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A_γ Eigenvalues of Zero Divisor Graph of Integer Modulo and Von Neumann Regular Rings

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Abstract: The A_γ matrix of a graph G is determined by $A_\gamma(G) = (1-\gamma)A(G) + \gamma D(G)$, where $0 \le \gamma \le 1$, $A(G)$ and $D(G)$ are the adjacency and the diagonal matrices of node degrees, respectively. In this case, the A_γ matrix brings together the spectral theories of the adjacency, the Laplacian, and the signless Laplacian matrices, and many more γ adjacency-type matrices. In this paper, we obtain the A_γ eigenvalues of zero divisor graphs of the integer modulo rings and the von Neumann rings. These results generalize the earlier published spectral theories of the adjacency, the Laplacian and the signless Laplacian matrices of zero divisor graphs.

Keywords: zero divisor graphs; adjacency matrix; Laplacian (signless) eigenvalues; A_γ matrix; von Neumann rings

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1. Introduction

Throughout this study, we discuss only undirected, connected, finite, and simple graphs. A graph G is represented by the pair $G(V(G), E(G))$, whereas $V(G)$ and $E(G)$ represent the node and the edge sets of G, respectively. The *size* and *order* of G are, respectively, the cardinalities of $E(G)$ and $V(G)$. The *degree* of $w \in V(G)$ is indicated by $d_G(w)$ (or simply by d_w) and is equal to the number of edges incident on w. The *neighborhood* $N(w)$ of $w \in V(G)$ is the collection of nodes of G connected to w, so d_w is the same as $|N(w)|$. If each node is of same degree, then G is said to be *regular*.

Consider the diagonal matrix $D(G) = diag(d_{v_1}, d_{v_2}, \dots, d_{v_n})$ of node degrees d_{v_i} of G, when $i = 1, 2, \dots, n$. The *adjacency* matrix $A(G) = (a_{jk})_{n\times n}$ is a real symmetric matrix, where (jk)-th entry is 1, if v_j is connected to v_k and 0 otherwise. The matrices $Q(G) = D(G) + A(G)$ and $L(G) = D(G) - A(G)$ are, respectively, the *signless Laplacian* as well as the *Laplacian* matrices of G. Their multiset of eigenvalues is the *signless Laplacian* and the *Laplacian spectrums* of G, respectively. The Laplacian and the signless Laplacian are positive real semi-definite matrices, so their spectrum is real, and they are ordered as $\lambda_n(G) \le \lambda_{n-1}(G) \le \cdots \le \lambda_1(G)$ and $\mu_n(G) \le \mu_{n-1}(G) \le \cdots \le \mu_1(G)$, respectively. Further details about these matrices can be seen in [1,2].

Nikiforov [3] suggested to investigate the symmetrical configurations $A_\gamma(G)$ of $D(G)$ and $A(G)$, and it is specified as $A_\gamma(G) = \gamma D(G) + (1-\gamma)A(G)$, whereas $1 \ge \gamma \ge 0$. Certainly, $A(G) = A_0(G)$, $D(G) = A_1(G)$ and $Q(G) = A(G) + D(G) = 2A_{\frac{1}{2}}(G)$. Thus, $A_\gamma(G)$ is a generalization of $A(G)$ as well as $Q(G)$ of G. Due to the fact that $A_\gamma(G)$ is symmetric and real, so its eigenvalues are ordered as $\lambda_1(A_\gamma(G)) \ge \lambda_2(A_\gamma(G)) \ge \cdots \ge$

$\lambda_n(A_\gamma(G))$, whenever $\lambda_1(A_\gamma(G))$ is referred to as the *generalized adjacency spectral radius* of G. Moreover, $A_\gamma(G)$ ($\gamma \neq 1$) is irreducible and non-negative for connected graph G. As a result, $\lambda_1(A_\gamma(G))$ is a simple eigenvalue (Perron–Frobenius theorem), and its associated eigenvector Y with positive entries is the generalized adjacency Perron vector of G. The spectral properties of $A_\gamma(G)$ are described in [3–7] and the references listed therein.

Consider a commutative ring R, with multiplicative identity $1 \neq 0$. If there exists $x_2 \in R$ ($x_2 \neq 0$) such that $x_1x_2 = 0$, then $x_1 \in R$ ($x_1 \neq 0$) is referred to as a zero divisor of R. The collection of zero divisors is symbolized by $Z(R)$, while $Z(R) \setminus \{0\} = Z^*(R)$ is the collection of non-zero zero divisors of R. The *zero divisor graph* $\Gamma(R)$ of R is a graph, where $Z^*(R)$ is its node set and two different nodes $y, z \in Z^*(R)$ are connected whenever $yz = zy = 0$. Beck [8] established such graphs over commutative rings; in his concept, he incorporated the identity and was primarily concerned with the coloring of commutative rings. Following that, the authors of [9] updated the concept of $\Gamma(R)$ by omitting the identity of R. The finite field of order n is represented by $\mathbb{F}_n$ and a ring of integers modulo n by $\mathbb{Z}_n$. The order of $\Gamma(\mathbb{Z}_n)$ is $n - 1 - \phi(n)$, whereas ϕ is Euler's phi function. The graph theoretic characteristics of $\Gamma(\mathbb{Z}_n)$ are widely investigated [10,11].

In [12], the authors showed that $\Gamma(\mathbb{Z}_n)$ is a Laplacian integral, where n is some prime power. According to [13], whenever a connected graph is bipartite, its lowest signless Laplacian eigenvalue equals 0, and the multiplicity of the eigenvalue 0 equals the number of bipartite components. Afkhami et al. [14] defined the normalized Laplacian as well as the signless Laplacian spectrums of $\Gamma(\mathbb{Z}_n)$ and evaluated such spectra over a range of n values. In addition, they have identified certain bounds for various eigenvalues of the normalized and signless Laplacian matrices of $\Gamma(\mathbb{Z}_n)$. In [15], the authors examined the adjacency spectrum of $\Gamma(\mathbb{Z}_n)$. Furthermore, the normalized (signless) Laplacian eigenvalues were discussed in [14,16–24] carried out the Laplacian and the adjacency spectral analysis. We apply the standard, symbol K_n for the complete graph, $\overline{K}_n$ as its complement, and $K_{a,b}$ for the complete bipartite graph. Additional unexplained terminologies and notations may be found in [25].

We have investigated many articles for the spectral graph theory to learn in depth the applications and use of chemical substances. From the applications point of view, the use of the eigenvalues and especially in the Laplacian matrix plays a vital role in the computer algorithms, where they play a foundational role in machine learning. In addition, it can also be used for load balancing in in these algorithms. In computers, nowadays, the image processing is very important for the security point as well as other archaeological points of view. In these processes, the adjacency matrix plays a key role in the visualization and other zooming purposes. In addition, there is a build up of a strong inter-network connection for certain topologies that the algebraic graph theory can play in such a circumstances. The connections inside the super computers are based on certain topologies, and its working rule is based on famous Cayley graphs that use the concept of symmetry.

There are still significant gaps in the existing work about the identification of certain A_γ eigenvalues of zero divisor graphs for commutative and von Neumann rings. The apparent reason for this is that neither the construction of zero divisor graphs over rings is well specified nor it is feasible to derive convenient formulas of graph characteristics for wide classes of rings. We make an attempt in this article to examine one of these problems.

The remaining article is organized as follows: Section 2 begins with some fundamental findings that will be employed to compute the A_γ eigenvalues of $\Gamma(\mathbb{Z}_n)$. Section 3 discusses the A_γ eigenvalues of zero divisor graphs over von Neumann regular rings. Section 4 contains the paper's conclusion and future work.

2. A_γ Eigenvalues of the Zero Divisor Graph

We begin this section with a couple of definitions and well-known outcomes, which we use to demonstrate our main results.

Definition 1. (Joined Union) *Let us assume that a connected graph G with $V = \{u_1, u_2, \ldots, u_n\}$, and G_i, where $i = 1, 2, \ldots, n$ are n_i order disjoint graphs. The* joined union *of G denoted by $G[G_1, G_2, \ldots, G_n]$ is obtained from G by substituting every u_i by G_i and connecting every node of G_i with each node of G_j, when u_i and u_j are adjacent in G.*

Consider the $n \times n$ matrix

$$M = \begin{pmatrix} t_{1,1} & t_{1,2} & \cdots & t_{1,l} \\ t_{2,1} & t_{2,2} & \cdots & t_{2,l} \\ \vdots & \vdots & \ddots & \vdots \\ t_{l,1} & t_{l,2} & \cdots & t_{l,l} \end{pmatrix},$$

such that the columns of M and rows M are partitioned as per the partition $P = \{\pi_1, \ldots, \pi_l\}$ of the $\pi = \{1, 2, \ldots, l\}$ set. The matrix $\mathcal{Q}$ is an $l \times l$ matrix with entries equal to the mean column or rows sum of the $t_{i,j}$ blocks of M; such matrix is known as the quotient matrix, see [1,26]. If every $t_{i,j}$ has a fixed column (row) sum, the P is known as regular, and $\mathcal{Q}$ is said to be the *regular quotient matrix*. For generality, the eigenvalues of $\mathcal{Q}$ and M are the same.

Let G_i be regular graphs, the subsequent result from [27] indicates the A_γ spectrum of the joined union of G_i in relation with the adjacency spectrum of G_i, for $n \geq i \geq 1$, together with the eigenvalues of $\mathcal{Q}$.

Theorem 1 ([27]). *Consider graph G of order $n \geq 2$. Suppose G_i are r_i-regular graphs having n_i order and $\lambda_{in_i} \leq \lambda_{i(n_i-1)} \leq \cdots \leq \lambda_{i2} \leq \lambda_{i1} = r_i$, where $1 \leq i \leq n$ are their adjacency eigenvalues. The A_γ spectrum of the joined union $G[G_1, \ldots, G_n]$ comprises $(1-\gamma)\lambda_{ik}(G_i) + \gamma(r_i + \gamma_i)$ eigenvalues, for $k = 2, 3, \ldots, n_i$, $i = 1, \ldots, n$, where $\gamma_i = \sum_{u_j \in N_G(u_i)} n_i$ is the sum of the orders of $G_j, i \neq j$ that correspond to the neighbors of $u_i \in G$. The other n eigenvalues of $G[G_1, \ldots, G_n]$ correspond to the eigenvalues of the matrix specified below:*

$$M = \begin{pmatrix} \xi_{11} & (1-\gamma)\xi_{12} & \ldots & (1-\gamma)\xi_{1n} \\ (1-\gamma)\xi_{21} & \xi_{22} & \ldots & (1-\gamma)\xi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ (1-\gamma)\xi_{n1} & (1-\gamma)\xi_{n2} & \ldots & \xi_{nn} \end{pmatrix}, \tag{1}$$

where $1 \leq i \leq n$, $\xi_{ii} = \gamma\gamma_i + r_i$, and $j \neq i$, $\xi_{ij} = n_j$, when u_i and u_j are connected, and 0 otherwise.

Assume that Υ_n is the simple connected graph, where $d_1, \ldots, d_s$ of n is a set of proper divisors with two distinct nodes being adjacent whenever n divides $d_i d_j$.

For $s \geq i \geq 1$, consider

$$C_{d_i} = \{z \in \mathbb{Z}_n : (z, n) = d_i\},$$

where (z, n) represents the G.C.D of z and n. Notice that $C_{d_i} \cap C_{d_j} = \phi$, when $j \neq i$; this implies that $C_{d_1}, C_{d_2}, \ldots, C_{d_s}$ are disjoint and partitions $V(\Gamma(\mathbb{Z}_n))$ as:

$$V(\Gamma(\mathbb{Z}_n)) = C_{d_1} \cup C_{d_2} \cup \cdots \cup C_{d_s}.$$

According to the C_{d_i} definition, a node of C_{d_i} is edge connected to the node of C_{d_j} in $\Gamma(\mathbb{Z}_n)$ when n divides $d_i d_j$, where $j, i \in \{1, 2, \ldots, s\}$. In addition, the order of C_{d_i} is $\phi\left(\frac{n}{d_i}\right)$, where $s \geq i \geq 1$, (see [22]).

The succeeding lemma presents important properties about the subgraphs that are either null graphs or cliques.

Lemma 1 ([12]). *Assume that d_i is its proper divisor and $n \in \mathbb{N}$. Then, the subsequent holds.*

(i) *For any $i \in \{1, 2, \ldots, s\}$, the subgraph induced by $\Gamma(C_{d_i})$ of $\Gamma(\mathbb{Z}_n)$ is either $K_{\phi\left(\frac{n}{d_i}\right)}$ or $\overline{K}_{\phi\left(\frac{n}{d_i}\right)}$. Furthermore, $\Gamma(C_{d_i})$ is $K_{\phi\left(\frac{n}{d_i}\right)}$ if $n|d_i^2$.*

(ii) *For $j, i \in \{1, 2, \ldots, s\}$ $(j \neq i)$, a node of C_{d_i} is connected to either none or all of the nodes in C_{d_j} of $\Gamma(\mathbb{Z}_n)$.*

The sequel results give the structure of $\Gamma(\mathbb{Z}_n)$.

Lemma 2 ([12]). *For $1 \leq i \leq s$, suppose $\Gamma(A_{d_i})$ is the subgraph induced by $\Gamma(\mathbb{Z}_n)$ of C_{d_i}. Then*

$$\Gamma(\mathbb{Z}_n) = \Upsilon_n[\Gamma(C_{d_1}), \Gamma(C_{d_2}), \ldots, \Gamma(C_{d_s})].$$

Lemma 3 ([21]). *The following properties hold for $\Gamma(\mathbb{Z}_n)$.*

(i) *If $n = q^{2m}$, whenever q is prime and $m \in \mathbb{N}$, we have*

$$\Gamma(\mathbb{Z}_n) = \Upsilon_n\left[\overline{K}_{\phi(q^{2m-1})}, \ldots, \overline{K}_{\phi(q^{m+1})}, K_{\phi(q^m)}, \ldots, K_{\phi(q)}\right].$$

(ii) *If $n = q^{2m+1}$, where q is prime and $m \in \mathbb{N}$, we have*

$$\Gamma(\mathbb{Z}_n) = \Upsilon_n\left[\overline{K}_{\phi(q^{2m})}, \ldots, \overline{K}_{\phi(q^{m+1})}, K_{\phi(q^m)}, \ldots, K_{\phi(q)}\right].$$

Furthermore, we examine the A_γ eigenvalues of $\Gamma(\mathbb{Z}_n)$ of $\mathbb{Z}_n$.

Theorem 2. *The A_γ spectrum of $\Gamma(\mathbb{Z}_n)$ contains the eigenvalues $(1-\gamma)\lambda_{ik}(\Gamma(A_{d_i})) + \gamma(r_i + \gamma_i)$ having multiplicity $\phi\left(\frac{n}{d_i}\right) - 1$, also the eigenvalues of M presented in Equation (1).*

Proof. The proof directly follows from Theorem 1. □

Corollary 1. *If $n = p_1 p_2 \ldots p_l$, where $l \geq 2$ and $p_l > p_{l-1} > \cdots > p_1$ are distinct prime numbers, then the A_γ spectrum of $\Gamma(\mathbb{Z}_n)$ contains the $\gamma(\gamma_i)$ eigenvalues, where $i = 1, 2, \ldots, l$ with multiplicity $\phi\left(\frac{n}{d_i}\right) - 1$ together with the eigenvalues of M presented in Equation (1).*

Next, we discuss the A_γ spectrum for some special classes of zero divisor graph. As a reminder, K_n has an adjacency spectrum $\{n-1, (-1)^{[n-1]}\}$ and that of $\overline{K}_n$ is $\{0^{[n]}\}$.

Lemma 4. *If $n = q^2$, then A_γ spectrum of $\Gamma(\mathbb{Z}_n)$ is given as:*

$$\left\{(\gamma\phi(q) - 1)^{[\phi(q)-1]}, \phi(q) - 1\right\},$$

where q is prime.

Proof. For prime q, the zero divisor graph $\Gamma(\mathbb{Z}_{q^2}) \cong K_{q-1}$ and its A_γ spectrum is already known. □

Lemma 5. *For primes $p_1 < p_2$, the A_γ spectrum of $\Gamma(\mathbb{Z}_{p_1 p_2})$ is specified below:*

$$\Big\{(\gamma\phi(p_2))^{[\phi(p_1)-1]}, (\gamma\phi(p_1))^{[\phi(p_2)-1]},$$
$$\frac{1}{2}\Big(\gamma(\phi(p_1) + \phi(p_2)) \pm \sqrt{(\gamma(\phi(p_1) + \phi(p_2)))^2 - 4(2\gamma - 1)\phi(p_1)\phi(p_2)}\Big)\Big\}.$$

Proof. Suppose $n = p_1p_2$, whereas $p_1 < p_2$ are prime numbers. Then, by Lemma 3, $\Gamma(\mathbb{Z}_{p_1p_2}) \cong K_2[\overline{K}_{\phi(p_1)}, \overline{K}_{\phi(p_2)}] = \overline{K}_{\phi(p_1)} \vee \overline{K}_{\phi(p_2)}$ and by Theorem 1, $r_1 = r_2 = 0$ and $\gamma_1 = \phi(p_2), \gamma_2 = \phi(p_1)$. The A_γ spectrum of $\Gamma(\mathbb{Z}_{p_1p_2})$ contains the eigenvalue

$$\gamma(r_1 + \gamma_1) + (1-\gamma)\lambda_{1i}(\overline{K}_{\phi(p_1)}) = \gamma\phi(p_2),$$

whose multiplicity is $\phi(p_1) - 1$. Similarly, $\gamma(r_2 + \gamma_2) + (1-\gamma)\lambda_{2i}(\overline{K}_{\phi(p_2)}) = \gamma\phi(p_1)$ is another A_γ eigenvalue of $\Gamma(\mathbb{Z}_{p_1p_2})$ whose multiplicity is $\phi(p_2) - 1$. The remaining two A_γ eigenvalues of $\Gamma(\mathbb{Z}_{p_1p_2})$ are the eigenvalues of the matrix presented below:

$$\begin{pmatrix} \gamma\phi(p_2) & (1-\gamma)\phi(p_2) \\ (1-\gamma)\phi(p_1) & \gamma\phi(p_1) \end{pmatrix},$$

and its characteristic polynomial is $\phi(p_1)\phi(p_2)(2\gamma - 1) - \gamma(\phi(p_1) + \phi(p_2))x + x^2$. □

Lemma 6. *For prime q, the A_γ spectrum of $\Gamma(\mathbb{Z}_{q^3})$ is*

$$\left\{ \Big(\gamma\phi(q)\Big)^{[\phi(q^2)-1]}, \Big(\gamma(\phi(q) + \phi(q^2)) - 1\Big)^{[\phi(q)-1]}, \frac{1}{2}\Big(\gamma(\phi(q^2) + \phi(q)) + \phi(q) - 1 \pm \sqrt{\Delta}\Big) \right\},$$

where $\Delta = \big(\phi(q) + \gamma(\phi(q^2) + \phi(q)) - 1\big)^2 - 4\big(\gamma(\phi(q^2) - \phi(q)) + \phi(q)\phi(q^2)(2\gamma - 1)\big)$.

Proof. By Lemma 3, then the graph $\Gamma(\mathbb{Z}_{q^3})$ of $\mathbb{Z}_{q^3}$ is specified as:

$$\Gamma(\mathbb{Z}_{q^3}) = K_2[\overline{K}_{\phi(q^2)}, K_{\phi(q)}] = \overline{K}_{\phi(q^2)} \vee K_{\phi(q)},$$

that is, the complete split graph with the clique number $\phi(q)$ as well as the independence number is $\phi(q^2)$. Using Theorem 1, the A_γ spectrum of $\Gamma(\mathbb{Z}_{q^3})$ contains the eigenvalue

$$\gamma(r_1 + \gamma_1) + (1-\gamma)\lambda_{1i}(\overline{K}_{\phi(q^2)}) = \gamma\phi(q),$$

with multiplicity $\phi(q^2) - 1$, the eigenvalue

$$\gamma(r_2 + \gamma_2) + (1-\gamma)\lambda_{2i}(K_{\phi(q)}) = \gamma(\phi(q^2) + \phi(q)) - 1,$$

whose multiplicity is $\phi(q) - 1$. The other two A_γ eigenvalues of $\mathbb{Z}_{q^3}$ correspond to the eigenvalues of the sequel matrix:

$$\begin{pmatrix} \gamma\phi(q) & (1-\gamma)\phi(q) \\ (1-\gamma)\phi(q^2) & \gamma\phi(q^2) + \phi(q) - 1 \end{pmatrix}.$$

□

Theorem 3. *Suppose $n = q^{2m}$, where $m \in \mathbb{N}$ and q is any prime. Then, the A_γ spectrum of $\Gamma(\mathbb{Z}_n)$ comprises the eigenvalue $\gamma(q^i - 1)$ whose multiplicity is $\phi(q^{2m-i}) - 1$, whenever $1 \le i \le m-1$, the eigenvalues $\gamma(q^i - 1) - 1$ with multiplicity $\phi(q^{2m-i}) - 1$, whenever $m \le i \le 2m-1$ and the eigenvalues of the matrix in Equation (2).*

Proof. Applying Lemma 3, the structure of $\Gamma(\mathbb{Z}_n)$ is given as:

$$\Gamma(\mathbb{Z}_n) = \Upsilon_n\big[\overline{K}_{\phi(q^{2m-1})}, \ldots, \overline{K}_{\phi(q^{m+1})}, K_{\phi(q^m)}, \ldots, K_{\phi(q)}\big].$$

Now, we need to know the structure of Υ_n. For that, note that $\{q, q^2, \ldots, q^m, \ldots, q^{2m-1}\}$ divides n properly. Thus, by definition of $\Upsilon_{q^{2m}}$, the node q^i is connected to q^j if $2m - i \le j$ and $j \ne i$ where $1 \le i \le 2m-1$. In addition, $r_1 = r_2 = \cdots = r_{m-1} = 0$ and

$r_i = \phi(q^{2m-i}) - 1$, where $m \le i \le 2m-1$. Then, $\gamma_2 = \phi(q^2) + \phi(q) = q^2 - 1$, $\gamma_1 = \phi(q)$. In general, using the fact that $\sum_{i=1}^{r} \phi(q^r) = q^r - 1$, we have

$$\gamma_i = \phi(q^i) + \phi(q^{i-1}) + \cdots + \phi(q^2) + \phi(q) = q^i - 1,$$

for $i = 1, 2, \ldots, m-1$.

Next, we find the remaining γ_i's

$$\begin{aligned}\gamma_m =& \phi(q^{m-1}) + \phi(q^{m-2}) + \cdots + \phi(q^2) + \phi(q)\\ =& \phi(q^{m-1}) + \phi(q^{m-2}) + \cdots + \phi(q^2) + \phi(q) + \phi(q^m) - \phi(q^m)\\ =& q^m - 1 - \phi(q^m).\end{aligned}$$

More generally, for $i = m, \ldots, 2m-1$, add and subtract $\phi(q^{2m-i})$, so γ_i values take the simple form

$$\gamma_i = \sum_{k=1}^{t} \phi(q^t) - \phi(q^{2m-i}) = q^i - 1 - \phi(q^{2m-i}).$$

Applying Theorem 1, the A_γ spectrum of $\Gamma(\mathbb{Z}_n)$ are the eigenvalues:

$$(1-\gamma)\lambda_{ik}(G_i) + \gamma(r_i + \gamma_i) = \gamma\gamma_i = \gamma(q^i - 1),$$

where $i = 1, \ldots, (m-1)$. Likewise, as $i = m, \ldots, (2m-1)$ and using the values of r_i, G_i, and γ_i, the other A_γ eigenvalues are:

$$\begin{aligned}(1-\gamma)\lambda_{ik}(G_i) + \gamma(r_i + \gamma_i) =& \gamma(\phi(q^{2m-i}) - 1 + q^i - 1 - \phi(q^{2m-i})) + (1-\gamma)(-1)\\ =& \gamma(q^i - 1) - 1,\end{aligned}$$

with $\phi(q^{2m-i}) - 1$ multiplicities. The remaining A_γ eigenvalues of $\Gamma(\mathbb{Z}_n)$ are actually the subsequent matrix eigenvalues:

$$\begin{pmatrix} A_m & & B_{m\times(m-1)} & & \\ & d_{m+1} & \cdots & (1-\gamma)\phi(q^2) & (1-\gamma)\phi(q) \\ C_{m-1\times m} & \vdots & \ddots & \vdots & \vdots \\ & (1-\gamma)\phi(q^{m-1}) & \cdots & d_{2m-2} & (1-\gamma)\phi(q) \\ & (1-\gamma)\phi(q^{m-1}) & \cdots & (1-\gamma)\phi(q^2) & d_{2m-1} \end{pmatrix}, \tag{2}$$

where $A_m = diag\big(\gamma\gamma_1, \gamma\gamma_2, \ldots, \gamma\gamma_{m-1}, \gamma(q^m - 1 - \phi(q^m)) + \phi(q^m) - 1\big)$,

$$B = \begin{pmatrix} 0 & \ldots & 0 & (1-\gamma)\phi(q) \\ 0 & \ldots & (1-\gamma)\phi(q^2) & (1-\gamma)\phi(q) \\ \vdots & \ddots & \vdots & \vdots \\ (1-\gamma)\phi(q^{m-1}) & \ldots & (1-\gamma)\phi(q^2) & (1-\gamma)\phi(q) \\ (1-\gamma)\phi(q^{m-1}) & \ldots & (1-\gamma)\phi(q^2) & (1-\gamma)\phi(q) \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & \ldots & 0 & \phi(q)(1-\gamma) \\ 0 & 0 & \ldots & \phi(q^2)(1-\gamma) & \phi(q)(1-\gamma) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi(q^{2m-1})(1-\gamma) & \phi(q^{2m-2})(1-\gamma) & \ldots & \phi(q^2)(1-\gamma) & \phi(q)(1-\gamma) \\ \phi(q^{2m-1})(1-\gamma) & \phi(q^{2m-2})(1-\gamma) & \ldots & \phi(q^2)(1-\gamma) & \phi(q)(1-\gamma) \end{pmatrix}$$

and $d_i = \gamma(q^i - 1 - \phi(q^{2m-i})) + \phi(q^{2m-i}) - 1$, where $i = m+1, \ldots, 2m-1$. □

Following the steps as in Theorem 3, we can prove the odd case.

Theorem 4. *If $n = q^{2m+1}$ where $m \geq 2$, then the A_γ spectrum of $\Gamma(\mathbb{Z}_n)$ contains $\gamma(q^i - 1)$ eigenvalues whose multiplicity is $\phi(q^{2m+1-i}) - 1$, for $i = 1, 2, \ldots, m$, the eigenvalues $\gamma(q^i - 1) - 1$ with multiplicity $\phi(q^{2m+1-i}) - 1$, where $m + 1 \leq i \leq 2m$, and the eigenvalues of the matrix below:*

$$\begin{pmatrix} A_{m+1} & & B_{(m+1)\times(m-1)} & & \\ & d_{m+2} & \cdots & (1-\gamma)\phi(q^2) & (1-\gamma)\phi(q) \\ C_{(m-1)\times(m+1)} & \vdots & \ddots & \vdots & \vdots \\ & (1-\gamma)\phi(q^{m-1}) & \cdots & d_{2m-1} & \phi(q)(1-\gamma) \\ & \phi(q^{m-1})(1-\gamma) & \cdots & (1-\gamma)\phi(q^2) & d_{2m} \end{pmatrix},$$

where $A_{m+1} = diag\big(\gamma\gamma_1, \gamma\gamma_2, \ldots, \gamma\gamma_{m-1}, \gamma\gamma_m, \gamma(q^{m+1} - \phi(q^{m+1}) - 1) - 1 + \phi(q^{m+1})\big)$,

$$B = \begin{pmatrix} 0 & \ldots & 0 & \phi(q)(1-\gamma) \\ 0 & \ldots & \phi(q^2)(1-\gamma) & \phi(q)(1-\gamma) \\ \vdots & \ddots & \vdots & \vdots \\ \phi(q^{m-1})(1-\gamma) & \ldots & \phi(q^2)(1-\gamma) & \phi(q)(1-\gamma) \\ \phi(q^{m-1})(1-\gamma) & \ldots & \phi(q^2)(1-\gamma) & \phi(q)(1-\gamma) \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & \ldots & 0 & \phi(q)(1-\gamma) \\ 0 & 0 & \ldots & \phi(q^2)(1-\gamma) & \phi(q)(1-\gamma) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi(q^{2m})(1-\gamma) & \phi(q^{2m-1})(1-\gamma) & \ldots & \phi(q^2)(1-\gamma) & \phi(q)(1-\gamma) \\ \phi(q^{2m})(1-\gamma) & \phi(q^{2m-1})(1-\gamma) & \ldots & \phi(q^2)(1-\gamma) & \phi(q)(1-\gamma) \end{pmatrix}$$

and $d_i = \gamma(q^i - \phi(q^{2m+1-i}) - 1) + \phi(q^{2m+1-i}) - 1$, *where* $i = m+2, \ldots, 2m-1, 2m$.

The next result gives the A_γ eigenvalues of $\Gamma(\mathbb{Z}_n)$, where n is the multiplication of *three* prime numbers.

Proposition 1. *The A_γ spectrum of $\Gamma(\mathbb{Z}_{p_1p_2p_3})$ contains the eigenvalues $\gamma(p_1 - 1)$, $\gamma(p_2 - 1)$, $\gamma(p_3 - 1)$, $\gamma(p_1p_2 - 1)$, $\gamma(p_1p_3 - 1)$, and $\gamma(p_2p_3 - 1)$ whose multiplicities are $\phi(p_2p_3) - 1$, $\phi(p_1p_3) - 1$, $\phi(p_1p_2) - 1$, $\phi(p_3) - 1$, $\phi(p_2) - 1$, and $\phi(p_1) - 1$, respectively. The leftover A_γ eigenvalues of $\Gamma(\mathbb{Z}_{p_1p_2p_3})$ are actually the eigenvalues of the matrix presented in Equation (3).*

Proof. Figure 1 illustrates the proper divisor graph $\Upsilon_{p_1p_2p_3}$. By expanding the divisor sequence while using Lemma 2 to the nodes, we obtain the following zero divisor graph:

$$\Gamma(\mathbb{Z}_{p_1p_2p_3}) = \Upsilon_{p_1p_2p_3}[\overline{K}_{p_2p_3}, \overline{K}_{p_1p_3}, \overline{K}_{p_1p_2}, \overline{K}_{p_3}, \overline{K}_{p_2}, \overline{K}_{p_1}].$$

By Theorem 1, the values of γ_i are presented as:

$$\begin{aligned} &\gamma_1 = \phi(p_1) = p_1 - 1, \gamma_2 = \phi(p_2) = p_2 - 1, \gamma_3 = \phi(p_3) = p_3 - 1, \\ &\gamma_4 = \phi(p_1p_2) + \phi(p_1) + \phi(p_2) = p_1p_2 - 1, \gamma_5 = \phi(p_1p_3) + \phi(p_1) + \phi(p_3) = p_1p_3 - 1, \\ &\gamma_6 = \phi(p_2p_3) + \phi(p_2) + \phi(p_3) = p_2p_3 - 1. \end{aligned}$$

As every component of $\Gamma(\mathbb{Z}_{p_1p_2p_3})$ is an empty graph, therefore, the A_γ spectrum of $\Gamma(\mathbb{Z}_{p_1p_2p_3})$ comprises the eigenvalue

$$\gamma(0 + \gamma_1) + (1-\gamma)0 = \gamma(p_1 - 1),$$

with multiplicity $\phi(p_2p_3) - 1$. Likewise, the other A_γ eigenvalues of $\Gamma(\mathbb{Z}_{p_1p_2p_3})$ can be calculated as given in the statement. The remaining six A_γ eigenvalues of $\Gamma(\mathbb{Z}_{p_1p_2p_3})$ correspond to the matrix as specified below:

$$\begin{pmatrix}
\gamma\phi(p_1) & 0 & 0 & 0 & 0 & (1-\gamma)\phi(p_1) \\
0 & \gamma\phi(p_2) & 0 & 0 & (1-\gamma)\phi(p_2) & 0 \\
0 & 0 & \gamma\phi(p_3) & -(\gamma-1)\phi(p_2) & 0 & 0 \\
0 & 0 & (1-\gamma)\phi(p_1p_2) & \gamma(p_1p_2-1) & -(\gamma-1)\phi(p_2) & -(\gamma-1)\phi(p_1) \\
0 & (1-\gamma)\phi(p_1p_3) & 0 & (1-\gamma)\phi(p_3) & \gamma(p_1p_3-1) & -(\gamma-1)\phi(p_1) \\
(1-\gamma)\phi(p_2p_3) & 0 & 0 & (1-\gamma)\phi(p_3) & -(\gamma-1)\phi(p_2) & \gamma(p_2p_3-1)
\end{pmatrix}. \quad (3)$$

□

By putting $\gamma = 0$ and $\gamma = \frac{1}{2}$ in Theorem 2 and its consequences, we obtain the adjacency spectrum while the signless Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ is obtained in [14,17,20]. Similarly, using the fact $(\gamma - \beta)L(G) = (\gamma - \beta)(D(G) - A(G)) = A_\gamma(G) - A_\beta(G)$ and Theorem 2 along with its consequences, we obtain the Laplacian eigenvalues, which were earlier obtained in [12,21].

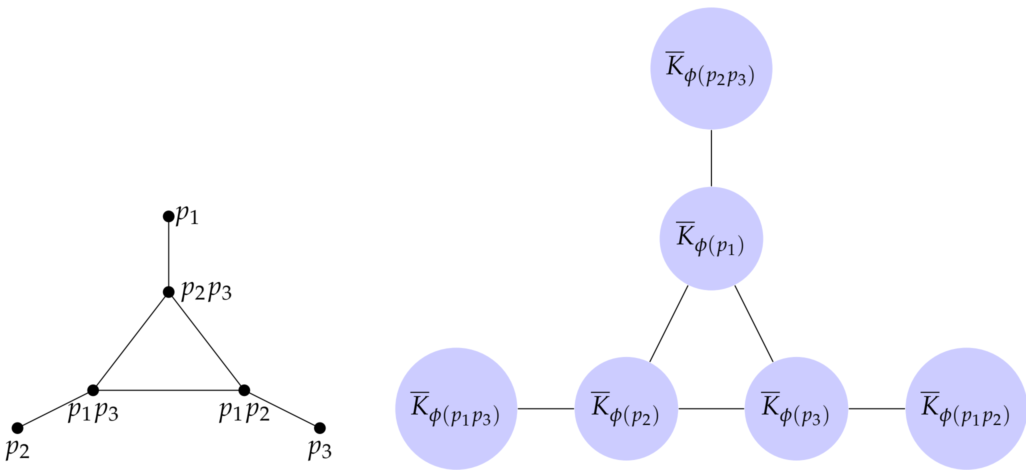

Figure 1. Proper divisor graph $\Upsilon_{p_1p_2p_3}$ and $\Gamma(\mathbb{Z}_{p_1p_2p_3})$.

3. A_γ Eigenvalues of Zero Divisor Graphs of Von Nuemann Regular Rings

A ring R is known as *von Neumann regular* if there exists $z \in R$ so that $y = y^2z$ for each $y \in R$. The collection of idempotents of R is represented by $B(R)$, and its zero divisor graph is represented by $\Gamma(B(R))$. In [28–32], researchers examined the zero divisor graphs of von Neumann regular rings and the adjacency spectrum was recently given in [33].

If $r_1 \in R$, then the annihilator of r_1 is denoted by $Ann(r_1)$ and is defined as $Ann(r_1) = \{r_2 \in R : r_1r_2 = 0\}$. Define a relation $r_1 \sim r_2$ on R, if $Ann(r_1) = Ann(r_2)$ and $\sim$ is clearly an equivalence relation. In [29], the authors show the graph isomorphism, and the equivalence class has a particular idempotent if R is a von Neumann regular. Patil and Shinde [33] proved that for every non-trivial idempotent, the equivalence class of e has an independent subgraph and two nodes $a, b \in \Gamma(R)$ are edge connected whenever e_a and e_b are edge connected in $\Gamma(B(R))$. They also showed that for a non-trivial idempotent $e = (e_1, e_2, \dots, e_k)$ in $\mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_k$, the cardinality of A_e is $\prod\limits_{e_i \neq 0} (|\mathbb{F}_i| - 1)$, where $\times$ is the usual product of rings (fields).

The structure of $\Gamma(R)$ of the von Neumann regular rings R is obtained by the following result.

Lemma 7 ([33])**.** *Assume that $e_1, e_2, \dots, e_t$ are the non-trivial idempotents in R. Then,*

$$\Gamma(R) = \Gamma(B(R))[\Gamma(A_{e_1}), \Gamma(A_{e_2}), \dots, \Gamma(A_{e_t})].$$

Now, we discuss the A_γ eigenvalues of R.

Theorem 5. *Suppose R is a finite von Neumann regular ring whose non-trivial idempotents are $e_1, e_2, \dots, e_t$. Then the A_γ spectrum of $\Gamma(R)$ consists of $\gamma(\gamma_i)$ eigenvalues with multiplicity $|A_{e_i}| - 1$, for $i = 1, 2, \dots, t$, together with the eigenvalues of M given in (1).*

Proof. This proof directly follows by Theorem 1 and Lemma 7. □

If $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_k$, for every $\mathbb{F}_i \cong \mathbb{Z}_{p_i}$ and $p_i, i = 1, \dots, k$ are distinct primes, the A_γ spectrum of $\Gamma(R)$ is presented by Corollary 1. Thus, by Theorem 5, we may determine the spectrum of more general classes of zero divisors graphs of rings.

Next, we discuss some consequence of Theorem 5. First, we will find the A_γ spectrum of $\mathbb{F}_1 \times \mathbb{F}_2$, whereas $\mathbb{F}_1$ and $\mathbb{F}_2$ are finite fields. If $\mathbb{F}_1 \cong \mathbb{Z}_p$ and $\mathbb{F}_1 \cong \mathbb{Z}_q$, whereas $p < q$ are primes, then A_γ eigenvalues are as in Lemma 5; otherwise, the A_γ spectrum is presented by the sequel result.

Corollary 2. *Suppose $R \cong \mathbb{F}_1 \times \mathbb{F}_2$. Then, the A_γ spectrum of $\Gamma(R)$ contains the eigenvalues $\gamma(|\mathbb{F}_2| - 1)$ and $\gamma(|\mathbb{F}_1| - 1)$ with multiplicities $|\mathbb{F}_1| - 2$ and $|\mathbb{F}_2| - 2$, respectively, and the two zeros of the following polynomial:*

$$\lambda^2 - \lambda\big(\gamma(|\mathbb{F}_2| + |\mathbb{F}_1|) - 2\gamma\big) + \gamma^2(|\mathbb{F}_1| - 1)(|\mathbb{F}_2| - 1) - (1-\gamma)^2(|\mathbb{F}_1| - 1)(|\mathbb{F}_2| - 1).$$

Proof. For $R \cong \mathbb{F}_1 \times \mathbb{F}_2$, the non-trivial idempotent set is $B(R) = \{e_1 = (1,0), e_2 = (0,1)\}$ and $A_{e_1} = \{(x,0) : x \in \mathbb{F}_1 \setminus \{0\}\}$ and $A_{e_2} = \{(0,y) : x \in \mathbb{F}_2 \setminus \{0\}\}$, with $|A_{e_i}| = |\mathbb{F}_i| - 1$, $i = 1,2$. Thus, by the definition of $\Gamma(B(R))$ also by Lemma 7, $\Gamma(R) \cong K_2[\overline{K}_{|\mathbb{F}_1|-1}, \overline{K}_{|\mathbb{F}_2|-1}]$. Therefore, from Theorem 5, the A_γ eigenvalues of $\Gamma(R)$ are the eigenvalue $\gamma(\gamma_1) = \gamma(|\mathbb{F}_2| - 1)$ with multiplicity $|\mathbb{F}_1| - 1$ and the eigenvalues $\gamma(\gamma_2) = \gamma(|\mathbb{F}_1| - 1)$ with multiplicity $|\mathbb{F}_2| - 1$. The other A_γ eigenvalues are actually the eigenvalues of the subsequent matrix:

$$\begin{pmatrix} (|\mathbb{F}_2| - 1)\gamma & (|\mathbb{F}_2| - 1)(1-\gamma) \\ (|\mathbb{F}_1| - 1)(1-\gamma) & (|\mathbb{F}_1| - 1)\gamma \end{pmatrix}.$$

□

If $R \cong \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$, when $r > q > p$ are distinct primes, as a result, Proposition 3 yields the A_γ eigenvalues of $\Gamma(R)$. For $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3$. As a consequence, we obtain the following.

Corollary 3. *Suppose $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3$. We have that the A_γ spectrum of $\Gamma(R)$ contains the eigenvalues $\gamma(|\mathbb{F}_3||\mathbb{F}_2| - 1)$, $\gamma(|\mathbb{F}_3||\mathbb{F}_1| - 1)$, $\gamma(|\mathbb{F}_2||\mathbb{F}_1| - 1)$, $\gamma(|\mathbb{F}_3| - 1)$, $\gamma(|\mathbb{F}_2| - 1)$, $\gamma(|\mathbb{F}_1| - 1)$ with multiplicities $|\mathbb{F}_1| - 2$, $|\mathbb{F}_2| - 2$, $|\mathbb{F}_3| - 2$, $(n_1 - 1)$, $(n_2 - 1)$, $(n_3 - 1)$, respectively, and the other A_γ eigenvalues of $\Gamma(R)$ are of Equation (4).*

Proof. For $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3$, the non-trivial idempotent set is $B(R) = \{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1), e_4 = (1,1,0), e_5 = (1,0,1), e_6 = (0,1,1),\}$ and $A_{e_1} = \{(x,0) : x \in \mathbb{F}_1 \setminus \{0\}\}$, $\Gamma(B(R))$ is shown in Figure 2. Likewise, the graph $\Gamma(R)$ is expressed on the right side of Figure 2, where $n_1 = (|\mathbb{F}_2| - 1)(|\mathbb{F}_3| - 1), n_2 = (|\mathbb{F}_1| - 1)(|\mathbb{F}_3| - 1)$ and $n_3 = (|\mathbb{F}_1| - 1)(|\mathbb{F}_2| - 1)$ and by Theorem 5, the γ_i values are:

$$\begin{aligned}
\gamma_1 &= (|\mathbb{F}_2| - 1)(|\mathbb{F}_3| - 1) + |\mathbb{F}_2| + |\mathbb{F}_3| - 2 = |\mathbb{F}_2||\mathbb{F}_3| - 1, \\
\gamma_2 &= (|\mathbb{F}_1| - 1)(|\mathbb{F}_3| - 1) + |\mathbb{F}_1| + |\mathbb{F}_3| - 2 = |\mathbb{F}_1||\mathbb{F}_3| - 1, \\
\gamma_3 &= |\mathbb{F}_1| + |\mathbb{F}_2| - 2 + (|\mathbb{F}_1| - 1)(|\mathbb{F}_2| - 1) = |\mathbb{F}_1||\mathbb{F}_2| - 1, \\
\gamma_4 &= |\mathbb{F}_3| - 1,\ \gamma_5 = |\mathbb{F}_2| - 1,\ \gamma_6 = |\mathbb{F}_1| - 1.
\end{aligned}$$

As a result, Theorem 5 states that the A_γ eigenvalues of $\Gamma(R)$ consist of the eigenvalues $\gamma(\gamma_1) = \gamma(|\mathbb{F}_2||\mathbb{F}_3| - 1)$ with multiplicity $|\mathbb{F}_1| - 2$, and the other A_γ eigenvalues are as stated. The remaining six A_γ eigenvalues correspond to the eigenvalues of the matrix given below:

$$\begin{pmatrix} \gamma\gamma_1 & (1-\gamma)(|\mathbb{F}_2|-1) & (1-\gamma)(|\mathbb{F}_3|-1) & 0 & 0 & (1-\gamma)n_1 \\ (1-\gamma)(|\mathbb{F}_1|-1) & \gamma\gamma_2 & (1-\gamma)(|\mathbb{F}_3|-1) & 0 & (1-\gamma)n_2 & 0 \\ (1-\gamma)(|\mathbb{F}_1|-1) & (1-\gamma)(|\mathbb{F}_2|-1) & \gamma\gamma_3 & (1-\gamma)n_3 & 0 & 0 \\ 0 & 0 & (1-\gamma)(|\mathbb{F}_3|-1) & \gamma\gamma_4 & 0 & 0 \\ 0 & (1-\gamma)(|\mathbb{F}_2|-1) & 0 & 0 & \gamma\gamma_5 & \\ (1-\gamma)(|\mathbb{F}_1|-1) & 0 & 0 & 0 & 0 & \gamma\gamma_6 \end{pmatrix}. \tag{4}$$

□

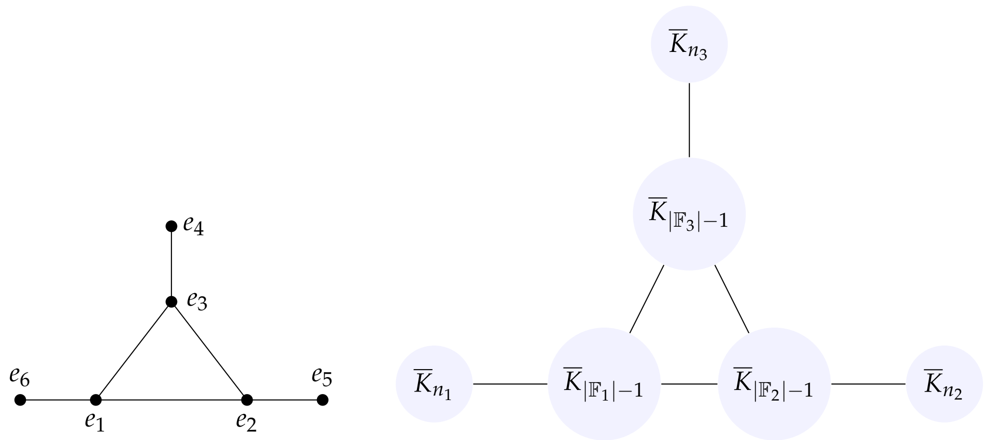

Figure 2. Idempotent zero divisor graph and zero divisor graph of $\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3$, where $n_1 = (|\mathbb{F}_2|-1)(|\mathbb{F}_3|-1), n_2 = (|\mathbb{F}_1|-1)(|\mathbb{F}_3|-1)$ and $n_3 = (|\mathbb{F}_1|-1)(|\mathbb{F}_2|-1)$.

We note that for $\gamma = 0$ in Theorem 5, we obtain the adjacency eigenvalues of the von Neumann regular rings obtained by Patil and Shinde [33]. Furthermore, from $A_\gamma(G) - A_\beta(G) = (\gamma-\beta)L(G)$, applying Theorem 5, we derive the Laplacian spectrum originally determined in [33]. For $\gamma = \frac{1}{2}$, we obtain the signless Laplacian eigenvalues of $\Gamma(R)$, where R is the von Neumann regular rings and they are given below.

Proposition 2. *Assume $R \cong \mathbb{F}_1 \times \mathbb{F}_2$. The signless Laplacian spectrum of $\Gamma(R)$ comprises the eigenvalues $|\mathbb{F}_2|-1$, $|\mathbb{F}_1|-1$ whose multiplicities are $|\mathbb{F}_1|-2$, and $|\mathbb{F}_2|-2$, respectively. The leftover two A_γ eigenvalues of $\Gamma(R)$ are the eigenvalues given below:*

$$\begin{pmatrix} |\mathbb{F}_2|-1 & |\mathbb{F}_2| \\ |\mathbb{F}_1| & |\mathbb{F}_1|-1 \end{pmatrix}.$$

For $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3$, we obtain the following result.

Proposition 3. *Suppose $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3$. The signless Laplacian spectrum of $\Gamma(R)$ contains the eigenvalues $|\mathbb{F}_2||\mathbb{F}_3|-1$, $|\mathbb{F}_1||\mathbb{F}_3|-1$, $|\mathbb{F}_1||\mathbb{F}_2|-1$, $|\mathbb{F}_3|-1$, $|\mathbb{F}_2|-1$, and $|\mathbb{F}_1|-1$ with multiplicities $|\mathbb{F}_1|-2$, $|\mathbb{F}_2|-2$, $|\mathbb{F}_3|-2$, n_1-1, n_2-1, and n_3-1, respectively. The leftover six A_γ eigenvalues of $\Gamma(R)$ are the eigenvalues given below:*

$$\begin{pmatrix} \gamma_1 & |\mathbb{F}_2|-1 & |\mathbb{F}_3|-1 & 0 & 0 & n_1 \\ |\mathbb{F}_1|-1 & \gamma_2 & |\mathbb{F}_3|-1 & 0 & n_2 & 0 \\ |\mathbb{F}_1|-1 & |\mathbb{F}_2|-1 & \gamma_3 & n_3 & 0 & 0 \\ 0 & 0 & |\mathbb{F}_3|-1 & \gamma_4 & 0 & 0 \\ 0 & |\mathbb{F}_2|-1 & 0 & 0 & \gamma_5 & \\ |\mathbb{F}_1|-1 & 0 & 0 & 0 & 0 & \gamma_6 \end{pmatrix}.$$

4. Conclusions

The present articles studied the A_γ eigenvalues of zero divisor graphs of various commutative rings. Therefore, we derived the adjacency, Laplacian, and the signless

Laplacian eigenvalues of such graphs. The field of theoretical chemistry is significant. We study a large number of articles on spectral graph theory in order to investigate chemical substances. Another useful application for the adjacency matrix is the spectral embedding of graphs in the plane. In machine learning, the eigenvalues of the Laplacian matrix provide the foundation for spectral clustering algorithms. In addition, computer scientists incorporate it into load-balancing algorithms. Algebraic graph theory can be used to build and study the topologies of interconnection networks. The topologies used to integrate processors in a supercomputer are typically Cayley graphs with a high degree of symmetry.

However, some eigenvalues of these graphs remain unknown in terms of the eigenvalues of the quotient matrix, which are hard to find. At large, the A_γ eigenvalues of zero divisors graphs of other commutative rings are yet to be discussed, and extremal characterizations in terms of various spectral invariants are open and may be discussed in future.

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