpha) = \Phi_\eta^{-1}(\alpha).$$

Definition 13. *(Li and Cai [34]) Let S be a linear function formed by multiple RSTIT2-FVs $S_i, i = 1,2,\cdots,n$, and S is a strictly increasing function with regard to $S_1, S_2, \cdots, S_t$ and a strictly decreasing function with regard to $S_{t+1}, S_{t+2}, \cdots, S_n$. Then, its ICD can be defined as*

$$\Phi_S^{-1}(\alpha) = f\left(\Phi_{S_1}^{-1}(\alpha), \cdots, \Phi_{S_t}^{-1}(\alpha), \Phi_{S_{t+1}}^{-1}(1-\alpha), \cdots, \Phi_{S_n}^{-1}(1-\alpha)\right). \tag{9}$$

Definition 14. *(Li and Cai [34]) Suppose that S is an RSTIT2-FV or a linear function derived from multiple RSTIT2-FVs, then the expected value of S can be defined as*

$$\begin{aligned} E[S] &= \int_0^{+\infty} \mathrm{Cr}\{S \ge x\}\mathrm{d}x - \int_{-\infty}^{0} \mathrm{Cr}\{S \le x\}\mathrm{d}x \\ &= \int_0^1 \Phi_S^{-1}(\alpha)\mathrm{d}\alpha = c. \end{aligned} \tag{10}$$

Definition 15. *(Li and Cai [34]) The RSTIT2-FVs $S_i, i = 1,2,\cdots,n$, are said to be mutually independent if*

$$\mathrm{Cr}\{S_i \in B_i, i = 1,2,\cdots,n\} = \min_{1 \le i \le n} \mathrm{Cr}\{S_i \in B_i\}$$

for any subsets $B_1, B_2, \cdots, B_n$.

Definition 16. *(Li and Liu [40]) Let δ be a continuous fuzzy number with a credibility function (CF) v. Then, the entropy of δ can be defined as*

$$H[\delta] = \int_{-\infty}^{+\infty} S(v(x))\mathrm{d}x,$$

where the function $S(t) = -t\ln t - (1-t)\ln(1-t)$, as shown in Figure 2.

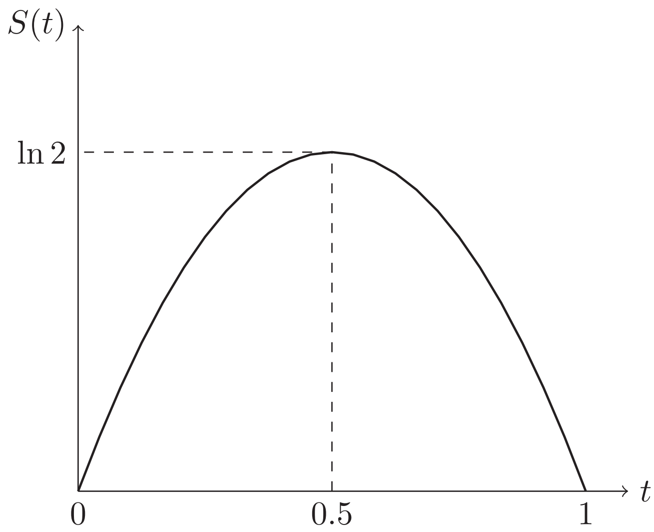

Figure 2. The function $S(t)$ in Definition 16.

Definition 17. *(Zhou et al. [33]) Given that δ is a continuous fuzzy number and its entropy exists, then the entropy is*

$$H[\delta] = \int_{-\infty}^{+\infty} S(\Phi(x))\mathrm{d}x,$$

where Φ is the CD of δ.

Definition 18. *(Zhou et al. [33]) Let δ be a regular LR fuzzy number. Then, the entropy of δ can be calculated as*

$$H[\delta] = \int_0^1 \Phi^{-1}(\gamma)\ln\frac{\gamma}{1-\gamma}\mathrm{d}\gamma, \tag{11}$$

where Φ^{-1} is the ICD of δ.

Definition 19. *(Zhou et al. [33]) Suppose that $\delta_1, \delta_2, \cdots, \delta_n$ are independent regular LR fuzzy numbers, and $f(x_1, x_2, ..., x_n) = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n$, where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are real numbers, then the entropy of $\delta = f(\delta_1, \delta_2, \cdots, \delta_n)$ is*

$$H[\delta] = |\lambda_1|H[\delta_1] + |\lambda_2|H[\delta_2] + \cdots + |\lambda_n|H[\delta_n]. \tag{12}$$

Definition 20. *(Zhou et al. [33]) Let δ be a continuous fuzzy number with the expected value e and CF v. Then, its lower semi-entropy can be expressed as*

$$H_S[\delta]^- = \int_{-\infty}^{+\infty} S(v(x)^-)\mathrm{d}x,$$

where $S(t) = -t\ln t - (1-t)\ln(1-t)$, and

$$v(x)^- = \begin{cases} v(x), & \text{if } x \le e \\ 0, & \text{if } x > e. \end{cases}$$

Since $S(0) = 0$, the lower semi-entropy of δ can be simplified as

$$H_S[\delta]^- = \int_{-\infty}^{e} S(v(x))\mathrm{d}x.$$

Similarly, the upper semi-entropy can be obtained following the above steps. Since $S(0) = 0$, the upper semi-entropy of the fuzzy number δ can be simplified as the following specification

$$H_S[\delta]^+ = \int_{e}^{+\infty} S(v(x))\mathrm{d}x.$$

Definition 21. *(Zhou et al. [33]) Let δ be a regular LR fuzzy number with the ICD Φ^{-1} and expected value e. Then, the lower and upper semi-entropies of δ can be calculated as*

$$H_S[\delta]^- = \begin{cases} \displaystyle\int_0^{\Phi(e)} (\Phi^{-1}(\gamma) - e) \ln \frac{\gamma}{1-\gamma}\mathrm{d}\gamma, & \text{if } e \le 0 \\ \displaystyle\int_0^{\Phi(e)} \Phi^{-1}(\gamma) \ln \frac{\gamma}{1-\gamma}\mathrm{d}\gamma + e\int_{\Phi(e)}^{1} \ln \frac{\gamma}{1-\gamma}\mathrm{d}\gamma, & \text{if } e > 0, \end{cases} \tag{13}$$

$$H_S[\delta]^+ = \begin{cases} \displaystyle e\int_0^{\Phi(e)} \ln \frac{\gamma}{1-\gamma}\mathrm{d}\gamma + \int_{\Phi(e)}^{1} \Phi^{-1}(\gamma) \ln \frac{\gamma}{1-\gamma}\mathrm{d}\gamma, & \text{if } e \le 0 \\ \displaystyle\int_{\Phi(e)}^{1} (\Phi^{-1}(\gamma) - e) \ln \frac{\gamma}{1-\gamma}\mathrm{d}\gamma, & \text{if } e > 0. \end{cases} \tag{14}$$

3. Entropy and Semi-Entropies of an RSTIT2-FV

By using the above-mentioned definitions and theorems, we can redefine and calculate the entropy and semi-entropies of an RSTIT2-FV via their ICD. To verify the performance of the proposed formulas, some examples are illustrated.

3.1. The Entropy of an RSTIT2-FV

Based on the entropy of a regular LR fuzzy number from Zhou et al. [33], the credibility-based type-2 entropy is defined in this paper as follows.

Theorem 2. *Given that V is an RSTIT2-FV and its entropy exists, then the entropy is*

$$H[V] = \int_{-\infty}^{+\infty} S(\Phi_V(\alpha))\mathrm{d}\alpha, \tag{15}$$

where Φ is the CD of V.

Proof of Theorem 2. Denote $\Phi_V(\alpha)$ and $v(\alpha)$ as the CD and CF of an RSTIT2-FV V, respectively. By virtue of Definitions 11 and 16, we have

$$\begin{aligned} H[V] &= \int_{-\infty}^{+\infty} S\big(v(\alpha)\big)\mathrm{d}\alpha \\ &= \int_{-\infty}^{c-l_U} S\big(v(\alpha)\big)\mathrm{d}\alpha + \int_{c-l_U}^{c-l_L} S\big(v(\alpha)\big)\mathrm{d}\alpha + \int_{c-l_L}^{c+l_L} S\big(v(\alpha)\big)\mathrm{d}x\alpha \\ &+ \int_{c+l_L}^{c+l_U} S\big(v(\alpha)\big)\mathrm{d}\alpha + \int_{c+l_U}^{+\infty} S\big(v(\alpha)\big)\mathrm{d}\alpha \\ &= \int_{-\infty}^{+\infty} S(\Phi_V(\alpha))\mathrm{d}\alpha. \end{aligned}$$

□

Theorem 3. *Let V be an RSTIT2-FV. Then, the entropy of V can be calculated as*

$$H[V] = \int_0^1 \Phi_V^{-1}(\alpha) \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha, \tag{16}$$

where Φ_V^{-1} is the ICD of α.

Proof of Theorem 3. With a view to Definition 16 and Theorem 2, the entropy can be expressed as

$$H[V] = \int_{-\infty}^{+\infty} S\big(\Phi_V(\alpha)\big)\mathrm{d}\alpha = \int_{-\infty}^{0}\int_{0}^{\Phi_V(\alpha)} S'(\alpha)\mathrm{d}\alpha\mathrm{d}\alpha + \int_0^{\infty}\int_{\Phi_V(\alpha)}^{1} -S'(\alpha)\mathrm{d}\alpha\mathrm{d}\alpha,$$

where $S'(\alpha) = (-\alpha\ln\alpha - (1-\alpha)\ln(1-\alpha))' = -\ln\frac{\alpha}{1-\alpha}$. By using the Fubini theorem [41], we have

$$\begin{aligned} H[V] &= \int_0^{\Phi_V(0)}\int_{\Phi_V^{-1}(\alpha)}^{0} S'(\alpha)\mathrm{d}\alpha\mathrm{d}\alpha + \int_{\Phi_V(0)}^{1}\int_0^{\Phi_V^{-1}(\alpha)} -S'(\alpha)\mathrm{d}\alpha\mathrm{d}\alpha \\ &= -\int_0^1 \Phi_V^{-1}(\alpha)S'(\alpha)\mathrm{d}\alpha \\ &= \int_0^1 \Phi_V^{-1}(\alpha)\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha. \end{aligned}$$

□

Theorem 4. *Suppose that V is an RSTIT2-FV with the ICD Φ_V^{-1} and medium of η, then its entropy is*

$$H[V] = \int_0^1 \Phi_\eta^{-1}(\alpha)\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha. \tag{17}$$

Proof of Theorem 4. According to Remark 2, we have Equation (17) immediately. □

Remark 3. *According to Definition 11 and Theorem 4, the specific formula of the RSTIT2-FVs' entropy is*

$$\begin{aligned} H[V] &= \int_0^{\frac{l_U - l_L}{4l_U}} (4l_U\alpha + c - l_U)\ln\frac{\alpha}{1-\alpha}d\alpha + \int_{\frac{l_U-l_L}{4l_U}}^{1-\frac{l_U-l_L}{4l_U}} \left(\frac{4l_U l_L\alpha}{l_U + l_L}\right)\ln\frac{\alpha}{1-\alpha}d\alpha \\ &+ \int_{1-\frac{l_U-l_L}{4l_U}}^{1}(4l_U\alpha + c - 3l_U)\ln\frac{\alpha}{1-\alpha}d\alpha \end{aligned} \tag{18}$$

Example 1. *According to Theorem 3, if we denote an RSTIT2-FV as $V_1 = \begin{pmatrix} 2 & 5 & 8 \\ 3 & 5 & 7\end{pmatrix}$, then it follows Equation (18)*

$$H[V_1] = \int_0^{\frac{1}{12}}(12\alpha+2)\ln\frac{\alpha}{1-\alpha}d\alpha + \int_{\frac{1}{12}}^{\frac{11}{12}}\left(\frac{24\alpha}{5}\right)\ln\frac{\alpha}{1-\alpha}d\alpha + \int_{\frac{11}{12}}^{1}(12\alpha-4)\ln\frac{\alpha}{1-\alpha}d\alpha$$

$$= \ln\frac{144\cdot 11^{\frac{5}{8}}}{121} + \frac{297\cdot\ln 11}{40} - 8\cdot\ln 12 + 3$$

3.2. The Semi-Entropies of an RSTIT2-FV

In the type-1 fuzzy set theory, the semi-entropies usually are used in controlling cost and investment risk, which can be regarded as an emerging and effective tool to manage the uncertainty of projected revenue for a company. In reality, these risks are difficult to accurately describe, so it is appropriate to catch these situations with fuzzy variables, and apply the semi-entropies to predict them.

In the light of the definitions proposed by Zhou et al. [33], we can also define the semi-entropies of an RSTIT2-FV via the CD and ICD.

Definition 22. *Let V be an RSTIT2-FV with the ICD of Φ^{-1}, expected value of e and medium of η. Then, its credibility-based type-2 semi-entropies can be expressed as*

$$H[V]^- = \begin{cases} \int_0^{\Phi_\eta(e)} (\Phi_\eta^{-1}(\alpha) - e) \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha, & \text{if } e \leq 0 \\ \int_0^{\Phi_\eta(e)} \Phi_\eta^{-1}(\alpha) \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha + e \int_{\Phi_\eta(e)}^1 \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha, & \text{if } e > 0, \end{cases} \tag{19}$$

$$H[V]^+ = \begin{cases} e \int_0^{\Phi_\eta(e)} \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha + \int_{\Phi_\eta(e)}^1 \Phi_\eta^{-1}(\alpha) \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha, & \text{if } e \leq 0 \\ \int_{\Phi_\eta(e)}^1 (\Phi_\eta^{-1}(\alpha) - e) \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha, & \text{if } e > 0. \end{cases} \tag{20}$$

Remark 4. *By using Definitions 11 and 22, we can easily obtain the lower and upper semi-entropies of an RSTIT2-FV as*

$$H[V]^- = \begin{cases} \int_0^{\frac{l_U - l_L}{4l_U}} (4l_U\alpha - l_U) \ln \frac{\alpha}{1-\alpha} d\alpha + \\ \int_{\frac{l_U - l_L}{4l_U}}^{\frac{1}{2}} \left(\frac{4l_U l_L \alpha - 2l_U l_L}{l_U + l_L} \right) \ln \frac{\alpha}{1-\alpha} d\alpha, & \text{if } e \leq 0 \\ \int_0^{\frac{l_U - l_L}{4l_U}} (4l_U\alpha + c - l_U) \ln \frac{\alpha}{1-\alpha} d\alpha + c \int_{\frac{1}{2}}^1 \ln \frac{\alpha}{1-\alpha} d\alpha + \\ \int_{\frac{l_U - l_L}{4l_U}}^{\frac{1}{2}} \left(\frac{4l_U l_L \alpha - 2l_U l_L}{l_U + l_L} + c \right) \ln \frac{\alpha}{1-\alpha} d\alpha, & \text{if } e > 0, \end{cases} \tag{21}$$

$$H[V]^+ = \begin{cases} c \int_0^{\frac{1}{2}} \ln \frac{\alpha}{1-\alpha} d\alpha + \int_{1-\frac{l_U - l_L}{4l_U}}^1 (4l_U\alpha + c - 3l_U) \ln \frac{\alpha}{1-\alpha} d\alpha + \\ \int_{\frac{1}{2}}^{1-\frac{l_U - l_L}{4l_U}} \left(\frac{4l_U l_L \alpha - 2l_U l_L}{l_U + l_L} + c \right) \ln \frac{\alpha}{1-\alpha} d\alpha, & \text{if } e \leq 0 \\ \int_{\frac{1}{2}}^{1-\frac{l_U - l_L}{4l_U}} \frac{4l_U l_L \alpha - 2l_U l_L}{l_U + l_L} \ln \frac{\alpha}{1-\alpha} d\alpha + \\ \int_{1-\frac{l_U - l_L}{4l_U}}^1 (4l_U\alpha - 3l_U) \ln \frac{\alpha}{1-\alpha} d\alpha. & \text{if } e > 0, \end{cases} \tag{22}$$

Corollary 1. *Let V be an RSTIT2-FV. Then, we have*

$$H[V] = H[V]^- + H[V]^+. \tag{23}$$

Proof of Corollary 1. Following from Equations (19) and (20), we have

$$
\begin{aligned}
&H[V]^- + H[V]^+ \\
&= \int_0^{\Phi_\eta(e)} \Phi_\eta^{-1}(\alpha) \ln \frac{\alpha}{1-\alpha} d\alpha + e \int_{\Phi_\eta(e)}^1 \ln \frac{\alpha}{1-\alpha} d\alpha + \int_{\Phi_\eta(e)}^1 (\Phi_\eta^{-1}(\alpha) - e) \ln \frac{\alpha}{1-\alpha} d\alpha \\
&= \int_{\Phi_\eta(e)}^1 \Phi_\eta^{-1}(\alpha) \ln \frac{\alpha}{1-\alpha} d\alpha + \int_0^{\Phi_\eta(e)} \Phi_\eta^{-1}(\alpha) \ln \frac{\alpha}{1-\alpha} d\alpha \\
&= \int_0^1 \Phi_\eta^{-1}(\alpha) \ln \frac{\alpha}{1-\alpha} d\alpha
\end{aligned}
$$

which is exactly equal to $H(V)$ of Equation (17) in Theorem 4. □

Example 2. *Assume that the deficit probability of a bank's real estate investment project can be described as an RSTIT2-FV* $V_2 = \begin{pmatrix} 0 & 0.2 & 0.4 \\ 0.1 & 0.2 & 0.3 \end{pmatrix}$*, its investment risk can be further accurately depicted by the lower semi-entropy proposed in this paper. In the light of Equations (6)–(8) and the MF of its medium* η_2*, the CD and ICD of* V_2 *are as follows:*

$$
\mu_{V_2}(x) = \begin{cases} \frac{5}{2}x, & x \in \left[0, \frac{1}{10}\right) \\ \frac{15}{2}x - \frac{1}{2}, & x \in \left[\frac{1}{10}, \frac{1}{5}\right) \\ -\frac{15}{2}x + \frac{5}{2}, & x \in \left[\frac{1}{5}, \frac{3}{10}\right) \\ -\frac{5}{2}x + \frac{3}{4}, & x \in \left[\frac{3}{10}, \frac{2}{5}\right] \\ 0, & otherwise, \end{cases}
$$

$$
\Phi_{\eta_2}(x) = \begin{cases} 0, & x \in (-\infty, 0] \\ \frac{5}{4}x, & x \in \left(0, \frac{1}{10}\right] \\ \frac{15}{4}x - \frac{1}{4}, & x \in \left(\frac{1}{10}, \frac{3}{10}\right] \\ -\frac{5}{4}x + \frac{1}{2}, & x \in \left(\frac{3}{10}, \frac{2}{5}\right] \\ 1, & x \in \left(\frac{2}{5}, +\infty\right), \end{cases}
$$

$$
\Phi_{\eta_2}^{-1}(\alpha) = \begin{cases} \frac{4}{5}\alpha, & \alpha \in \left[0, \frac{1}{8}\right) \\ \frac{4}{15}\alpha + \frac{2}{15}, & \alpha \in \left[\frac{1}{8}, \frac{7}{8}\right) \\ \frac{4}{5}\alpha - \frac{2}{5}, & \alpha \in \left[\frac{7}{8}, 1\right]. \end{cases}
$$

Finally, the lower semi-entropy of the bank's investment risk can be derived as

$$\begin{aligned} H[V_2]^- &= \int_0^{\frac{1}{8}} (0.8\alpha) \ln \tfrac{\alpha}{1-\alpha} d\alpha + \int_{\frac{1}{8}}^{\frac{1}{2}} \left(\tfrac{0.08\alpha - 0.04}{0.3} + 0.2 \right) \ln \tfrac{\alpha}{1-\alpha} d\alpha \\ &+0.2 \int_{\frac{1}{2}}^{1} \ln \tfrac{\alpha}{1-\alpha} d\alpha \\ &= \tfrac{\ln 2}{5} + \ln \tfrac{2^{\frac{4}{5}} \cdot 7^{\frac{63}{160}}}{4} + \ln \tfrac{2^{\frac{2}{5}} \cdot 7^{\frac{389}{480}}}{7} + \tfrac{1}{10}. \end{aligned}$$

Similarly, the risk that costs go over budgets can be described as an RSTIT2-FV. By using the upper semi-entropy, we can give a measure in this kind of issues.

Example 3. *Given that* $V_3 = \begin{pmatrix} 0.4 & 0.7 & 1 \\ 0.6 & 0.7 & 0.8 \end{pmatrix}$ *is an RSTIT2-FV, then the MF of its medium* η_3*, the credibility distribution and inverse credibility distribution are as follows:*

$$\mu_{V_3}(x) = \begin{cases} \dfrac{5}{3}x - \dfrac{2}{3}, & x \in \left[\dfrac{2}{5}, \dfrac{3}{5}\right) \\ \dfrac{20}{3}x - \dfrac{11}{3}, & x \in \left[\dfrac{3}{5}, \dfrac{7}{10}\right) \\ -\dfrac{20}{3}x + \dfrac{17}{3}, & x \in \left[\dfrac{7}{10}, \dfrac{4}{5}\right) \\ -\dfrac{5}{3}x + \dfrac{5}{3}, & x \in \left[\dfrac{4}{5}, 1\right] \\ 0, & \textit{otherwise}, \end{cases}$$

$$\Phi_{\eta_3}(x) = \begin{cases} 0, & x \in \left(-\infty, \dfrac{2}{5}\right] \\ \dfrac{5}{6}x - \dfrac{1}{3}, & x \in \left(\dfrac{2}{5}, \dfrac{3}{5}\right] \\ \dfrac{10}{3}x - \dfrac{11}{6}, & x \in \left(\dfrac{3}{5}, \dfrac{4}{5}\right] \\ \dfrac{5}{6}x + \dfrac{1}{6}, & x \in \left(\dfrac{4}{5}, 1\right] \\ 1, & x \in (1, +\infty), \end{cases}$$

$$\Phi_{\eta_3}^{-1}(\alpha) = \begin{cases} \dfrac{6}{5}\alpha + \dfrac{2}{5}, & \alpha \in \left[0, \dfrac{1}{6}\right) \\ \dfrac{3}{10}\alpha + \dfrac{11}{20}, & \alpha \in \left[\dfrac{1}{6}, \dfrac{5}{6}\right) \\ \dfrac{6}{5}\alpha - \dfrac{1}{5}, & \alpha \in \left[\dfrac{5}{6}, 1\right]. \end{cases}$$

Accordingly, the upper semi-entropy of the cost control risk can be shown as

$$H[V_3]^+ = \int_{\frac{1}{2}}^{\frac{5}{6}} \frac{0.12\alpha - 0.06}{0.4} \ln \frac{\alpha}{1-\alpha} d\alpha + \int_{\frac{5}{6}}^{1} (1.2\alpha - 0.9) \ln \frac{\alpha}{1-\alpha} d\alpha$$

$$= \ln \frac{5^{\frac{1}{3}} \cdot 6^{\frac{7}{10}}}{6} - \ln \frac{5}{48} + \frac{3}{20}.$$

4. Entropy and Semi-Entropies of Linear Function of RSTIT2-FVs

In order for verifying the desirable property of linearity in the credibility-based type-2 entropy, the formula for calculating the entropy and semi-entropies of a linear function constructed by the RSTIT2-FVs is considered in this section.

4.1. The Entropy of a Linear Function of RSTIT2-FVs

Theorem 5. *Suppose that $V_i, i = 1, 2, \cdots, n$ are mutually independent RSTIT2-FVs with the mediums of η_i. If the function $f(x_1, \cdots, x_t, x_{t+1}, \cdots, x_n)$ is strictly increasing with respect to $x_i, i = 1, 2, \cdots, t$ and strictly decreasing with respect to $x_i, i = t+1, t+2, \cdots, n$, then*

$$V = f(V_1, \cdots, V_t, V_{t+1}, \cdots, V_n),$$

has the ICD of

$$\Phi_V^{-1}(\alpha) = f\left(\Phi_{V_1}^{-1}(\alpha), \cdots, \Phi_{V_t}^{-1}(\alpha), \Phi_{V_{t+1}}^{-1}(1-\alpha), \cdots, \Phi_{V_n}^{-1}(1-\alpha)\right) \tag{24}$$

Proof of Theorem 5. It follows from Theorem 1 immediately. □

Theorem 6. *Suppose that $V_1, V_2, \cdots, V_n$ are independent RSTIT2-FVs, and $f(V_1, V_2, ..., V_n) = \lambda_1 V_1 + \lambda_2 V_2 + \cdots + \lambda_n V_n$, where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are real numbers, then the entropy of $V = f(V_1, V_2, \cdots, V_n)$ is*

$$H[V] = |\lambda_1| H[V_1] + |\lambda_2| H[V_2] + \cdots + |\lambda_n| H[V_n]. \tag{25}$$

Proof of Theorem 6. Suppose that the mediums of $V_i (i = 1, 2, \cdots, n)$ are, respectively, η_i, and the function $f(x_1, \cdots, x_t, x_{t+1}, \cdots, x_n)$ is strictly increasing with respect to $x_i, i = 1, 2, \cdots, t$ and strictly decreasing with respect to $x_i, i = t+1, t+2, \cdots, n$,

$$V = f(V_1, \cdots, V_t, V_{t+1}, \cdots, V_n).$$

$\lambda_1, \cdots, \lambda_t > 0$, and $\lambda_{t+1}, \cdots, \lambda_n < 0$. Then, on the basis of Theorem 5, we have

$$\begin{aligned}\Phi_V^{-1}(\alpha) &= f\left(\Phi_{V_1}^{-1}(\alpha), \cdots, \Phi_{V_t}^{-1}(\alpha), \Phi_{V_{t+1}}^{-1}(1-\alpha), \cdots, \Phi_{V_n}^{-1}(1-\alpha)\right) \\ &= \left(\Phi_{V_1}^{-1}(\alpha) + \cdots + \Phi_{V_t}^{-1}(\alpha) + \Phi_{V_{t+1}}^{-1}(1-\alpha) + \cdots + \Phi_{V_n}^{-1}(1-\alpha)\right).\end{aligned}$$

According to Theorem 6, it can be derived that

$$\begin{aligned}H[f(V_1, V_2, \cdots, V_n)] &= \int_0^1 (\lambda_1 \Phi_{V_1}^{-1}(\alpha) + \cdots + \lambda_t \Phi_{V_t}^{-1}(\alpha) + \lambda_{t+1} \Phi_{V_{t+1}}^{-1}(1-\alpha) \\ &\quad + \cdots + \lambda_n \Phi_{V_n}^{-1}(1-\alpha)) \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha \\ &= \int_0^1 (\lambda_1 \Phi_{\eta_1}^{-1}(\alpha) + \cdots + \lambda_t \Phi_{\eta_t}^{-1}(\alpha) + \lambda_{t+1} \Phi_{\eta_{t+1}}^{-1}(1-\alpha) \\ &\quad + \cdots + \lambda_n \Phi_{\eta_n}^{-1}(1-\alpha)) \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha \\ &= \sum_{i=1}^{t} \lambda_i \int_0^1 \Phi_{\eta_i}^{-1}(\alpha) \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha + \sum_{i=t+1}^{n} \lambda_i \int_0^1 \Phi_{\eta_i}^{-1}(1-\alpha) \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha \\ &= \sum_{i=1}^{t} \lambda_i \int_0^1 \Phi_{\eta_i}^{-1}(\alpha) \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha - \sum_{i=t+1}^{n} \lambda_i \int_0^1 \Phi_{\eta_i}^{-1}(\alpha) \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha \\ &= \sum_{i=1}^{t} \lambda_i H[V_i] - \sum_{i=t+1}^{n} \lambda_i H[V_i] \\ &= \sum_{i=1}^{n} |\lambda_i| H[V_i].\end{aligned}$$

□

Remark 5. *Assume that $V_i, i = 1, 2, \cdots, n$, are RSTIT2-FVs with the mediums of η_i, and f is a function from $\mathbb{R}^n$ to $\mathbb{R}$, then the entropy of the linear function, $V = f(V_1, V_2, \cdots, V_n)$ satisfy*

$$H(V) = H(\eta),$$

where $\eta = f(\eta_1, \eta_2, \cdots, \eta_n)$ is the medium of V.

Proof of Remark 5. It follows from Remark 2 and Theorem 3 immediately. □

Example 4. *Given that $V_4 = \begin{pmatrix} 0 & 2 & 4 \\ 1 & 2 & 3 \end{pmatrix}$ and $V_5 = \begin{pmatrix} 2 & 5 & 8 \\ 3 & 5 & 7 \end{pmatrix}$ are two RSTIT2-FVs, then the entropy of $H[f(V_4, V_5)] = 3H[V_4] - 5H[V_5]$ is*

$$\begin{aligned}
&H[f(V_4, V_5)] = 3H[V_4] - 5H[V_5] \\
&\quad = 3\left(\int_0^{\frac{1}{8}} (8\alpha) \ln \frac{\alpha}{1-\alpha} d\alpha + \int_{\frac{1}{8}}^{\frac{7}{8}} (\frac{8\alpha}{3}) \ln \frac{\alpha}{1-\alpha} d\alpha + \int_{\frac{7}{8}}^{1} (8\alpha - 4) \ln \frac{\alpha}{1-\alpha} d\alpha\right) \\
&\quad + 5\left(\int_0^{\frac{1}{12}} (12\alpha + 2) \ln \frac{\alpha}{1-\alpha} d\alpha + \int_{\frac{1}{12}}^{\frac{11}{12}} (\frac{24\alpha}{5}) \ln \frac{\alpha}{1-\alpha} d\alpha + \int_{\frac{11}{12}}^{1} (12\alpha - 4) \ln \frac{\alpha}{1-\alpha} d\alpha\right) \\
&\quad = 3 \cdot \ln \frac{343 \cdot 7^{\frac{15}{16}}}{4096} + 21 \cdot \frac{\ln 7}{48} + 5 \cdot \ln \frac{144 \cdot 11^{\frac{5}{8}}}{121} + \frac{297 \cdot \ln 11}{8} - 40 \ln 12 + 21.
\end{aligned}$$

4.2. The Semi-Entropies of a Linear Function of RSTIT2-FVs

Based on Theorem 5, we can obtain the semi-entropies of a linear function contributed by multiple RSTIT2-FVs. This paper provides two examples as follows.

Example 5. *According to Remark 5, if we denote two RSTIT2-FVs as $V_6 = \begin{pmatrix} 4 & 9 & 14 \\ 6 & 9 & 12 \end{pmatrix}$, $V_7 = \begin{pmatrix} 0 & 3 & 6 \\ 1 & 3 & 5 \end{pmatrix}$ and we have $V_8 = V_6 - V_7$, then it follows Equations (19), (20) and (24) that*

$$\Phi_{V_8}^{-1}(\alpha) = \Phi_{V_6}^{-1}(\alpha) - \Phi_{V_7}^{-1}(1-\alpha) = \begin{cases} 8\alpha + 4, & \alpha \in \left[0, \frac{1}{8}\right) \\ \frac{8}{3}\alpha + \frac{14}{3}, & \alpha \in \left[\frac{1}{8}, \frac{7}{8}\right) \\ 8\alpha, & \alpha \in \left[\frac{7}{8}, 1\right]. \end{cases}$$

$$\begin{aligned}
H[V_8]^- &= \int_0^{\frac{1}{8}} (8\alpha + 4) \ln \frac{\alpha}{1-\alpha} d\alpha + \int_{\frac{1}{8}}^{\frac{1}{2}} (\frac{8\alpha - 4}{3} + 6) \ln \frac{\alpha}{1-\alpha} d\alpha + 6 \int_{\frac{1}{2}}^{1} \ln \frac{\alpha}{1-\alpha} d\alpha \\
&= \ln \frac{4096 \cdot 7^{\frac{29}{48}}}{117649} - 18 \cdot \ln 2 + \frac{119 \cdot \ln 7}{16} + 1 \\
H[V_8]^+ &= \int_{\frac{1}{2}}^{\frac{7}{8}} (\frac{8\alpha - 4}{3}) \ln \frac{\alpha}{1-\alpha} d\alpha + \int_{\frac{7}{8}}^{1} (8\alpha - 6) \ln \frac{\alpha}{1-\alpha} d\alpha \\
&= \ln \frac{49 \cdot 7^{\frac{3}{16}}}{64} + \frac{7 \ln 7}{48} + 1
\end{aligned}$$

Example 6. *Similarly, if we denote three RSTIT2-FVs as* $V_9 = \begin{pmatrix} 0 & 2 & 4 \\ 1 & 2 & 3 \end{pmatrix}$, $V_{10} = \begin{pmatrix} 2 & 5 & 8 \\ 3 & 5 & 7 \end{pmatrix}$ *and* $V_{11} = \begin{pmatrix} 0 & 3 & 6 \\ 1 & 3 & 5 \end{pmatrix}$, *and assume that* $V_{12} = V_9 + V_{10} + V_{11}$, *then it follows Equation (24) that*

$$\Phi^{-1}_{V_{12}}(\alpha) = \Phi^{-1}_{V_9}(\alpha) + \Phi^{-1}_{V_{10}}(\alpha) + \Phi^{-1}_{V_{11}}(\alpha) = \begin{cases} 32\alpha + 2, & \alpha \in \left[0, \dfrac{3}{32}\right) \\ \dfrac{160}{13}\alpha + \dfrac{50}{13}, & \alpha \in \left[\dfrac{3}{32}, \dfrac{29}{32}\right) \\ 32\alpha - 14, & \alpha \in \left[\dfrac{29}{32}, 1\right]. \end{cases}$$

$$H[V_{12}]^- = \int_0^{\frac{2}{32}} (32\alpha + 5) \ln \frac{\alpha}{1-\alpha} d\alpha + \int_{\frac{3}{32}}^{\frac{1}{2}} (\frac{160\alpha - 80}{13} + 6) \ln \frac{\alpha}{1-\alpha} d\alpha + 6 \int_{\frac{1}{2}}^{1} \ln \frac{\alpha}{1-\alpha} d\alpha$$

$$= \frac{39 \ln 3}{64} - 99 \ln 2 + \frac{1305 \cdot \ln 29}{64} + \ln \frac{1.68 \cdot 10^7 \cdot 3^{\frac{223}{832}} \cdot 29^{\frac{289}{832}} \cdot 87^{\frac{9}{13}}}{5.17 \cdot 10^{10}} + 4,$$

$$H[V_{12}]^+ \int_{\frac{1}{2}}^{\frac{29}{32}} (\frac{160\alpha - 80}{13}) \ln \frac{\alpha}{1-\alpha} d\alpha + \int_{\frac{29}{32}}^{1} (32\alpha - 24) \ln \frac{\alpha}{1-\alpha} d\alpha$$

$$= \ln \frac{5 \cdot 10^{11} \cdot 3^{\frac{25}{64}} \cdot 29^{\frac{39}{64}}}{3.3 \cdot 10^{12}} + \ln \frac{3^{\frac{435}{832}} \cdot 29^{\frac{397}{832}}}{29} + 4$$

Compared with the previous study, in terms of calculating the type-2 semi-entropies, the formulas presented in this paper have an advantage in reducing computational complexity. This study makes it possible to evaluate the indicators that are formed by a set of RSTIT2-FVs in related areas, such as business investment and cost control. Theoretically, Remark 4 provided a way to help enterprises and other entities in operation management.

5. Conclusions

This paper proposed the concepts and formulas of the credibility-based type-2 fuzzy entropy and semi-entropies for the interval type-2 fuzzy variables, and we even explored these in a linear function. Based on the inverse credibility distribution, they can be efficiently figured out without complex calculations. Furthermore, this paper put forward that for independent type-2 fuzzy variables, the entropy and semi-entropies can be calculated even if the variable distribution formed by their linear combination is not known.

However, there are still certain limitations in this paper that can be explored in future studies. Firstly, it is worth noting that due to the complicated calculation of type-2 fuzzy variables and entropy, this paper only applied regular symmetric triangular interval type-2 fuzzy variables (RSTIT2-FVs) proposed by Li and Cai [34] to numerical verification, but the formulas can be extended to the general interval type-2 variable, which is about to be expanded in its application prospects and deepening theoretical research. Moreover, this paper only sets up several numerical experiments for possible application fields in real situations. The research on its application in this paper is still vacant, such as cost and risk control in operation management and the portfolio decisions in financial investment [42], the market share problem [43], and even the bio-gas implementation problem [44]. The above issues are waiting for more in-depth empirical research in future work.

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Abbreviations

The following abbreviations are used in the manuscript:

CD	credibility distribution
CF	credibility function
FOU	footprint of uncertainty
FV	fuzzy variable
ICD	inverse credibility distribution
IT2-FV	interval type-2 fuzzy variable
LMF	lower membership function
MF	membership function
T1-FV	type-1 fuzzy variable
T2-FV	type-2 fuzzy variable
RSTIT2-FV	regular symmetric triangular Interval type-2 fuzzy variable
UMF	upper membership function

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